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On Local Spectral Properties of Extended Hamilton Operators

Wurichaihu Bai[a,c], Alatancang Chen[b,c]

[a] *School of Mathematical Sciences, Inner Mongolia Normal University, Hohhot 010022, P.R.China*
[b] *Inner Mongolia Normal University, Hohhot 010022, P.R.China*
[c] *Inner Mongolia Center for Applied Mathematics, Hohhot 010022, P.R.China*

Abstract. This paper deals with local spectral properties of Extended Hamilton operators and their adjoint operators. The relationship between the local spectral properties (strongly decomposability, hyperinvariant subspace problem, etc.) of Extended Hamilton operators and the corresponding properties of their adjoint operators is obtained.

1. Introduction

The Hamiltonian system is an important branch in dynamical systems, and has various applications in our daily life. While infinite dimensional Hamiltonian operators come from the corresponding infinite dimensional Hamiltonian systems, and have deep mechanical background, their spectral theory is the theoretical foundation of the separation of the variables method solving mechanical problems, and plays a significant role in elasticity mechanics and other related fields[6,9,12].

The various results on infinite dimensional Hamiltonian operators frequently appear. In [2], the authors study the symmetry with respect to imaginary axis of the spectrum of infinite dimensional Hamiltonian operators; in the proof process, some properties between operators and their adjoint operators are applied. In[7], the decomposability, Weyl type theorems and invariant subspace problem of Hamilton operators and the similar properties with their adjoint operators are studied. In[8], extended Hamilton operator is introduced and studied, and various properties of extended Hamilton operators are obtained. In [11], the strongly decomposability, Weyl type theorems and hyperinvariant subspace problem of Hamilton operators and the similar properties with their adjoint operators are given. In this paper, local spectral properties of extended Hamilton operators and their adjoint operators are studied. The relationship between the local spectral properties (strongly decomposability, hyperinvariant subspace problem, etc.) of extended Hamilton operators and the corresponding properties of their adjoint operators is obtained.

This paper is organized as follows. In section 2, we state some definitions and notations. The main results and examples of this paper, together with their proofs, are presented in section 3.

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Email addresses: wurichaihu19@163.com (Wurichaihu Bai), alatanca@imu.edu.cn (Alatancang Chen)

2. Preliminaries

Let X be an infinite dimensional Hilbert space. Throughout this paper, an operator is always bounded. According to [8], extended Hamilton operators and bounded Hamilton operators can be defined as follows.

Definition 2.1. *Let $H : X \times X \longrightarrow X \times X$ be a bounded operator. If $(JH)^* = JH$, then H is called an infinite dimensional Hamilton operator, where $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ with I being the identity operator on X, 0 the zero operator on X, and $(JH)^*$ the adjoint operator of JH.*

Remark 2.2. *Evidently $J^* = -J$.*

Definition 2.3. *A bounded operator $T : X \times X \longrightarrow X \times X$ is called extended Hamilton operator, provided there is an antilinear unitary operator J on $X \times X$ for which $J^2 = \begin{bmatrix} -I & 0 \\ 0 & -I \end{bmatrix}$ and $(JT)^* = JT$. At this time, T is called extended Hamilton operators with J as antilinear unitary operator.*

Definition 2.4. *We say that T satisfies*
(1) property (h) if $\sigma(T) \setminus \sigma_{SF_+^-}(T) = \pi_{00}^a(T)$, where $\pi_{00}^a(T) = \{\lambda \in iso\sigma_a(T) : 0 < \alpha(T-\lambda) < \infty\}$.
(2) property (gh) if $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E^a(T)$, where $E^a(T) = \{\lambda \in iso\sigma_a(T) : 0 < \alpha(T-\lambda)\}$.

Remark 2.5. *The definition of $\sigma_{SF_+^-}(T), \sigma_{SBF_+^-}(T), \alpha(T-\lambda)$ is introduced in [1,3,4].*

Definition 2.6. [5]*A linear subspace Y of X is said to be T- hyperinvariant if $SY \subset Y$ for every bounded linear operator S on X that commutes with T.*

According to [5], the local resolvent set $\rho_T(x)$ of T at point $x \in X$ is defined as the union of all open subset U of $\mathbb{C}$ for which there is an analytic function $f : U \to X$ which satisfies $(T-\lambda)f(\lambda) = x$ for all $\lambda \in U$. The local spectrum $\sigma_T(x)$ of T at x is then defined as $\sigma_T(x) = \mathbb{C} \backslash \rho_T(x)$. The local spectral subspace of T is defined as $X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}$ for all sets $F \subseteq \mathbb{C}$.

Lemma 2.7. [5] *A bounded operator T on X, is strongly decomposable if and only if T is decomposable and $X_T(F) = X_T(F \cap \overline{U}_1) + \cdots + X_T(F \cap \overline{U}_m)$ for every open cover $\{U_1, ..., U_m\}$ of an arbitrary closed set $F \subseteq \mathbb{C}$.*

3. Main results

Lemma 3.1. *Let T be an extended Hamilton operator with J as antilinear unitary operator.Then*
(1) $\sigma_{T^*}(Jx) = -\sigma_T(x)^*, \sigma_T(J^*x)^* = -\sigma_{T^*}(x)$ for all $x \in X$.
(2) $X_{T^*}(F) = JX_T(-F^*)$ for all $F \subseteq \mathbb{C}$.

Proof. (1) Let $\lambda_0 \in \rho_T(x)$, then there exists an analytic function $f : U \to X$ (U is a neighborhood of λ_0)which satisfies $(T-\lambda)f(\lambda) = x$ for every $\lambda \in U$. Hence $(T^*+\bar{\lambda})J^*f(\lambda) = J(T-\lambda)f(\lambda) = Jx$ on U. So $(T^*-\lambda)J^*f(-\bar{\lambda}) = Jx$ on $-U^*$. Since $J^*f(-\bar{\lambda})$ is analytic on $-U^*$, then $-\bar{\lambda}_0 \in \rho_{T^*}(Jx)$. Hence $\sigma_{T^*}(Jx) \subset -\sigma_T(x)^*$. By the similiar way ,we can obtain $\sigma_{T^*}(Jx) \supset -\sigma_T(x)^*$. we can obtain the second equality, by the similiar fashion.

(2) Let $x \in X_{T^*}(F)$, then $\sigma_{T^*}(x) \subseteq F$. It follows from (1) that $\sigma_T(J^*x) \subseteq -F^*$ and so $J^*x \in X_T(-F^*)$. Hence $X_{T^*}(F) \subseteq JX_T(-F^*)$. By the similiar way ,we can obtain $X_{T^*}(F) \supseteq JX_T(-F^*)$. Therefore $X_{T^*}(F) = JX_T(-F^*)$. □

Lemma 3.2. *Let T be an extended Hamilton operator with J as antilinear unitary operator.Then*
(1) $\sigma(T)^* = -\sigma(T^*), \pi_{00}^a(T)^* = -\pi_{00}^a(T^*), E^a(T)^* = -E^a(T^*)$.
(2) $\sigma_{SF_+^-}(T)^* = -\sigma_{SF_+^-}(T^*), \sigma_{SBF_+^-}(T)^* = -\sigma_{SBF_+^-}(T^*)$.

Proof. (1) Since $T^* - \lambda = J(T + \bar{\lambda})J$ and J is bijection, then $\sigma(T)^* = -\sigma(T^*)$, $\sigma_a(T)^* = -\sigma_a(T^*)$ and $\alpha(T + \bar{\lambda}) = \alpha(T^* - \lambda)$. Therefore $\pi^a_{00}(T)^* = -\pi^a_{00}(T^*), E^a(T)^* = -E^a(T^*)$.

(2) If λ is not belong to $\sigma_{SF^-_+}(T^*)$, then $T^* - \lambda$ is upper semi-Weyl operator[1], i.e. $\alpha(T^* - \lambda) < \infty, R(T^* - \lambda)$ is closed and $ind(T^* - \lambda) \leq 0$. Since $T^* - \lambda = J(T + \bar{\lambda})J$, then $\alpha(T + \bar{\lambda}) < \infty, R(T + \bar{\lambda})$ is closed and $ind(T + \bar{\lambda}) \leq 0$, and hence $T + \bar{\lambda}$ is upper semi-Weyl operator, so $-\bar{\lambda}$ is not belong to $\sigma_{SF^-_+}(T)$. i.e. $-\sigma_{SF^-_+}(T)^* \subseteq \sigma_{SF^-_+}(T^*)$.Replacing T by T^* shows that $-\sigma_{SF^-_+}(T)^* \supseteq \sigma_{SF^-_+}(T^*)$. Therefore $\sigma_{SF^-_+}(T)^* = -\sigma_{SF^-_+}(T^*)$.

If λ is not belong to $\sigma_{SBF^-_+}(T^*)$, then $T^* - \lambda$ is upper semi B-Weyl operator[1], i.e. for some $n \geq 0$, $\alpha((T^* - \lambda)_{[n]}) < \infty, R((T^* - \lambda)_{[n]}), R((T^* - \lambda)^n)$ are closed and $ind((T^* - \lambda)_{[n]}) \leq 0$. Since $T^* - \lambda = J(T + \bar{\lambda})J$, then $\alpha((T + \bar{\lambda})_{[n]}) < \infty, R((T + \bar{\lambda})_{[n]}), R((T + \bar{\lambda})^n)$ are closed and $ind((T + \bar{\lambda})_{[n]}) \leq 0$, and hence $T + \bar{\lambda}$ is upper semi B-Weyl operator, so $-\bar{\lambda}$ is not belong to $\sigma_{SBF^-_+}(T)$. i.e. $-\sigma_{SBF^-_+}(T)^* \subseteq \sigma_{SBF^-_+}(T^*)$.Replacing T by T^* shows that $-\sigma_{SBF^-_+}(T)^* \supseteq \sigma_{SBF^-_+}(T^*)$. Therefore $\sigma_{SBF^-_+}(T)^* = -\sigma_{SBF^-_+}(T^*)$. □

In the following theorem we give a duality theorem of strongly decomposable operators. In general, the strongly decomposability of T^* is not transmitted to operator T([10]).

Theorem 3.3. *Let T be an extended Hamilton operator with J as antilinear unitary operator. Then T is strongly decomposable if and only if T^* is strongly decomposable.*

Proof. If T is strongly decomposable, then T is decomposable, by Lemma 2.7. By Theorem 3.4 of [8], it follows that T has property (β) or property (δ). Then we know from Theorem 2.2.5 of [5] that, T^* has property (δ), so T^* is decomposable. Now we consider an arbitrary closed set $F \subseteq \mathbb{C}$ and a finite open cover $\{U_1, ..., U_m\}$ of F. Then $\{-U^*_1, ..., -U^*_m\}$ is a cover of $-F^*$. Given any $x \in X_{T^*}(F)$, we have $-\sigma_T(J^*x)^* \subseteq F$, by Lemma 3.1 and the definition of $X_{T^*}(F)$. Moreover $J^*x \in X_T(-F^*)$. The strong decomposability of T leads to $J^*x \in X_T((-F)^* \cap (-\overline{U}^*_1)) + \cdots + X_T((-F)^* \cap (-\overline{U}^*_m))$, it is immediate that $x \in J(X_T((-F)^* \cap (-\overline{U}^*_1)) + \cdots + X_T((-F)^* \cap (-\overline{U}^*_m))) = X_{T^*}(F \cap \overline{U}_1) + \cdots + X_{T^*}(F \cap \overline{U}_m)$, therefore $X_{T^*}(F) \subseteq X_{T^*}(F \cap \overline{U}_1) + \cdots + X_{T^*}(F \cap \overline{U}_m)$. To show the opposite inclusion, let $x \in X_{T^*}(F \cap \overline{U}_1) + \cdots + X_{T^*}(F \cap \overline{U}_m)$ be arbitrary. Then $J^*x \in J^*X_{T^*}(F \cap \overline{U}_1) + \cdots + J^*X_{T^*}(F \cap \overline{U}_m) = X_T(-(F \cap \overline{U}_1)^*) + \cdots + X_T(-(F \cap \overline{U}_m)^*)$. Moreover $J^*x \in X_T(-F^*)$, so $x \in X_{T^*}(F)$.therefore $X_{T^*}(F \cap \overline{U}_1) + \cdots + X_{T^*}(F \cap \overline{U}_m) \subseteq X_{T^*}(F)$. Thus $X_{T^*}(F \cap \overline{U}_1) + \cdots + X_{T^*}(F \cap \overline{U}_m) = X_{T^*}(F)$. By Lemma 2.7, this establishes the strong decomposability of T^*.

For the reverse implication replace T by T^*. □

Theorem 3.4. *Let T be an extended Hamilton operator with J as antilinear unitary operator. Then λY is T-hyperinvariant if and only if $\bar{\lambda}J^*Y$ is T^*- hyperinvariant, where $\lambda \in \mathbb{C}$.*

Proof. Let S be a bounded linear operator on X and $ST^* = T^*S$, then $JSJ^*T = TJSJ^*$. Since λY is T-hyperinvariant, we know from Definition 2.6 that $JSJ^*\lambda Y \subset \lambda Y$. Therefore $SJ^*\lambda Y \subset J^*\lambda Y$. i.e. $J^*\lambda Y$ is T^*-hyperinvariant.

To see the converse, suppose that $J^*\lambda Y$ is T^*- hyperinvariant. Let S be a bounded linear operator on X and $ST = TS$, then $JSJ^*T^* = T^*JSJ^*$. Since $J^*\lambda Y$ is T^*- hyperinvariant, we know from Definition 2.6 that $JSJ^*J^*\lambda Y \subset J^*\lambda Y$. Therefore $S\lambda Y \subset \lambda Y$. i.e. λY is T- hyperinvariant. □

In the following theorems we give the necessary and sufficient conditions for extended Hamilton operator which satisfies property (h) and (gh).

Theorem 3.5. *Let T be an extended Hamilton operator with J as antilinear unitary operator. Then T satisfies property (h) if and only if T^* satisfies property (h).*

Proof. Let T satisfies property (h), then $\sigma(T) \setminus \sigma_{SF^-_+}(T) = \pi^a_{00}(T)$. Given any $\lambda \in \sigma(T^*) \setminus \sigma_{SF^-_+}(T^*)$, we have $-\bar{\lambda} \in \sigma(T) \setminus \sigma_{SF^-_+}(T)$, by Lemma 3.2 . Since T satisfies property (h), then $-\bar{\lambda} \in \pi^a_{00}(T)$, and hence $\lambda \in \pi^a_{00}(T^*)$, therefore $\sigma(T^*) \setminus \sigma_{SF^-_+}(T^*) \subseteq \pi^a_{00}(T^*)$. To show the opposite inclusion, let $\lambda \in \pi^a_{00}(T^*)$, then $\lambda \in iso\sigma_a(T^*)$ and $0 < \alpha(T^* - \lambda) < \infty$, and therefore $-\bar{\lambda} \in \pi^a_{00}(T)$. Since T satisfies property (h), then $-\bar{\lambda} \in \sigma(T) \setminus \sigma_{SF^-_+}(T)$. We conclude from Lemma 3.2 that $\lambda \in \sigma(T^*) \setminus \sigma_{SF^-_+}(T^*)$. Hence $\pi^a_{00}(T^*) \subseteq \sigma(T^*) \setminus \sigma_{SF^-_+}(T^*)$. So T^* satisfies property (h).

A similar argument shows that T^* satisfies property (h), then T satisfies property (h). □

Theorem 3.6. *Let T be an extended Hamilton operator with J as antilinear unitary operator. Then T satisfies property (gh) if and only if T^* satisfies property (gh).*

Proof. Let T satisfies property (gh), then $\sigma(T) \setminus \sigma_{SBF_+^-}(T) = E^a(T)$. Given any $\lambda \in \sigma(T^*) \setminus \sigma_{SBF_+^-}(T^*)$, we have $-\bar{\lambda} \in \sigma(T) \setminus \sigma_{SBF_+^-}(T)$, by Lemma 3.2. Since T satisfies property (gh), then $-\bar{\lambda} \in E^a(T)$, and hence $\lambda \in E^a(T^*)$, therefore $\sigma(T^*) \setminus \sigma_{SBF_+^-}(T^*) \subseteq E^a(T^*)$. To show the opposite inclusion, let $\lambda \in E^a(T^*)$, then $\lambda \in iso\sigma_a(T^*)$ and $0 < \alpha(T^* - \lambda)$, and therefore $-\bar{\lambda} \in E^a(T)$. Since T satisfies property (gh), then $-\bar{\lambda} \in \sigma(T) \setminus \sigma_{SBF_+^-}(T)$. We conclude from Lemma 3.2 that $\lambda \in \sigma(T^*) \setminus \sigma_{SBF_+^-}(T^*)$. Hence $E^a(T^*) \subseteq \sigma(T^*) \setminus \sigma_{SBF_+^-}(T^*)$. So T^* satisfies property (gh).

A similar argument shows that T^* satisfies property (gh), then T satisfies property (gh). □

Remark 3.7. *In general, the results of Theorem 3.5 and 3.6 do not hold if we replace extended Hamilton operator by bounded operator.*

Example 3.8. *Let T be defined for each* $x = (x_i) \in \ell^2$ *by* $T(x_1, x_2, x_3, \cdots, x_n, \cdots) = (\frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, \cdots, \frac{1}{n}x_n, \cdots)$. *Then* $\sigma(T) = \sigma_a(T^*) = \sigma_w(T) = \sigma_{SF_+^-}(T^*) = \pi_{00}(T) = \{0\}$, $\pi_{00}^a(T^*) = \emptyset$. *Hence* $\sigma(T) \setminus \sigma_w(T) = \emptyset \neq \{0\} = \pi_{00}(T)$, *i.e. T does not satisfy Weyl's theorem. Then T does not obey a-Weyl's theorem. Hence T does not satisfy property (h), and therefore does not satisfy property (gh). But* $\sigma_a(T^*) \setminus \sigma_{SF_+^-}(T^*) = \pi_{00}^a(T^*) = \emptyset$, *i.e.* T^* *satisfies a-Weyl's theorem. Then* T^* *obeys property (h) and property (gh).*

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