Filomat 36:12 (2022), 4069–4086
https://doi.org/10.2298/FIL2212069S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Note on Pointwise Semi-Slant Submanifold of Para-Cosymplectic Manifolds

S. K. Srivastava[a], M. Dhiman[a], K. Sood[a], A. Kumar[a], F. Mofarreh[b], A. Ali[c]

[a] *Srinivasa Ramanujan Department of Mathematics,*
Central University of Himachal Pradesh, Dharamshala-176215, Himachal Pradesh, INDIA.
[b] *Mathematical Science Department, Faculty of Science Princess Nourah bint Abdulrahman University. Riyadh 11546 Saudi Arabia.*
[c] *Department of Mathematics, College of Science, King Khalid University, 9004 Abha,*
Saudi Arabia.

Abstract. This article covers the geometric study of pointwise slant and pointwise semi-slant submanifolds of a para-Cosymplectic manifold $\overline{M}^{2m+1}$ with the semi-Riemannian metric. We give an advanced definition of these type of submanifolds for the spacelike and timelike vector fields. We obtain the characterization results for the involutive and totally geodesic foliation for such type of manifold $\overline{M}^{2m+1}$.

1. Introduction

Analogous to the contact structure, the geometry of paracontact Riemannian structure has been vigorously studied by several researchers since 1976, when it was introduced by I. Sato[13]. Since the Riemannian geometric approach may not found suitable for the theory of spacetime and black holes where the metric may not be Riemannian. Thus, the study of paracontact structure with semi-Riemannian metric became a topic of investigation.

Takahashi [28] was the first who studied contact structure endowed with semi-Riemannian metric as the direct generalization for contact Riemannian metric structure. In addition, B. Y. Chen in [6] generalizes complex and totally real submanifolds such as the advance class for submanifolds named *slant submanifolds* at an almost Hermitian manifolds. He introduced the slant submanifolds as the submanifolds possessing the constant Wirtinger angle θ (i.e., the angle between the φX_1 and the tangent space of submanifold) for every vector field X_1. Many researchers forwarded this concept to different manifolds and structures with Riemannian as well as semi-Riemannian setting. For example, Chen and Mihai defined it for Lorentzian complex space forms[8], Alegre in [21] studied the same submanifolds for Lorentzian and Lorentzian para-Sasakian manifolds.
But later he analyzed and found some difficulties in defining the slant submanifolds for semi-Riemannian manifolds. In [22] authors have given a more generalized and improved definition of slant submanifolds for the para-Hermitian manifolds where the metric is semi-Riemannian. The authors in [22] defined the

---

2020 *Mathematics Subject Classification*. Primary: 53C12, 53C25, 53D15; Secondary: 53B25, 53B30
Keywords. Pointwise semi-slant submanifold, Pointwise slant submanifold, para-Cosymplectic manifold
Received: 09 March 2021; Revised: 23 June 2022; Accepted: 24 August 2022
Communicated by Dragan S. Djordjević
Email addresses: sachin@cuhimachal.ac.in (S. K. Srivastava), memudhiman@gmail.com (M. Dhiman), soodkanika1212@gmail.com (K. Sood), kumaranuj9319@gmail.com (A. Kumar), fyalmofarrah@pnu.edu.sa (F. Mofarreh), akramali133@gmail.com (A. Ali)

submanifold M to be slant submanifold in case of all spacelike or timelike tangent vector field X_1, $\frac{g(PX_1,PX_1)}{g(\varphi X_1,\varphi X_1)}$ is constant. They have taken a two-dimensional case as an example to distinguish the slant submanifolds of three different types (later named type 1, type 2 and type 3 slant submanifolds for the generalized case). Recently, the same study has been done on an almost paracontact semi-Riemannian manifold by S. K. Chanyal [25]. Papaghuic, Neculai introduced the notion of semi-slant submanifolds of a Kaehlerian manifold [20] and Cabrerizo with co-authors did the same in an almost contact environment [15]. P. Alegre and A. Carriazo also studied semi-slant submanifolds of para-Kaehler manifolds in the paper [23].
Further, Chen-Garay [9] generalizes the concept of slant submanifolds to pointwise slant submanifolds of an almost Hermitian and Kaehler manifolds. Such submanifolds were earlier studied by Etayo [11] with the name quasi-slant submanifold in almost Hermitian manifolds. Since then many differential geometers have studied the theory of pointwise slant submanifolds in different ambient manifolds [17, 19]. Recently, the authors in [26] extended the theory of pointwise slant submanifolds to pointwise semi-slant submanifolds. Because of its numerous applications to mathematical physics, several researcher found interest and studied these concepts in different settings [1, 5]. Pointwise semi-slant submanifolds for Kaehler manifold introduced by B. Sahin in [3] and for contact manifolds it is studied by K. S Park [17]. Motivated by these works and by considering the slant submanifolds defined in [22], we present the theory of pointwise slant submanifold and pointwise semi-slant submanifolds for the semi-Riemannian structure which can be taken as the generalization case for slant, semi-invariant, semi-slant submanifolds.
Sectional study of this paper includes: At Sect.[2], many basics of paracontact metric manifold, para-Cosymplectic manifold, geometry of submanifolds are recalled and some characterizations for such submanifolds are derived. Sect.[3] includes the definition of pointwise slant submanifold along with example and some related theorems showing the slant conditions. In Sect.[4], we first give the definition of pointwise slant distributions and derived some results for these type of distributions in para-Cosymplectic manifold. Finally in Sect. [5], we defined the pointwise semi-slant submanifolds in addition to derived the totally geodesic and involutive conditions for the involved distributions.

2. Preliminaries

Definition 2.1. *An almost paracontact manifold is an odd-dimensional smooth manifold,* $\overline{M}^{2m+1}$*, furnished with a structure* (φ, ξ, η)*, where* φ *is a* $(1,1)$*-tensor field,* ξ *is called characteristic vector field and* η *is a globally defined* 1*-form on* $\overline{M}^{2m+1}$ *satisfying:*

$$\varphi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{1}$$

where I *denotes an identity transformation of tangent space of* $\overline{M}$ *and* $\otimes$ *denotes tensor product. The Eqs.*(1) *leads to follow the given conditions*

$$\eta \circ \varphi = 0, \quad \varphi\xi = 0 \quad \text{and} \quad rank(\varphi) = 2m. \tag{2}$$

A semi-Riemannian metric of type $(0,2)$, g, with signature $(n+1, n)$ is called compatible with the structure (φ, ξ, η) if following condition holds

$$g(\cdot,\cdot) = -g(\varphi\cdot,\varphi\cdot) + \eta(\cdot)\,\eta(\cdot). \tag{3}$$

Also,

$$g(\cdot,\xi) = \eta(\cdot). \tag{4}$$

Therefore, the structure (φ, ξ, η, g) named *an almost paracontact semi-Riemannian structure* as well as the manifold $\overline{M}^{2m+1}$ together with this structure named the *almost paracontact semi-Riemannian manifold* $\overline{M}(\varphi, \xi, \eta, g)$ [18, 27]. In light of Eqs. (1) to (3), it is clear that

$$g(\varphi\cdot,\cdot) + g(\cdot,\varphi\cdot) = 0. \tag{5}$$

In addition to the above properties, an almost paracontact semi-Riemannian structure also holds

$$d\eta(X_1, X_2) = g(X_1, \varphi X_2), \tag{6}$$

for all vector fields X_1, X_2 at $\overline{M}^{2m+1}$. The almost paracontact semi-Riemannian manifold turns to the *paracontact semi-Riemannian manifold* if the fundamental 2-form Φ on $\overline{M}^{2m+1}$ satisfies $d\eta = \Phi$. Moreover, we have that

$$\left(\overline{\nabla}_{X_3}\Phi\right)(X_1, X_2) = g\left(\left(\overline{\nabla}_{X_3}\varphi\right)X_1, X_2\right) = \left(\overline{\nabla}_{X_3}\Phi\right)(X_2, X_1), \tag{7}$$

for any vector field X_3 and Levi-Civita connection $\overline{\nabla}$ on $\overline{M}^{2m+1}$.

Normality. A *normal almost paracontact manifold* is one on which the Nijenhuis tensor becomes zero identically. Equivalently, satisfies the following condition

$$2d\eta \otimes \xi + [\varphi, \varphi] = 0.$$

Basis. Considering an almost paracontact semi-Riemannian manifold, it always appears the $\varphi -$ *basis* which is the specific type of the local pseudo-orthonormal basis $\{E_i, E_i^\star, \xi\}$; such that E_i, ξ define space-like vector fields as well as $E_i^\star = \varphi E_i$ define timelike vector fields.

Definition 2.2. *An almost paracontact metric manifold* $\overline{M}^{2m+1}$ *named as:*

(i) an almost para-Cosymplectic submanifold, if the forms η *as well as* Φ *are closed on* $\overline{M}^{2m+1}$,

$$d\eta = 0 \quad \text{and} \quad d\Phi = 0. \tag{8}$$

(ii) para-Cosymplectic submanifold, if the forms η *as well as* Φ *are parallel respecting Levi-Civita connection* $\overline{\nabla}$ *at* $\overline{M}^{2m+1}$,

$$\overline{\nabla}\eta = 0 \quad \text{and} \quad \overline{\nabla}\Phi = 0. \tag{9}$$

Next result follows directly with the use of above definition, Eq. (2) and covariant differentiation formula.

Lemma 2.3. *If the structure vector field* $\xi \in \Gamma(T\overline{M}^{2m+1})$, *then para-Cosymplectic manifold* $\overline{M}^{2m+1}$ *satisfies:*

$$\overline{\nabla}_{X_1}\xi = 0, \tag{10}$$

for any $X_1 \in \Gamma(T\overline{M}^{2m+1})$.

2.1. Geometry of submanifold

Suppose M is the real submanifold which is immersed isometrically in para-Cosymplectic manifold $\overline{M}^{2m+1}$ with an induced non-degenerate metric g (denoted metric by same symbol as on $\overline{M}^{2m+1}$). Denoting $\Gamma(TM)$ and $\Gamma(TM^\perp)$ as the sections for tangent bundle TM and the set of normal vector fields for M in the same order. Thus, for every $X_1, X_2 \in \Gamma(TM)$ having $\zeta \in \Gamma(TM^\perp)$, the Gauss and Weingarten formulas can be given by

$$\overline{\nabla}_{X_1}X_2 = \nabla_{X_1}X_2 + h(X_1, X_2), \tag{11}$$

$$\overline{\nabla}_{X_1}\zeta = -A_\zeta X_1 + \nabla^\perp_{X_1}\zeta, \tag{12}$$

where ∇ is the Levi-Civita connection induced at M, $\nabla^\perp$ defines normal connection at normal bundle $\Gamma(TM^\perp)$, h defines second fundamental form as well A_ζ is the shape operator related to the normal section ζ. The metric relation of A_ζ and h is given by

$$g(A_\zeta X_1, X_2) = g(h(X_1, X_2), \zeta). \tag{13}$$

For every $X_1 \in \Gamma(TM)$ as well $\zeta \in \Gamma(TM^{\perp})$, we decompose

$$\varphi X_1 = tX_1 + nX_1, \tag{14}$$
$$\varphi\zeta = t'\zeta + n'\zeta, \tag{15}$$

where tX_1 in addition to $t'\zeta$ (nX_1 and $n'\zeta$) are the tangential part (normal part) for φX_1 and $\varphi\zeta$ respectively. Based on Eq. (14), the submanifold M is classified as *invariant* (*anti-invariant*) if n is identically zero (t is identically zero) on M. After using Eq. (14) in Eq. (5) for all $X_1 \in \Gamma(TM)$, we get

$$g(X_1, tX_2) = -g(tX_1, X_2). \tag{16}$$

In view of (5), (14) and (15), it is obtained that

$$g(n'\zeta_1, \zeta_2) + g(\zeta_1, n'\zeta_2) = 0, \; g(t'\zeta_1, X_1) + g(\zeta_1, nX_1) = 0, \tag{17}$$

for all $X_1 \in \Gamma(TM)$ and $\zeta_1, \zeta_2 \in \Gamma(TM^{\perp})$.

Lemma 2.4. *Let M be an isometrically immersed submanifold in $\overline{M}^{2m+1}$ having the structure vector field $\xi \in \Gamma(TM)$. Then*

$$\nabla_{X_1}\xi = \nabla_{\xi}X_1 = \nabla_{\xi}\xi = 0 \text{ and } h(X_1, \xi) = 0,$$
$$A_{\zeta}\xi = 0 \text{ and } A_{\zeta}X_1 \perp \xi,$$

for all $X_1 \in \Gamma(TM)$ and $\zeta \in \Gamma(TM^{\perp})$.

3. Pointwise slant submanifolds

In [22], the authors stated that the Wirtinger angle has no meaning in semi-Riemannian manifold where the vector fields are lightlike and defined slant submanifolds for the para -Hermitian case. On the parallel lines, S.K. Chanyal [25] defined slant submanifolds for para contact metric manifold. Thus, motivating with the concept[22], we generalize the slant submanifolds to pointwise slant submanifolds in our ambient semi-Riemannian manifold.

Definition 3.1. *An isometrically immersed submanifold M for the almost paracontact manifold $\overline{M}^{2m+1}$ named pointwise slant if for every point $p \in M$, the quotient $\frac{g(tX_1, tX_1)}{g(\varphi X_1, \varphi X_1)} = \lambda(p)$ is independent of every non-zero selection for spacelike or timelike vector $X_1 \in M_p$, where $M_p = \{X_1 \in T_pM : g(X_1, \xi) = 0\}$ and we call $\lambda(p)$ a slant coefficient which depends on the slant function $\theta(p) : M \to [0, \infty)$.*

Remark 3.2. *The value of $\lambda(p)$ can be derived*

(i) $\lambda(p) = \cosh^2\theta(p) \in [1, \infty)$ *for* $\frac{|tX_1|}{|\varphi X_1|} \geq 1$, *$tX_1$ is timelike or spacelike of each spacelike or timelike vector field X_1 in addition to $\theta(p) > 0$.*

(ii) $\lambda(p) = \cos^2\theta(p) \in [0, 1]$ *for* $\frac{|tX_1|}{|\varphi X_1|} \leq 1$, *$tX_1$ is timelike or spacelike of each spacelike or timelike vector field X_1 in addition to $0 \leq \theta(p) \leq 2\pi$.*

(iii) $\lambda(p) = -\sinh^2\theta(p) \in (-\infty, 0]$ *for tX_1 is timelike or spacelike for any timelike or spacelike vector field X_1 and for a slant function $\theta(p) > 0$.*

Remark 3.3. *Some particular cases:*

- *If $\lambda(p)$ is constant throughout M for a constant slant function $\theta(p)$ then a submanifold M is slant [6, 22].*
- *A point $p \in M$ is defined a totally real point if $t \equiv 0$ or equivalently $\lambda(p) = 0$.*
- *The point $p \in M$ is defined the complex point if $t \equiv \varphi$ or equivalently $\lambda(p) = 1$.*

Thus, it is clear that on the pointwise slant submanifold, totally real point makes the slant coefficient equals 0 and complex point makes the slant coefficient to 1.
Unambigously, totally real submanifold is one whose every point is totally real point and complex submanifold is one whose every point is complex point. If the pointwise slant submanifold is none of the above two, then the submanifold named proper pointwise slant submanifold.

Furthermore, we have the union of all tangent vectors in M_p and denoting as the following

$$T^{*}M = \bigcup_{p\in M}\left\{X_1 \in M_p \mid g\left(X_1, \xi\right) = 0\right\}. \tag{18}$$

Now, we mention the following useful characterization of pointwise slant submanifolds of para-Cosymplectic manifold $\overline{M}^{2m+1}$.

Lemma 3.4. *A submanifold M of $\overline{M}^{2m+1}$ defines the pointwise slant submanifold if and only if for all points $p \in M$ and for all spacelike or timelike vector field $X_1 \in M_p$, there exists real valued function $\lambda(p)$ such that $t^2X_1 = \lambda\left(p\right)\left(X_1 - \eta\left(X_1\right)\xi\right)$.*

Proof. Consider M is a pointwise slant submanifold of $\overline{M}^{2m+1}$. From definition, for every $p \in M$ and $X_1 \in T_pM$, we have

$$g\left(tX_1, tX_1\right) = \lambda\left(p\right) g\left(\varphi X_1, \varphi X_1\right). \tag{19}$$

With the use of Eqs. (5), (16) and the condition that $X_1 \in T_pM$ in Eq. (19), we get the desired result. □

Thus, one can see that in accordance with the definition, we have the simpler form of result with respect to para-Hermitian manifold.

Remark 3.5. *Proceeding further with some results which are not hard to prove that any proper pointwise slant submanifold M of $\overline{M}^{2m+1}$ satisfies*

(i) $g\left(tX_1, tX_2\right) = \lambda\left(p\right) g\left(\varphi X_1, \varphi X_2\right) = -\lambda\left(p\right) g\left(X_1, X_2\right)$,
(ii) $g\left(nX_1, nX_2\right) = \left(1 - \lambda\left(p\right)\right) g\left(\varphi X_1, \varphi X_2\right) = -\left(1 - \lambda\left(p\right)\right) g\left(X_1, X_2\right)$,
(iii) $\left(\nabla_{X_1} t\right) X_2 = A_{nX_2}X_1 + t'h\left(X_1, X_2\right)$,
(iv) $\left(\nabla_{X_1} n\right) X_2 = -h\left(X_1, tX_2\right) + n'h\left(X_1, X_2\right)$,

for $X_1, X_2 \in T^{*}M$.

Example 3.6. *Considering $\overline{M} = \mathbb{R}^4\times\mathbb{R}_+ \subset \mathbb{R}^5$ to be a 5-dimensional manifold having standard Cartesian coordinates (x_1, x_2, y_1, y_2, s). Define the structure (φ, ξ, η, g) by*

$$\begin{cases} \varphi e_1 = e_3,\ \varphi e_2 = e_4,\ \varphi e_3 = e_1,\ \varphi e_4 = e_2, \varphi e_5 = 0,\ \xi = e_5,\ \eta = ds, \\ g(e_1, e_1) = g(e_2, e_2) = g(e_5, e_5) = -g(e_3, e_3) = -g(e_4, e_4) = 1 \text{ and} \\ g(e_i, e_j) = 0 \text{ for } i \neq j. \end{cases} \tag{20}$$

where $\{e_1, e_2, e_3, e_4, e_5\}$ is the local orthonormal basis frame for the TM and $e_i = \frac{\partial}{\partial x_i}$ for $i = \{1, 2\}$, $e_i = \frac{\partial}{\partial y_i}$ for $i = \{3, 4\}$ and $e_5 = \frac{\partial}{\partial s}$. With straightforward calculations it is easy to see that $\overline{M}\left(\varphi, \xi, \eta, g\right)$ is a para-Cosymplectic manifold. Let an isometrically immersed submanifold M with semi-Riemannian metric defined by

$$M\left(u, v, s\right) = \left(u^2, v, v, u + v, s\right)$$

where u, v, s are real valued functions such that $u \neq -\frac{1}{2}$ then M defines the pointwise slant submanifold with slant coefficient $\left(\frac{2u-1}{2u+1}\right)$.

Proposition 3.7. *A submanifold M of $\overline{M}^{2m+1}$ defines as a pointwise slant submanifold with slant coefficient $\lambda(p)$ if and only if*

(i) $t'nX_1 = (1-\lambda(p))X_1$ *and* $n\,tX_1 = -n'nX_1$ *for non-lightlike tangent vector field* X_1 *on* M*.*

(ii) $(n')^2 V = \lambda(p)V$ *for non-lightlike normal vector field* V*.*

Proof. Assume M is the pointwise slant submanifold of $\overline{M}^{2m+1}$. Then for every $p \in M$ and $X_1 \in T^*M$, we have $\varphi^2 X_1 = X_1$. Therefore, $\varphi X_1 = tX_1 + nX_1$ implies that

$$X_1 = t^2X_1 + n\,tX_1 + t'\,nX_1 + n'\,nX_1.$$

Equalizing tangential and normal parts and using Lemma 3.4, we can abtain the result (*i*).
Since $V \in \Gamma(TM^\perp)$, thus there exists $X_1 \in \Gamma(T^*M)$ as M is a pointwise slant submanifold such that $nX_1 = V$.

Now, $(n')^2 V = n'\,n'\,nX_1 = -n'\,n\,tX_1 = n\,t^2X_1 = \lambda(p)V$. The converse can be easily derived by using same equations. The proof of (*ii*) is completed. □

Considering Remark 3.2, a natural question arises that under which geometric condition, the pointwise slant submanifold can be a slant submanifold? To find the answer, we give the result as follows:

Theorem 3.8. *If a connected pointwise slant submanifold M of $\overline{M}^{2m+1}$ is totally geodesic, then M is slant submanifold.*

Proof. A smooth curve γ joining points $p, q \in M$ because of M is connected. Let $\beta(s)$ be the parallel transport of a vector $X_1 \in T_pM$ to a non-zero vector $X_2 \in T_qM$. Then using the condition that M is totally geodesic that gives $h(\gamma', \beta(s)) = 0$. Thus

$$\overline{\nabla}_{\gamma'}\beta(s) = \nabla_{\gamma'}\beta(s) = 0. \tag{21}$$

Using Lemma 2.3 in above equation and the fact that $X_1 \in T_pM$ which means $g(X_1, \xi) = 0$, this leads to following

$$g(\beta(s), \xi) = \text{constant},$$

which implies

$$g(X_2, \xi) = 0 \Rightarrow X_2 \in T_qM.$$

As φ is parallel in $\overline{M}^{2m+1}$ so with the use of covariant differentiation, we find that $\varphi(\beta(s))$ is a parallel transport along the curve γ in $\overline{M}^{2m+1}$ with

$$\varphi(\beta(0)) = \varphi X_1 \quad \text{and} \quad \varphi(\beta(1)) = \varphi X_2.$$

By defining a map $\mathsf{F} : T_p\overline{M}^{2m+1} \to T_q\overline{M}^{2m+1}$ such that $\mathsf{F}(X_3) = X_4$ for $X_3 \in T_p\overline{M}^{2m+1}$ and $X_4 \in T_q\overline{M}^{2m+1}$, and taking parallel transport α of a vector X_3 to a vector X_4, we get that F is isometry and

$$\mathsf{F}\left(T_pM\right) = T_qM \quad \text{and} \quad \mathsf{F}\left(T_pM^\perp\right) = T_pM^\perp.$$

This implies

$$\mathsf{F}(\varphi X_1) = \varphi X_2 \Rightarrow \mathsf{F}(tX_1) = tX_2.$$

Hence

$$\lambda(p) = \frac{\| tX_1 \|^2}{\| \varphi X_1 \|^2} = \frac{\| tX_2 \|^2}{\| \varphi X_2 \|^2} = \lambda(q).$$

Therefore, M is a slant submanifold. □

Next, we give some properties of pointwise slant submanifold as follows:

Theorem 3.9. *The pointwise slant submanifold M of $\overline{M}^{2m+1}$ satisfies*

$$tX_1 = \sqrt{-\lambda(p)}X_1^*,$$

where $X_1 \in T^*M$ and X_1^* is the orthogonal unit vector field, both unitary and $\lambda(p)$ as a slant coefficient of M.

Proof. From definition, it well known that for all non-zero spacelike or timelike unit vector field $X_1 \in T^*M$, we have

$$|tX_1| = \sqrt{-\lambda(p)}|\varphi X_1| = \sqrt{-\lambda(p)}|X_1| = \sqrt{-\lambda(p)}.$$

Now, as in the same direction of tX_1, we have a unit vector field $X_1^* = \frac{tX_1}{|tX_1|}$, then $tX_1 = \sqrt{-\lambda(p)}X_1^*$. Also, since

$$g(\varphi X_1, X_1) = 0 \Rightarrow g(tX_1, X_1) = 0,$$

which means that

$$g(tX_1, X_1) = g\left(X_1^*|tX_1|, X_1\right) = |tX_1|g\left(X_1^*, X_1\right) = 0 \Rightarrow g\left(X_1^*, X_1\right) = 0.$$

Thus, X_1^* and X_1 are orthogonal to each other. The proof is completed. □

The following theorem as similar way can be achieved as the last theorem.

Theorem 3.10. *A pointwise slant submanifold M of $\overline{M}^{2m+1}$ satisfies*

$$nX_1 = \sqrt{(\lambda(p) - 1)}X_1^*,$$

where $X_1 \in T^*M$ and X_1^* is the orthogonal unit vector field, both unitary and $\lambda(p)$ as a slant coefficient of M.

Proposition 3.11. *A pointwise slant submanifold M of $\overline{M}^{2m+1}$ is slant if and only if the shape operator A insures the following equality*

$$A_{nX_1}tX_1 = A_{n\,tX_1}X_1,$$

for $X_1 \in \Gamma(TM)$.

Proof. Considering Eqs. (11), (14) as well as (15) with the following

$$\overline{\nabla}_{X_2}\varphi X_1 = \varphi\overline{\nabla}_{X_2}X_1,$$

for any $X_1, X_2 \in \Gamma(TM)$. We left with

$$\varphi\overline{\nabla}_{X_2}X_1 = t\nabla_{X_2}X_1 + t'h(X_1, X_2) + n'h(X_2, X_1) + n\nabla_{X_2}X_1. \tag{22}$$

On the other side, using Eq. (14) and Theorem 3.9, we have

$$\overline{\nabla}_{X_2}\varphi X_1 = \sqrt{-\lambda(p)}\,\overline{\nabla}_{X_2}\varphi X_1^* + \sqrt{-\lambda(p)}\,h\left(X_2, X_1^*\right) + \lambda'(p)(X_2\theta)X_1^* - A_nX_1 + \nabla^{\perp}_{X_2}nX_1,$$

where $\lambda'(p)$ is the first derivative of $\lambda(p)$. Using the comparison of tangential parts, then taking the inner product with X_1^* and again using theorem 3.9 gives the desired result. □

Definition 3.12. *A totally umbilical submanifold M of $\overline{M}^{2m+1}$ satisfies the following equality between the second fundamental form h and the mean curvature H*

$$h(X_1, X_2) = g(X_1, X_2) H, \tag{23}$$

for $X_1, X_2 \in \Gamma(TM)$.

As consequences of Proposition 3.11 and Eq. (23), the next result can be proved:

Theorem 3.13. *Any non-totally geodesic and totally umbilical proper pointwise slant submanifold M of $\overline{M}^{2m+1}$ is non-slant.*

Proof. If M is a slant then we have necessary and sufficient condition

$$g(A_{nX_1} tX_1, X_2) - g(A_{ntX_1} X_1, X_2) = 0, \tag{24}$$

for any non-null $X_1, X_2 \in \Gamma(TM)$. In the above equation, using Eq. (13) and (23) along with M is totally geodesic, the condition arrived that $tX_1 = 0$ which means $t \equiv 0$, this leads to contradicting the assumption of proper pointwise slant submanifold. □

Now, t mentioned in equation (14) is an endomorphism and we put $t^2 = Q$, which is a self-adjoint endomorphism on tangent bundle of M, T^*M based on which we have our next result.

Proposition 3.14. *If M is a pointwise slant submanifold of $\overline{M}^{2m+1}$, then*

$$\left(\nabla_{X_2} t^2\right) X_1 = (\nabla_{X_2} Q) X_1 = \lambda'(p)(X_2\theta) X_1,$$

for any non-lightlike vector field $X_1, X_2 \in T^*M$ and where $\lambda'(p)$ is the first derivative of $\lambda(p)$ and θ is a slant function on M.

Proof. From Lemma 3.4, we have

$$t^2 X_1 = QX_1 = \lambda(p) X_1, \tag{25}$$

for $X_1 \in T^*M$. Consequently, for any $X_2 \in T^*M$

$$t^2(\nabla_{X_2} X_1) = Q(\nabla_{X_2} X_1) = \lambda(p)(\nabla_{X_2} X_1). \tag{26}$$

Also,

$$\nabla_{X_2}(QX_1) = \nabla(\lambda(p) X_1) = \lambda(p) \nabla_{X_2} X_1 + \lambda'(p)(X_2\theta) X_1. \tag{27}$$

Subtracting Eq. (26) from Eq. (27), we left with the result

$$(\nabla_{X_2} Q) X_1 = \lambda'(p)(X_2\theta) X_1. \tag{28}$$

This is required proof. □

From the above result, it is clear that for a slant submanifold $(\nabla_{X_2} Q) X_1$ equals to zero. Now, for $\xi \in \Gamma(TM)$ one can choose a set of orthonormal basis
$\{e_1, e_2 \cdots e_k, e_1', \cdots e_k', \xi\}$ $(k \leq m)$ of T_pM at any given point p for a pointwise slant submanifold M such that

$$e_i' = \frac{1}{\sqrt{-\lambda(p)}} te_i, \tag{29}$$

for $i \in \{1, \cdots, m\}$ and $\lambda(p) \neq 0$. For the 3-dimensional proper pointwise slant submanifold, the above Proposition becomes as follows:

Proposition 3.15. *Assuming that M is a 3-dimensional proper pointwise slant submanifold of $\overline{M}^{2m+1}$ with $\xi \in \Gamma(TM)$ then*

$$(\nabla_{X_2} t) X_1 = \lambda'(p)(X_2\theta)[g(X_1, e_1) e_2 - g(X_1, e_2) e_1],$$

for $X_1, X_2 \in \Gamma(TM)$, $\{e_1, e_2, \xi\}$ is basis for M and $\lambda'(p)$ is the first derivative of slant coefficient $\lambda(p)$ and θ is slant function on M.

Proof. For 3-dimensional case with $\xi \in \Gamma(TM)$, we have $\{e_1, e_2, \xi\}$ as the orthonormal frame for T_pM for any given point $p \in M$. Then

$$\nabla_{X_2} e_i = \Sigma_{i=1}^{j} \alpha_i^j (X_2) e_j, \tag{30}$$

where α_i^j are the associated structural 1-forms and $X_2 \in T_pM$. Using Lemma 2.3, we have

$$(\nabla_{X_2} t)\xi = \nabla_{X_2} t\xi + t(\nabla_{X_2}\xi) = 0.$$

In the similar way along with the use of the properties that $\alpha_i^j = 0$ for $i = j$ and $\alpha_i^j = -\alpha_j^i$, we get

$$(\nabla_{X_2} t) e_1 = \nabla_{X_2} te_1 + t(\nabla_{X_2} e_1) = \lambda'(p)(X_2\theta) e_2,$$
$$(\nabla_{X_2} t) e_2 = \nabla_{X_2} te_2 + t(\nabla_{X_2} e_2) = \lambda'(p)(X_2\theta) e_1.$$

Now, for any non-zero $X_1 \in T_pM$, we have

$$X_1 = g(X_1, e_1) e_1 + g(X_1, e_2) e_2 + \eta(X_1)\xi, \tag{31}$$

which implies

$$(\nabla_{X_2} t) X_1 = g(X_1, e_1)(\nabla_{X_2} t) e_1 + g(X_1, e_2)(\nabla_{X_2} t) e_2 + \eta(X_1)(\nabla_{X_2} t)\xi.$$

Using above results we get,

$$(\nabla_{X_2} t) X_1 = \lambda'(p)(X_2\theta)[g(X_1, e_1) e_2 - g(X_1, e_2) e_1].$$

The proof is completed. □

Next, we provide result related to the particular case of pointwise slant submanifold which is stated as:

Theorem 3.16. *For a pointwise slant submanifold M of $\overline{M}^{2m+1}$, the next two statements are equivalent:*

(i) M defines a slant submanifold.

(ii) $\left(\nabla_{X_2} t^2\right) X_1 = (\nabla_{X_2} Q) X_1 = 0$, for all non-lightlike vector fields $X_1, X_2 \in T^*M$.

Proof. (ii) $\Longrightarrow$ (i): Clear from the Proposition 3.14. On the other hand, (i) $\Longrightarrow$ (ii): let M is the slant submanifold i.e θ is constant. We have for all non-lightlike vector field $X_1 \in \Gamma(T^*M)$

$$g(tX_1, tX_1) = \lambda(p) g(\varphi X_1, \varphi X_1).$$

Using equations (14), (16) and covariant differentiation respecting to X_2, the above equation gives

$$g\left(\nabla_{X_2} t^2 X_1, X_1\right) + g\left(t^2 X_1, \nabla_{X_2} X_1\right) = \lambda(p)\left\{g\left(\nabla_{X_2}\varphi^2 X_1, X_1\right) + g\left(\varphi^2 X_1, \nabla_{X_2} X_1\right)\right\}. \tag{32}$$

Also,

$$g\left(t^2 \nabla_{X_2} X_1, X_1\right) = \lambda(p) g\left(\varphi^2 \nabla_{X_2} X_1, X_1\right). \tag{33}$$

After subtracting the above two equations and using Eq. (9), we arrive at

$$\begin{aligned} g\left(\left(\nabla_{X_2} t^2\right) X_1, X_1\right) =& \lambda(p) g\left(\varphi^2 X_1, \nabla_{X_2} X_1\right) - g\left(t^2 X_1, \nabla_{X_2} X_1\right) \\ =& \lambda(p)\left\{g(X_1, \nabla_{X_2} X_1) - g(X_1, \nabla_{X_2} X_1)\right\}, \end{aligned} \tag{34}$$

which directly implies the result. □

Now proceeding as [25], consider a para-Hermitian manifold $\left(\overline{M}, J, g\right)$ with structure J. An almost paracontact structure (φ, ξ, η, g) on a product manifold $\overline{M} \times \mathbb{R}$ is given by

$$\varphi\left(X_1, s\frac{d}{du}\right) = (JX_1, 0), \quad \xi = \left(0, \frac{d}{du}\right), \quad \eta = du, \tag{35}$$

where u is coordinate on $\mathbb{R}$. Let (M, f) be an immersed submanifold of $\overline{M}$ with immersion f and denote $M_0 = (M, f_0)$ and $M_1 = (M \times \mathbb{R}, f_1)$ as an immersed submanifolds of $\overline{M} \times \mathbb{R}$ with immersions

$$f_0 : M \to \overline{M} \times \mathbb{R} \quad \text{such that } f_0(p) = (f(p), 0),$$
$$f_1 : M \times \mathbb{R} \to \overline{M} \times \mathbb{R} \quad \text{such that } f_1(p, u) = (f(p), u),$$

We can see that,

$$\forall\, p \in M_0, \quad T_pM_0 = T_pM \times \{0\} \text{ and } T_pM_0^{\perp} = T_pM^{\perp} \times \mathbb{R} \tag{36}$$
$$\forall\, (p, u) \in M_1, \quad T_{(p,u)}M_1 = T_pM \times \mathbb{R} \text{ and } T_{(p,u)}M_1^{\perp} = T_pM^{\perp} \times \{0\}. \tag{37}$$

Further, for $p \in M$ and $X_1 \in T_pM$, we write

$$JX_1 = tX_1 + nX_1, \quad \varphi(X_1, 0) = t_0(X_1, 0) + n_0(X_1, 0), \tag{38}$$
$$\varphi\left(X_1, s\frac{d}{du}\right) = t_1\left(X_1, s\frac{d}{du}\right) + n_1\left(X_1, s\frac{d}{du}\right), \tag{39}$$

for which $tX_1 \in T_pM$, $nX_1 \in T_pM^{\perp}$, $t_0(X_1, 0) \in T_pM_0$, $n_0(X_1, 0) \in T_pM_0^{\perp}$ and $t_1\left(X_1, s\frac{d}{du}\right) \in T_pM_1$, $n_1\left(X_1, s\frac{d}{du}\right) \in T_pM_1^{\perp}$.
Further, there are some important results to recall.

Theorem 3.17. *[24] Let M be an almost para-Cosymplectic manifold then the following statements are equivalent:*

1 *M is para-Cosymplectic.*
2 *The Nijenhius tensor $N = 0$.*
3 *The φ is parallel.*
4 *M is locally a product of an open interval and a para-Kaehlerian. manifold.*

Theorem 3.18. *[29] Let M be a submanifold of para-Kaehler manifold $\left(\overline{M}, J, g\right)$ then M is pointwise slant submanifold if and only if there exists a function λ such that $t^2X_1 = \lambda X_1$, where λ can have values $\cosh^2\theta$, $\cos^2\theta$ or $-\sinh^2\theta$ for some slant function θ.*

Thus, we have our next result.

Proposition 3.19. *Assume (M, f) be a pointwise slant submanifold of para-Kaehler manifold $\left(\overline{M}, J, g\right)$ with a slant coefficient $\lambda(p)$ and let M_0, M_1 are the immersed submanifolds of $\overline{M} \times \mathbb{R}$ set as above. Then*

1. *the characteristic vector field ξ of $\overline{M} \times \mathbb{R}$ is normal to M_0 and is tangent to M_1.*
2. *for $\lambda(p) \in (-\infty, \infty)$, the following statement are equivalent:*
   - *(i) M is pointwise slant in $\overline{M}$ with slant coefficient $\lambda(p)$.*
   - *(ii) M_0 is pointwise slant in $\overline{M} \times \mathbb{R}$ with slant coefficient $\lambda(p)$.*
   - *(iii) M_1 is pointwise slant in $\overline{M} \times \mathbb{R}$ with pointwise slant coefficient $\lambda(p)$.*

Further, all these submanifolds M and M_0, M_1 of manifolds $\overline{M}$ and $\overline{M} \times \mathbb{R}$, respectively possesses same slant coefficient.

Proof. 1. Directly follows from equations (35), (36) and (37).
2. (i) $\Longrightarrow$ (ii). Since, for every point $p \in M$ and $(X_1, 0) \in T_pM_0$ from equations (35) and (38), we have

$$t_0^2 (X_1, 0) = \left(t^2 X_1, 0\right)$$

Thus, if M is pointwise slant submanifold then M_0 is pointwise slant submanifold with same slant coefficient and vice-versa.
(i) $\Longrightarrow$ (iii). Similarly, for every point $p \in M$ and $\left(X_1, s\frac{d}{du}\right) \in T_{(p,u)}M_1$ using equations (35) and (39), we get

$$t_1^2 \left(X_1, s\frac{d}{du}\right) = \left(t^2 X_1, 0\right). \tag{40}$$

If (iii) is true and we denote H as a orthogonal complement of ξ_p in $T_{(p,u)}M_1$ defined as $H = \{(X_1, 0) \mid X_1 \in T_pM\}$. Then

$$\forall X_1 \in T_pM, \quad t^2 X_1 = \lambda(p) X_1. \tag{41}$$

Conversely, suppose (i) holds then from (40)

$$\forall \left(X_1, s\frac{d}{dt}\right) \in T_{(p,t)}M_1, \quad t_1^2 X_1 = \lambda(p) X_1. \tag{42}$$

Hence result follows. □

4. Pointwise slant distributions

Analogous to [23], we generalize slant distributions by defining pointwise slant distributions in $\overline{M}^{2m+1}$. Furthermore, we study some basic characterizations for the distributions on our ambient manifold.

Definition 4.1. *A differentiable distribution $\mathfrak{D}$ at $\overline{M}^{2m+1}$ is defined as a pointwise slant distribution in case for all given point $p \in M$, the quotient $\frac{g(t_{\mathfrak{D}}X_1, t_{\mathfrak{D}}X_1)}{g(\varphi X_1, \varphi X_1)} = \lambda_{\mathfrak{D}}(p)$ is independent of any selection for spacelike or timelike vector field $X_1 \in \mathfrak{D}_p$. Then here*

(i) $\mathfrak{D}_p$ is the distribution at point $p \in M$.
(ii) $t_{\mathfrak{D}}X_1$ is the projection of φX_1 at the distribution $\mathfrak{D}$.
(iii) $\lambda_{\mathfrak{D}}(p)$ is the slant coefficient corresponding to the distribution $\mathfrak{D}$ on M for slant function $\theta(p) : M \to [0, \infty)$.

Remark 4.2. *Any pointwise slant distribution $\mathfrak{D}$ is called invariant if $t_{\mathfrak{D}}X_1 \equiv \varphi X_1$ and $\lambda_{\mathfrak{D}}(p) = 1$ whereas it is anti-invariant for $t_{\mathfrak{D}}X_1 \equiv 0$ and $\lambda_{\mathfrak{D}}(p) = 0$. Other than these two cases, we call the distribution a proper pointwise slant distribution. Similar to the remark 3.2, the slant coefficient $\lambda_{\mathfrak{D}}(p)$ on the distribution $\mathfrak{D}$ can have the value $\cosh^2\theta$, $\cos^2\theta$ or $-\sinh^2\theta$ for slant function $\theta(p)$.*

Next, we have one characterization result for these types of distributions.

Theorem 4.3. *A distribution $\mathfrak{D}$ of submanifold M of $\overline{M}^{2m+1}$ is defined as pointwise slant distribution if and only if there exist $\lambda_{\mathfrak{D}}(p) \in (-\infty, \infty)$ on M such that $(t_{\mathfrak{D}})^2 X_1 = \lambda_{\mathfrak{D}}(p) X_1$, for any non-lightlike vector field $X_1 \in \mathfrak{D}_p \subset T_pM$ and for slant function $\theta(p)$.*

Proof. It is proofed similarly as Lemma 3.4. □

Remark 4.4. *The distribution $\mathfrak{D}$ at M named [7, 16]:*

- *Totally geodesic, in case its second fundamental form vanishes identically.*
- *Umbilical in the direction of the normal vector field ζ (called the umbilical section) on M, if $A_\zeta = \pi I$, for certain function π on M.*
- *Totally umbilical, in case M is umbilical respecting to each (local) normal vector field.*
- *Involutive, in case for all $X_1, X_2 \in \mathfrak{D}$, $[X_1, X_2] \in \mathfrak{D}$.*

5. Pointwise semi-slant submanifold

Definition 5.1. *A submanifold M of $\overline{M}^{2m+1}$ is defined as a pointwise semi-slant submanifold if the set of complementary orthogonal distributions $\{\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_{\lambda}\}$ exists on M and fulfills the listed conditions:*

(i) $TM = \mathfrak{D}_{\mathfrak{T}} \oplus \mathfrak{D}_{\lambda}$.

(ii) The distribution $\mathfrak{D}_{\mathfrak{T}}$ is invariant considering φ which means $\mathfrak{D}_{\mathfrak{T}} \subseteq \mathfrak{D}_{T}$.

(iii) The $\mathfrak{D}_{\lambda}$ distribution is pointwise slant distribution under $\lambda(p)$ as a slant function for $\theta(p) : M \to [0, \infty)$.

In particular, we have the following classifications:

(i) If $\mathfrak{D}_{\mathfrak{T}}, \mathfrak{D}_{\lambda} \neq \{0\}$ and $\lambda(p)$ is not a constant for any $\theta(p) \geq 0$, therefore M is *proper pointwise semi-slant submanifold*.
(ii) In case $\mathfrak{D}_{\mathfrak{T}} = \{0\}$ and $\mathfrak{D}_{\lambda} \neq \{0\}$ with $\lambda(p)$ globally constant for $\theta(p) \geq 0$, then M is a *proper slant submanifold* [21].
(iii) If $\mathfrak{D}_{\mathfrak{T}} \neq \{0\}$ and $\mathfrak{D}_{\lambda} \neq \{0\}$ such that $tX_1 \equiv 0$ for any $X_1 \in \Gamma(\mathfrak{D}_{\lambda})$, therefore M is a *proper semi-invariant submanifold* [22].
(iv) In case $\mathfrak{D}_{\lambda} = \{0\}$, therefore M is an *invariant submanifold* [21].
(v) In case $\mathfrak{D}_{\mathfrak{T}} = \{0\}$ and $tX_1 \equiv 0$ for any $X_1 \in \Gamma(\mathfrak{D}_{\lambda})$, therefore M defines the *anti-invariant submanifold* [21].

Remark 5.2. *Now we denote an another distribution $\mathfrak{D}_{\mathfrak{T}}'$ such that $\mathfrak{D}_{\mathfrak{T}}' = \{X_1 \in \mathfrak{D}_{\mathfrak{T}} : g(X_1, \xi) = 0\} \subseteq \mathfrak{D}_{\mathfrak{T}}$. Subsequently, we followed with the two cases:*

(i) For $\xi \in \Gamma(TM^{\perp})$, clearly $TM = \mathfrak{D}_{\mathfrak{T}}' \oplus \mathfrak{D}_{\lambda}$.
(ii) For $\xi \in \Gamma(TM)$, we have $TM = \langle \xi \rangle \oplus \mathfrak{D}_{\mathfrak{T}}' \oplus \mathfrak{D}_{\lambda}$, which means that if ξ is tangent at any point $p \in M$ then it should belong to $\mathfrak{D}_{\mathfrak{T}}$ decomposing it in the above form.

Thus, we have either $\mathfrak{D}_{\mathfrak{T}} = \mathfrak{D}_{\mathfrak{T}}'$ or $\mathfrak{D}_{\mathfrak{T}} = \langle \xi \rangle \oplus \mathfrak{D}_{\mathfrak{T}}'$ [17]. Now, we present one example of proper pointwise semi-slant submanifolds.

Example 5.3. *Suppose $\overline{M} = \mathbb{R}^8 \times \mathbb{R}_+ \subset \mathbb{R}^9$ to be a 9-dimensional manifold having standard Cartesian coordinates $(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z)$. Define the structure (φ, ξ, η, g) by*

$$\begin{cases} \varphi e_1 = e_2,\ \varphi e_2 = e_1,\ \varphi e_3 = e_4,\ \varphi e_4 = e_3, \varphi e_5 = e_6,\ \varphi e_6 = e_5,\ \varphi e_7 = e_8, \\ \varphi e_8 = e_7,\ \varphi e_9 = 0,\ \xi = e_9,\ \eta = dz, \\ g(e_i, e_i) = 1 \text{ for } \{i = 1, 2, 3, 4\},\ g(e_i, e_i) = -1 \text{ for } \{i = 5, 6, 7, 8\} \text{ and } g(e_i, e_j) = 0 \text{ for } i \neq j. \end{cases} \tag{43}$$

where $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}$ is the local orthonormal basis frame for the $T\overline{M}$ and $e_i = \frac{\partial}{\partial x_i}$ for $i = \{1, 2, 3, 4\}$, $e_i = \frac{\partial}{\partial y_i}$ for $i = \{5, 6, 7, 8\}$ and $e_9 = \frac{\partial}{\partial z}$. With straightforward calculations it is easy to see that $\overline{M}(\varphi, \xi, \eta, g)$ is a para-Cosymplectic manifold.
Let an isometrically immersed submanifold M with semi-Riemannian metric defined by

$$M(u, v, r, s, z) = \left(v, 2, u, \frac{v^2}{2}, \frac{u^2}{2}, 2, 2r, s, z\right)$$

where u, v, r, s and z are real valued functions on M. Therefore the following vector fields

$$X_u = e_3 + ue_5,\ X_v = e_1 + ve_4,\ X_r = 2e_7,\ X_s = e_8,\ X_z = e_9. \tag{44}$$

generates the tangent bundle TM of M. Therefore, a submanifold M defines the pointwise semi-slant submanifold with the distributions $\mathfrak{D}_{\mathfrak{T}}$ and $\mathfrak{D}_{\lambda}$ characterized by the span $\{X_r, X_s\}$ and span $\{X_u, X_v\}$, respectively. The distribution $\mathfrak{D}_{\mathfrak{T}}$ is an invariant and $\mathfrak{D}_{\lambda}$ defines the pointwise slant distribution with $t^2 = \frac{v^2}{(1-u^2)(1+v^2)} I$.

Further, if $\mathcal{P}_T$ and $\mathcal{P}_\lambda$ denote the projections on the distributions $\mathfrak{D}_\mathfrak{T}$ and $\mathfrak{D}_\lambda$, respectively. Therefore, for all $X_3 \in \Gamma(TM)$, we have

$$X_3 = \mathcal{P}_T X_3 + \mathcal{P}_\lambda X_3. \tag{45}$$

Previous equation by operating φ and using Eq. (14), it becomes

$$\varphi X_3 = t\,\mathcal{P}_T X_3 + t\,\mathcal{P}_\lambda X_3 + n\,\mathcal{P}_\lambda X_3. \tag{46}$$

Thus, from the expression, we concluded that

$$t\,\mathcal{P}_T X_3 \in \Gamma(\mathfrak{D}_\mathfrak{T}), \quad n\,\mathcal{P}_T X_3 = 0,$$

and

$$t\mathcal{P}_\lambda X_3 \in \Gamma(\mathfrak{D}_\lambda), \quad n\mathcal{P}_\lambda X_3 \in \Gamma(TM^\perp).$$

Using Eq. (14) and above expressions in Eq. (46), we deduce that

$$t\,X_3 = t\mathcal{P}_T X_3 + t\mathcal{P}_\lambda X_3, \quad n\,X_3 = n\mathcal{P}_\lambda X_3.$$

Since, $\mathfrak{D}_\lambda$ is pointwise slant distribution, by the consequences of Theorem 4.3, we obtain that

$$t^2 X_3 = \lambda(p)\,X_3, \tag{47}$$

for $X_3 \in \Gamma(\mathfrak{D}_\lambda)$ and some real-valued function $\lambda(p)$ on M. Clearly, for any point $p \in M$ if $\xi \in T_pM$, then

$$\varphi X_3 = t\mathcal{P}_{\mathcal{T}}{}' X_3 + t\mathcal{P}_\lambda X_3 + n\mathcal{P}_\lambda X_3,$$

where $\mathcal{P}_{\mathcal{T}}{}'$ is the projection on the distribution $\mathfrak{D}_\mathfrak{T}{}'$. But this does not effect our result as ξ disappears when φ operates on X_3, so instead of latter equation we use Eq. (46). Moreover, the normal bundle of M can be expressed in the following form

$$TM^\perp = n\mathfrak{D}_\lambda \oplus \mu, \tag{48}$$

where μ is a φ-invariant subspace of normal bundle.
Now, by virtue of above construction, we have some important results of pointwise semi-slant submanifold as follows:

Proposition 5.4. *Let M be a proper pointwise semi-slant submanifold of $\overline{M}^{2m+1}$. Then for any $\xi \in \Gamma(TM)$, $\zeta \in \Gamma(TM^\perp)$ and $X_3 \in \Gamma(\mathfrak{D}_\lambda)$, the tensor field n is parallel if and only if the shape operator A insures*

$$A_\zeta X_3 = -\frac{1}{\lambda(p)} A_{n'\zeta} t X_3. \tag{49}$$

Proof. By the (*iv*) part of Remark 3.5, we have

$$-h(X_1, tX_3) + n'h(X_1, X_3) = 0.$$

Now interchange X_3 by tX_3 into above equation, we have

$$-h(X_1, t^2X_3) + n'h(X_1, tX_3) = 0.$$

In view of Eq. (47) above relation reduces into the following form

$$-\lambda(p)h(X_1, X_3) + n'h(X_1, tX_3) = 0.$$

By the consequence of (17), we have

$$-\lambda(p)g(h(X_1, X_3), \zeta) - g(h(X_1, tX_3), n'\zeta) = 0.$$

If we applying (13) into the above expression, we get (49). □

Lemma 5.5. *If M is the proper pointwise semi-slant submanifold of* $\overline{M}^{2m+1}$*, then*

$$g(tX_3, tX_4) = \lambda(p)\, g(\varphi X_3, \varphi X_4) \tag{50}$$

$$g(nX_3, nX_4) = (1 - \lambda(p))\, g(\varphi X_3, \varphi X_4), \tag{51}$$

for all $X_3, X_4 \in \Gamma(\mathfrak{D}_\lambda)$.

Proof. Using Eq. (14), we have

$$g(tX_3, tX_4) = g(\varphi X_3 - nX_3, tX_4),$$

Hence,

$$g(tX_3, tX_4) = -g(X_3, \varphi tX_4).$$

Using Eqs. (3) and (47), we obtain Eq. (50). Again using Eq. (50) we get Eq. (51). □

Lemma 5.6. *For a proper pointwise semi-slant submanifold M of* $\overline{M}^{2m+1}$*, both the distributions* $\mathfrak{D}_{\mathfrak{T}}$ *and* $\mathfrak{D}_\lambda$ *are t-invariant.*

Proof. Since $\mathfrak{D}_{\mathfrak{T}}$ is φ-invariant so

$$\varphi\mathfrak{D}_{\mathfrak{T}} \subset \mathfrak{D}_T \Rightarrow t\mathfrak{D}_{\mathfrak{T}} \subset \mathfrak{D}_{\mathfrak{T}}.$$

Now in view of Eq. (46) if $X_3 \in \Gamma(\mathfrak{D}_\lambda)$ and for any $X_1 \in \Gamma(\mathfrak{D}_{\mathfrak{T}})$, we have

$$g(tP_T X_3, X_1) = g(\varphi X_3, X_1) = -g(X_3, \varphi X_1) = 0.$$

Moreover,

$$g(tP_T X_3, X_2) = 0 \quad \forall\, X_2 \in \Gamma(\mathfrak{D}_{\mathfrak{T}}),$$

which implies $tP_T X_3 = 0$, therefore $tX_3 = tP_\lambda X_3$. □

Proposition 5.7. *Let M be a proper totally umbilical pointwise semi-slant submanifold of* $\overline{M}^{2m+1}$*. Then for any* $\xi \in \Gamma(TM)$ *and* $X_1, X_2 \in \Gamma(\mathfrak{D}_{\mathfrak{T}}{}' \oplus \langle\xi\rangle)$*, then* $H \in \Gamma(n\mathfrak{D}_\lambda)$*.*

Proof. By consequence of (11), we have

$$\overline{\nabla}_{X_1}\varphi X_2 = \nabla_{X_1}\varphi X_2 + h(X_1, \varphi X_2).$$

Since $\overline{\nabla}_{X_1}\varphi X_2 = \varphi\overline{\nabla}_{X_1}X_2$, then by the use of (11), (14) and (15), we achieve

$$t\overline{\nabla}_{X_1}X_2 + n\overline{\nabla}_{X_1}X_2 + t'h(X_1, X_2) + n'h(X_1, X_2) = \nabla_{X_1}\varphi X_2 + h(X_1, \varphi X_2).$$

If we taking inner product with $\zeta \in \Gamma(\mu)$, then we achieve

$$g(h(X_1, \varphi X_2), \zeta) = g(n'h(X_1, X_2), \zeta).$$

By applying equations (17) and (23) into above relation, we have

$$g(X_1, \varphi X_2)g(H, \zeta) = -g(X_1, X_2)g(H, n'\zeta). \tag{52}$$

Now replace X_1 with X_2 into above relation, we get

$$-g(X_1, \varphi X_2)g(H, \zeta) = -g(X_1, X_2)g(H, n'\zeta). \tag{53}$$

Adding (52) and (53), we have

$$g(X_1, X_2)g(H, n'\zeta) = 0.$$

By the direct application (48), we get the desired result. □

Next, we will find the necessary and sufficient conditions of involutive and totally geodesic foliations for such involved distributions.

Theorem 5.8. *If M is the proper pointwise semi-slant submanifold of $\overline{M}^{2m+1}$, then for $\xi \in \Gamma(TM)$, $X_1, X_2 \in \Gamma(\mathfrak{D}_{\mathfrak{T}}' \oplus \langle\xi\rangle)$ and $X_3 \in \Gamma(\mathfrak{D}_\lambda)$, the distribution $\mathfrak{D}_{\mathfrak{T}}' \oplus \langle\xi\rangle$ is*

(i) integrable in case $h(X_1, tX_2) = h(tX_1, X_2)$, where h is the second fundamental form of M.
(ii) totally geodesic if $A_{n\,tX_3}X_2, = A_{nX_3}tX_2$, where A is the shape operator.

Proof. (i) For any $X_1, X_2 \in \Gamma(\mathfrak{D}_{\mathfrak{T}}' \oplus \langle\xi\rangle)$ and $X_3 \in \Gamma(\mathfrak{D}_\lambda)$ and using Eq. (9), we have

$$g([X_1, X_2], X_3) = -g\left(\varphi\left(\overline{\nabla}_{X_1}X_2 - \overline{\nabla}_{X_2}X_1\right), \varphi X_3\right) + \eta([X_1, X_2])\,\eta(X_3). \tag{54}$$

Using Eq. (14) for the φX_3 in Eq. (54) and followed by using Eq. (5), we have

$$g([X_1, X_2], X_3) = g\left(\overline{\nabla}_{X_1}X_2 - \overline{\nabla}_{X_2}X_1, \varphi tX_3\right) + g\left(\varphi\left(\overline{\nabla}_{X_1}X_2 - \overline{\nabla}_{X_2}X_1\right), nX_3\right). \tag{55}$$

Further using Eqs. (14), (9) (11) and Lemma 3.4 in Eq. (55), gives

$$g([X_1, X_2], X_3) = \lambda(p)\,g([X_1, X_2], X_3) + g(h(X_1, tX_2) - h(X_2, tX_1), nX_3), \tag{56}$$

which implies

$$(1 - \lambda(p))\,g([X_1, X_2], X_3) = g(h(X_1, tX_2) - h(X_2, tX_1), nX_3). \tag{57}$$

Using remark 3.3 as M is the proper pointwise semi slant submanifold and X_1, X_2, X_3 are non-null vector fields, the $[X_1, X_2] \in \Gamma(\mathfrak{D}_{\mathfrak{T}}' \oplus \langle\xi\rangle)$ if and only if

$$h(X_1, tX_2) = h(X_2, tX_1).$$

(ii) For any $X_1, X_2 \in \Gamma(\mathfrak{D}_{\mathfrak{T}}' \oplus \langle\xi\rangle)$ and $X_3 \in \Gamma(\mathfrak{D}_\lambda)$, from Gauss formula we have

$$g(\nabla_{X_1}X_2, X_3) = g\left(\overline{\nabla}_{X_1}X_2, X_3\right).$$

Employing Eqs. (3), (9), (11) and (14) in above expression, we obtain that

$$g(\nabla_{X_1}X_2, X_3) = -g\left(\overline{\nabla}_{X_1}X_2, t^2X_3\right) - g(h(X_1, X_2), n\,tX_3) + g(h(X_1, tX_2), nX_3). \tag{58}$$

Using Eqs. (47) and (13) in Eq. (58), we arrive at

$$g(\nabla_{X_1}X_2, X_3) = \lambda(p)\,g(\nabla_{X_1}X_2, X_3) - g(A_{n\,tX_3}X_2, X_1) + g(A_{nX_3}tX_2, X_1), \tag{59}$$

we conclude from above equation that

$$(1 - \lambda(p))(\nabla_{X_1}X_2, X_3) = -g(A_{n\,tX_3}X_2, X_1) + g(A_{n\,X_3}tX_2, X_1). \tag{60}$$

Thus, from (60), we deduce that $\nabla_{X_1}X_2 \in \Gamma(\langle\xi\rangle \oplus \mathfrak{D}_{\mathfrak{T}}')$ if and only if

$$-g(A_{n\,tX_3}X_2, X_1) + g(A_{nX_3}tX_2, X_1) = 0.$$

For M to become proper pointwise semi-slant submanifold and X_1, X_2, X_3 are non-null vector fields, the proof directly follows. □

Theorem 5.9. *In case M is the proper pointwise semi-slant submanifold of $\overline{M}^{2m+1}$. For $\xi \in \Gamma(TM^\perp)$, $X_1, X_2 \in \Gamma(\mathfrak{D}_{\mathfrak{T}}')$ and $X_3 \in \Gamma(\mathfrak{D}_\lambda)$, the distribution $\mathfrak{D}_{\mathfrak{T}}'$ is* *(i) integrable in case the second fundamental form h of M insures*

$$h(X_1, tX_2) = h(tX_1, X_2).$$

(ii) totally geodesic if metric g at M insures

$$g(A_{ntX_3}X_2, X_1) = g(A_{nX_3}tX_2, X_1),$$

where A is the shape operator.

Proof. The result is gained similarly to Theorem 5.8. □

Theorem 5.10. *Let M be a proper pointwise semi-slant submanifold of* $\overline{M}^{2m+1}$. *Then for any* $\xi \in \Gamma(TM)$, $X_1 \in \Gamma(\mathfrak{D}_{\mathfrak{T}}{}' \oplus \langle\xi\rangle)$ *and* $X_3, X_4 \in \Gamma(\mathfrak{D}_\lambda)$, *the pointwise slant distribution* $\mathfrak{D}_\lambda$ *is*

(i) involutive if and only if metric g of M fulfills

$$g(A_{nX_4}X_3 - A_{nX_3}X_4, tX_1) = g(A_{ntX_3}X_4 - A_{ntX_4}X_3, X_1).$$

(ii) totally geodesic if and only if metric g on M satisfies

$$g(A_{nX_4}tX_1, X_3) = g(A_{ntX_4}X_1, X_3).$$

Proof. (i) For any $X_1 \in \Gamma(\mathfrak{D}_{\mathfrak{T}}{}' \oplus \langle\xi\rangle)$ and $X_3, X_4 \in \Gamma(\mathfrak{D}_\lambda)$, using Eq. (3) we have

$$g([X_3, X_4], X_1) = -g(\varphi[X_3, X_4], \varphi X_1) + \eta([X_3, X_4])\eta(X_1). \tag{61}$$

Solving separately $\{-g(\varphi[X_3, X_4], \varphi X_1)\}$ and by the use of equations (5), (9) and (14), it is obtained that

$$-g(\varphi[X_3, X_4], \varphi X_1) = g\left(\overline{\nabla}_{X_3}\varphi(tX_4) - \overline{\nabla}_{X_4}\varphi(tX_3), X_1\right) - g(-A_{nX_4}X_3 + A_{nX_3}X_4, \varphi X_1). \tag{62}$$

Again using (5), (12) and (47) in above equation, we find

$$\begin{aligned} -g(\varphi[X_3, X_4], \varphi X_1) =& g\left(\overline{\nabla}_{X_3}(\lambda(p)X_4) - \overline{\nabla}_{X_4}(\lambda(p)X_3), X_1\right) + g(-A_{ntX_4}X_3 + A_{ntX_3}X_4, X_1) \\ &- g(-A_{nX_4}X_3 + A_{nX_3}X_4, \varphi X_1), \end{aligned} \tag{63}$$

which implies

$$\begin{aligned} -g(\varphi[X_3, X_4], \varphi X_1) =& \lambda(p)\, g\left(\overline{\nabla}_{X_3}X_4 - \overline{\nabla}_{X_4}X_3, X_1\right) - g(-A_{nX_4}X_3 + A_{nX_3}X_4, \varphi X_1) \\ &+ g\left(\lambda'(p)(X_3\theta)X_4 - \lambda'(p)(X_4\theta)X_3, X_1\right) + g(-A_{ntX_4}X_3 + A_{ntX_3}X_4, X_1), \end{aligned} \tag{64}$$

where $\lambda'(p)$ is the first derivative of $\lambda(p)$. Substitute Eq. (64) in Eq. (61) leads to following

$$\begin{aligned} (1 - \lambda(p))([X_3, X_4], X_1) =& g(-A_{ntX_4}X_3 + A_{ntX_3}X_4, X_1) + \eta([X_3, X_4])\eta(X_1) \\ &+ g(A_{nX_4}X_3 - A_{nX_3}X_4, \varphi X_1). \end{aligned} \tag{65}$$

Since, $\xi \in \Gamma(TM)$ one can replace X_1 by ξ in the above equation and consequently we get

$$\begin{aligned} (1 - \lambda(p))([X_3, X_4], \xi) =& g(-A_{ntX_4}X_3 + A_{ntX_3}X_4, \xi) + \eta([X_3, X_4]), \\ -\lambda(p)\, g([X_3, X_4], \xi) =& g(h(X_3, \xi), ntX_4) - g(h(X_4, \xi), ntX_3). \end{aligned} \tag{66}$$

Using Lemma 2.4 in Eq. (66) in addition to M is the proper pointwise slant, resulted in

$$g([X_3, X_4], \xi) = 0 \Rightarrow \eta([X_3, X_4]) = 0.$$

Therefore, in Eq. (65) using the facts that M is the proper pointwise slant submanifold with non-null vector fields X_1, X_3, X_4 in M and $\mathfrak{D}_{\mathfrak{T}}{}'$ is $\varphi-$invariant, we arrived at the desired result.
(ii) For any $X_3, X_4 \in \Gamma(\mathfrak{D}_\lambda)$ and $X_1 \in \Gamma(\mathfrak{D}_{\mathfrak{T}}{}' \oplus \langle\xi\rangle)$, from Gauss formula we have

$$g(\nabla_{X_3}X_4, X_1) = g\left(\overline{\nabla}_{X_3}X_4, X_1\right). \tag{67}$$

Employing Eqs. (3), (9), (11) and (14) in above expression, we obtain that

$$g(\nabla_{X_3}X_4, X_1) = g\left(\overline{\nabla}_{X_3}t^2X_4, X_1\right) + g(h(X_1, X_3), n\,tX_4) - g(h(X_3, tX_1), nX_4) + \eta\left(\overline{\nabla}_{X_3}X_4\right)\eta(X_1). \tag{68}$$

Using Eq. (13) and (47) in equation (68), we arrive at

$$\begin{aligned} g(\nabla_{X_3}X_4, X_1) =& \lambda(p)\, g(\nabla_{X_1}X_2, X_3) + g\left(\lambda'(p)(X_3\theta)X_4, X_1\right) + g(A_{n\,tX_4}X_1, X_3) \\ &- g(A_{nX_4}tX_1, X_3)\,\eta\left(\overline{\nabla}_{X_3}X_4\right)\eta(X_1). \end{aligned} \tag{69}$$

Since, $\xi \in \Gamma(TM)$, we can replace X_1 by ξ in Eq. (69) and consequently we get

$$\begin{aligned} (1-\lambda(p))\,\eta(\nabla_{X_4}X_3) &= -\,g(A_{n\,tX_3}\xi, X_4) + \eta(\nabla_{X_4}X_3), \\ (-\lambda(p))\,\eta(\nabla_{X_4}X_3) &= -\,g(A_{n\,tX_3}\xi, X_4). \end{aligned}$$

Using Lemma 2.4 in above expression, we get

$$\eta(\nabla_{X_4}X_3) = \eta\left(\overline{\nabla}_{X_4}X_3\right) = 0,$$

we conclude from above equation that

$$(1-\lambda(p))\, g(\nabla_{X_3}X_4, X_1) = g(A_{n\,tX_4}X_1, X_3) - g(A_{nX_4}tX_1, X_3). \tag{70}$$

Thus, from (70), we deduce that $\nabla_{X_3}X_4 \in \Gamma(\mathfrak{D}_\lambda)$ if and only if

$$g(A_{n\,tX_4}X_1, X_3) - g(A_{nX_4}tX_1, X_3) = 0.$$

This proves the result (*ii*). □

Theorem 5.11. *Let M be a proper pointwise semi-slant submanifold of $\overline{M}^{2m+1}$. Then for any $\xi \in \Gamma(TM^\perp)$, $X_1 \in \Gamma(\mathfrak{D}_{\mathfrak{T}}{}')$ and $X_3, X_4 \in \Gamma(\mathfrak{D}_\lambda)$, the pointwise slant distribution $\mathfrak{D}_\lambda$ is*

(i) involutive if and only if metric g on M fulfils

$$g(A_{nX_4}X_3 - A_{nX_3}X_4, tX_1) = g(A_{n\,tX_3}X_4 - A_{n\,tX_4}X_3, X_1).$$

(ii) totally geodesic if and only if metric g at M insures

$$g(A_{nX_4}tX_1, X_3) = g(A_{n\,tX_4}X_1, X_3).$$

Proof. Similar to the proof of Theorem 5.10. □

Acknowledgement The authors are grateful to the anonymous referee(s) for careful reading of the manuscript and valuable suggestions that improved the presentation of the work.

The last author extends his appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through the research group program under grant number R.G.P.2/130/43. The authors also express their gratitude to the Princess Nourah Bint Abdulrahman University Researchers Supporting Project Number (PNURSP2022R27), Princess Nourah Bint Abdulrahman University, Riyadh, Saudi Arabia.

References

[1] A. Ali, C. Ozel, Geometry of warped product pointwise semi-slant submanifolds of cosymplectic manifolds and its applications, International Journal of Geometric Methods in Modern Physics 14 (3)(2017) 1750042.
[2] B. O'Neill, Semi-Riemannian geometry with applications to relativity, Academic Press, New york, 1983.
[3] B. Sahin, Warped product pointwise semi-slant submanifolds of Kaehler manifolds, Portugaliae Mathematica 70(3)(2013) 251-268.
[4] B. Y. Chen, Geometry of submanifolds. Marcel Dekker, Inc., New York, 1973.
[5] B. Y. Chen, Geometry of warped product submanifolds: A survey, J. Adv. Math. Stud.6(2)(2013) 1-8.
[6] B. Y. Chen, Slant immersion, Bull. Austral. Mat. Soc. 41(1)(1990) 135-147.
[7] B. Y. Chen, Pseudo-Riemannian geometry, δ-invariants and applications, Word Scientific, 2011.
[8] B. Y. Chen, I. Mihai, Classification of quasi-minimal slant surfaces in Lorentzian complex space forms, Acta Mathematica Hungarica. 122(4) (2009) 307-328.
[9] B. Y. Chen, O. J. Garay, Pointwise slant submanifolds in almost Hermitian manifolds, Turkish J. Math. 36(4)(2012) 630-640.
[10] D. E. Blair, Riemannian geometry of contact and symplectic manifolds, Progr. Math., Birkhäuser, Boston, 2002.
[11] F. Etayo, On quasi-slant submanifolds of an almost Hermitian manifold, Publ. Math. Debrecen. 53(1-2)(1998) 217-223.
[12] F. R. Al-Solamy, V. A. Khan, S. Uddin, Geometry of warped product semi-slant submanifolds of nearly Kaehler manifolds, Results. Math. 71(3)(2017) 783-799.
[13] I. Sato, On a structure similar to the almost contact structure, Tensor, NS 30(1976) 219-224.
[14] I. Hasegawa, I. Mihai, Contact CR-warped product submanifolds in Sasakian manifolds, Geometriae Dedicata. 102(1)(2003) 143-150.
[15] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez, M. Fernandez, Semi-slant submanifolds of a Sasakian manifold, Geometriae Dedicata 78(2)(1999) 183-199.
[16] K. L. Duggal, Lorentzian geometry of CR submanifolds, Acta Applicandae Mathematica. 17(2) (1989) 171-193.
[17] K. S. Park, Pointwise slant and pointwise semi-slant submanifolds in almost contact metric manifolds, arXiv preprint arXiv:1410.5587(2014).
[18] K. Srivastava, S. K. Srivastava, On a class of α-para Kenmotsu manifolds, Mediterr. J. Math. 13(1)(2016), 391-399.
[19] M. B. K. Balgeshir, Point-wise slant submanifolds in almost contact geometry, Turkish J. Math. 40(3) (2016), 657-664.
[20] N. Papaghiuc, Semi-slant submanifolds of Kaehlerian manifolds, Ann. St. Univ. Iasi. Tom., 9,(1994) 55-61.
[21] P. Alegre, Slant submanifolds of Lorentzian Sasakian and para Sasakian manifolds, Taiwanese J. Math. 17(3)(2013), 897-910.
[22] P. Alegre and A. Carriazo: Slant submanifolds of para-Hermitian manifolds, Mediterr. J. Math. 14(5)(2017), 214.
[23] P. Alegre and A. Carriazo, Bi-slant submanifolds of para Hermitian manifolds, Mathematics, 7(7) (2019), 618.
[24] P. Dacko, On almost para-cosymplectic manifolds, Tsukuba Journal of Mathematics, 28(1) (2004), 193-213.
[25] S. K. Chanyal, Slant submanifolds in an almost paracontact metric manifold, An. Stiint. Univ. Al. I. Cuza Iasi. Mat.(NS). 67(2)(2021), 213-229.
[26] S. K. Srivastava, A. Sharma, Pointwise pseudo-slant warped product submanifolds in a Kähler manifold, Mediterr. J. Math. 14(1) (2017).
[27] S. Zomkovoy, On quasi-para-Sasakian manifolds, arXiv:171103008v2 [math.DG] (2018).
[28] T. Takahashi, Sasakian manifold with pseudo-Riemannian metrics, Tohoku Math. J. 21(2), 271-290 (1969).
[29] Y. Gündüzalp, Warped product pointwise hemi-slant submanifolds of a Para-Kaehler manifold, Filomat, 36(1) (2022), 275-288.