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Properties of Dual Toeplitz Operator on the Orthogonal Complement of the Pluriharmonic Bergman Space of the Unit Ball

Yinyin Hu[a], Yufeng Lu[b], Yixin Yang[b]

[a]*Dalian Maritime university*
[b]*Dalian University of Technology*

Abstract. In this paper, we characterize the hyponormal dual Toeplitz operators with special symbols on the orthogonal complement of the pluriharmonic Bergman space of the unit ball. Also we completely characterize the pluriharmonic symbols for (semi)commuting dual Toeplitz operators.

1. Introduction

For any integer $n > 1$, let B_n denote the open unit ball in $\mathbb{C}^n$. The boundary of B_n is the sphere S_n and the closure of B_n with the Euclidean metric on $\mathbb{C}^n$ is denoted by $\overline{B_n}$. Let dv denote the Lebesgue measure on the unit ball B_n of $\mathbb{C}^n$, normalized so that the measure of B_n equals 1. The space $L^2 = L^2(B_n, dv)$ is the completion of the collection of all functions f on B_n for which

$$\|f\| = [\int_{B_n} |f(z)|^2 dv(z)]^{\frac{1}{2}} < \infty,$$

equipped with the inner product

$$\langle f, g\rangle = \int_{B_n} f(z)\overline{g(z)}dv(z).$$

The Bergman space $A^2 = A^2(B_n, dv)$ is the closed subspace of $L^2(B_n, dv)$ consisting of all holomorphic functions, and let P denote the orthogonal projection from $L^2(B_n, dv)$ onto $A^2 = A^2(B_n, dv)$. Then P is an integral operator represented by

$$P(f)(w) = \langle f, K_w\rangle = \int_{B_n} f(z)\overline{K_w(z)}dv,$$

where $K_w(z) = K(z, w)$ is the reproducing kernel of $A^2 = A^2(B_n, dv)$. By computation, we know

$$K(z, w) = 1 + \sum_{\alpha\in\mathbb{N}^n-\{0\}} \frac{(|\alpha| + n)!}{n!\alpha!} z^\alpha \bar{w}^\alpha,$$

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Email addresses: huyinyin@dlmu.edu.cn (Yinyin Hu), lyfdlut@dlut.edu.cn (Yufeng Lu), yangyixin@dlut.edu.cn (Yixin Yang)

where $\{0\} = (0, \cdots, 0)$, $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, $|\alpha| = \sum_{i=1}^n \alpha_i$ and $\mathbb{N}$ is the set of nonnegative integers. The pluriharmonic Bergman space $A_h^2 = A_h^2(B_n, dv)$ is the closed subspace of $L^2(B_n, dv)$ consisting of all pluriharmonic functions. Let Q denote the orthogonal projection from L^2 onto A_h^2, then $(Qf)(z) = \langle f, R_z \rangle$, where $R_z = K_z + \bar{K}_z - 1$. In fact,

$$(Qf)(z) = (Pf)(z) + \overline{(P\bar{f})(z)} - (Pf)(0).$$

Given a function $f \in L^\infty(B_n)$, the multiplication operator M_f, the Toeplitz operator T_f, the Hankel operator H_f, the dual Toeplitz operator S_f and dual Hankel operator R_f with symbol f are defined respectively by

$$\begin{aligned}
&M_f : L^2 \to L^2,\ M_f(h) = fh,\ h \in L^2;\\
&T_f : A_h^2 \to A_h^2,\ T_f(h) = Q(fh),\ h \in A_h^2;\\
&H_f : A_h^2 \to (A_h^2)^\perp,\ H_f(h) = (I - Q)(fh),\ h \in A_h^2;\\
&S_f : (A_h^2)^\perp \to (A_h^2)^\perp,\ S_f(h) = (I - Q)(fh),\ h \in (A_h^2)^\perp;\\
&R_f : (A_h^2)^\perp \to A_h^2, R_f(h) = Q(fh),\ h \in (A_h^2)^\perp.
\end{aligned}$$

They are all bounded linear operators. Under the decomposition $L^2 = A_h^2 \oplus (A_h^2)^\perp$, the multiplication operator M_f is represented as

$$\begin{pmatrix} T_f & R_f \\ H_f & S_f \end{pmatrix}.$$

Since$M_f M_g = M_g M_f$, we have

$$\begin{aligned}
T_{fg} &= T_f T_g + R_f H_g;\\
S_{fg} &= S_f S_g + H_f R_g.
\end{aligned}$$

This shows close relationships among the above four types of operators. Many studies for dual Toeplitz operators offer some insights into the study for Toeplitz operators. So it is reasonable to focus on the dual Toeplitz operators. Although dual Toeplitz operators differ in many ways from Toeplitz operators, they do have some analogous properties. The general problem that we are interested in is the following: what is the relationship between their symbols when two dual Toeplitz operators commute?

For Toeplitz operators, this problem has been studied for a long time. In the case of the classical Hardy space, A. Brown and P. R. Halmos [5] showed that two Toeplitz operators with general bounded symbols commute if and only if either both symbols are analytic, or both symbols are conjugate analytic, or a nontrivial linear combination of the symbols is constant.

Initiated by Brown and Halmos's pioneering work, the problem of characterizing when two Toeplitz operators commute has been one of the topics of constant interest in the study of Toeplitz operators on classical function spaces over various domains. On the Bergman space of the unit disk, S. Axler and Z. Čučković [3] studied commuting Toeplitz operators with harmonic symbols, and obtained the similar result to Brown and Halmos's. K. Stroethoff [25] later extended that result to essentially commuting Toeplitz operators. S. Axler et al. [4] showed that if two Toeplitz operators commute and the symbol of one of them is nonconstant analytic, then the other one must be analytic. Z. Čučković and N. Rao [6] studied Toeplitz operators that commute with Toeplitz operators with monomial symbols. On the Bergman space of several complex variables, by making use of $\mathcal{M}$-harmonic function theory, D. Zheng [31] characterized commuting Toeplitz operators with pluriharmonic symbols on the Bergman space of the unit ball. B. Choe and Y. Lee [10, 11, 16] studied commuting and essentially commuting Toeplitz operators with pluriharmonic symbols on the unit ball. Y. Lu [18] characterized commuting Toeplitz operators on the bidisk with pluriharmonic symbols. B. Choe et al. [12] obtained characterizations of (essentially) commuting Toeplitz operators with pluriharmonic symbols on the Bergman space of the polydisk.

The fact that the product of two harmonic functions is no longer harmonic adds some mystery in the study of operators on harmonic Bergman space. Many methods which work for the operators on analytic

Bergman space lose their effectiveness on harmonic Bergman space. On the harmonic Bergman space of the unit disk, S. Ohno [34] first characterized the commutativity of T_f and T_z, where f is a analytic function. B. Choe and Y. Lee [35] studied commuting Toeplitz operator with harmonic symbols and one of the symbols is a polynomial. In [14], B. Choe and Y. Lee proved that if $f, g \in H^\infty$ and suppose one of them is noncyclic, then $T_f T_{\bar{g}} = T_g T_{\bar{f}}$ if and only if either f or g is constant. On the pluriharmonic Bergman space of the unit ball, commuting Toeplitz operators was studied in [15, 17].

However, the study on the problem for dual Toeplitz operators started recently. K. Stroethoff and D. Zheng [24] characterized the commutativity of dual Toeplitz operators with bounded symbols on the orthogonal complement of the Bergman space of the unit disk and studied algebraic and spectral properties of dual Toeplitz operators. On the Bergman space of the unit ball and the polydisk, commuting dual Toeplitz operators was studied in [19–21]. J. Yang and Y. Lu [26] gave complete characterization for the (semi)commuting dual Toeplitz operators with harmonic symbols on harmonic Bergman space.

In recent years the Dirichlet space has received a lot of attention from mathematicians in the areas of modern analysis, probability and statistical analysis. Many mathematicians are interested in function theory and operator theory on the Dirichlet space. T. Yu and S. Wu [28, 29] investigated commuting dual Toeplitz operators with harmonic symbols on the Dirichlet space. T. Yu [30] obtained the commutativity of dual Toeplitz operators with general symbols on Dirichlet space.

A bounded operator T is said to be hyponormal if $[T^*, T] = T^*T - TT^* \geq 0$, where T^* denotes the adjoint of T. An equivalent definition of hyponormality is $\|Tu\| \geq \|T^*u\|$ for all vectors u. Such operators are of interest because of Putnam's inequality (see Theorem 1 in [34]), which says that hyponormal operators satisfy

$$\|[T^*, T]\| \leq \frac{|\sigma(T)|}{\pi},$$

where $\sigma(T)$ is the spectrum of T.

We are interested in understanding what symbols φ yield dual Toeplitz operators S_φ that are hyponormal. An analogous question can be asked in the setting of the Hardy space of the unit disk and it was answered by Cowen [33].

There are several obvious examples of hyponormal Toeplitz operators acting on the Bergman space. For instance, $T_{|z|^2}$ is hyponormal because (recalling the fact that $T_f^* = T_{\bar{f}}$), it is self-adjoint. The operator T_z is also hyponormal because if $f \in A^2(D)$, then

$$\|T_z f\|^2 = \int_D |zf|^2 dz = \int_D |\bar{z} f|^2 dz \geq \int_D |P(\bar{z} f)|^2 dz = \|T_z^* f\|^2.$$

The same reasoning shows that T_g is hyponormal for any $g \in H^\infty$.

While a complete characterization of hyponormal Toeplitz operators acting on the Bergman space has remained elusive, there has been a substantial amount of work on understanding the case when f is a polynomial in z and $\bar{z}$.

A pluriharmonic function in the unit ball is the sum of a holomorphic function and the conjugate of a holomorphic function. It is clear that all pluriharmonic functions on B_n are $\mathcal{M}$-harmonic. A good reference for the function theory of the unit ball is Rudin's book [23]. In this paper, we want to characterize the hyponormal and commuting dual Toeplitz operators with pluriharmonic symbols on the orthogonal complement of the pluriharmonic Bergman space of the unit ball.

We state our main result now. We postpone the proofs of these theorems until Section 3 and 4.

Theorem 1.1. *Suppose that φ is a bounded holomorphic function on B_n, S_φ is hyponormal if and only if φ is a constant function.*

Theorem 1.2. *Suppose that $f, g \in L^\infty(B_n)$ are pluriharmonic functions, then $S_{fg} = S_f S_g$ if and only if one of the following statements holds:*

(1) Both f and g are holomorphic;

(2) Both $\overline{f}$ and $\overline{g}$ are holomorphic;

(3) Either f or g is constant.

Theorem 1.3. *Suppose $f, g \in L^\infty(B_n)$ are pluriharmonic functions, then $S_g S_f = S_f S_g$ if and only if one of the following statement holds:*

(1) Both f and g are holomorphic;

(2) Both $\overline{f}$ and $\overline{g}$ are holomorphic;

(3) There are constants α and β, not both zero, such that $\alpha f + \beta g$ is constant.

2. Some Lemmas

For two multi-indexes $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$, the notation $\alpha \succ \beta$ means that

$$\alpha \neq \beta, \quad \text{and} \quad \alpha_i \geq \beta_i, \quad i = 1, \cdots, n.$$

The standard orhonormal basis for $\mathbb{C}^n$ consists of the vectors $d_1, d_2, \cdots, d_n$, where d_k is the ordered n-tuple that has 1 in the k-th spot and 0 everywhere else. A direct computation gives that

$$Q(z^\alpha \bar{z}^\beta) = \begin{cases} \frac{\alpha!}{(\alpha-\beta)!}\frac{(n+|\alpha|-|\beta|)!}{(n+|\alpha|)!} z^{\alpha-\beta}, & \alpha \succ \beta; \\ \frac{n!\alpha!}{(n+|\alpha|)!}, & \alpha = \beta; \\ \frac{\beta!}{(\beta-\alpha)!}\frac{(n+|\beta|-|\alpha|)!}{(n+|\beta|)!} \bar{z}^{\beta-\alpha}, & \alpha \prec \beta; \\ 0, & else. \end{cases}$$

Let $\mathcal{N} = span\{z^\alpha \bar{z}^\beta - Q(z^\alpha \bar{z}^\beta) : \alpha, \beta \geq 0\}$ and we have the following Lemma.

Lemma 2.1. *Set $\mathcal{N}$ is dense in $(A_h^2)^\perp$.*

Proof. Since polynomials are dense in L^2 and $I - Q$ is a bounded operator, we get that $\mathcal{N}$ is dense in $(A_h^2)^\perp$. □

The following Lemma will be useful for the proof of the main theorem.

Lemma 2.2. *Suppose $f \in L^\infty(B_n)$ is holomorphic, then we have $R_f((A_h^2)^\perp) \subset A^2$, $R_{\bar{f}}((A_h^2)^\perp) \subset \overline{A^2}$.*

Proof. Since $\mathcal{N}$ is dense in $(A_h^2)^\perp$, it suffices to prove $R_f[z^\alpha \bar{z}^\beta - Q(z^\alpha \bar{z}^\beta)] \in A^2$ for $\alpha, \beta \in \mathbb{N}^n - \{0\}$. Since $f \in L^\infty$ is holomorphic, we have $f = \sum_{|m| \geq 0} a_m z^m$. For $\alpha = \beta$, it follows

$$\begin{aligned} &R_f\left[z^\alpha \bar{z}^\alpha - Q(z^\alpha \bar{z}^\alpha)\right] \\ &= R_f\left[z^\alpha \bar{z}^\alpha - \frac{n!\alpha!}{(n+|\alpha|)!}\right] \\ &= Q\left[\sum_{|m|\geq 0} a_m z^{m+\alpha}\bar{z}^\alpha - \frac{n!\alpha!}{(n+|\alpha|)!}\sum_{|m|\geq 0} a_m z^m\right] \\ &= \sum_{|m|\geq 0} a_m \left[\frac{(m+\alpha)!}{m!}\frac{(n+|m|)!}{(n+|m|+|\alpha|)!} - \frac{n!\alpha!}{(n+|\alpha|)!}\right] z^m \in A^2. \end{aligned}$$

For $\alpha \succ \beta$, a direct computation gives that

$$\begin{aligned} &R_f\left[z^\alpha \bar{z}^\beta - \frac{\alpha!}{(\alpha-\beta)!}\frac{(n+|\alpha|-|\beta|)!}{(n+|\alpha|)!} z^{\alpha-\beta}\right] \\ &= \sum_{|m|\geq 0} a_m \left[\frac{(\alpha+m)!(n+|\alpha|+|m|-|\beta|)!}{(\alpha+m-\beta)!(n+|\alpha|+|m|)!} - \frac{\alpha!(n+|\alpha|-|\beta|)!}{(\alpha-\beta)!(n+|\alpha|)!}\right] z^{\alpha+m-\beta}, \end{aligned}$$

which is also in A^2. For $\alpha \prec \beta$, it is obtained that

$$\begin{aligned} &R_f\left[z^\alpha\bar{z}^\beta - \frac{\beta!}{(\beta-\alpha)!}\frac{(n+|\beta|-|\alpha|)!}{(n+|\beta|)!}\bar{z}^{\beta-\alpha}\right] \\ &= Q\left[\sum_{|m|\geq 0} a_m z^{\alpha+m}\bar{z}^\beta - \frac{\beta!}{(\beta-\alpha)!}\frac{(n+|\beta|-|\alpha|)!}{(n+|\beta|)!}\sum_{|m|\geq 0} a_m z^m\bar{z}^{\beta-\alpha}\right] \\ &= \sum_{m>\beta-\alpha} c(m,\beta,\alpha)a_m z^{m+\alpha-\beta}, \end{aligned}$$

where $c(m,\beta,\alpha) = \frac{(m+\alpha)!(n+|m|+|\alpha|-|\beta|)!}{(m+\alpha-\beta)!(n+|\alpha|+|m|)!} - \frac{\beta!}{(\beta-\alpha)!}\frac{(n+|\beta|-|\alpha|)!}{(n+|\beta|)!}\frac{m!(n+|m|+|\alpha|-|\beta|)!}{(m+\alpha-\beta)!(n+|m|)!}$.

The last case is similar, we omit the proof. Hence we get that if $f \in L^\infty$ and f is holomorphic, we have $R_f((A_h^2)^\perp) \subset A^2$. By a similar discussion, we can deduce that $R_{\bar{f}}((A_h^2)^\perp) \subset \overline{A^2}$. □

The standard orhonormal basis for $\mathbb{C}^n$ consists of the vectors $d_1, d_2, \cdots, d_n$, where d_k is the ordered n-tuple that has 1 in the k-th spot and 0 everywhere else. In the following proposition, we give an answer to the question that when a dual Toeplitz operator equals to zero.

Proposition 2.3. *Suppose $f \in L^\infty$ is a pluriharmonic function. Then $S_f = 0$ if and only if $f \equiv 0$.*

Proof. Assume that $S_f = 0$. Let

$$h_1 = z^{d_1}\bar{z}^{d_1} + \cdots + z^{d_n}\bar{z}^{d_n} - \frac{n}{n+1} \in (H_h^2)^\perp.$$

Put $f(z) = \sum_{|\alpha|=0}^\infty a_\alpha z^\alpha$, a direct computation gives that

$$\begin{aligned} (S_f h_1)(z) &= (I-Q)(fh_1)(z) \\ &= f(z)|z|^2 - \sum_{|\alpha|=0}^\infty a_\alpha z^\alpha \frac{n+|\alpha|}{n+|\alpha|+1} \\ &= \sum_{|\alpha|=0}^\infty a_\alpha z^\alpha[|z|^2 - \frac{n+|\alpha|}{n+|\alpha|+1}] = 0. \end{aligned}$$

Since z is arbitrary, it follows that $f \equiv 0$. The converse part is easy to see. □

If f, g, h, and k are holomorphic functions in B_n, when is $f\bar{g} - h\bar{k}$ $\mathcal{M}$-harmonic? In [31], Zheng give a necessary and sufficient condition for this question. In the following lemma, we give a generalization. For $z, w \in \mathbb{C}^n$, the inner product of z and w is defined by $\langle z, w\rangle_{C^n} = \sum_{j=1}^n z_j\bar{w}_j$. The following lemma is important to the proof of commuting dual Toeplitz operators.

Lemma 2.4. *[35] Suppose $f_1, \cdot, f_N$ and $g_1, \cdots, g_N$ are holomorphic functions. Then $f_1\overline{g_1} + \cdots + f_N\overline{g_N}$ is pluriharmonic if and only if there is a $N \times N$ unitary matrix*

$$U = \begin{pmatrix} \overline{u_{11}} & \cdots & \overline{u_{1N}} \\ \vdots & \ddots & \vdots \\ \overline{u_{N1}} & \cdots & \overline{u_{NN}} \end{pmatrix} = \begin{pmatrix} \overline{u_1} \\ \vdots \\ \overline{u_N} \end{pmatrix}$$

and some $1 \leq k \leq N+1$ such that $\langle (f_1, \cdots, f_N), u_j\rangle_{C^N}$ are constants for $1 \leq j \leq k-1$, and $\langle (g_1, \cdots, g_N), u_j\rangle_{\mathbb{C}^N}$ are constants for $k \leq j \leq N$.

3. Hyponormal dual Toeplitz operator

Let H be a complex Hilbert space and T be a bounded linear operator acting on H with adjoint T^*. Operator T is said to be hyponormal if $[T^*, T] = T^*T - TT^* \geq 0$. That is for all $u \in H$,

$$\langle [T^*, T]u, u\rangle \geq 0.$$

Lemma 3.1. *S_φ is hyponormal if and only if $\|R_\varphi u\|^2 \geq \|R_{\overline{\varphi}} u\|^2$ for all $u \in (A_h^2)^\perp$.*

Proof. Since $\langle [S_\varphi^*, S_\varphi]\rangle = S_\varphi^* S_\varphi - S_\varphi S_\varphi^* = S_{\overline{\varphi}} S_\varphi - S_\varphi S_{\overline{\varphi}} = H_{\overline{\varphi}} R_\varphi - H_\varphi R_{\overline{\varphi}}$, it follows that for all $u \in (A_h^2)^\perp$,

$$\begin{aligned}\langle [S_\varphi^*, S_\varphi]u, u\rangle &= \langle (H_{\overline{\varphi}} R_\varphi - H_\varphi R_{\overline{\varphi}})u, u\rangle = \langle \overline{\varphi} Q(\varphi u) - \varphi Q(\overline{\varphi} u), u\rangle \\ &= \langle Q(\varphi u), \varphi u\rangle - \langle Q(\overline{\varphi} u), \overline{\varphi} u\rangle = \|R_\varphi u\|^2 - \|R_{\overline{\varphi}} u\|^2.\end{aligned}$$

The proof is completed. □

With this lemma, we have the following proposition and theorem.

Proposition 3.2. *Let $\varphi(z) = z^m$ where m is a nonzero multi-index, then S_φ is not hyponormal.*

Proof. Given $u(z) = z^{d_1}\overline{z}^{\alpha+d_1} - \frac{\alpha_1+1}{n+|\alpha|+1}\overline{z}^\alpha$, where neither $\alpha \succ m$ nor $\alpha \prec m$. A direct computation gives

$$R_\varphi u = Q[z^{m+d_1}\overline{z}^{\alpha+d_1} - \frac{\alpha_1+1}{n+|\alpha|+1} z^m \overline{z}^\alpha] = 0,$$

and

$$\begin{aligned} R_{\overline{\varphi}} u &= Q[z^{d_1}\overline{z}^{(m+\alpha+d_1)} - \frac{\alpha_1+1}{n+|\alpha|+1}\overline{z}^{m+\alpha}] \\ &= [\frac{m_1+\alpha_1+1}{n+|m|+|\alpha|+1} - \frac{\alpha_1+1}{n+|\alpha|+1}]\overline{z}^{m+\alpha}. \end{aligned}$$

Choose a multi-index α such that $[\frac{m_1+\alpha_1+1}{n+|m|+|\alpha|+1} - \frac{\alpha_1+1}{n+|\alpha|+1}] \neq 0$, it follows $\|R_{\overline{\varphi}} u\| > \|R_\varphi u\|$. Hence S_φ is not hyponormal. □

Theorem 3.3. *Suppose that $\varphi(z) = z^\alpha \overline{z}^\beta$, then S_φ is hyponormal if and only if $\alpha = \beta$.*

Proof. First assume that $\alpha = \beta$, then $S_\varphi = S_\varphi^*$, and S_φ is self-adjoint. It follows that $[S_\varphi, S_\varphi^*] = 0$ which implies that S_φ is normal operator.

Secondly, if $\alpha \succ \beta$. Given $u = z^{d_1}\overline{z}^{m+d_1} - \frac{m_1+1}{n+|m|+1}\overline{z}^m$, where neither $m + \beta \succ \alpha$ nor $m + \beta \prec \alpha$.

$$R_\varphi u = Q[z^{\alpha+d_1}\overline{z}^{m+\beta+d_1} - \frac{m_1+1}{n+|m|+1} z^\alpha \overline{z}^{m+\beta}] = 0,$$

and

$$\begin{aligned} R_{\overline{\varphi}} u &= Q[z^{\beta+d_1}\overline{z}^{m+\alpha+d_1} - \frac{m_1+1}{n+|m|+1} z^\beta \overline{z}^{\alpha+m}] \\ &= z^{m+\alpha-\beta}\left[\frac{(m+\alpha+d_1)!}{(m+\alpha-\beta)!}\frac{(n+|m|+|\alpha|-|\beta|)!}{(n+|m|+|\alpha|+1)!}\right. \\ &\left. - \frac{m_1+1}{n+|m|+1}\frac{(m+\alpha)!}{(m+\alpha-\beta)!}\frac{(n+|m|+|\alpha|-|\beta|)!}{(n+|m|+|\alpha|)!}\right] \\ &= \frac{(m+\alpha)!(n+|m|+|\alpha|-|\beta|)!}{(m+\alpha-\beta)!(n+|m|+|\alpha|)!}\left[\frac{m_1+\alpha+1}{n+|m|+|\alpha|+1} - \frac{m_1+1}{n+|m|+1}\right] z^{m+\alpha-\beta}, \end{aligned}$$

which implies that S_φ is not hyponormal with the condition $\alpha \succ \beta$.

The proof of the case $\alpha \prec \beta$ is similar.

If neither $\alpha \succ \beta$ nor $\alpha \prec \beta$, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n), \beta = (\beta_1, \beta_2, \cdots, \beta_n)$, without loss of generality, $\alpha_1 > \beta_1, \alpha_2 < \beta_2$. Put $m = (m_1, 0, m_3, \cdots, m_n)$, which $m_1, m_3, \cdots, m_n$ large enough that $m_1 + \beta_1 > \alpha_1, m_3 + \alpha_3 > \beta_3, m_3 + \beta_3 > \alpha_3 \cdots, m_n + \alpha_n > \beta_n, m_n + \beta_n > \alpha_n$. Let $u = z^{m+d_1}\overline{z}^{d_1} - \frac{m_1+1}{n+|m|+1} z^m$, a direct computation gives

$$R_\varphi u = Q(z^{\alpha+m+d_1}\overline{z}^{\beta+d_1} - \frac{m_1+1}{n+|m|+1} z^{\alpha+m}\overline{z}^{\beta}) = 0,$$

and

$$\begin{aligned} R_{\overline{\varphi}} u &= Q(z^{\beta+m+d_1}\overline{z}^{\alpha+d_1} - \frac{m_1+1}{n+|m|+1} z^{\beta+m}\overline{z}^{\alpha}) \\ &= \frac{(\beta+m)!(n+|\beta|+|m|-|\alpha|)!}{(\beta+m-\alpha)!(n+|\beta|+|m|)!}\left[\frac{\beta_1+m_1+1}{n+|\beta|+|m|+1} - \frac{m_1+1}{n+|m|+1}\right] z^{m+\beta-\alpha}. \end{aligned}$$

Choose a m such that $\frac{\beta_1+m_1+1}{n+|\beta|+|m|+1} - \frac{m_1+1}{n+|m|+1} \neq 0$, it follows that $\|R_{\overline{\varphi}} u\| > \|R_\varphi u\|$ which implies that the operator is not hyponormal. □

Theorem 3.4. *Suppose that φ is a bounded holomorphic function on B_n, S_φ is hyponormal if and only if φ is a constant function.*

Proof. If φ is a constant, it follows that $S_\varphi = S_\varphi^*$, which implies that S_φ is normal. It suffices to prove that S_φ is not hyponormal when φ is not a constant holomorphic function. By lemma 3.1, we only need to prove that there exist a function $u \in (A_h^2)^\perp$ such that $\|R_{\overline{\varphi}} u\| > \|R_\varphi u\|$.

Let $u = (I-Q)(\Sigma_{j=1}^n z^{d_j}\overline{z}^{m+d_j}) = \Sigma_{j=1}^n z^{d_j}\overline{z}^{m+d_j} - \frac{n+|m|}{n+|m|+1}\overline{z}^m$, where m is a nonzero multi-index. Suppose that $\varphi = \sum_{|\alpha|=0}^\infty a_\alpha z^\alpha$, a direct computation gives

$$\begin{aligned} R_{\overline{\varphi}} u &= Q\left[\sum_{|\alpha|=0}^\infty \overline{a_\alpha}\overline{z}^\alpha (\Sigma_{j=1}^n z^{d_j}\overline{z}^{m+d_j} - \frac{n+|m|}{n+|m|+1}\overline{z}^m)\right] \\ &= \sum_{|\alpha|=0}^\infty \overline{a_\alpha}\left[\Sigma_{j=1}^n \frac{m_j+\alpha_j+1}{n+|m|+|\alpha|+1}\overline{z}^{\alpha+m} - \frac{n+|m|}{n+|m|+1}\overline{z}^{\alpha+m}\right] \\ &= \sum_{|\alpha|=0}^\infty \overline{a_\alpha}\left[\frac{n+|m|+|\alpha|}{n+|m|+|\alpha|+1} - \frac{n+|m|}{n+|m|+1}\right]\overline{z}^{\alpha+m} \\ &= \sum_{|\alpha|=0}^\infty \overline{a_\alpha}\frac{|\alpha|\overline{z}^{\alpha+m}}{(n+|m|+|\alpha|+1)(n+|m|+1)}, \end{aligned}$$

and

$$\begin{aligned} R_\varphi u &= Q\left[\sum_{|\alpha|=0}^\infty a_\alpha z^\alpha (\Sigma_{j=1}^n z^{d_j}\overline{z}^{m+d_j} - \frac{n+|m|}{n+|m|+1}\overline{z}^m)\right] \\ &= \sum_{\alpha \succ m} a_\alpha \frac{\alpha!}{(\alpha-m)!}\frac{(n+|\alpha|-|m|)!}{(n+|\alpha|)!}\left[\frac{n+|\alpha|}{n+|\alpha|+1} - \frac{n+|m|}{n+|m|+1}\right] z^{\alpha-m} \\ &= \sum_{\alpha \succ m} a_\alpha \frac{\alpha!}{(\alpha-m)!}\frac{(n+|\alpha|-|m|)!}{(n+|\alpha|)!}\frac{|\alpha|-|m|}{(n+|\alpha|+1)(n+|m|+1)} z^{\alpha-m}. \end{aligned}$$

It follows that

$$\begin{aligned}\|R_{\overline{\varphi}}u\|^2 &= \sum_{|\alpha|=0}^{\infty} |a_\alpha|^2 \frac{|\alpha|^2}{(n+|m|+|\alpha|+1)^2(n+|m|+1)^2} \frac{n!(m+\alpha)!}{(n+|m|+|\alpha|)!} \\ &= \sum_{|\alpha|=0}^{\infty} \frac{|a_\alpha|^2 n!\alpha!}{(n+|\alpha|)!(n+|m|+1)^2} \frac{|\alpha|^2}{(n+|m|+|\alpha|+1)^2} C_1\end{aligned}$$

where $C_1 = \frac{(\alpha_1+1)\cdots(\alpha_1+m_1)\cdots(\alpha_n+1)\cdots(\alpha_n+m_n)}{(n+|\alpha|+1)\cdots(n+|\alpha|+|m|)}$. Also we have

$$\|R_\varphi u\|^2 = \sum_{\alpha > m} \frac{|a_\alpha|^2 n!\alpha!}{(n+|\alpha|)!(n+|m|+1)^2} \frac{(|\alpha|-|m|)^2}{(n+|\alpha|+1)^2} C_2,$$

where $C_2 = \frac{\alpha!}{(\alpha-m)!} \frac{(n+|\alpha|-|m|)!}{(n+|\alpha|)!}$. A direct computation gives

$$\frac{|\alpha|}{n+|m|+|\alpha|+1} - \frac{|\alpha|-|m|}{n+|\alpha|+1} = \frac{|m|(n+|m|+1)}{(n+|\alpha|+1)(n+|m|+|\alpha|+1)} > 0,$$

hence

$$\frac{|\alpha|^2}{(n+|m|+|\alpha|+1)^2} > \frac{(|\alpha|-|m|)^2}{(n+|\alpha|+1)^2}.$$

Let $m = d_j$, $j = 1, 2, \cdots, n$, it follows

$$C_1 = \frac{\alpha_j + 1}{n+|\alpha|+1}, C_2 = \frac{\alpha_j}{n+|\alpha|}.$$

Hence

$$\begin{aligned}C_1 - C_2 &= \frac{\alpha_j+1}{n+|\alpha|+1} - \frac{\alpha_j}{n+|\alpha|} \\ &= \frac{(\alpha_j+1)(n+|\alpha|) - \alpha_j(n+|\alpha|+1)}{(n+|\alpha|+1)(n+|\alpha|)} \\ &= \frac{n+|\alpha|-\alpha_j}{(n+|\alpha|+1)(n+|\alpha|)} > 0.\end{aligned}$$

Then we have $\|R_{\overline{\varphi}}u\| > \|R_\varphi u\|$ which implies that S_φ is not hyponormal where φ is not a constant holomorphic function. The proof is complete. □

4. Commuting dual Toeplitz operators

In this section, we will present the proof of the main results.

Theorem 4.1. *Suppose that $f, g \in L^\infty(B_n)$ are pluriharmonic functions, then $S_{fg} = S_f S_g$ if and only if one of the following statements holds:*

(1) Both f and g are holomorphic;

(2) Both $\overline{f}$ and $\overline{g}$ are holomorphic;

(3) Either f or g is constant.

Proof. If (1) holds, we have that $R_g((A_h^2)^\perp)$ is contained in $A^2(B_n)$. It follows that $H_fR_g = 0$. The desired result follows from the equation $S_{fg} = H_fR_g + S_fS_g$. The case (2) is similar. The case (3) is easy to get the desired result.

To prove the necessity, suppose that $S_{fg} = S_fS_g$. Then we have $H_fR_g = 0$. Since f and g are pluriharmonic functions, there exist holomorphic functions f_1, f_2, g_1, g_2 such that $f = f_1 + \overline{f_2},\ \ g = g_1 + \overline{g_2}$. Without loss of generality, we assume that $f(0) = g(0) = 0$. And $g_1 = \sum_{|\alpha|>0} a_\alpha z^\alpha,\ g_2 = \sum_{|\beta|>0} b_\beta z^\beta$. Let

$$h_1 = z^{d_1}\bar{z}^{d_1} + \cdots + z^{d_n}\bar{z}^{d_n} - \frac{n}{n+1} \in ((A_h^2)^\perp.$$

By a direct calculation, we have

$$\begin{aligned} Q(g_1h_1) &= Q\left[\sum_{|\alpha|>0} a_\alpha(z^{\alpha+d_1}\bar{z}^{d_1} + \cdots + z^{\alpha+d_n}\bar{z}^{d_n} - \frac{n}{n+1}z^\alpha)\right] \\ &= \sum_{|\alpha|>0} a_\alpha z^\alpha\left[\frac{(\alpha+d_1)!(n+|\alpha|)!}{\alpha!(n+|\alpha|+1)!} + \cdots + \frac{(\alpha+d_n)!(n+|\alpha|)!}{\alpha!(n+|\alpha|+1)!} - \frac{n}{n+1}\right] \\ &= \sum_{|\alpha|>0} a_\alpha z^\alpha \frac{|\alpha|}{(n+|\alpha|+1)(n+1)}. \end{aligned}$$

Similarly, we have

$$Q(\overline{g_2}h_1) = \sum_{|\beta|>0} b_\beta z^\beta \frac{|\beta|}{(n+|\beta|+1)(n+1)}.$$

Since $H_fR_gh_1 = 0$, it follows

$$(I-Q)[(f_1+\overline{f_2})(Q(g_1h_1) + Q(\overline{g_2}h_1))] = 0.$$

It is obtained $f_1Q(\overline{g_2}h_1) + Q(g_1h_1)\overline{f_2} \in A_h^2$. By Theorem 5.6 in [31], we have $f_1Q(\overline{g_2}h_1) + Q(g_1h_1)\overline{f_2} \in A_h^2$ implies that one of the following statements holds:

(1) Both f and g are holomorphic ;

(2) Both $\overline{f}$ and $\overline{g}$ are holomorphic;

(3) Either f or g is constant;

(4) There is a nonzero constant t_1 such that $f_1 - t_1Q(g_1h_1)$ and $f_2 + \overline{t_1}Q(\overline{g_2}h_1)$ are constants.

Then it suffices to prove that $t_1 = 0$ in condition (4) when both $Q(g_1h_1)$ and $Q(\overline{g_2}h_1)$ are not constants. It follows that both g_1 and g_2 are not constant. Let

$$h_2 = z^{m+d_1}\bar{z}^{d_1} - \frac{m_1+1}{n+|m|+1}z^m \in (A_h^2)^\perp,$$

m can be chosen such that

$$Q(g_1h_2) = z^m \sum_{|\alpha|\geq 1} a_\alpha[\frac{\alpha_1+m_1+1}{n+|m|+|\alpha|+1} - \frac{m_1+1}{n+|m|+1}]z^\alpha$$

is not a constant function as g_1 is not a constant function. Since $H_fR_gh_2 = 0$, a direct computation gives that

$$(I-Q)[(f_1+\overline{f_2})(Q(g_1h_2) + Q(\overline{g_2}h_2))] = 0.$$

It follows $(t_1Q(g_1h_1) - t_1\overline{Q(g_2h_1)})[Q(g_1h_2) + Q(\overline{g_2}h_2)] \in A_h^2$.

Case 1. If $Q(\overline{g_2}h_2)$ is a constant function, then $t_1\overline{Q(g_2h_1)}Q(g_1h_2)$ is pluriharmonic. Since $Q(g_2h_1)$ and $Q(g_1h_2)$ are not constant functions, which is a contradiction.

Case 2. If $Q(\overline{g_2}h_2)$ is not a constant function, it follows there exist a constant t_2 such that

$$f_1 = t_2Q[g_1h_2] = t_2z^m\sum_{|\alpha|\geq 1} a_\alpha[\frac{\alpha_1+m_1+1}{n+|m|+|\alpha|+1}-\frac{m_1+1}{n+|m|+1}]z^\alpha.$$

Since m can be sufficient large , then $f_1 = 0$. Hence we get the desired result. □

Theorem 4.2. *Suppose $f, g \in L^\infty(B_n)$ are pluriharmonic functions, then $S_gS_f = S_fS_g$ if and only if one of the following statement holds:*

(1) Both f and g are holomorphic;

(2) Both $\overline{f}$ and $\overline{g}$ are holomorphic;

(3) There are constants α and β, not both zero, such that $\alpha f + \beta g$ is constant.

Proof. From the equation $S_{fg} = H_fR_g + S_fS_g$, it follows that

$$S_fS_g - S_gS_f = H_gR_f - H_fR_g.$$

Then $S_fS_g = S_gS_f$ if and only if $H_gR_f = H_fR_g$.

Assume that $S_fS_g = S_gS_f$. Then for any $v \in (A_h^2)^\perp$, we have $H_gR_fv = H_fR_gv$. It is obtained that

$$(I-Q)[(f_1+\overline{f_2})Q(g_1v+\overline{g_2}v)] = (I-Q)[(g_1+\overline{g_2})Q(f_1v+\overline{f_2}v)]. \tag{1}$$

By Lemma 2.2, we have $Q(g_1v), Q(f_1v)$ are holomorphic and $\overline{Q(\overline{g_2}v)}, \overline{Q(\overline{f_2}v)}$ are holomorphic. Then we get

$$(I-Q)[f_1Q(g_1v)+\overline{f_2}Q(\overline{g_2}v)] = 0$$

and

$$(I-Q)[g_1Q(f_1v)+\overline{g_2}Q(\overline{f_2}v)] = 0.$$

It follows that

$$(I-Q)[f_1Q(\overline{g_2}v)+\overline{f_2}Q(g_1v)-g_1Q(\overline{f_2}v)-\overline{g_2}Q(f_1v)] = 0. \tag{2}$$

If one of $f_1, g_1, \overline{f_2}, \overline{g_2}$ is a constant function, without loss of generality, assume that f_1 is a constant function, it follows for any $v \in (A_h^2)^\perp$, we get

$$(I-Q)[\overline{f_2}Q(g_1v)-g_1Q(\overline{f_2}v)] = 0.$$

We have $\overline{f_2}Q(g_1v)-g_1Q(\overline{f_2}v)$ is pluriharmonic for all $v \in (A_h^2)^\perp$. By Theorem 5.6 in [31], one of the following holds:

(1) Both g_1 and $\overline{f_2}$ are constants ;

(2) Both g_1 and $Q(g_1v)$ are constants;

(3) Both $Q(\overline{f_2}v)$ and $\overline{f_2}$ are constants;

(4) Both $Q(\overline{f_2}v)$ and $Q(g_1v)$ are constants;

(5) There is a nonzero constant t such that $g_1 - tQ(g_1v)$ and $\overline{f_2} - tQ(\overline{f_2}v)$ are constants.

If g_1 is a constant function, we have both $\overline{f}$ and $\overline{g}$ are holomorphic. If $\overline{f_2}$ is a constant function, then f is a constant function. Assume that neither $\overline{f_2}$ nor g_1 is constant. Then for all $v \in (A^2_h)^\perp$, $\overline{f_2}Q(g_1 v) - g_1 Q(\overline{f_2}v)$ is pluriharmonic if and only if one of the following holds:

(1) Both $Q(\overline{f_2}v)$ and $Q(g_1 v)$ are constants;

(2) There is a nonzero constant t such that $g_1 - tQ(g_1 v)$ and $\overline{f_2} - tQ(\overline{f_2}v)$ are constants.

Since g_1 is holomorphic, $g_1 = \sum_{m\geq 0} a_m z^m$. And g_1 is not a constant, there exists a multi-index $\beta > 0$ such that $a_\beta \neq 0$. For any multi-index $\alpha > \beta$, let $v_\alpha = z^{\alpha+d_1}\bar{z}^{d_1} - \frac{\alpha_1+1}{n+|\alpha|}z^\alpha \in (A^2_h)^\perp$. A direct computation gives

$$Q(g_1 v_\alpha) = z^\alpha \sum_{m\geq 0} a_m \left[\frac{m_1+\alpha_1+1}{n+|m|+|\alpha|+1} - \frac{\alpha_1+1}{n+|\alpha|+1}\right] z^m.$$

We choose a $\alpha' > \beta$ such that $\frac{\beta_1+\alpha'_1+1}{n+|\beta|+|\alpha'|+1} - \frac{\alpha'_1+1}{n+|\alpha'|+1} \neq 0$. Since $a_\beta \neq 0$, it follows that $Q(g_1 v_{\alpha'})$ is not a constant. Then we get that there is a nonzero constant t such that $g_1 - tQ(g_1 v_{\alpha'})$ is constant. Since $\alpha' > \beta$, from the fact that $g_1 - tQ(g_1 v_{\alpha'})$ is constant, we get $a_\beta = 0$, which is a contradiction. Hence if f_1 is a constant function, we have either both $\overline{f}$ and $\overline{g}$ are holomorphic or f is a constant function.

In the following proof , assume that none of $f_1, g_1, \overline{f_2}, \overline{g_2}$ is a constant function. It follows that $f_1 Q(\overline{g_2}v) + \overline{f_2}Q(g_1 v) - g_1 Q(\overline{f_2}v) - \overline{g_2}Q(f_1 v) \in A^2_h$. By Lemma 2.4, we get there is a 4×4 unitary matrix U_v such that for some $1 \leq k \leq 5$, $\langle (f_1, Q(g_1 v), g_1, -Q(f_1 v)), u_j\rangle_{C^4}$ are constants for $1 \leq j \leq k-1$, and $\langle (Q(\overline{g_2}v), \overline{f_2}, -Q(\overline{f_2}v), \overline{g_2}), u_j\rangle_{C^4}$ are constants for $k \leq j \leq 4$.

Case 1. If there exists a $v \in (A^2_h)^\perp$ such that $k = 1$ or $k = 5$, it follows that f_1, g_1 are constants or $\overline{f_2}$, $\overline{g_2}$ are constants since U is a unitary matrix. Hence we get both f and g are holomorphic or both $\overline{f}$ and $\overline{g}$ are holomorphic.

Case 2. If there exists a $v \in (A^2_h)^\perp$ such that $k = 2$ or $k = 4$. We just prove the case of $k = 4$, $k = 2$ is similar. Since $\langle (f_1, Q(g_1 v), g_1, -Q(f_1 v)), u_j\rangle_{C^4}$ are constants for $1 \leq j \leq 3$, it follows that there exist a nonzero constant t_1 and a constant c_1 such that

$$f_1(z) = t_1 g_1(z) + c_1.$$

Then by (2), we get

$$(I-Q)[t_1 g_1 Q(\overline{g_2}v) + \overline{f_2}Q(g_1 v) - g_1 Q(\overline{f_2}v) - t_1\overline{g_2}Q(g_1 v)] = 0,$$

which implies

$$g_1 Q[(t_1\overline{g_2} - \overline{f_2})v)] + Q(g_1 v)[\overline{f_2} - t_1\overline{g_2}]$$

is pluriharmonic. By Theorem 5.6 in [31], one of the following holds:

(1) Both $Q[(t_1\overline{g_2} - \overline{f_2})v)]$ and $Q(g_1 v)$ are constants;

(2) Both $Q[(t_1\overline{g_2} - \overline{f_2})v)]$ and $\overline{f_2} - t_1\overline{g_2}$ are constants;

(3) There is a nonzero constant t_2 such that $Q(g_1 v) - t_2 g_1$ and $Q[(t_1\overline{g_2} - \overline{f_2})v)] + t_2[\overline{f_2} - t_1\overline{g_2}]$ are constants.

If $\overline{f_2} - t_1\overline{g_2}$ is a constant, it follows easily that $f = t_1 g + c$. Assume that $\overline{f_2} - t_1\overline{g_2}$ is not a constant. Then for all $v \in (A^2_h)^\perp$, $g_1 Q[(t_1\overline{g_2} - \overline{f_2})v)] + Q(g_1 v)[\overline{f_2} - t_1\overline{g_2}]$ is pluriharmonic if and only if one of the following holds:

(1) Both $Q[(t_1\overline{g_2} - \overline{f_2})v)]$ and $Q(g_1 v)$ are constants;

(2) There is a nonzero constant t_2 such that $Q(g_1 v) - t_2 g_1$ and $[t_1 Q(\overline{g_2}v) - Q(\overline{f_2}v)] + t_2[\overline{f_2} - t_1\overline{g_2}]$ are constants.

Since g_1 is not a constant, similar to the previous proof, we can find a $v_{\alpha'} \in (A_h^2)^\perp$ such that neither $Q(g_1 v)$ nor $Q(g_1 v) - t_2 g_1$ is constant, which is a contradiction. Hence we get that $f = t_1 g + c$.

Case 3. For all $v \in (A_h^2)^\perp$, we have $k = 3$. For each v, there exist constants t_1, t_2 and c_1 such that

$$f_1 = t_1 Q(f_1 v) + t_2 Q(g_1 v) + c_1.$$

Suppose that $f_1 = \sum a_m z^m$ and $g_1 = \sum b_m z^m$. For multi-index α, let $v = z^{\alpha+d_1}\bar{z}^{d_1} - \frac{\alpha_1+1}{n+|\alpha|+1} z^\alpha$, there exist holomorphic functions h_1 and h_2 such that $Q(f_1 v) = z^\alpha h_1$ and $Q(g_1 v) = z^\alpha h_2$. Then for all multi-index $m < \alpha$, we get $a_m = 0$. Since α can be chosen sufficient large enough, hence we get $f_1 = 0$. Similarly, it follows that $g_1 = 0$ which implies both $\overline{f}$ and $\overline{g}$ are holomorphic. □

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