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Analysis for Unilateral Contact Problem with Coulomb's Friction in Thermo-Electro-Visco-Elasticity

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Abstract. The aim of this paper is to study a Signorini's problem with Coulomb's friction between a thermo-electro-viscoelasticity body and an electrically and thermally conductive foundation. The materiel's behavior is described by the linear thermo-electro-viscoelastic constitutive laws. The variational formulation is written as nonlinear quasivariational inequality for the displacement field, a nonlinear family elliptic variational equations for the electric potential and a nonlinear parabolic variational equations for the temperature field. We prove under some assumption existence of a weak solution to the problem. The thermo-electro-viscoelastic law with a some temperature parameter $\boldsymbol{\alpha} > 0$ is considered. Then we prove its unique solution as well as the convergence of its solution to the solution of the original problem as the temperature parameter $\boldsymbol{\alpha} \to 0$.

1. Introduction

Contact problems with friction between materials are a very common and important phenomenon in mechanical models, engineering, which is why scientists have tried to study and model it. Mathematical modeling of quasi-static contact problem with normal compliance between a piezoelectric body and deformable conductive foundation, can be found in [10], and with rigid foundation in [11]. Analysis contact problems with friction for viscoelastic body was made in [1, 5]. The dynamic evolution with the Tresca model for the frictional problem of viscoelastic body have been studied in [9, 14].

An excellent reference of new model describes the quasi-static process of unilateral contact and friction between a thermo-electro-viscoelastic body and a conductive foundation is [3, 4, 8]. Numerical analysis and error estimates for Signorini's contact problem in thermo-electro-viscoelasticity with non conductive foundation is referred in [7], and with penalized normal compliance contact condition in [4]. A large number of studies the continuous dependence result of the solution on perturbation of the contact conditions, see [1, 5, 9, 11, 12].

In this paper, we investigate to study the processes of quasi-static frictional contact between a thermo-eletro-viscoelastic body and electrically and thermally conductive foundation. We use the Coulomb's friction, and we model the contact by Singorini's condition law. We also establish the existence of a weak

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solution to the **Problem (PV1)**. We introduce the quasi-static problem for an electro-viscoelastic material with the same conditions of contact and friction, we derive its variational formulation, this problem denoted by **Problem (P2)**, we present its existence and uniqueness of a weak solution. The novelty of this paper to study the behavior of the solution of the small temperature parameter when the convergence to zero, and to establish the link to the corresponding solution.

The rest of the paper is structured as follows. In section 2, we present the model of the equilibrium process of the thermo-electro-viscoelastic body in frictional contact with a conductive rigid foundation in **Problem (P1)**. We introduce the notations, list the assumptions on the problem data, derive the variational formulation of the problem and state the result of a weak solution. We present the quasi-static frictional contact problem for viscoelastic material denoted by **Problem (P2)** with the result, Theorem (2.2). We introduce a new contact problem for thermo-electro-viscoelastic with a small temperature parameter, **Problem (P3)**, and its existence and convergence result in Theorem (2.3). The proofs are established in Section 3.

2. Mechanical and weak formulation

2.1. The physical setting

The mechanical setting of our problem is as follows: An piezoelectric body occupying, in its reference configuration the domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) which is supposed to be bounded, and with a sufficiently regular boundary $\partial\Omega = \Gamma$ as showed in Figure 1. We suppose that is divided into three open disjoint measurable parts Γ_D, Γ_N, and Γ_C on one hand and a partition of $\Gamma_D \cup \Gamma_N$ into two open parts Γ_a and Γ_b on the other hand, such that $meas(\Gamma_D) > 0$ and $meas(\Gamma_a) > 0$.

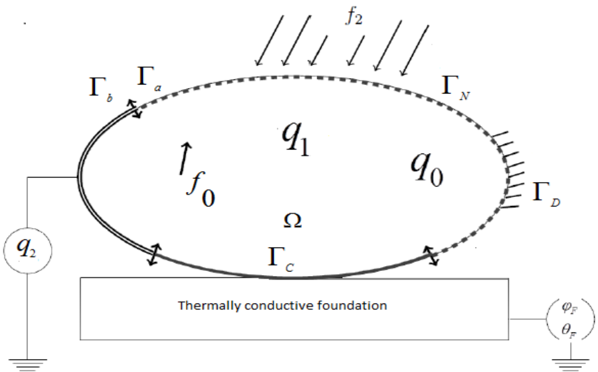

Figure 1: Domain in the initial configuration.

Let $[0; T]$ time interval of interest, where $T > 0$. The body is assumed to be clamped in $\Gamma_D \times (0, T)$ and is submitted to a volume force f_0 in $\Omega \times (0, T)$. A volume electric charge of density q_0 in $\Omega \times (0, T)$, a heat source of constant strength q_1 in $\Omega \times (0, T)$. It also submitted to mechanical, electrical and thermal constants on the boundary. A density of traction forces f_2 in $\Gamma_N \times (0, T)$, and a surface electrical charge of density q_2 in $\Gamma_b \times (0, T)$. The electric potential and the variation of temperature are assumed to be zero, respectively on $\Gamma_a \times (0, T)$ and $(\Gamma_D \cup \Gamma_N) \times (0, T)$. Moreover the body may come in contact over $\Gamma_C \times (0, T)$ an electrically thermally conductive foundation. Assume that its potential and its temperature are maintained at φ_F and θ_F. An unilateral contact is frictional and there may be electrical charges and heat transfer on the contact surface. The normalized gap between $\Gamma_C \times (0, T)$ and the rigid foundation is denoted by g.
We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$, we define the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ by

$$u.v = u_i v_i, \qquad \|v\| = (v.v)^{\frac{1}{2}}, \qquad \forall u, v \in \mathbb{R}^d,$$

$$\sigma.\tau = \sigma_{ij}\tau_{ij}, \qquad \|\tau\| = (\tau.\tau)^{\frac{1}{2}}, \qquad \forall \sigma, \tau \in \mathbb{S}^d.$$

We recall that the usual notation for normal and tangential components of the displacement vector and stress

$$v_\nu = v.\nu, \quad v_\tau = v - v_\nu.\nu \quad \text{and} \quad \sigma_\nu = \sigma\nu.\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu\nu,$$

where ν denote the outward normal vector on Γ. Also, we denote

$$\Sigma = \Omega \times (0,T), \quad \Sigma_a = \Gamma_a \times (0,T), \quad \Sigma_b = \Gamma_b \times (0,T),$$

$$\Sigma_D = \Gamma_D \times (0,T), \quad \Sigma_N = \Gamma_N \times (0,T), \quad \Sigma_C = \Gamma_C \times (0,T).$$

Throughout the article, we adopt the following notation $u : \Sigma \longrightarrow \mathbb{R}^d$ the displacement field, $\sigma = (\sigma_{ij}) : \Omega \longrightarrow \mathbb{S}^d$ the stress tensor, $E(\varphi) = (E_i(\varphi))$ the electric vector field, where $\varphi : \Sigma \longrightarrow \mathbb{R}$ is the electric potential and $D = (Di) : \Omega \longrightarrow \mathbb{R}^d$ the electric displacement field. We also denote $\theta : \Sigma \longrightarrow \mathbb{R}^d$ the temperature, $q = (q_i) : \Omega \longrightarrow \mathbb{R}^d$, the heat flux vector.
Moreover, let $\varepsilon(u) = (\varepsilon_{ij}(u))$ denote the linearized strain tensor given by $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$, and "$Div$" and "$div$" denote the divergence operators for tensor and vector valued functions, respectively, i.e., $Div\ \sigma = (\sigma_{ij,j})$ and $div\ \xi = (\xi_{j,j})$.
• **Problem (P1)**: Find a displacement field $u : \Sigma \longrightarrow \mathbb{R}^d$, an electric potential $\varphi : \Sigma \longrightarrow \mathbb{R}$, and a temperature $\theta : \Sigma \longrightarrow \mathbb{R}$ such that

$$\begin{cases} \sigma(t) = \mathcal{F}\varepsilon(u(t)) - \mathcal{E}^* E(\varphi(t)) - \theta(t)\mathcal{M} + C\varepsilon(\dot{u}(t)) \quad \text{in} \quad \Sigma, & (1)\\ D(t) = \mathcal{E}\varepsilon(u(t)) + \beta E(\varphi(t)) - \theta(t)\mathcal{P} \quad \text{in} \quad \Sigma, & (2)\\ q(t) = -\mathcal{K}\nabla\theta(t) \quad \text{in} \quad \Sigma, & (3) \end{cases}$$

$$\begin{cases} Div\ \sigma(t) = -f_0(t) \quad \text{in} \quad \Sigma, & (4)\\ div\ D(t) = q_0(t) \quad \text{in} \quad \Sigma, & (5)\\ \dot{\theta}(t) + div\ q(t) = q_1(t) \quad \text{in} \quad \Sigma, & (6) \end{cases}$$

$$\begin{cases} u = 0 \quad \text{on} \quad \Sigma_D, & (7)\\ \sigma(t)\nu = f_2(t) \quad \text{on} \quad \Sigma_N, & (8)\\ \varphi = 0 \quad \text{on} \quad \Sigma_a, & (9)\\ D.\nu = q_2 \quad \text{on} \quad \Sigma_b, & (10)\\ \theta = 0 \quad \text{on} \quad (\Sigma_D \cup \Sigma_N), & (11)\\ u(0,x) = u_0, \ \ \theta(0,x) = \theta_0, \quad \text{in} \quad \Omega, & (12) \end{cases}$$

$$\sigma_\nu(u(t)) \le 0, \quad u_\nu(t) - g \le 0, \quad \sigma_\nu(u(t))(u_\nu(t) - g) = 0, \quad \text{on} \quad \Sigma_C, \tag{13}$$

$$\left.\begin{array}{l} \|\sigma_\tau(t)\| \le \mu\left(\|u_\nu(t)\|\right)|B\sigma_\nu(u(t))|, \\ \|\sigma_\tau(t)\| < \mu\left(\|u_\tau(t)\|\right)|B\sigma_\nu(u(t))| \Longrightarrow \dot{u}_\tau(t) = 0, \\ \|\sigma_\tau(t)\| = \mu\left(\|u_\tau(t)\|\right)|B\sigma_\nu(u(t))| \Longrightarrow \exists\lambda \neq 0 \ / \ \sigma_\tau(t) = -\lambda\dot{u}_\tau(t), \end{array}\right\} \text{on } \Sigma_C, \tag{14}$$

$$\begin{cases} D(t).\nu = \psi(u_\nu(t) - g)\phi_L(\varphi(t) - \varphi_F) \quad \text{on} \quad \Sigma_C, & (15)\\ \dfrac{\partial q(t)}{\partial \nu} = k_c(u_\nu(t) - g)\phi_L(\theta(t) - \theta_F) \quad \text{on} \quad \Sigma_C. & (16) \end{cases}$$

The equation (1)-(2) represent the thermo-electro-viscoelastic constitutive law of the material in which denotes $\mathcal{F} = (f_{ijkl})$, $\mathcal{E} = (e_{ijk})$, $\mathcal{M} = (m_{ij})$, $\beta = (\beta_{ij})$, $\mathcal{P} = (p_i)$, and $C = (c_{ijkl})$ are respectively, elastic, piezoelectric, thermal expansion, electric permittivity, pyroelectric tensor, and (fourth-order) viscosity tensor. $\mathcal{E}^*$ is the transpose of $\mathcal{E}$ given by

$$\mathcal{E}^* = (e^*_{ijk}), \text{ where } e^*_{ijk} = e_{kij},$$

and

$$\mathcal{E}\sigma v = \sigma\mathcal{E}^* v, \qquad \forall\sigma \in \mathbb{S}^d, \quad v \in \mathbb{R}^d. \tag{17}$$

The heat flux field q is defined through the thermal conductivity tensor $\mathcal{K} = (k_{ij})$ by the Fourier law of heat conduction (3). The relations (4)-(6) represent the equilibrium equations for the stress, the electric displacement and the heat flux fields. The equations (7)-(11) represent the mechanical, the electrical and the thermal boundary conditions. Moreover (12) represents the initial condition of the problem. Relation (13) models the frictional contact on Γ_C with Signorini's conditions. The unilateral boundary condition (14) represents the Coulomb's friction law in which μ is the coefficient of friction and B is a regularization operator. Finally the equation (15) represents the regularization electrical contact condition on Γ_C (see [10]), and (16) represents the heat flux condition (see [6]) such that:

$$\phi_L(s) = \begin{cases} -L & \text{if} \quad s < -L, \\ s & \text{if} \quad -L \le s \le L, \\ L & \text{if} \quad s > L, \end{cases} \quad \psi(r) = \begin{cases} 0 & \text{if} \quad r < 0, \\ k_e \delta r & \text{if} \quad 0 \le r \le \frac{1}{\delta}, \\ k_e & \text{if} \quad r > \frac{1}{\delta}. \end{cases}$$

Where L is a large positive constant, $\delta > 0$ is a small parameter, and $k_e \ge 0$ is the electrical conductivity coefficient such that the thermal conductance function $k_c : r \longrightarrow k_c(r)$ is supposed to be zero for $r < 0$ and positive otherwise, non-decreasing and Lipschitz continuous.

2.2. Formulation and uniqueness result

To order a variational formulation for the problem (1)-(16), we need to use the following notations:

$$\mathcal{H} = \left\{\sigma \in \mathbb{S}^d \ : \ \sigma = \sigma_{ij},\ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\},$$

$$\mathcal{W} = \left\{D = (D)_i \in \mathbf{H}^1(\Omega) \ : \ D_i \in L^2(\Omega),\ \ div\ D \in L^2(\Omega)\right\},$$

$$\mathbf{L}^2(\Omega) = L^2(\Omega)^d,\ \mathbf{H}^1(\Omega) = H^1(\Omega)^d,\ \text{and}\ \ \mathbf{H}(\Omega) = \mathcal{H}(\Omega)^d.$$

Endowed with the inner products

$$(\sigma, \tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij}dx, \quad (\sigma,\ \tau)_{\mathbf{H}} = (\sigma,\ \tau)_{\mathcal{H}} + (Div\ \sigma,\ Div\ \tau)_{\mathbf{L}^2(\Omega)},$$

$$(D,\ E)_{\mathcal{W}} = (D,\ E)_{\mathbf{L}^2(\Omega)} + (div\ D,\ div\ E)_{L^2(\Omega)},$$

$$(u, v)_{\mathbf{L}^2(\Omega)} = \int_\Omega u_i v_i dx, \qquad (u, v)_{\mathbf{H}^1(\Omega)} = (u, v)_{\mathbf{L}^2(\Omega)} + (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}.$$

Keeping in mind the boundary conditions (7)-(11), we introduce the following function spaces

$$V = \left\{v \in \mathbf{H}^1(\Omega) \ : \ v = 0 \ \text{ on } \ \Gamma_D\right\}, \ \ W = \left\{\xi \in H^1(\Omega) \ : \ \xi = 0 \text{ on } \Gamma_a\right\},$$

$$Q = \left\{\eta \in H^1(\Omega) \ : \ \eta = 0 \text{ on } \Gamma_D \cup \Gamma_N\right\},$$

and the set of admissible displacement

$$V_{ad} = \{v \in V \ : \ v_\nu - g \le 0 \ \text{ on } \ \Gamma_C\},$$

endowed with the inner products

$$(u,\ v)_V = (\varepsilon(u),\ \varepsilon(v))_{\mathcal{H}},\ \ (\varphi,\ \xi)_W = (\nabla\varphi,\ \nabla\xi)_{\mathbf{L}^2(\Omega)}, \text{and}\ (\theta,\ \eta)_Q = (\nabla\theta,\ \nabla\eta)_{\mathbf{L}^2(\Omega)},$$

with the associated norms $\|v\|_V = \|\varepsilon(v)\|_{\mathcal{H}}$, $\|\xi\|_W = \|\nabla\xi\|_{\mathbf{L}^2(\Omega)}$ and $\|\eta\|_Q = \|\nabla\eta\|_{\mathbf{L}^2(\Omega)}$ are equivalent on V, W and Q respectively with the usual norms $\|.\|_{\mathbf{H}^1(\Omega)}$ and $\|.\|_{H^1(\Omega)}$.
Let $\gamma : \mathbf{H}^1(\Omega) \to \mathbf{H}^1_\Gamma$ be the trace map on Γ. Since $meas(\Gamma_D) > 0$, Korn's inequality holds

$$\|\varepsilon(v)\|_{\mathcal{H}} \ge c_K \|v\|_{\mathbf{H}^1(\Omega)}, \qquad \forall v \in V, \tag{18}$$

where c_K is a non-negative constant depending only on Ω and Γ_D.
Notice also that since $meas(\Gamma_a) > 0$, than the following Friedrichs-Poincaré inequalities hold on W and Q, for all $\xi \in W$ and $\eta \in Q$

$$\|\nabla\xi\|_{\mathcal{W}} \geq c_{F_1}\|\xi\|_W, \text{ and } \|\nabla\eta\|_{\mathbf{L}^2(\Omega)} \geq c_{F_2}\|\eta\|_Q, \tag{19}$$

where c_{F_1} and c_{F_2} are the positive constants which depends only on Ω, Γ_a, Γ_D, and Γ_N.
Moreover, using Sobolev's trace theorem, there exists a constant $c_d > 0$, $c_e > 0$ and $c_t > 0$ depends only Ω, Γ_C, Γ_D, Γ_N, and Γ_a such that, for all $v \in V$, $\xi \in W$ and $\eta \in Q$

$$\|v\|_{\mathbf{L}^2(\Gamma)} \leq c_d\|v\|_V, \ \|\xi\|_{L^2(\Gamma_C)} \leq c_e\|\xi\|_W, \text{ and } \|\eta\|_{L^2(\Gamma_C)} \leq c_t\|\eta\|_Q. \tag{20}$$

If $q,\ D \in \mathcal{W}$ are a sufficiently regular functions, the following Green's type formula holds

$$(D,\ \nabla\xi)_{\mathbf{L}^2(\Omega)} + (div\ D,\ \xi)_{L^2(\Omega)} = \int_\Gamma D.\nu\ \xi\ da, \quad \forall\xi \in H^1(\Omega), \tag{21}$$

$$(q,\ \nabla\eta)_{\mathbf{L}^2(\Omega)} + (div\ q,\ \eta)_{L^2(\Omega)} = \int_\Gamma q.\nu\ \eta\ da, \quad \forall\eta \in H^1(\Omega). \tag{22}$$

Finally, for every real Hilbert spaces X, and $1 \leq p \leq \infty$, we use the classical notation for the spaces $L^p(0,T;X)$, $C(0,T;X)$ and $W^{k,p}(0,T;X)$ with their standard norm.
In order to study the mechanical problem, we denote by

$$\begin{cases} a : V \times V \to \mathbb{R}, & (u,v) \mapsto a(u,v) := (\mathcal{F}\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \\ b : W \times W \to \mathbb{R}, & (\varphi,\xi) \mapsto b(\varphi,\xi) := (\beta\nabla\varphi, \nabla\xi)_{\mathbf{L}^2(\Omega)}, \\ c : V \times V \to \mathbb{R}, & (u,v) \mapsto c(u,v) := (C\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \\ d : Q \times Q \to \mathbb{R}, & (\theta,\eta) \mapsto d(\theta,\eta) := (\mathcal{K}\nabla\theta, \nabla\eta)_{\mathbf{L}^2(\Omega)}, \end{cases}$$

are bilinear and symmetric operators. Also denote by

$$\begin{cases} e : V \times W \to \mathbb{R}, & (v,\xi) \mapsto e(v,\xi) := (\mathcal{E}\varepsilon(v), \nabla\xi)_{\mathbf{L}^2(\Omega)} = (\mathcal{E}^*\nabla\xi, \varepsilon(v))_V, \\ m : Q \times V \to \mathbb{R}, & (\theta,v) \mapsto m(\theta,v) := (\theta\mathcal{M}, \varepsilon(v))_Q, \\ p : Q \times W \to \mathbb{R}, & (\theta,\xi) \mapsto p(\theta,\xi) := (\nabla(\mathcal{P}\theta), \nabla\xi)_{\mathbf{L}^2(\Omega)}. \end{cases}$$

are bilinear operators.
In the study of mechanical **Problem (P1)**, we make the following assumptions:

(HP1) The elasticity operator $\mathcal{F} = (f_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$, the electric permittivity tensor $\beta = (\beta_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, the viscosity tensor $C : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ and the thermal conductivity tensor $\mathcal{K} = (k_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfy the usual properties of symmetry, boundedness and ellipticity,

$$\begin{cases} f_{ijkl} = f_{jikl} = f_{lkij} \in L^\infty(\Omega), & \beta_{ij} = \beta_{ji} \in L^\infty(\Omega), \\ c_{ijkl} = c_{jikl} = c_{lkij} \in L^\infty(\Omega), & k_{ij} = k_{ji} \in L^\infty(\Omega), \end{cases}$$

and there exists that $m_{\mathcal{F}}, m_\beta, m_C, m_{\mathcal{K}} > 0$ such that

$$\begin{cases} f_{ijkl}(x)\xi_k\xi_l \geq m_{\mathfrak{F}}\|\xi\|^2, & c_{ijkl}(x)\xi_k\xi_l \geq m_C\|\xi\|^2, \quad \forall\xi \in \mathbb{S}^d, \quad \forall x \in \Omega, \\ \beta_{ij}\zeta_i\zeta_j \geq m_\beta\|\zeta\|^2, & k_{ij}\zeta_i\zeta_j \geq m_{\mathcal{K}}\|\zeta\|^2, \quad \forall\zeta \in \mathbb{R}^d. \end{cases}$$

(HP2) The piezoelectric tensor $\mathcal{E} = (e_{ijk}) : \Omega \times \mathbb{S}^d \to \mathbb{R}$, the thermal expansion tensor $\mathcal{M} = (m_{ij}) : \Omega \times \mathbb{R} \to \mathbb{R}$ and the pyroelectric tensor $\mathcal{P} = (p_i) : \Omega \to \mathbb{R}^d$ satisfy:

$$e_{ijk} = e_{ikj} \in L^\infty(\Omega), \qquad m_{ij} = m_{ji} \in L^\infty(\Omega), \qquad p_i \in L^\infty(\Omega).$$

(HP3) From the hypothesis (HP_1), we obtain

$$\begin{cases} |a(u,v)| \leq M_{\mathcal{F}} \|u\|_V \|v\|_V, \quad |b(\varphi,\xi)| \leq M_{\beta} \|\varphi\|_W \|\xi\|_W, \\ |c(u,v)| \leq M_{C} \|u\|_V \|v\|_V, \quad |d(\theta,\eta)| \leq M_{\mathcal{K}} \|\theta\|_Q \|\eta\|_Q, \\ |e(v,\xi)| \leq M_{\mathcal{E}} \|v\|_V \|\xi\|_W, \ |m(\theta,v)| \leq M_{\mathcal{M}} \|\theta\|_Q \|v\|_V, \\ |p(\theta,\xi)| \leq M_{\mathcal{P}} \|\theta\|_Q \|\xi\|_W. \end{cases}$$

(HP4) The surface electrical conductivity ψ and the thermal conductance k_c satisfy the following hypothesis:

$$\begin{cases} \psi : \Gamma_C \times \mathbb{R} \to \mathbb{R}^+, \\ \exists M_{\psi} > 0 \ \text{ such that } |\psi(x,u)| \leq M_{\psi}, \quad \forall u \in \mathbb{R}, \ a.e. \ x \in \Gamma_C, \\ x \to \psi(x,u) \text{ is measurable on } \Gamma_C \ \text{ for all } u \in \mathbb{R}, \\ x \to \psi(x,u) = 0 \text{ for all } u \leq 0, \\ \exists L_{\psi} > 0 \ \text{ such that } |\psi(.,u_1) - \psi(.,u_2)| \leq L_{\psi} |u_1 - u_2|, \ \forall u_1, \ u_2 \in \mathbb{R}. \end{cases}$$

$$\begin{cases} k_c : \Gamma_C \times \mathbb{R} \to \mathbb{R}^+, \\ \exists M_{k_c} > 0 \ \text{ such that } |k_c(x,u)| \leq M_{k_c}, \quad \forall u \in \mathbb{R}, \ a.e. \ x \in \Gamma_C, \\ x \to k_c(x,u) \text{ is measurable on } \Gamma_C \ \text{ for all } u \in \mathbb{R}, \\ x \to k_c(x,u) = 0 \text{ for all } u \leq 0, \\ \exists L_{k_c} > 0 \ \text{ such that } |k_c(.,u_1) - \psi(.,u_2)| \leq L_{k_c} |u_1 - u_2|, \ \forall u_1, \ u_2 \in \mathbb{R}. \end{cases}$$

(HP5) The coefficient of friction $\mu : \Gamma_C \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies

$$\begin{cases} \exists L_{\mu} > 0, \ \forall a, b \in \mathbb{R}^+, \ \left|\mu(.,a) - \mu(.,b)\right| < L_{\mu} |a - b| \ a.e. \text{ on } \Gamma_C. \\ \text{For all } a \in \mathbb{R}^+, \ \text{the mapping } x \mapsto \mu(x,a) \text{ is meaeurable on } \Gamma_C. \\ \text{For all } a \in \mathbb{R}^+, \ \text{the mapping } x \mapsto \mu(x,a) \text{ is } \mu^*\text{-bounded } a.e. \text{ on } \Gamma_C, \end{cases}$$

with

$$\mu^* = \sup_{t \in [0,T]} \|\mu\|_{L^{\infty}(\Gamma_C)}.$$

The mapping $B : H'_{\Gamma_C} \to L^{\infty}(\Gamma_C)$ is linear compact and continuous with $c_B = \|B\|$.

(HP6) The forces, the traction, the volume, the surfaces charge densities, the strength of the heat source satisfies

$$\begin{cases} f_0 \in W^{1,\infty}\left(0,T; L^2(\Omega)^d\right), \ \ f_2 \in W^{1,\infty}\left(0,T; L^2(\Gamma_N)^d\right), \\ \ q_0 \in W^{1,\infty}\left(0,T; L^2(\Omega)\right), q_2 \in W^{1,\infty}\left(0,T; L^2(\Gamma_b)\right), \ \ q_1 \in L^2\left(0,T; L^2(\Omega)\right). \end{cases}$$

The initial conditions, the friction bounded function, the gap function, the potential and temperature satisfy:

$$u_0 \in V_{ad}, \quad \theta_0 \in L^2(\Omega), \quad g \geq 0, \ g \in L^{\infty}(\Gamma_C), \ \varphi_F \in L^2(0,T;\Gamma_C), \text{ and } \theta_F \in L^2(0,T;\Gamma_C).$$

In addition, we assume that the initial conditions u_0, θ_0 satisfy the compatibility condition: there exist $\varphi_0 \in W$ such that

$$b(\varphi_0(t),\xi) - e(u_0(t),\xi) - p(\theta_0(t),\xi) + \ell(u_0(t),\varphi_0(t),\xi) = (q_e(t),\xi)_W, \ \ \forall \xi \in W. \tag{23}$$

This Problem, has a unique solution φ_0 by using the fixed point theorem.
Using the standard procedure based on Green's formula and the equality $E = -\nabla\varphi$, we obtain the following formulation of the problem (1)-(16).

• **Problem (PV1)**: Find a displacement field $u :]0;T[\to V_{ad}$, an electric potential $\varphi :]0;T[\to W$ and a

temperature field $\theta :]0;T[\rightarrow Q$ such that:
For all $v \in V$, $\xi \in W$, $\eta \in Q$ and a.e. $t \in]0;T[$

$$\begin{aligned} &a(u(t), v-\dot{u}(t)) + e(v-\dot{u}(t), \varphi(t)) - m(\theta(t), v-\dot{u}(t)) \\ &+c(\dot{u}(t), v-\dot{u}(t)) + j(u(t), v) - j(u(t), \dot{u}(t)) \geq (f(t), v-\dot{u}(t))_V, \end{aligned} \tag{24}$$

$$b(\varphi(t), \xi) - e(u(t), \xi) - p(\theta(t), \xi) + \ell(u(t), \varphi(t), \xi) = (q_e(t), \xi)_W, \tag{25}$$

$$d(\theta(t), \eta) + (\dot{\theta}(t), \eta)_Q + \chi(u(t), \theta(t), \eta) = (q_{th}(t), \eta)_Q, \tag{26}$$

$$u(0,x) = u_0(x), \ \theta(0,x) = \theta_0(x). \tag{27}$$

Where

$$\begin{cases} (f(t), v)_V = \displaystyle\int_\Omega f_0(t).v dx + \int_{\Gamma_N} f_2(t).v da, \quad \forall v \in V, & (28) \\ (q_e(t), \xi)_W = \displaystyle\int_\Omega q_0(t).\xi dx - \int_{\Gamma_b} q_2(t).\xi da, \quad \forall \xi \in W, & (29) \\ (q_{th}(t), \eta)_Q = \displaystyle\int_\Omega q_1(t).\eta dx, \quad \forall \eta \in Q, & (30) \end{cases}$$

and

$$\begin{cases} j(u(t), v) = \displaystyle\int_{\Gamma_C} \mu(\|u_\tau\|) |B\sigma_\nu(u(t))| \|v_\tau\| da, \quad \forall v \in V, & (31) \\ \ell(u(t), \varphi(t), \xi) = \displaystyle\int_{\Gamma_C} \psi(u_\nu(t) - g)\phi_L(\varphi(t) - \varphi_F)\xi da, \quad \forall \xi \in W, & (32) \\ \chi(u(t), \theta(t), \eta) = \displaystyle\int_{\Gamma_C} k_c(u_\nu(t) - g)\phi_L(\theta(t) - \theta_F)\eta da, \quad \forall \eta \in Q. & (33) \end{cases}$$

Now, we present the existence and uniqueness result of problem (24)-(27)

Theorem 2.1. *Assume that (HP1)-(HP6) and* (28)-(33) *hold. Then problem* (24)-(27) *has a unique solution. Moreover the solution satisfies*

$$u \in W^{2,\infty}(0,T;V), \ \theta \in L^2(0,T;Q) \cap C([0,T];Q) \text{ and } \varphi \in W^{1,\infty}(0,T;W).$$

2.3. A frictional electro-viscoelastic contact problem

In this short section, we study a new model by neglecting the effect of temperature in **Problem (P1)**, we obtain the following mechanical **Problem (P2)** defined by
• **Problem (P2)** : Find a displacement field $u :]0,T[\longrightarrow \mathbb{R}^d$ and an electric potential $\varphi :]0,T[\longrightarrow \mathbb{R}$ such that

$$\sigma(t) = \mathcal{F}\varepsilon(u(t)) - \mathcal{E}^* E(\varphi(t)) + C\varepsilon(\dot{u}(t)) \quad \text{in} \quad \Sigma, \tag{34}$$

$$D(t) = \mathcal{E}\varepsilon(u(t)) + \beta E(\varphi(t)) \quad \text{in} \quad \Sigma. \tag{35}$$

keeping a count the relations (4)-(5), (7)-(10) and (13)-(15) with the data condition

$$u(0,x) = u_0 \quad \text{in} \quad \Omega. \tag{36}$$

For all $v \in V_{ad}$, $\xi \in W$ and $t \in]0,T[$, we have the following variational formulation of this problem.
•**Problem (PV2)** : Find a displacement field $u :]0;T[\rightarrow V_{ad}$, and an electric potential $\varphi :]0;T[\rightarrow W$ such that:

$$\begin{aligned} &a(u(t), v-\dot{u}(t)) + e(v-\dot{u}(t), \varphi(t)) + c(\dot{u}(t), v-\dot{u}(t)) \\ &+j(u(t), v) - j(u(t), \dot{u}(t)) \geq (f(t), v-\dot{u}(t))_V, \ \forall v \in V, \end{aligned} \tag{37}$$

$$b(\varphi(t),\xi) - e(u(t),\xi) + \ell(u(t),\varphi(t),\xi) = (q_e(t),\xi)_W, \ \forall \xi \in W, \tag{38}$$

$$u(0,x) = u_0(x). \tag{39}$$

We have the following existence and uniqueness result.

Theorem 2.2. *Let (HP1)-(HP6),* (28)-(29) *and* (31)-(32) *hold. Then **Problem (PV2)** has a unique solution satisfies*

$$u \in W^{2,\infty}(0,T;V) \text{ and } \varphi \in W^{1,\infty}(0,T;W).$$

2.4. Asymptotic Behavior for Vanishing Temperature

In this section, we consider a temperature perturbation of **Problem (P)**. To this end let $\alpha > 0$ be a small temperature parameter. We replace the thermo-electro-viscoelastic law (1)-(3) in **Problem (P)** by the following linear constitutive equation

$$\begin{aligned} \sigma_\alpha(t) &= \mathcal{F}\varepsilon(u_\alpha(t)) - \mathcal{E}^* E(\varphi_\alpha(t)) - \alpha\theta_\alpha(t)\mathcal{M} + C\epsilon(\dot u_\alpha(t)) \quad \text{in} \quad \Sigma, & (40)\\ D_\alpha(t) &= \mathcal{E}\varepsilon(u_\alpha(t)) + \beta E(\varphi_\alpha(t)) - \alpha\theta_\alpha(t)\mathcal{P} \quad \text{in} \quad \Sigma, & (41)\\ q_\alpha(t) &= -\mathcal{K}\nabla(\alpha\theta_\alpha(t)) \quad \text{in} \quad \Sigma. & (42) \end{aligned}$$

By Riesz's representation theorem, we know that there exists $h(u_\nu,\theta)$ such that

$$\langle h(u_\nu(t),\theta(t)),\eta\rangle_{\Gamma_C} = \chi(u_\alpha(t),\theta_\alpha(t),\eta). \tag{43}$$

During the process of contact (i.e. $u_\nu \geq g$), the heat flux is supposed to be proportional to the difference between the temperature of the foundation and the body's surface temperature. So, when the temperature vanishes inside of the domain, at the surface, difference between the temperature of the foundation and the body's surface temperature must be equal to the body's and surface sources. That is

$$h(u_\nu(t),\theta(t)) = q_{th}(t). \tag{44}$$

Because of the conductivity of the foundation, we assume the following condition

$$\chi(u_\alpha(t),\theta_\alpha(t),\eta) \to (q_{th}(t),\eta)_Q \text{ when } \alpha \to 0. \tag{45}$$

Let **Problem (PV3)** denote the variational problem due to perturbed temperature parameter $\alpha\theta$ and $(u_\alpha,\varphi_\alpha,\theta_\alpha)$ its solution given by.

•**Problem (PV3)**: Find a displacement field $u_\alpha :]0;T[\to V_{ad}$, an electric potential $\varphi_\alpha :]0;T[\to W$ and a temperature field $\theta_\alpha :]0;T[\to Q$ such that:

For all $v \in V$, $\xi \in W$, $\eta \in Q$ and a.e. $t \in]0;T[$

$$\begin{aligned} &a(u_\alpha(t),v-\dot u_\alpha(t)) + e(v-\dot u_\alpha(t),\varphi_\alpha(t)) - \alpha m(\theta_\alpha(t),v-\dot u_\alpha(t)) \\ &+c(\dot u_\alpha(t),v-\dot u_\alpha(t)) + j(u_\alpha(t),v) - j(u_\alpha(t),\dot u_\alpha(t)) \geq (f(t),v-\dot u_\alpha(t))_V, \end{aligned} \tag{46}$$

$$b(\varphi_\alpha(t),\xi) - e(u_\alpha(t),\xi) - \alpha p(\theta_\alpha(t),\xi) + \ell(u_\alpha(t),\varphi_\alpha(t),\xi) = (q_e(t),\xi)_W, \tag{47}$$

$$d(\theta_\alpha(t),\eta) + (\dot\theta_\alpha(t),\eta)_Q + \chi(u_\alpha(t),\theta_\alpha(t),\eta) = (q_{th}(t),\eta)_Q, \tag{48}$$

$$u_\alpha(0,x) = u_0(x), \ \theta_\alpha(0,x) = \theta_0(x). \tag{49}$$

We have the following existence, uniqueness and convergence result.

Theorem 2.3. *Assume the assumptions stated in Theorem* (2.1) *hold, and the condition*

$$m_\beta \geq M_\psi c_e^2, \ m_{\mathcal{K}} \geq M_{k_c} L c_t^2.$$

Then, for all $\alpha > 0$.

(1) *The **Problem (PV3)** has a unique solution* $(u_\alpha,\ \varphi_\alpha,\ \theta_\alpha)$.

(2) *Let* (u,φ) *is a solution of **Problem (PV2)**. The solution* $(u,\ \varphi,\ 0)$ *is a limit when* α *converge to* 0 *of* $(u_\alpha,\ \varphi_\alpha,\ \theta_\alpha)$.

3. Proof of main results

3.1. Proof of Theorem (2.1)

The proof of Theorem (2.1) will be carried out in several steps, and this is based on the argument of nonlinear variational inequality and Banach fixed point.
In the first step, let $z_1 \in L^2(0,T;V)$ be given and consider the problem of finding $u_{z_1} : V_{ad} \to \mathbb{R}^d$ such that

$$c\left(\dot{u}_{z_1}(t), v - \dot{u}_{z_1}(t)\right) + a\left(u_{z_1}(t), v - \dot{u}_{z_1}(t)\right) + \left(z_1(t), v - \dot{u}_{z_1}(t)\right) \tag{50}$$
$$+ j\left(u_{z_1}(t), v\right) - j\left(u_{z_1}(t), \dot{u}_{z_1}(t)\right) \geq \left(f(t), v - \dot{u}_{z_1}(t)\right)$$
$$u_{z_1}(0,x) = u_0(x). \tag{51}$$

The unique solvability of this problem as follows

Lemma 3.1. *For all $v \in V_{ad}$ and for a.e. $t \in]0,T[$, the problem* (50)-(51) *has a unique solution $u_{z_1} \in W^{2,\infty}(0,T;V)$.*

Proof. We use the Riesz's representation theorem to define the operator

$$\left(f_{z_1}(t), v\right)_V = \left(f(t), v\right)_V - \left(z_1(t), v\right)_V. \tag{52}$$

Then, problem (50)-(51) can be written

$$c\left(\dot{u}_{z_1}(t), v - \dot{u}_{z_1}(t)\right) + a\left(u_{z_1}(t), v - \dot{u}_{z_1}(t)\right) \tag{53}$$
$$+ j\left(u_{z_1}(t), v\right) - j\left(u_{z_1}(t), \dot{u}_{z_1}(t)\right) \geq \left(f_{z_1}(t), v - \dot{u}_{z_1}(t)\right),$$
$$u_{z_1}(0,x) = u_0(x). \tag{54}$$

From assumption (HP3), combined with the regularity of f and z_1, it follows that $f_{z_1} \in W^{1,\infty}(0,T;V)$.
By (HP1)-(HP3), (31) and using the theorem 3.5 presented in [15, P. 67-68] we obtain the result. □

In the second step, let $z_2 \in L^2(0,T;Q)$ and we consider the following problem of the temperature.
Find $\theta_{z_2} \in Q$ for all $\eta \in Q$ and a.e., $t \in]0,T[$ such that

$$\left(\dot{\theta}_{z_2}(t), \eta\right) + d\left(\theta_{z_2}(t), \eta\right) + \left(z_2(t), \eta\right) = \left(q_{th}(t), \eta\right), \tag{55}$$
$$\theta_{z_2}(0,x) = \theta_0(x). \tag{56}$$

Lemma 3.2. *There exists a unique solution θ_{z_2} to the problem* (55)-(56)*. Moreover, the solution satisfies $\theta_{z_2} \in L^2(0,T;Q) \cap C([0,T];Q)$.*

Proof. Similar to (52) we define the operator

$$\left(q_{z_2}(t), \eta\right)_Q = \left(q_{th}(t), \eta\right)_Q - \left(z_2(t), \eta\right)_Q. \tag{57}$$

The problem (55)-(56) can be written as follows

$$\left(\dot{\theta}_{z_2}(t), \eta\right) + d\left(\theta_{z_2}(t), \eta\right) = \left(q_{z_2}(t), \eta\right), \tag{58}$$
$$\theta_{z_2}(0,x) = \theta_0(x). \tag{59}$$

From assumptions (HP1)-(HP3), operator d is a hemicontinuous and monotone.
Using (57) and the regularity of q_{th}, we find that $q_{z_2} \in L^2(0,T;Q)$.
Hence, in view of the Theorem presented in [14, P. 48], we have result. □

In the third step, we let $z_3 \in L^2(0,T;W)$ and we present the following problem of electric potential.
Find $\varphi_{z_3} \in W$ for all $\xi \in W$ and a.e., $t \in]0,T[$ such that

$$b\left(\varphi_{z_3}(t), \xi\right) + \left(z_3(t), \xi\right)_W = \left(q_e(t), \xi\right)_W. \tag{60}$$

Lemma 3.3. *For all $\xi \in W$ and for a.e., $t \in]0,T[$, the variational equality* (60) *has a unique solution $\varphi_{z_3} \in W^{1,\infty}(0,T;W)$.*

Proof. Thank to Riesz's representation theorem, we can define the function

$$(q_{z_3}(t), \xi)_Q = (q_e(t), \xi)_Q - (z_3(t), \xi)_Q \,. \tag{61}$$

The problem (60) can be written

$$b\left(\varphi_{z_3}(t), \xi\right) = \left(q_{z_3}(t), \xi\right). \tag{62}$$

From (HP6) and the regularity of u_{z_1}, θ_{z_2} and q_e, we conclude that $q_{z_3} \in W^{1,\infty}(0, T; W)$.
We apply now the Lax-Milgram theorem to deduce that exists a unique element φ_{z_3} satisfies (60). □

In the last step, we denote $z = (z_1, z_2, z_3) \in \mathcal{Z}$ where $\mathcal{Z} = V \times Q \times W$ and we define the operator

$$\Lambda(z)(t) := (\Lambda_1(z)(t), \Lambda_2(z)(t), \Lambda_3(z)(t)), \tag{63}$$

given by

$$\begin{cases} (\Lambda_1(z)(t), v) := e\left(v, \varphi_{z_3}(t)\right) - m\left(\theta_{z_2}(t), v\right), & (64)\\ (\Lambda_2(z)(t), \eta) := \chi\left(u_{z_1}(t), \theta_{z_2}(t), \eta\right), & (65)\\ (\Lambda_3(z)(t), \xi) := -e\left(u_{z_1}(t), \xi\right) - p\left(\theta_{z_2}(t), \xi\right) + \ell\left(u_{z_1}(t), \varphi_{z_3}(t), \xi\right). & (66) \end{cases}$$

We have the following result of operator Λ.

Lemma 3.4. *For $z \in L^2(0, T; \mathcal{Z})$, the operator Λ is continuous. Moreover, there exists a unique element $z^* \in L^2(0, T; \mathcal{Z})$ such that $\Lambda z^* = z^*$.*

Proof. Let $z \in L^2(0, T; \mathcal{Z})$ and $t_1, t_2 \in [0, T]$, using (64) and (HP3) we obtain

$$\|\Lambda_1(z)(t_1) - \Lambda_1(z)(t_2)\|_{\mathcal{Z}} \leq c\left(\|\varphi_{z_3}(t_1) - \varphi_{z_3}(t_2)\|_W + \|\theta_{z_2}(t_1) - \theta_{z_2}(t_2)\|_Q\right). \tag{67}$$

From the regularities of θ_{z_2} and φ_{z_3}, we deduce that $\Lambda_1(z) \in C([0, T], \mathcal{Z})$.
Here and below c denotes a positive generic constant whose value may change from line to line.
By (65) and (HP4), there exists a constant c depending only in c_t, L_{k_c} and M_{k_c} such that

$$\|\Lambda_2(z)(t_1) - \Lambda_2(z)(t_2)\|_{\mathcal{Z}} \leq c\left(\|\theta_{z_2}(t_1) - \theta_{z_2}(t_2)\|_Q + \|u_{z_1}(t_1) - u_{z_1}(t_2)\|_V\right). \tag{68}$$

Then, $\Lambda_2(z) \in C([0, T], \mathcal{Z})$.
In the same way, we have

$$\begin{aligned} \|\Lambda_3(z)(t_1) - \Lambda_3(z)(t_2)\|_{\mathcal{Z}} \quad &\leq \quad c\left(\|u_{z_1}(t_1) - u_{z_1}(t_2)\|_V + \|\theta_{z_2}(t_1) - \theta_{z_2}(t_2)\|_Q\right. \\ &+ \quad \|\varphi_{z_3}(t_1) - \varphi_{z_3}(t_2)\|_W). \end{aligned} \tag{69}$$

Which implies $\Lambda_3(z) \in C([0, T], \mathcal{Z})$. Consequently, the operator Λ is continuous.
Now, we prove that Λ has a unique fixed point, to this let $z, \widehat{z} \in L^2(0, T; \mathcal{Z})$ and $t \in [0, T]$, by similar way to (67)-(69) we find that

$$\left\|\Lambda(z)(t) - \Lambda(\widehat{z})(t)\right\|_{\mathcal{Z}}^2 \quad \leq \quad c\left(\|u_{z_1}(t) - u_{\widehat{z_1}}(t)\|_V^2 + \|\theta_{z_2}(t) - \theta_{\widehat{z_2}}(t)\|_Q^2 + \|\varphi_{z_3}(t) - \varphi_{\widehat{z_3}}(t)\|_W^2\right). \tag{70}$$

From (50) we obtain

$$\begin{aligned} &c\left(\dot{u}_{z_1}(t) - \dot{u}_{\widehat{z_1}}(t), \dot{u}_{z_1}(t) - \dot{u}_{\widehat{z_1}}(t)\right) + a\left(u_{z_1}(t) - u_{\widehat{z_1}}(t), \dot{u}_{z_1}(t) - \dot{u}_{\widehat{z_1}}(t)\right) + \left(z_1(t) - \widehat{z_1}(t), \dot{u}_{z_1}(t) - \dot{u}_{\widehat{z_1}}(t)\right) \\ &+ j\left(u_{z_1}(t), \dot{u}_{z_1}(t)\right) - j\left(u_{z_1}(t), \dot{u}_{\widehat{z_1}}(t)\right) - j\left(u_{\widehat{z_1}}(t), \dot{u}_{z_1}(t)\right) + j\left(u_{\widehat{z_1}}(t), \dot{u}_{\widehat{z_1}}(t)\right) \leq 0. \end{aligned} \tag{71}$$

Moreover, by (HP5) and (31) we get the following inequality

$$\begin{aligned} &\left| j\left(u_{z_1}(t), \dot{u}_{z_1}(t)\right) - j\left(u_{z_1}(t), \dot{u}_{\widehat{z_1}}(t)\right) - j\left(u_{\widehat{z_1}}(t), \dot{u}_{z_1}(t)\right) + j\left(u_{\widehat{z_1}}(t), \dot{u}_{\widehat{z_1}}(t)\right)\right| \\ &\leq c_B L_\mu c_d^2 \left\|u_{z_1}(t) - u_{\widehat{z_1}}(t)\right\|_V \left\|\dot{u}_{z_1}(t) - \dot{u}_{\widehat{z_1}}(t)\right\|_V. \end{aligned} \tag{72}$$

We integrate (71) to 0 at t, combined with (72), (HP1), (HP3) and the inequality

$$\left\|u_{z_1}(t) - u_{\widehat{z_1}}(t)\right\|_V^2 \leq c \int_0^t \left\|\dot{u}_{z_1}(s) - \dot{u}_{\widehat{z_1}}(s)\right\|_V^2 ds, \tag{73}$$

we have

$$\left\|u_{z_1}(t) - u_{\widehat{z_1}}(t)\right\|_V^2 \leq c \left(\int_0^t \left\|u_{z_1}(s) - u_{\widehat{z_1}}(s)\right\|_V^2 ds + \int_0^t \left\|z_1(s) - \widehat{z_1}(s)\right\|_V^2 ds \right). \tag{74}$$

By Gronwall inequality we conclude that

$$\left\|u_{z_1}(t) - u_{\widehat{z_1}}(t)\right\|_V^2 \leq c \int_0^t \left\|z_1(s) - \widehat{z_1}(s)\right\|_V^2 ds. \tag{75}$$

Similar to this inequality, and after some tedious algebraic manipulation, we have the following relations

$$\left\|\theta_{z_2}(t) - \theta_{\widehat{z_2}}(t)\right\|_Q^2 \leq c \int_0^t \left\|z_2(s) - \widehat{z_2}(s)\right\|_Q^2 ds, \tag{76}$$

and

$$\left\|\varphi_{z_3}(t) - \varphi_{\widehat{z_3}}(t)\right\|_W^2 \leq c \int_0^t \left\|z_3(s) - \widehat{z_3}(s)\right\|_W^2 ds. \tag{77}$$

Combining (70) and (75)-(77), are getting

$$\left\|\Lambda(z)(t) - \Lambda(\widehat{z})(t)\right\|_{\mathcal{Z}}^2 \leq c \int_0^t \left\|z(s) - \widehat{z}(s)\right\|_{\mathcal{Z}}^2 ds. \tag{78}$$

Iterating this inequality n times result in

$$\left\|\Lambda^n(z)(t) - \Lambda^n(\widehat{z})(t)\right\|_{\mathcal{Z}}^2 \leq \frac{(cT)^n}{n!} \left\|z(t) - \widehat{z}(t)\right\|_{L^2(0,T;\mathcal{Z})}. \tag{79}$$

Which implies that for n sufficiently large, Λ^n is a contraction operator in the Banach space $L^2(0,T;\mathcal{Z})$. Therefore Λ has a unique fixed point.

We are now have all the ingredients to proof of Theorem (2.1).

Existence: Let $z^* = (z_1^*, z_2^*, z_3^*) \in L^2(0,T;\mathcal{Z})$ be the fixed point of operator Λ. Denote by u^* the solution of (50)-(51) for $z_1 = z_1^*$, θ^* be the solution of (55)-(56) for $z_2 = z_2^*$ and φ^* be the solution of (60) for $z_3 = z_3^*$, using (64)-(66), we find that the triplet $(u^*, \theta^*, \varphi^*)$ is a solution of (24)-(27).

Uniqueness: The uniqueness of the solution follows from the uniqueness of the fixed point of the operator Λ. □

3.2. *Proof of Theorem* (2.2)

Theorem (2.2) has been proved in [10], using arguments of evolutionary variational inequalities and fixed points of operators.

3.3. Proof of Theorem (2.3)

Proof. [Proof of (1) in Theorem (2.3)]
The proof will be carried out in several steps.
For $\boldsymbol{\alpha} > 0$, the operators $\boldsymbol{\alpha} m$ and $\boldsymbol{\alpha} p$ satisfies the condition (HP3).
Then the existence and uniqueness of problem (46)-(49) is proved in the same way that in proof of Theorem (2.1). □

Proof. [Proof of (2) in Theorem (2.3)]
Estimate 1: We take $\eta = \theta_\alpha(t)$ in (48), we obtain

$$d\left(\theta_{\boldsymbol{\alpha}}(t), \theta_{\boldsymbol{\alpha}}(t)\right) + \left(\dot{\theta}_{\boldsymbol{\alpha}}(t), \dot{\theta}_{\boldsymbol{\alpha}}(t)\right) + \chi\left(u_{\boldsymbol{\alpha}}(t), \theta_{\boldsymbol{\alpha}}(t), \theta_{\boldsymbol{\alpha}}(t)\right) = \left(q_{th}(t), \theta_{\boldsymbol{\alpha}}(t)\right)_Q. \tag{80}$$

We use (20), (33) and the bounds $|k_c(u_{\boldsymbol{\alpha}_\nu}(t) - g)| \leq M_{k_c}$ and $|\phi_L(\theta_{\boldsymbol{\alpha}}(t) - \theta_F)| \leq L$, we find

$$\begin{aligned} |\chi\left(u_{\boldsymbol{\alpha}}(t), \theta_{\boldsymbol{\alpha}}(t), \theta_{\boldsymbol{\alpha}}(t)\right)| &\leq \int_{\Gamma_C} |k_c(u_{\boldsymbol{\alpha}_\nu}(t) - g)|.|\phi_L(\theta_{\boldsymbol{\alpha}}(t) - \theta_F)|.|\theta_{\boldsymbol{\alpha}}(t)| da \\ &\leq M_{k_c} L c_t^2 \|\theta_{\boldsymbol{\alpha}}(t)\|_Q^2. \end{aligned} \tag{81}$$

By the following Young's inequality

$$xy \leq \lambda x^2 + \frac{1}{4\lambda} y^2, \quad \forall \lambda > 0, \tag{82}$$

we deduce that

$$|\chi\left(u_{\boldsymbol{\alpha}}(t), \theta_{\boldsymbol{\alpha}}(t), \theta_{\boldsymbol{\alpha}}(t)\right)| \leq \frac{M^2}{4\lambda} + \lambda \|\theta_{\boldsymbol{\alpha}}(t)\|_Q^2, \text{ where } M = M_{k_c} L c_t^2 \text{ and } \lambda > 0. \tag{83}$$

Combining (81) and (83), we have

$$\left(m_{\mathcal{K}} - M_{k_c} L c_t^2\right) \|\theta_{\boldsymbol{\alpha}}(t)\|_Q^2 \leq \frac{1}{2} \frac{d}{dt} \|\theta_{\boldsymbol{\alpha}}(t)\|_Q^2 + \frac{1}{4\lambda} \|q_{th}(t)\|_Q^2 + \lambda \|\theta_{\boldsymbol{\alpha}}(t)\|_Q^2. \tag{84}$$

We integrate from 0 to t for almost all $t \in [0, T]$ and by Gronwall inequality, we have

$$\|\theta_{\boldsymbol{\alpha}}(t)\|_{L^2(0,T;Q)}^2 \leq c\left(\|\theta_{\boldsymbol{\alpha}}(0)\|_Q^2 + \|q_{th}(t)\|_{L^2(0,T;Q)}^2\right). \tag{85}$$

Using the regularity of q_{th} and $\theta_{\boldsymbol{\alpha}}(0)$, we find the following estimate: there exits $c > 0$ such that

$$\|\theta_{\boldsymbol{\alpha}}(t)\|_{L^2(0,T;Q)} \leq c. \tag{86}$$

Estimate 2:
We combine (46) with $v = \dot{u}(t)$ and (24) with $v = \dot{u}_\alpha(t)$, we have

$$\begin{aligned} &c(\dot{u}(t) - \dot{u}_{\boldsymbol{\alpha}}(t), \dot{u}(t) - \dot{u}_{\boldsymbol{\alpha}}(t)) + a(u(t) - u_{\boldsymbol{\alpha}}(t), \dot{u}(t) - \dot{u}_{\boldsymbol{\alpha}}(t)) \\ &+ e(\dot{u}(t) - \dot{u}_{\boldsymbol{\alpha}}(t), \varphi(t) - \varphi_{\boldsymbol{\alpha}}(t)) + \boldsymbol{\alpha} m(\theta_{\boldsymbol{\alpha}}(t), \dot{u}(t) - \dot{u}_{\boldsymbol{\alpha}}(t)) \\ &+ j(u_{\boldsymbol{\alpha}}(t), \dot{u}_{\boldsymbol{\alpha}}(t)) - j(u_{\boldsymbol{\alpha}}(t), \dot{u}(t)) + j(u(t), \dot{u}(t)) - j(u(t), \dot{u}_{\boldsymbol{\alpha}}(t)) \leq 0. \end{aligned} \tag{87}$$

By (20), (31) and (HP5), we obtain

$$\begin{aligned} &\left| j(u_{\boldsymbol{\alpha}}(t), \dot{u}_{\boldsymbol{\alpha}}(t)) - j(u_{\boldsymbol{\alpha}}(t), \dot{u}(t)) + j(u(t), \dot{u}(t)) - j(u(t), \dot{u}_{\boldsymbol{\alpha}}(t)) \right| \\ &\leq c_B L_\mu c_d^2 \|u(t) - u_{\boldsymbol{\alpha}}(t)\|_V . \|\dot{u}(t) - \dot{u}_{\boldsymbol{\alpha}}(t)\|_V. \end{aligned} \tag{88}$$

Using (HP3), (HP3) and (88), we find

$$\begin{aligned} &m_C \|\dot{u}(t) - \dot{u}_{\boldsymbol{\alpha}}(t)\|_V^2 + \frac{1}{2} \frac{d}{dt} a(u(t) - u_{\boldsymbol{\alpha}}(t), u(t) - u_{\boldsymbol{\alpha}}(t)) \leq M_{\mathcal{E}} \|\dot{u}(t) - \dot{u}_{\boldsymbol{\alpha}}(t)\|_V . \|\varphi(t) - \varphi_{\boldsymbol{\alpha}}(t)\|_W \\ &+ \boldsymbol{\alpha} M_{\mathcal{M}} \|\theta_{\boldsymbol{\alpha}}(t)\|_Q . \|\dot{u}(t) - \dot{u}_{\boldsymbol{\alpha}}(t)\|_V + c_B L_\mu c_d^2 \|u(t) - u_{\boldsymbol{\alpha}}(t)\|_V . \|\dot{u}(t) - \dot{u}_{\boldsymbol{\alpha}}(t)\|_V. \end{aligned} \tag{89}$$

By Young's inequality, there exists $\lambda > 0$ such that

$$\left(m_C - \lambda\left(M_{\mathcal{E}} - \boldsymbol{\alpha} M_{\mathcal{M}} + c_B L_\mu c_d^2\right)\right)\|\dot{u}(t) - \dot{u}_{\boldsymbol{\alpha}}(t)\|_V^2 + \frac{1}{2}\frac{d}{dt}a(u(t) - u_{\boldsymbol{\alpha}}(t), u(t) - u_{\boldsymbol{\alpha}}(t)) \tag{90}$$
$$\leq \frac{M_{\mathcal{E}}}{4\lambda}\|\varphi(t) - \varphi_{\boldsymbol{\alpha}}(t)\|_W^2 + \boldsymbol{\alpha}\frac{M_{\mathcal{M}}}{4\lambda}\|\theta_{\boldsymbol{\alpha}}(t)\|_Q^2 + \frac{c_B L_\mu c_d^2}{4\lambda}\|u(t) - u_{\boldsymbol{\alpha}}(t)\|_V^2.$$

We integrate from 0 to t for a.e. $t \in [0, T]$ and using the inequality

$$\|u(t) - u_{\boldsymbol{\alpha}}(t)\|_V \leq c\int_0^t \|\dot{u}(s) - \dot{u}_{\boldsymbol{\alpha}}(s)\|_V ds, \tag{91}$$

with $u_0 = u_{\boldsymbol{\alpha}}(0)$, it follows that

$$\|u(t) - u_{\boldsymbol{\alpha}}(t)\|_V^2 \leq \boldsymbol{\alpha}.c\|\theta_{\boldsymbol{\alpha}}(t)\|_Q^2 + c\int_0^t \|\varphi(s) - \varphi_{\boldsymbol{\alpha}}(s)\|_W ds. \tag{92}$$

Similar to (87), we find the relation

$$b\left(\varphi(t) - \varphi_{\boldsymbol{\alpha}}(t), \varphi(t) - \varphi_{\boldsymbol{\alpha}}(t)\right) - e\left(u(t) - u_{\boldsymbol{\alpha}}(t), \varphi(t) - \varphi_{\boldsymbol{\alpha}}(t)\right) + \boldsymbol{\alpha} p\left(\theta_{\boldsymbol{\alpha}}(t), \varphi(t) - \varphi_{\boldsymbol{\alpha}}(t)\right) \tag{93}$$
$$-\ell\left(u_{\boldsymbol{\alpha}}(t), \varphi_{\boldsymbol{\alpha}}(t), \varphi(t) - \varphi_{\boldsymbol{\alpha}}(t)\right) + \ell\left(u(t), \varphi(t), \varphi(t) - \varphi_{\boldsymbol{\alpha}}(t)\right) = 0.$$

Taking account the hypothesis (HP4) and (32), we obtain

$$\left|\ell\left(u(t), \varphi(t), \varphi(t) - \varphi_{\boldsymbol{\alpha}}(t)\right) - \ell\left(u_{\boldsymbol{\alpha}}(t), \varphi_{\boldsymbol{\alpha}}(t), \varphi(t) - \varphi_{\boldsymbol{\alpha}}(t)\right)\right| \tag{94}$$
$$= \left|\int_{\Gamma_C} \psi(u_\nu(t) - g)\phi_L(\varphi(t) - \varphi_F)(\varphi(t) - \varphi_{\boldsymbol{\alpha}}(t))da - \int_{\Gamma_C} \psi(u_{\boldsymbol{\alpha}_\nu}(t) - g)\phi_L(\varphi_{\boldsymbol{\alpha}}(t) - \varphi_F)(\varphi(t) - \varphi_{\boldsymbol{\alpha}}(t))da\right|$$
$$\leq \left|\int_{\Gamma_C}\left(\psi(u_\nu(t) - g)\phi_L(\varphi(t) - \varphi_F) - \psi(u_{\boldsymbol{\alpha}_\nu}(t) - g)\phi_L(\varphi_{\boldsymbol{\alpha}}(t) - \varphi_F)\right)\left(\varphi(t) - \varphi_{\boldsymbol{\alpha}}(t)\right)da\right|$$
$$\leq \left|\int_{\Gamma_C}\left(\psi(u_\nu(t) - g)\phi_L(\varphi(t) - \varphi_F) - \psi(u_{\boldsymbol{\alpha}_\nu}(t) - g)\phi_L(\varphi(t) - \varphi_F)\right.\right.$$
$$\left.\left.+\psi(u_{\boldsymbol{\alpha}_\nu}(t) - g)\phi_L(\varphi(t) - \varphi_F) - \psi(u_{\boldsymbol{\alpha}_\nu}(t) - g)\phi_L(\varphi_{\boldsymbol{\alpha}}(t) - \varphi_F)\right)\left(\varphi(t) - \varphi_{\boldsymbol{\alpha}}(t)\right)\right|da$$
$$\leq \int_{\Gamma_C}|\psi(u_\nu(t) - u_{\boldsymbol{\alpha}_\nu}(t))|.|\phi_L(\varphi(t) - \varphi_{\boldsymbol{\alpha}}(t))|.|\varphi(t) - \varphi_{\boldsymbol{\alpha}}(t)|da + \int_{\Gamma_C}|\psi(u_{\boldsymbol{\alpha}_\nu}(t) - g)|.|\varphi(t) - \varphi_{\boldsymbol{\alpha}}(t)|^2 da$$
$$\leq M_\psi c_e^2\|\varphi(t) - \varphi_{\boldsymbol{\alpha}}(t)\|_W^2 + LL_\psi c_d c_e\|u(t) - u_{\boldsymbol{\alpha}}(t)\|_V\|\varphi(t) - \varphi_{\boldsymbol{\alpha}}(t)\|_W.$$

From the proprieties of b, e, p and by (94), we conclude that

$$\left(m_\beta - M_\psi c_e^2\right)\|\varphi(t) - \varphi_{\boldsymbol{\alpha}}(t)\|_W^2 \leq \left(M_{\mathcal{E}} + LL_\psi c_d c_e\right)\|u(t) - u_{\boldsymbol{\alpha}}(t)\|_V\|\varphi(t) - \varphi_{\boldsymbol{\alpha}}(t)\|_W + \boldsymbol{\alpha} M_{\mathcal{P}}\|\theta_{\boldsymbol{\alpha}}(t)\|_Q\|\varphi(t) - \varphi_{\boldsymbol{\alpha}}(t)\|_W. \tag{95}$$

Then, there exists a positive constant $c > 0$ such that

$$\|\varphi(t) - \varphi_{\boldsymbol{\alpha}}(t)\|_W \leq c\|u(t) - u_{\boldsymbol{\alpha}}(t)\|_V + \boldsymbol{\alpha}.c\|\theta_{\boldsymbol{\alpha}}(t)\|_Q. \tag{96}$$

Combining (92), (96) and integrating from 0 to t for a.e. $t \in [0, T]$ we get

$$\|u(t) - u_{\boldsymbol{\alpha}}(t)\|_V^2 + \|\varphi(t) - \varphi_{\boldsymbol{\alpha}}(t)\|_W^2 \leq \boldsymbol{\alpha}.c\|\theta_{\boldsymbol{\alpha}}(t)\|_Q^2 + \int_0^t\left(\|u(s) - u_{\boldsymbol{\alpha}}(s)\|_V + \|\varphi(s) - \varphi_{\boldsymbol{\alpha}}(s)\|_W\right)ds. \tag{97}$$

By Gronwall inequality, we have the following estimate

$$\|u(t) - u_{\boldsymbol{\alpha}}(t)\|_{L^2(0,T;V)} + \|\varphi(t) - \varphi_{\boldsymbol{\alpha}}(t)\|_{L^2(0,T;W)} \leq \boldsymbol{\alpha}.c\|\theta_{\boldsymbol{\alpha}}(t)\|_Q. \tag{98}$$

Estimate 3:
Similar to (89), we find

$$m_C\|\dot{u}(t) - \dot{u}_{\boldsymbol{\alpha}}(t)\|_V \leq \boldsymbol{\alpha}.M_{\mathcal{M}}\|\theta_{\boldsymbol{\alpha}}(t)\|_Q + c\left(\|u(t) - u_{\boldsymbol{\alpha}}(t)\|_V + \|\varphi(t) - \varphi_{\boldsymbol{\alpha}}(t)\|_W\right). \tag{99}$$

From (98), we deduce the following estimate

$$\|\dot{u}(t) - \dot{u}_{\alpha}(t)\|_V \leq \boldsymbol{\alpha}.c\|\theta_{\alpha}(t)\|_Q. \tag{100}$$

Convergence:
From the estimates (86), (98) and (100), there exists a sub-sequence of u_{α}, φ_{α} and θ_{α} such that when $\boldsymbol{\alpha} \to 0$ we have

$$u_{\alpha} \rightharpoonup u \text{ in } L^2(0,T,V),\ \dot{u}_{\alpha} \rightharpoonup \dot{u} \text{ in } L^2(0,T,V^*),\ \varphi_{\alpha} \rightharpoonup \varphi \text{ in } L^2(0,T,W),\ \text{and } \theta_{\alpha} \rightharpoonup \theta \text{ in } L^2(0,T,Q). \tag{101}$$

Keeping in mind the assumption (45), we obtain that

$$\theta_{\alpha} \rightharpoonup 0,\ \text{when } \boldsymbol{\alpha} \to 0 \text{ in } L^2(0,T,Q). \tag{102}$$

Moreover, by the compactness of trace operator $\gamma : V \times W \times Q \to L^2(\Gamma_C)^d \times L^2(\Gamma_C) \times L^2(\Gamma_C)$, we deduce that

$$\begin{aligned} &u_{\alpha} \to u \text{ in } L^2(\Gamma_C)^d,\ \dot{u}_{\alpha} \to \dot{u} \text{ in } L^2(\Gamma_C)^d,\ \varphi_{\alpha} \to \varphi \text{ in } L^2(\Gamma_C), \\ &\theta_{\alpha} \to 0 \ \text{ in } L^2(\Gamma_C) \text{ and } \dot{\theta_{\alpha}} \to 0 \text{ in } L^2(\Gamma_C). \end{aligned} \tag{103}$$

Taken into account the hypothesis (HP4)-(HP5), (31)-(32) and (103), we find that

$$\begin{aligned} &(j(u_{\alpha}(t), v) - j(u_{\alpha}(t), \dot{u}_{\alpha}(t))) \to j(u(t), v) - j(u(t), \dot{u}(t)) \ \text{ in } \mathbb{R}, \\ &\ell(u_{\alpha}(t), \varphi_{\alpha}(t), \xi) \to \ell(u(t), \varphi(t), \xi) \text{ in } \mathbb{R}. \end{aligned} \tag{104}$$

In other word, using the condition (45) and (103), we have the limit

$$\left[d(\theta_{\alpha}(t), \eta) + (\dot{\theta}_{\alpha}(t), \eta)_Q + \chi(u_{\alpha}(t), \theta_{\alpha}(t), \eta) - (q_{th}(t), \eta)_Q\right] \to 0 \text{ when } \alpha \to 0. \tag{105}$$

Now, we combine (103)-(105) with the problem (46)-(49), we have the relations

$$\begin{aligned} &a(u(t), v - \dot{u}(t)) + e(v - \dot{u}(t), \varphi(t)) + c(\dot{u}(t), v - \dot{u}(t)) \\ &+ j(u(t), v) - j(u(t), \dot{u}(t)) \geq (f(t), v - \dot{u}(t))_V, \\ &b(\varphi(t), \xi) - e(u(t), \xi) + \ell(u(t), \varphi(t), \xi) = (q_e(t), \xi)_W, \\ &u(0, x) = u_0(x). \end{aligned} \tag{106}$$

We conclude that (u, φ) is the solution of Problem (PV). □

Appendix

In this paragraph, we present the result on the solvability of elliptic quasivariational inequalities that can find it as soon as the following references [15].

Theorem 3.5. *Let X be a Hilbert space and assume that*

1) $a : X \times X \to \mathbb{R}$ *is a bilinear form and there exits* $M > 0$ *such that*

$$|a(u,v)| \leq M\|u\|_X\|v\|_X,\ \forall u, v \in X.$$

2) $b : X \times X \to \mathbb{R}$ *is a bilinear symmetric form and*

   i) *there exits* $M^{'} > 0$ *such that*
   $$|b(u,v)| \leq M^{'}\|u\|_X\|v\|_X,\ \forall u, v \in X.$$

   ii) *there exits* $m^{'} > 0$ *such that*
   $$b(v,v) \geq m^{'}\|v\|_X^2,\ \forall v \in X.$$

3) $j : X \times X \to \mathbb{R}$ *and*

   i) *for all* $\eta \in X$, $j(\eta, .)$ *is convex and l.s.c. on X.*

ii) there exits $\beta > 0$ *such that*

$$\left|j(\eta_1, v_2) - j(\eta_1, v_1) + j(\eta_2, v_1) - j(\eta_2, v_2)\right| \leq \beta\|\eta_1 - \eta_2\|_X \|v_1 - v_2\|_X,\ \forall \eta_1, \eta_2, v_1, v_2 \in X.$$

4) $f \in W^{1,p}(0, T; X)$ *for some* $p \in [1, \infty]$.

5) $u_0 \in X$.
Then, the problem: Find $u : [0, T] \to X$ *such that*

$$a(u(t), v - \dot{u}(t)) + b(\dot{u}(t), v - \dot{u}(t)) + j(\dot{u}(t), v) - j(\dot{u}(t), \dot{u}(t)) \geq (f(t), v - \dot{u}(t)),\ \forall x \in X,\ t \in [0, T],$$
$$u(0) = 0,$$

has a unique solution $u \in W^{2,p}(0, T; X)$.

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