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Inequalities Involving Casorati Curvatures for Submanifolds of Real Space Forms with a Quarter-Symmetric Connection

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Abstract. In this paper, we obtain some inequalities based on Casorati curvature for submanifolds in a real space form with a special kind of quarter-symmetric connection.

1. Introduction

Hayden [10] introduced the notion of a semi-symmetric metric connection on a Riemannian manifold. Nakao [21] studied submanifolds of a Riemannian manifold with semi-symmetric metric connections. Agashe and Chafle [1, 2] introduced the notion of a semi-symmetric non-metric connection and studied some of its properties and submanifolds of a Riemannian manifold with a semi-symmetric non-metric connection.

Chen[4] obtained a necessary condition for the existence of minimal isometric immersion from a given Riemannian manifold into Euclidean space and established a sharp inequality for a submanifold in a real space form using intrinsic invariants(scalar curvature, sectional curvature) and extrinsic invariant(squared mean curvature). The inequalities in this direction are known as Chen inequalities [5]. Mihia and Özgür derived the Chen inequalities for submanifolds of real space from with semi-symmetric metric connection and semi-symmetric non-metric connection[18, 20]. The same authors extended the inequalities for complex space forms and sasakian space forms with semi-symmetric metric connections[19].

Casorati [3] introduced the notion of Casorati curvature(extrinsic invariant) defined as the normalized square length of the second fundamental form. The notion of Casorati curvature gives a better intuition of the curvature compared to Gaussian curvature. The geometrical meaning and the importance of the Casorati curvature discussed by some distinguished geometers [7, 8, 11]. During the last decade, it becomes attractive area of research for geometers to obtain the optimal inequalities based on Casorati curvatures for various submanifolds of different ambient spaces [6, 12–16, 24, 25, 27].

The concept of "quarter-symmetric" connection was originally introduced by S. Golab [9]. Recently, in [23], authors introduced the special quater-symmetric connection and investigated Einstein warped products and multiply warped products. In 2019, Wang [26] obtained Chen inequalities for submanifolds of complex space forms and Sasakian space forms with special quarter-symmetric connections. The author [17] proved some basic inequalities in quatenionic settings with special quarter-symmetric connections.

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In this paper, we obtain optimal inequalities for submanifolds in real space forms with special quarter-symmetric connection. The chronology of the paper is as follows. In Section 2, we give a brief introduction about the special quarter-symmetric connection. In the last section, we obtain some inequalities for generalized normalized δ-Casorati curvatures for submanifolds in real space forms with special quarter-symmetric connection.

2. Preliminaries

Let $\widetilde{M}$ be an m-dimensional Riemannian manifold with Riemannian metric g and $\overline{\widetilde{\nabla}}$ be the Levi-Civita connection on $\widetilde{M}$. Let $\overline{\nabla}$ be a linear connection defined by

$$\overline{\nabla}_X Y = \overline{\widetilde{\nabla}}_X Y + \Lambda_1 \pi(Y) X - \Lambda_2 g(X,Y) P, \tag{1}$$

for X, Y vector fields on $\widetilde{M}$, Λ_1, Λ_2 are real constants and P the vector field on $\widetilde{M}$ such that $\pi(X) = g(X, P)$, where π is 1-form. If $\overline{\nabla} g = 0$, then $\overline{\nabla}$ is known as quarter -symmetric metric connection and if $\overline{\nabla} g \neq 0$, then $\overline{\nabla}$ is known as quarter -symmetric non-metric connection. The special cases of (1) can be obtained as
(i) when $\Lambda_1 = \Lambda_2 = 1$, then the above connection reduces to semi-symmetric metric connection.
(ii) when $\Lambda_1 = 1$ and $\Lambda_2 = 0$, then the above connection reduces to semi-symmetric non metric connection.

The curvature tensor with respect to $\overline{\nabla}$ is defined as

$$\overline{R}(X,Y)Z = \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z. \tag{2}$$

Similarly, we can define the curvature tensor with respect to $\overline{\widetilde{\nabla}}$.

Now, using (1), the curvature tensor takes the following form [26]

$$\begin{aligned} \overline{R}(X,Y,Z,W) = \overline{\widetilde{R}}(X,Y,Z,W) &+ \Lambda_1 \alpha(X,Z) g(Y,W) - \Lambda_1 \alpha(Y,Z) g(X,W) \\ &+ \Lambda_2 g(X,Z) \alpha(Y,W) - \Lambda_2 g(Y,Z) \alpha(X,W) + \Lambda_2(\Lambda_1 - \Lambda_2) g(X,Z) \beta(Y,W) \\ &- \Lambda_2(\Lambda_1 - \Lambda_2) g(Y,Z) \beta(X,W), \end{aligned} \tag{3}$$

where

$$\alpha(X,Y) = (\overline{\widetilde{\nabla}}_X \pi)(Y) - \Lambda_1 \pi(X) \pi(Y) + \frac{\Lambda_2}{2} g(X,Y) \pi(P),$$

and

$$\beta(X,Y) = \frac{\pi(P)}{2} g(X,Y) + \pi(X)\pi(Y)$$

are $(0,2)$-tensors. For simplicity, we denote by $\mathrm{tr}(\alpha) = a$ and $\mathrm{tr}(\beta) = b$.

Let M be an n-dimensional submanifold of an m-dimensional real space form $\widetilde{M}(c)$. On the submanifold M, we consider the induced quarter-symmetric connection denoted by ∇ and the induced Levi-Civita connection denoted by $\widetilde{\nabla}$. Let R and $\widetilde{R}$ be the curvature tensors of ∇ and $\widetilde{\nabla}$. Decomposing the vector field P on M uniquely into its tangent and normal components $P^{\top}$ and $P^{\perp}$, respectively, then we have $P = P^{\top} + P^{\perp}$. The Gauss formulas with respect to ∇ and $\widetilde{\nabla}$ can be written as:

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad X, Y \in \Gamma(TM),$$

$$\overline{\widetilde{\nabla}}_X Y = \widetilde{\nabla}_X Y + \widetilde{h}(X,Y), \quad X, Y \in \Gamma(TM),$$

where $\widetilde{h}$ is the second fundamental form of M in N and

$$h(X,Y) = \widetilde{h}(X,Y) - \Lambda_2 g(X,Y) P^{\perp}.$$

In $\widetilde{M}(c)$ we can choose a local orthonormal frame $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_m\}$, such that, restricting to M, $\{e_1, \cdots, e_n\}$ are tangent to M^n. We write $h_{ij}^r = g(h(e_i, e_j), e_r)$. The squared length of h is $\|h\|^2 = \sum_{i,j=1}^n g(h(e_i, e_j), h(e_i, e_j))$ and the mean curvature vector of M associated to ∇ is $H = \frac{1}{n}\sum_{i=1}^n h(e_i, e_i)$. Similarly, the mean curvature vector of M associated to $\widetilde{\nabla}$ is $\widetilde{H} = \frac{1}{n}\sum_{i=1}^n \widetilde{h}(e_i, e_i)$. Let $\widetilde{M}(c)$ be an m-dimensional real space form of constant sectional curvature c endowed with a quarter-symmetric connection satisfying (1). The curvature tensor $\widetilde{\overline{R}}$ with respect to the Levi-Civita connection $\widetilde{\overline{\nabla}}$ on $\widetilde{M}(c)$ is expressed by

$$\widetilde{\overline{R}}(X, Y, Z, W) = c\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\}. \tag{4}$$

By (3) and (4), we get

$$\begin{aligned}\overline{R}(X, Y, Z, W) = {} & c\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\} + \Lambda_1\alpha(X, Z)g(Y, W)\\ & - \Lambda_1\alpha(Y, Z)g(X, W) + \Lambda_2 g(X, Z)\alpha(Y, W) - \Lambda_2 g(Y, Z)\alpha(X, W)\\ & + \Lambda_2(\Lambda_1 - \Lambda_2)g(X, Z)\beta(Y, W) - \Lambda_2(\Lambda_1 - \Lambda_2)g(Y, Z)\beta(X, W).\end{aligned} \tag{5}$$

The Gauss equation takes the following form

$$\begin{aligned}\overline{R}(X, Y, Z, W) = {} & R(X, Y, Z, W) - g(h(X, W), h(Y, Z)) + g(h(Y, W), h(X, Z))\\ & + (\Lambda_1 - \Lambda_2)g(h(Y, Z), P)g(X, W) + (\Lambda_2 - \Lambda_1)g(h(X, Z), P)g(Y, W).\end{aligned} \tag{6}$$

For a Riemannian manifold M^n, we denote by $K(\pi)$ the sectional curvature of M^n associated with a plane section $\pi \subset T_pM^n, p \in M^n$. For an orthonormal basis $\{e_1, e_2, ..., e_n\}$ of the tangent space T_pM^n, the scalar curvature τ is defined by

$$\tau = \sum_{i<j} K_{ij},$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i and e_j. The normalized scalar curvature ρ is defined as

$$\rho = \frac{2\tau}{n(n-1)}.$$

The norm of the squared mean curvature of the submanifold is defined by

$$\|H\|^2 = \frac{1}{n^2}\sum_{\gamma=n+1}^{n+p}\Big(\sum_{i=1}^n h_{ii}^\gamma\Big)^2,$$

and the squared norm of second fundamental form h is denoted by $\mathcal{C}$ defined as

$$\mathcal{C} = \frac{1}{n}\sum_{\gamma=n+1}^{n+p}\sum_{i,j=1}^n \left(h_{ij}^\gamma\right)^2,$$

known as Casorati curvature of the submanifold.

If we suppose that L be an s-dimensional subspace of TM, $s \geq 2$, and $\{e_1, e_2, \ldots, e_s\}$ is an orthonormal basis of L. then the scalar curvature of the s-plane section L is given as

$$\tau(L) = \sum_{1\leq\gamma<\beta\leq s} K(e_\gamma \wedge e_\beta)$$

and the Casorati curvature C of the subspace L is as follows

$$C(L) = \frac{1}{s}\sum_{\gamma=n+1}^{n+p}\sum_{i,j=1}^{s}\left(h_{ij}^{\gamma}\right)^2.$$

A point $p \in M$ is said to be an *invariantly quasi-umbilical point* if there exist p mutually orthogonal unit normal vectors $\xi_{n+1},\ldots,\xi_{n+p}$ such that the shape operators with respect to all directions ξ_γ have an eigenvalue of multiplicity $n-1$ and that for each ξ_γ the distinguished Eigen direction is the same. The submanifold is said to be an *invariantly quasi-umbilical submanifold* if each of its points is an invariantly quasi-umbilical point.

The normalized δ-Casorati curvature $\delta_c(n-1)$ and $\widetilde{\delta}_c(n-1)$ are defined as

$$[\delta_c(n-1)]_p = \frac{1}{2}C_p + \frac{n+1}{2n}\inf\{C(L)|L : \text{a hyperplane of } T_pM\} \tag{7}$$

and

$$[\widetilde{\delta}_c(n-1)]_p = 2C_p + \frac{2n-1}{2n}\sup\{C(L)|L : \text{a hyperplane of } T_pM\}. \tag{8}$$

For a positive real number $t \neq n(n-1)$, the generalized normalized δ-Casorati curvatures $\delta_c(t;n-1)$ and $\widetilde{\delta}_c(t;n-1)$ are given as

$$[\delta_c(t;n-1)]_p = tC_p + \frac{(n-1)(n+t)(n^2-n-t)}{nt}\inf\{C(L)|L : \text{a hyperplane of } T_pM\}$$

if $0 < t < n^2-n$, and

$$[\widetilde{\delta}_c(t;n-1)]_p = tC_p + \frac{(n-1)(n+t)(n^2-n-t)}{nt}\sup\{C(L)|L : \text{a hyperplane of } T_pM\}.$$

if $t > n^2-n$.

Now, we recall the following lemmas, which plays an important role for the proof of the main results.

Oprea[22] gives new direction to prove the Chen inequalities using optimization techniques. For a submanifold (M,g) of a Riemannian manifold $(\widetilde{M},\widetilde{g})$ and $\mathcal{F} : \widetilde{M} \to \mathbf{R}$ be a differential function. If we have a constrained problem

$$\min_{x\in M}\mathcal{F}(x) \tag{9}$$

then the following result holds

Lemma 2.1. *[22] Let $x_\circ \in M$ is the solution of the problem (9), then*
(i) $(grad(\mathcal{F}))(x_\circ) \in T_{x_\circ}^{\perp}M$
(ii) the bilinear form
$\mathcal{B} : T_{x_\circ}M \times T_{x_\circ}M \to \mathbf{R}$
$\mathcal{B}(X,Y) = Hess_{\mathcal{F}}(X,Y) + \widetilde{g}(h(X,Y),(grad(\mathcal{F})(x_\circ))$
is positive semi-definite, where h is the second fundamental form of M in $\widetilde{M}$ and $grad(\mathcal{F})$ if the gradient of $\mathcal{F}$.

3. Inequalities for generalized normalized δ-Casorati curvatures

Theorem 3.1. *Let M be an n-dimensional submanifold of an m-dimension real space form $\widetilde{M}(c)$ endowed with a connection $\overline{\nabla}$, then*

(i) The generalized normalized δ-Casorati curvature $\delta_c(t;n-1)$ satisfies

$$\rho \;\leq\; \frac{\delta_c(t;n-1)}{n(n-1)} + c - \frac{(\Lambda_1+\Lambda_2)}{n}a - \frac{\Lambda_2(\Lambda_1-\Lambda_2)}{n}b - (\Lambda_1-\Lambda_2)\pi(H) \tag{10}$$

for any real number t such that $0 < t < n(n-1)$.

(ii) The generalized normalized δ*-Casorati curvature* $\widetilde{\delta_c}(t; n-1)$ *satisfies*

$$\rho \leq \frac{\widetilde{\delta_c}(t; n-1)}{n(n-1)} + c - \frac{(\Lambda_1 + \Lambda_2)}{n}a - \frac{\Lambda_2(\Lambda_1 - \Lambda_2)}{n}b - (\Lambda_1 - \Lambda_2)\pi(H) \tag{11}$$

for any real number $t > n(n-1)$. *Moreover , the equality holds in (10) and (11) iff* M *is an invariantly quasi-umbilical submanifold with trivial normal connection in* $\widetilde{M}$, *such that with respect to suitable tangent orthonormal frame* $\{e_1, \ldots, e_n\}$ *and normal orthonormal frame* $\{e_{n+1}, \ldots, e_m\}$, *the shape operator* $A_\gamma \equiv A_{e_\gamma}$, $\gamma \in \{n+1, \ldots, m\}$, *take the following form*

$$A_{n+1} = \begin{pmatrix} h^\gamma_{11} & 0 & 0 & \ldots & 0 & 0 \\ 0 & h^\gamma_{22} & 0 & \ldots & 0 & 0 \\ 0 & 0 & h^\gamma_{33} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & h^\gamma_{n-1n-1} & 0 \\ 0 & 0 & 0 & \ldots & 0 & \frac{n(n-1)}{t}h^\gamma_{nn} \end{pmatrix}, \tag{12}$$

$$A_{n+2} = \cdots = A_m = 0.$$

Proof. Let $\{e_1, e_2, \ldots, e_n\}$ and $\{e_{n+1}, e_{n+2}, \ldots, e_m\}$ be the orthonormal bases of T_pM and $T^\perp_pM$ respectively at a point $p \in M$. Using (5), we have

$$\begin{aligned} 2\tau(p) = {} & n(n-1)c - (\Lambda_1 + \Lambda_2)(n-1)a - \Lambda_2(\Lambda_1 - \Lambda_2)(n-1)b \\ & -(\Lambda_1 - \Lambda_2)(n-1)n\pi(H) + n^2\|H\|^2 - n\mathcal{C}. \end{aligned} \tag{13}$$

Consider a polynomial $\mathcal{Q}$ in the components of second fundamental form h defined as

$$\begin{aligned} \mathcal{Q} = {} & t\mathcal{C} + \frac{(n-1)(n+t)(n^2-n-t)}{nt}\mathcal{C}(L) - 2\tau(p) + n(n-1)c \\ & -(\Lambda_1 + \Lambda_2)(n-1)a - \Lambda_2(\Lambda_1 - \Lambda_2)(n-1)b - (\Lambda_1 - \Lambda_2)(n-1)n\pi(H), \end{aligned}$$

where L is hyperplane of tangent space at a point p. We assume that L is spanned by $e_1, e_2, \ldots, e_{n-1}$ and $\mathcal{Q}$ has an expression of the form

$$\begin{aligned} \mathcal{Q} = {} & \frac{t}{n}\sum_{\gamma=n+1}^{m}\sum_{i,j=1}^{n}(h^\gamma_{ij})^2 + \frac{(n+t)(n^2-n-t)}{nt}\sum_{\gamma=n+1}^{m}\sum_{i,j=1}^{n-1}(h^\gamma_{ij})^2 \\ & -2\tau(p) + n(n-1)c - (\Lambda_1 + \Lambda_2)(n-1)a \\ & -\Lambda_2(\Lambda_1 - \Lambda_2)(n-1)b - (\Lambda_1 - \Lambda_2)(n-1)n\pi(H). \end{aligned} \tag{14}$$

From (13) and (14), we arrive at

$$\begin{aligned} \mathcal{Q} = {} & \sum_{\gamma=n+1}^{m}\sum_{i=1}^{n-1}\Big[\Big(\frac{(n^2+nt-n-2t)}{t}\Big)(h^\gamma_{ii})^2 + \frac{2(n+t)}{n}(h^\gamma_{in})^2\Big] \\ & + \sum_{\gamma=n+1}^{m}\Big[2\Big(\frac{2(n+t)(n-1)}{t}\Big)\sum_{(i<j)=1}^{n}(h^\gamma_{ij})^2 - 2\sum_{(i<j)=1}^{n}h^\gamma_{ii}h^\gamma_{jj} + \frac{t}{n}(h^\gamma_{nn})^2\Big] \\ \geq {} & \sum_{\gamma=n+1}^{m}\sum_{i=1}^{n-1}\Big[\Big(\frac{(n^2+n(t-1)-2t)}{t}\Big)(h^\gamma_{ii})^2 - 2\sum_{(i<j)=1}^{n}h^\gamma_{ii}h^\gamma_{jj} + \frac{t}{n}(h^\gamma_{nn})^2\Big]. \end{aligned} \tag{15}$$

For $t = n+1,\dots,m$, lets us have a quadratic form $\mathcal{F}_\gamma : \mathbf{R}^n \to \mathbf{R}$ defined as

$$\mathcal{F}_\gamma(h_{11}^\gamma,\dots,h_{nn}^\gamma) = \sum_{i=1}^{n-1} \frac{n^2+n(t-1)-2t}{t}(h_{ii}^\gamma)^2 - 2\sum_{(i<j)=1}^{n} h_{ii}^\gamma h_{jj}^\gamma + \frac{t}{n}(h_{nn}^\gamma)^2$$

and the optimization problem

$$\begin{aligned} &\min \quad \mathcal{F}_\gamma \\ &\text{subject to} \quad G : h_{11}^\gamma + \cdots + h_{nn}^\gamma = c^\gamma \end{aligned}$$

where c^γ is a real constant. The partial derivatives of $\mathcal{F}_\gamma$ are

$$\begin{cases} \dfrac{\partial \mathcal{F}_\gamma}{\partial h_{ii}^\gamma} = \dfrac{2(n+t)(n-1)}{t} h_{ii}^\gamma - 2\sum_{l=1}^{n} h_{ll}^\gamma \\ \frac{\partial \mathcal{F}_\gamma}{\partial h_{nn}^\gamma} = \dfrac{2t}{n} h_{nn}^\gamma - 2\sum_{l=1}^{n-1} h_{ll}^\gamma \end{cases} \tag{16}$$

where $i = \{1,2,\dots,n-1\}, i \neq j$, and $\gamma \in \{n+1,\dots,m\}$.

The vector $grad\mathcal{F}_\gamma$ is normal at G for the optimal $(h_{11}^\gamma,\dots,h_{nn}^\gamma)$ of the problem. that is, it is collinear with the vector $(1,1,\dots,1)$. Using (16), the critical point of the corresponding problem has the form

$$\begin{cases} h_{ii}^\gamma = \dfrac{t}{n(n-1)} v^\gamma, i \in \{1,\dots,n-1\}, \\ h_{nn}^\gamma = v^\gamma \end{cases} \tag{17}$$

By use of (17) and $\sum_{i=1}^{\gamma} h_{ii}^\gamma = c^\gamma$, we arrive at

$$\begin{cases} h_{ii}^\gamma = \dfrac{t}{(n+t)(n-1)} c^\gamma, i \in \{1,\dots,n-1\} \\ h_{nn}^\gamma = \dfrac{n}{(n+t)} c^\gamma. \end{cases} \tag{18}$$

For an arbitrary fixed point $p \in G$, the 2-form $\mathcal{B} : T_pG \times T_pG \to \mathbf{R}$ has the following form

$$\mathcal{B}(X,Y) = Hess(\mathcal{F}_\gamma(X,Y)) + \langle h(X,Y), (grad(\mathcal{F}))(x_\circ)\rangle \tag{19}$$

where h and $\langle,\rangle$ are the second fundamental form of G in $\mathbf{R}^n$ and standard inner product on $\mathbf{R}^n$ respectively. The Hessian matrix of $\mathcal{F}_\gamma$ is of the form

$$Hess(\mathcal{F}_\gamma) = \begin{pmatrix} 2\frac{(n+t)(n-1)}{t} - 2 & -2 & \dots & -2 & -2 \\ -2 & 2\frac{(n+t)(n-1)}{t} - 2 & \dots & -2 & -2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & \dots & 2\frac{(n+t)(n-1)}{t} - 2 & -2 \\ -2 & -2 & \dots & -2 & \frac{2t}{n}. \end{pmatrix}.$$

Though G is totally geodesic in $\mathbf{R}^n$, take a tangent vector $X = (X_1,\dots,X_n)$ at any arbitrary point p on G, verifying the relation $\sum_{i=1}^n X_i = 0$, we have the following

$$\begin{aligned} \mathcal{B}(X,X) &= \frac{2(n^2-n+tn-2t)}{t}\sum_{i=1}^{n-1} X_i^2 + \frac{2t}{n}X_n^2 - 2\Big(\sum_{i=1}^{n} X_i\Big)^2 \\ &= \frac{2(n^2-n+tn-2t)}{t}\sum_{i=1}^{n-1} X_i^2 + \frac{2t}{n}X_n^2 \\ &\geq 0. \end{aligned} \tag{20}$$

Hence the point $(h^{\gamma}_{11}, \ldots, h^{\gamma}_{nn})$ is the global minimum point by Lemma 2.1 and $\mathcal{F}_{\gamma}(h^{\gamma}_{11}, \ldots, h^{\gamma}_{nn}) = 0$. Thus, we have $Q \geq 0$ and hence

$$\begin{aligned} 2\tau \quad \leq \quad & tC + \frac{(n-1)(n+t)(n^2-n-t)}{nt}C(L) + n(n-1)c \\ & -(\Lambda_1+\Lambda_2)(n-1)a - \Lambda_2(\Lambda_1-\Lambda_2)(n-1)b - (\Lambda_1-\Lambda_2)(n-1)n\pi(H), \end{aligned}$$

whereby, we obtain

$$\begin{aligned} \rho \quad \leq \quad & \frac{t}{n(n-1)}C + \frac{(n+t)(n^2-n-t)}{n^2 t}C(L) + c \\ & -\frac{(\Lambda_1+\Lambda_2)}{n}a - \frac{\Lambda_2(\Lambda_1-\Lambda_2)}{n}b - (\Lambda_1-\Lambda_2)\pi(H), \end{aligned}$$

for every tangent hyperplane L of M. If we take the infimum over all tangent hyperplanes L, the result trivially follows. Moreover the equality sign holds iff

$$h^{\gamma}_{ij} = 0, \ \forall\, i, j \in \{1, \ldots, n\},\ i \neq j \text{ and } \gamma \in \{n+1, \ldots, m\} \tag{21}$$

and

$$\begin{aligned} h^{\gamma}_{nn} = \frac{n(n-1)}{t}h^{\gamma}_{11} = \cdots = & \frac{n(n-1)}{t}h^{\gamma}_{n-1n-1}, \\ & \forall \gamma \in \{n+1, \ldots, m\}. \end{aligned} \tag{22}$$

From (21) and (22), we obtain that the equality holds if and only if the submanifold is invariantly quasi-umbilical with normal connections in $\widetilde{M}$, such that the shape operator takes the form (12) with respect to the orthonormal tangent and orthonormal normal frames.

In the same way, we can prove (ii).

□

Now, if we put $\Lambda_1 = \Lambda_2 = 1$, we get the obtained *Theorem*2.1 by in [13]

Corollary 3.2. *Let M be an n-dimensional submanifold of an m-dimension real space form $\widetilde{M}(c)$ endowed with a semi-symmetric metric connection $\overline{\nabla}$, then*

(i) The generalized normalized δ-Casorati curvature $\delta_c(t; n-1)$ satisfies

$$\rho \quad \leq \quad \frac{\delta_c(t; n-1)}{n(n-1)} + c - \frac{2}{n}\Lambda, \tag{23}$$

for any real number t such that $0 < t < n(n-1)$.

(ii) The generalized normalized δ-Casorati curvature $\widetilde{\delta_c}(t; n-1)$ satisfies

$$\rho \quad \leq \quad \frac{\widetilde{\delta_c}(t; n-1)}{n(n-1)} + c - \frac{2}{n}\Lambda, \tag{24}$$

for any real number $t > n(n-1)$. Moreover , the equality holds in (23) and (24) iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\widetilde{M}$, such that with respect to suitable tangent orthonormal frame $\{e_1, \ldots, e_n\}$ and normal orthonormal frame $\{e_{n+1}, \ldots, e_m\}$, the shape operator $A_{\gamma} \equiv A_{e_{\gamma}}$, $\gamma \in \{n+1, \ldots, m\}$, take the *following form*

$$A_{n+1} = \begin{pmatrix} h^{\gamma}_{11} & 0 & 0 & \dots & 0 & 0 \\ 0 & h^{\gamma}_{22} & 0 & \dots & 0 & 0 \\ 0 & 0 & h^{\gamma}_{33} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & h^{\gamma}_{n-1n-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{n(n-1)}{t}h^{\gamma}_{nn} \end{pmatrix}, \tag{25}$$
$$A_{n+2} = \dots = A_m = 0.$$

In the similar way , if we put $\Lambda_1 = 1$ and $\Lambda_2 = 0$, we get the following result.

Corollary 3.3. *Let M be an n-dimensional submanifold of a real space form $\widetilde{M}(c)$ of dimension (m) endowed with semi-symmetric non-metric connection $\overline{\nabla}$, then*

(i) The generalized normalized δ-Casorati curvature $\delta_c(t; n-1)$ satisfies

$$\rho \leq \frac{\delta_c(t; n-1)}{n(n-1)} + c - \frac{\Lambda}{n} - \pi(H), \tag{26}$$

for any real number t such that $0 < t < n(n-1)$.

(ii) The generalized normalized δ-Casorati curvature $\widetilde{\delta_c}(t; n-1)$ satisfies

$$\rho \leq \frac{\widetilde{\delta_c}(t; n-1)}{n(n-1)} + c - \frac{\Lambda}{n} - \pi(H), \tag{27}$$

for any real number $t > n(n-1)$. Moreover , the equality holds in (26) and (27) iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\widetilde{M}$, such that with respect to suitable tangent orthonormal frame $\{e_1, \dots, e_n\}$ and normal orthonormal frame $\{e_{n+1}, \dots, e_m\}$, the shape operator $A_\gamma \equiv A_{e_\gamma}$, $\gamma \in \{n+1, \dots, m\}$, take the following form

$$A_{n+1} = \begin{pmatrix} h^{\gamma}_{11} & 0 & 0 & \dots & 0 & 0 \\ 0 & h^{\gamma}_{22} & 0 & \dots & 0 & 0 \\ 0 & 0 & h^{\gamma}_{33} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & h^{\gamma}_{n-1n-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{n(n-1)}{t}h^{\gamma}_{nn} \end{pmatrix}, \tag{28}$$
$$A_{n+2} = \dots = A_m = 0.$$

Now, if we put $\Lambda_1 = \Lambda_2 = 0$, we have the following result for real space forms.

Corollary 3.4. *Let M be an n-dimensional submanifold of an m-dimension real space form $\widetilde{M}(c)$, then*

(i) The generalized normalized δ-Casorati curvature $\delta_c(t; n-1)$ satisfies

$$\rho \leq \frac{\delta_c(t; n-1)}{n(n-1)} + c, \tag{29}$$

for any real number t such that $0 < t < n(n-1)$.

(ii) The generalized normalized δ-Casorati curvature $\widetilde{\delta_c}(t; n-1)$ satisfies

$$\rho \leq \frac{\widetilde{\delta_c}(t; n-1)}{n(n-1)} + c, \tag{30}$$

for any real number $t > n(n-1)$. Moreover , the equality holds in (29) and (30) iff M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\widetilde{M}$, such that with respect to suitable tangent orthonormal frame $\{e_1, \ldots, e_n\}$ and normal orthonormal frame $\{e_{n+1}, \ldots, e_m\}$, the shape operator $A_\gamma \equiv A_{e_\gamma}$, $\gamma \in \{n+1, \ldots, m\}$, take the following form

$$A_{n+1} = \begin{pmatrix} h_{11}^{\gamma} & 0 & 0 & \cdots & 0 & 0 \\ 0 & h_{22}^{\gamma} & 0 & \cdots & 0 & 0 \\ 0 & 0 & h_{33}^{\gamma} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & h_{n-1n-1}^{\gamma} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{n(n-1)}{t} h_{nn}^{\gamma} \end{pmatrix}, \quad (31)$$

$$A_{n+2} = \cdots = A_m = 0.$$

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