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Conditional Distributivity of Semi-t-operators Over Conjunctive Uninorms

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Abstract.
The conditional distributivity, that is distributivity equation with additional restriction imposed on the domain of aggregation operations, is an issue of interest for many different theoretical and practical areas, special for integration theory and utility theory. This paper presents new results on this specific form of distributivity for semi-t-operators over conjunctive uninorms. Since the observed class of uninorms is rather wide and includes not only the continuous case, the presented research is an extension of some well-known results and provides wider classes of solutions.

1. Introduction

The applicability of aggregation operations is significant in many different fields, both theoretical and practical, from mathematics and natural sciences to economics and social science. Therefore, aggregation operations are being intensively studied during the last decades from the purely theoretical point of view (see [2, 3, 10]). The researchers are showing increasing interest in characterization of pairs of aggregation operations that are satisfying distributivity laws. Investigation of this problem has roots in [1], and current focus of researchers is on finding solutions for many known classes of aggregation operations. Results for t-norms and t-conorms can be found in [17], for quasi-arithmetic means in [33], for semi-t-operators and uninorms in [5, 28, 29], for semi-t-operator and semi-nullnorms in [4, 6, 12], for uninorms in [26, 27], for 2-uninorms in [7, 35], for uni-nullnorms in [30, 32], etc. It is interesting that the distributivity with additional restriction, i.e., distributivity observed under additional conditions imposed on the domain of aggregation operations, ensures a wider variety of solutions. This fact is due to the elimination of rigid points for the domain (see [11, 14, 15, 17–21, 30, 31]).

The aim of this paper is to extend research from [14] towards conjunctive uninorms. The uninorms in question are generalization of uninorms from the class $U_{\min}$, i.e., the wider class of uninorms is now being considered. Also, since semi-t-operators are generalization of nullnorms, the research presented here

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generalizes the one presented in [19]. Therefore, the main concern of this paper is how to solve functional equations

$$F(x, U(y,z)) = U(F(x,y), F(x,z)), \quad x,y,z \in [0,1]$$

and

$$F(U(y,z), x) = U(F(y,x), F(z,x)), \quad x,y,z \in [0,1]$$

under additional restriction $U(y,z) < 1$, where F is a semi-t-operator, and U is a conjunctive uninorm.

Since uninorms are generalizations of t-norms and t-conorms, there are three options for the condition in conditional distributivity: $U(y,z) < 1$, $U(y,z) > 0$, and $0 < U(y,z) < 1$. As stated above, the first one is used in this paper. The second one will provide analogous results. The third one was used in [11], however for a more narrow class of operations, i.e. for nullnorms over specific uninorms. The structure of operations from [11] are simpler than those observed in this paper, and the third type of restriction did provide results for that case. However, it is not clear whether the results would be significant for operations of the more complex structure, such as semi-t-operators and general uninorms.

The results presented in this paper are providing characterization of some new pairs of aggregation operations that fulfill conditional distributivity. In addition, this is a continuation of research from [16] where the focus was on continuous semi-t-operators and disjunctive uninorms with continuous underlining t-norms and t-conorms. Although topics seem to be analogous, is not the case of mere duality. Constructions and results vary, and therefore the detailed presentation is needed. The main difference between this paper and [16] lies in the fact that more than half of the presented results hold without assumption of continuity for F and U. Therefore, much wider classes of operations are now being considered.

Some basic preliminary notions concerning aggregation operations, semi-t-operators, uninorms and conditional distributivity are given in Section 2. Section 3 consists of results on conditional distributivity, from the left and from the right, for semi-t-operators. The inner aggregation operation is now a conjunctive uninorm that need not be from the class $\mathcal{U}_{\min}$. Since the behavior of the inner aggregation operation and construction itself significantly differ for analogous problem for disjunctive uniorms ([16]), proves are provided.

2. Preliminaries

A short overview of the basic notions that are needed for the presented research is given in this section (see [4, 9, 10, 17, 22, 23, 34]).

2.1. Aggregation operations

The bare-boon structure of an aggregation operation in $[0,1]^n$ is given by the following definition.

Definition 2.1. *([10]) An aggregation operation in* $[0,1]^n$ *is a function* $A^{(n)} : [0,1]^n \to [0,1]$ *that is nondecreasing in each variable and that fulfills the following boundary conditions*

$$A^{(n)}(0,\dots,0) = 0 \quad \textit{and} \quad A^{(n)}(1,\dots,1) = 1.$$

Depending on the problem that is being addressed by aggregation operations, the previous definition can be extended to an arbitrary real interval $\mathbb{I}$. The integer n represents number of input values of the observed aggregation operation. Since the topic of this paper are the binary aggregation operations, they will be denoted simply by A instead of $A^{(2)}$.

The aggregation operations of interest for this research are uninorms of a specific class and semi-t-operators.

2.1.1. Uninorms

Definition 2.2. *([34]) A uninorm $U : [0,1]^2 \to [0,1]$ is a binary aggregation operation that is commutative, associative, and for which there exists a neutral element $e \in [0,1]$, i.e., $U(x,e) = x$ for all $x \in [0,1]$.*

For all uninorms it holds $U(0,1) \in \{0,1\}$. If $U(0,1) = 0$, the uninorm U is conjunctive and, if $U(0,1) = 1$ the uninorm U is a disjunctive one. The focus of this paper is on conjunctive ones.

If $e = 1$, the uninorm U becomes a t-norm denoted by T, and if $e = 0$, U is a t-conorm denoted by S. Commutativity and associativity can be omitted and, in that case, t-seminorms and t-semiconorms are obtained, respectively. Further on, the working assumption is that $e \in (0,1)$, that is, uninorms that are not t-norms nor t-conorms are considered. For that case, the following result is of significance.

Theorem 2.3. *([9]) Let U be a uninorm with a neutral element $e \in (0,1)$. Then there exists a t-norm T_U, a t-conorm S_U, and an increasing operator $C : [0,e) \times (e,1] \cup (e,1] \times [0,e) \to [0,1]$ that fulfils $\min \leq C \leq \max$, such that U is given by*

$$U(x,y) = \begin{cases} eT_U\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x,y) \in [0,e]^2, \\ e + (1-e)S_U\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if } (x,y) \in [e,1]^2, \\ C(x,y) & \text{otherwise.} \end{cases} \tag{1}$$

A t-norm T_U and a t-conorm S_U from the previous theorem are the underlying t-norm and the underlying t-conorm of U. For the given T_U, S_U and the neutral element $e \in (0,1)$, determining the corresponding uninorm U means finding operation C so that (1) holds. In general, this is still an open problem. The most studied classes of uninorms can be found in, e.g., [8, 9, 23, 25]). The focus of this paper is on the following classes:

- Uninorms in $\mathcal{U}_{\min}$ ($\mathcal{U}_{\max}$), i.e., uninorms given by minimum (maximum) in $A(e) = [0,e) \times (e,1] \cup (e,1] \times [0,e)$.
- Uninorms with continuous underlying t-norms and continuous underlying t-conorms.
- Idempotent uninorms, i.e., uninorms that satisfy $U(x,x) = x$ for all $x \in [0,1]$.
- Locally internal uninorms, i.e., uninorms such that $U(x,y) \in \{x,y\}$ for all $(x,y) \in A(e)$.

The next result, that is being used in Section 3 of this paper, provides a partial answer on the intersection between classes of locally internal uninorms and uninorms with continuous underlying t-norm and t-conorm.

Theorem 2.4. *([19]) Let U be a uninorm with a neutral element $e \in (0,1)$, and with continuous underlying t-norm and t-conorm.*
(i) If $S_U = \max$, then U is local internal uninorm.
(ii) If $T_U = \min$, then U is local internal uninorm.

The idempotent uninorms are characterized by terminology of Id-symmetrical functions (see [25]) in the following manner.

Definition 2.5. *([25]) Let $g : [0,1] \to [0,1]$ be any decreasing function and let G be the graph of g, i.e., $G = \{(x, g(x)) | x \in [0,1]\}$. For any discontinuity s of g, let us denote by s^- and s^+ the corresponding lateral limits, that are $s^- = \lim_{x \to s^-} g(x)$ and $s^+ = \lim_{x \to s^+} g(x)$. Then we define the completed graph of g as the set*

$$F_g = G \cup \{(0,y) | y > g(0)\} \cup \{(1,y) | y < g(1)\} \cup \{(s,y) | s^+ \leq y \leq s^-\},$$

for all discontinuity points s of g.

Definition 2.6. *([25]) A subset A of $[0,1]^2$ is Id-symmetrical if for all $(x,y) \in [0,1]^2$ it holds that $(x,y) \in A \Leftrightarrow (y,x) \in A$.*

A decreasing function $g : [0,1] \to [0,1]$ is called Id-symmetrical if its completed graph F_g is Id-symmetrical.

Theorem 2.7. *([25]) Let $e \in (0,1)$. The following claims are equivalent:*

(i) U is an idempotent uninorm with the neutral element e.

(ii) There is a decreasing function $g : [0,1] \to [0,1]$ with fixed point e, which is Id-symmetrical, such that U is given by

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } y < g(x) \text{ or } y = g(x) \text{ and } x < g^2(x), \\ \max(x,y) & \text{if } y > g(x) \text{ or } y = g(x) \text{ and } x > g^2(x), \\ x \text{ or } y & \text{if } y = g(x) \text{ and } x = g^2(x), \end{cases} \tag{2}$$

and is commutative on the set of points $(x, g(x))$ for which holds $x = g^2(x)$.

Through this paper the function g from the previous theorem will be referred as the associated function of U. Also, a characterization of all locally internal uninorms with continuous underlying t-norms and t-conorms based on Id-symmetrical functions can be found in [8] (see Theorem 17 from [8]). This result is being used in proves in Section 3 of this paper.

Remark 2.8. The first uninorms, considered by Yager and Rybalov in [34], are the idempotent uninorms U_e^{min} and U_e^{max} from classes $U_{\min}$, and $U_{\max}$, respectively, of the following form

$$U_e^{min} = \begin{cases} \max & \text{on } [e,1]^2, \\ \min & \text{otherwise,} \end{cases} \quad \text{with} \quad g(x) = \begin{cases} 1 & \text{if } x \in [0,e), \\ e & \text{if } x \in [e,1], \end{cases}$$

and

$$U_e^{max} = \begin{cases} \min & \text{on } [0,e]^2, \\ \max & \text{otherwise,} \end{cases} \quad \text{with} \quad g(x) = \begin{cases} e & \text{if } x \in [0,e], \\ 0 & \text{if } x \in (e,1]. \end{cases}$$

2.1.2. Semi-t-operators

Definition 2.9. *([4]) A semi-t-operator $F : [0,1]^2 \to [0,1]$ is an associative binary aggregation operation such that functions F_0, F_1, F^0, F^1, where $F_0(x) = F(0,x)$, $F_1(x) = F(1,x)$, $F^0(x) = F(x,0)$, $F^1(x) = F(x,1)$, are continuous.*

If the operator F from the previous definition is, in addition, commutative, it becomes t-operator, i.e., nullnorm with absorbing element $k = F(0,1)$. Therefore, the class of nullnorms is a subclass of semi-t-operators.

Let $\mathcal{F}_{a,b}$ denote the family of all semi-t-operators such that $F(0,1) = a$, $F(1,0) = b$. The following theorem provides the structure of operations from $\mathcal{F}_{a,b}$ and is crucial for the results that follow.

Theorem 2.10. *([4]) Let $F : [0,1]^2 \to [0,1]$, $F(0,1) = a$, $F(1,0) = b$. The operation $F \in \mathcal{F}_{a,b}$ if and only if there exists an associative t-seminorm T_F and an associative t-semiconorm S_F such that*

$$F(x,y) = \begin{cases} aS_F\left(\frac{x}{a}, \frac{y}{a}\right) & \text{if } (x,y) \in [0,a]^2, \\ b + (1-b)T_F\left(\frac{x-b}{1-b}, \frac{y-b}{1-b}\right) & \text{if } (x,y) \in [b,1]^2, \\ a & \text{if } x \le a \le y, \\ b & \text{if } y \le b \le x, \\ x & \text{otherwise,} \end{cases} \tag{3}$$

for $a \le b$ and

$$F(x,y) = \begin{cases} bS_F\left(\frac{x}{b},\frac{y}{b}\right) & \textit{if } (x,y) \in [0,b]^2, \\ a + (1-a)T_F\left(\frac{x-a}{1-a},\frac{y-a}{1-a}\right) & \textit{if } (x,y) \in [a,1]^2, \\ a & \textit{if } x \le a \le y, \\ b & \textit{if } y \le b \le x, \\ y & \textit{otherwise}, \end{cases} \tag{4}$$

for $b \le a$.

If F is a continuous semi-t-operator, T_F and S_F from Theorem 2.10 are continuous, and according to [22], are also commutative, i.e., T_F and S_F are a continuous t-norm and a continuous t-conorm, respectively. More on operations of this type can be found in [22].

2.2. Conditional distributivity

Since the problem of distributivity of a t-norm over a t-conorm gives us only an idempotent solution, i.e., t-conorm has to be $S_M = \max$, the relaxed version of this concept was introduced in [17].

As discussed in Introduction, that concept can be extended to some more general aggregation operations.

Definition 2.11. *([16]) Let $F \in \mathcal{F}_{a,b}$ and U be a uninorm with neutral element $e \in (0,1)$.*

- *F is conditionally distributive from the left (CDl) over U if*
$$F(x, U(y,z)) = U(F(x,y), F(x,z))$$
for all $x, y, z \in [0,1]$ *such that* $U(y,z) < 1$.
- *F is conditionally distributive from the right (CDr) over U if*
$$F(U(y,z), x) = U(F(y,x), F(z,x))$$
for all $x, y, z \in [0,1]$ *such that* $U(y,z) < 1$.
- *F is conditionally distributive (CD) over U if both (CDl) and (CDr) are fulfilled simultaneously.*

If operator F is commutative, (CDl) and (CDr) coincide and F is conditionally distributive over U.

This type of distributivity is also known as the restricted distributivity [10]. Although the domain is restricted, the class of pairs of operators that fulfill (CD) is much wider (see [17] (p. 138-140)).

3. Conditional distributivity for semi-t-operators

Let F be a semi-t-operator from $\mathcal{F}_{a,b}$, such that $a \neq b$.

Since semi-t-operators in general are not commutative, the left and the right conditional distributivity is considered separately. Conditional distributivity from the left (CDl) is investigated for $a < b$, and conditional distributivity from the right (CDr) for $a > b$. The joint study of (CDl) and (CDr) is also possible. However, again, the same two cases, $a < b$ and $b > a$, have to be separated (see [29]).

The following two theorems from [16] are the starting point of the research both in [16] and this paper. The considered cases are the borderline cases when $a = 0$, $b = 1$ and $b = 0$, $a = 1$.

Theorem 3.1. *([16]) A semi-t-operator $F \in \mathcal{F}_{0,1}$, and a uninorm U with a neutral element $e \in (0,1)$ satisfy (CDl) if and only if U is idempotent.*

Theorem 3.2. *([16]) A semi-t-operator $F \in \mathcal{F}_{1,0}$, and a uninorm U with a neutral element $e \in (0,1)$ satisfy (CDr) if and only if U is idempotent.*

As seen from the previous two theorems, results hold without assumption of continuity for the semi-t-operator F, and the underlying t-norm and t-conorm of uninorm U. Also the previous results show that in this borderline cases, despite the restricted domain, only idempotent solutions are obtained. Therefore, the left (right) distributivity without restrictions and (CDl) ((CDr)) are equivalent for $a = 0$ and $b = 1$ ($a = 1$ and $b = 0$).

3.1. Case I: conditional distributivity from the left over a conjunctive uninorm

The following lemma will explain relation between neutral element e of the conjunctive uninorm U and parameters a, b of the semi-t-operator F.

Lemma 3.3. *Let $F \in \mathcal{F}_{a,b}$, $a < b$, be a semi-t-operator, and U be a conjunctive uninorm with a neutral element $e \in (0, 1)$.*

(i) If F and U satisfy (CDl) and $b < 1$, then $e > b$.

(ii) If F and U satisfy (CDl) and $b = 1$, then $e \geq a$.

Proof : (i) Let F and U satisfy (CDl), $b < 1$, and suppose the opposite, i.e., $e \leq b$.

- If $e < a$, then for $x = e,\ y = 0,\ z = 1$ from (CDl) follows

$$e = F(e, 0) = F(e, U(0, 1)) = U(F(e, 0), F(e, 1)) = U(e, a) = a,$$

which is a contradiction.

- If $e \in [a, b]$, then for $x = 1,\ y = 0,\ z = 1$ from (CDl) follows

$$b = F(1, 0) = F(1, U(0, 1)) = U(F(1, 0), F(1, 1)) = U(b, 1) = 1,$$

which is again a contradiction.

Therefore, $e > b$.

(ii) Let F and U satisfy (CDl), $b = 1$, and $a > 0$ ($a = 0$ and $b = 1$ is considered in Theorem 3.1). If the opposite is supposed, i.e., if $e < a$, using the same arguments as in (i), a contradiction is obtained. Therefore, $e \geq a$. □

Again, the previous holds without the assumption of continuity for semi-t-operator F, and the underlying t-norm and t-conorm of the uninorm U,

Further on, two lines of investigation have to be distinguished: $b = 1$ and $b < 1$.

3.1.1. $b = 1$

Theorem 3.4. *Let $F \in \mathcal{F}_{a,b}$, $a < b = 1$, be a semi-t-operator, and U be a conjunctive uninorm with a neutral element $e \in (0, 1)$. F is conditionally distributive over U from the left if and only if $e \geq a$ and U is an idempotent uninorm given by*

$$U(x, y) = \begin{cases} \max(x, y) & \textit{if } (x, y) \in [e, 1]^2, \\ \min(x, y) & \textit{if } (x, y) \in [0, e]^2 \cup [0, a) \times (e, 1) \cup (e, 1) \times [0, a), \\ 0 & \textit{if } (x, y) \in \{(0, 1), (1, 0)\}. \end{cases} \tag{5}$$

Proof : ($\Rightarrow$) Let F be a semi-t-operator, and U be a conjunctive uninorm such that satisfy (CDl) condition. Then from Lemma 3.3 follows $e \geq a$.

(i) The idempotency of U can be proved as in Theorem 3.1 (see [16]).

(ii) The next step is to show that function g of U satisfies $g(z) = 1$ for all $z < a$. Let us suppose opposite, i.e., there exists $z_0 < a$ such that $g(z_0) < 1$. For $x = 0,\ y \in (g(z_0), 1),\ z = z_0$ from (CDl) follows

$$a = F(0, y) = F(0, U(y, z_0)) = U(F(0, y), F(0, z_0)) = U(a, z_0) = \min(a, z_0) = z_0$$

which is a contradiction.

Thus, (i) and (ii) insures that U is an idempotent uninorm given by (5).

($\Leftarrow$) Now, let U be an idempotent uninorm given by (5), and $e \geq a$. In order to prove (CDl) condition, the following cases have to be considered.

- If $x \geq a$, then $F(x, y) = x$ for all $y \in [0, 1]$. Therefore,

$$F(x, U(y, z)) = x = U(x, x) = U(F(x, y), F(x, z)).$$

- If $x < a \leq y, z$, then $U(y, z) \geq a,\ F(x, y) = a,\ F(x, z) = a$. Therefore,

$$F(x, U(y, z)) = a = U(a, a) = U(F(x, y), F(x, z)).$$

- If $x, z < a \leq y$, then $U(y, z) = \min(y, z) = z,\ F(x, y) = a$ and $F(x, z) \leq a$. Therefore,

$$F(x, U(y, z)) = F(x, z) = \min(F(x, y), F(x, z)) = U(F(x, y), F(x, z)).$$

- If $x, y < a \leq z$, the (CDl) is proved analogously to the previous.
- If $x, y, z < a$, then $U(y, z) = \min(y, z),\ F(x, y) \leq a,\ F(x, z) \leq a$. Therefore, since F is increasing we obtain

$$F(x, U(y, z)) = F(x, \min(y, z)) = \min(F(x, y), F(x, z)) = U(F(x, y), F(x, z)).$$

□

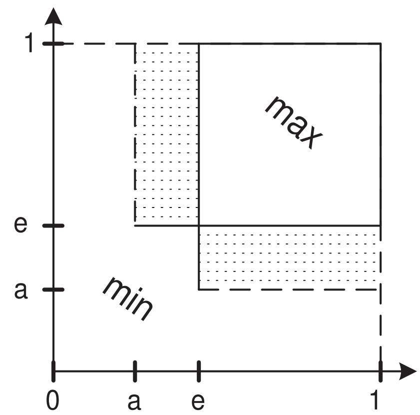

Figure 1. Uninorm (5).

Remark 3.5. *The structure of uninorm in the previous theorem is not determined in full (see Figure 1), however this does not have any effect on the claim itself, the idempotency is proved. Therefore,* $U(x, y) \in \{x, y\}$ *for* $(x, y) \in [a, e) \times (e, 1] \cup (e, 1] \times [a, e)$.

Remark 3.6.

- *As seen from the proof, Theorem 3.4 holds without the assumption of continuity for F and U. This means that a wider class of aggregation operations satisfy (CDl) condition, while in [16] all results hold only with the assumption of continuity for F and U.*
- *If U is a uninorm from the class* $\mathcal{U}_{\min}$, *Lemma 3.3 coincides with Lemma 22 from [14].*
- *Theorem 3.4 confirms the fact that (CDl) is a very strong condition. Although the restricted domain was considered, only an idempotent uninorm as solution is obtained. Therefore, the left distributivity and (CDl) are equivalent for* $b = 1$ *(see Theorem 11 from [5]).*

3.1.2. $b < 1$

In the following theorem the assumption of continuity has to be added because its proof is based on the result concerning the conditional distributivity of continuous t-norms over continuous t-conorms (see Theorem 5.21 from [17]), as well as on Theorem 2.4. Also, for the next result the crucial is Theorem 17 from [8] that gives a characterization of all locally internal uninorms with continuous underlying t-norms and continuous underlying t-conorms.

Theorem 3.7. *Let $F \in \mathcal{F}_{a,b}$, $a < b < 1$, be a continuous semi-t-operator, and U be a conjunctive uninorm with a continuous underlying t-norm and t-conorm. F is conditionally distributive over U from the left if and only if $e > b$ and exactly one of the following cases is fulfilled:*

(i) U is an idempotent uninorm given by

$$U(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in [e,1]^2 \text{ or } x > g(1),\ y = 1 \text{ or } y > g(1),\ x = 1, \\ 1 \text{ or } g(1) & \text{if } (x,y) \in \{(1,g(1)),(g(1),1)\}, \\ \min(x,y) & \text{otherwise,} \end{cases} \tag{6}$$

and F is given by

$$F(x,y) = \begin{cases} aS\left(\frac{x}{a},\frac{y}{a}\right) & \text{if } (x,y) \in [0,a]^2, \\ b + (e-b)T_1\left(\frac{x-b}{e-b},\frac{y-b}{e-b}\right) & \text{if } (x,y) \in [b,e]^2, \\ e + (1-e)T\left(\frac{x-e}{1-e},\frac{y-e}{1-e}\right) & \text{if } (x,y) \in [e,1]^2, \\ a & \text{if } x \le a \le y, \\ b & \text{if } y \le b \le x, \\ x & \text{if } a \le x \le b, \\ \min(x,y) & \text{otherwise,} \end{cases} \tag{7}$$

(ii) there is $c \in [e,1)$ such that U and F are given by

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } (x,y) \in [0,e) \times [0,1] \bigcup [0,1] \times [0,e), \\ c + (1-c)S^*\left(\frac{x-c}{1-c},\frac{y-c}{1-c}\right) & \text{if } (x,y) \in [c,1]^2, \\ \max(x,y) & \text{otherwise,} \end{cases} \tag{8}$$

and

$$F(x,y) = \begin{cases} aS\left(\frac{x}{a},\frac{y}{a}\right) & \text{if } (x,y) \in [0,a]^2, \\ b + (e-b)T_1\left(\frac{x-b}{e-b},\frac{y-b}{e-b}\right) & \text{if } (x,y) \in [b,e]^2, \\ e + (c-e)T_2\left(\frac{x-e}{c-e},\frac{y-e}{c-e}\right) & \text{if } (x,y) \in [e,c]^2, \\ c + (1-c)T^*\left(\frac{x-c}{1-c},\frac{y-c}{1-c}\right) & \text{if } (x,y) \in [c,1]^2, \\ a & \text{if } x \le a \le y, \\ b & \text{if } y \le b \le x, \\ x & \text{if } a \le x \le b, \\ \min(x,y) & \text{otherwise,} \end{cases} \tag{9}$$

where S is a continuous t-conorm and T, T_1 and T_2 are continuous t-norms, S^* is a t-conorm isomorphic to Lukasiewicz t-conorm $S_L(x,y) = \min\{x+y,1\}$, and T^* is a t-norm isomorphic to product t-norm $T_P(x,y) = x \cdot y$.

Proof : (⇒) Let F be a continuous semi-t-operator and let U be a conjunctive uninorm with a continuous underlying t-norm and t-conorm such that satisfy (CDl) condition. Then, from Lemma 3.3, follows $e > b$. As in Theorem 25 from [14], the following can be proved.

- For $x \leq b$, holds $U(x, x) = x$.
- e is an idempotent element of F, and F is of the form (7) on $[b, 1]^2$.
- $U(x, x) = x$ for all $x \in [0, e]$, i.e., on the square $[0, e]^2$ $U = \min$.
- If $c \in [e, 1)$ is an idempotent element of U, then all elements from $[e, c]$ are idempotent elements for U. Hence, either all elements from $[e, 1]$ are idempotent elements for U and U is an idempotent uninorm, or there is the largest nontrivial idempotent element $c \in [e, 1)$ of U, such that U and F are given by (8) and (9), respectively, on the square $[e, 1]^2$.

It remains to describe the structure of uninorm U on the set $A(e)$.

- The first step is to consider the case when U is an idempotent uninorm. Let us prove that associated function g is given by

$$g(z) = \begin{cases} 1 & \text{if } z \in [0, e), \\ e & \text{if } z \in [e, 1). \end{cases} \tag{10}$$

  Let $z_0 \in [0, e)$ and suppose that $g(z_0) < 1$. For $x = e,\ y \in (g(z_0), 1),\ z = z_0$ from (CDl) and known structures of F and U follows

$$e = F(e, y) = F(e, U(y, z)) = U(F(e, y), F(e, z)) = F(e, z) \in \{z, b\}.$$

  This is contradicts with $z < e$ and $e > b$. Therefore $g(z) = 1$ for all $z \in [0, e)$, and since g is Id-symmetrical, the form (10) is obtained. Now, from Theorem 2.7, follows that uninorm U is given by (6).

- The next step is to suppose that there is the largest nontrivial idempotent element $c \in [e, 1)$ of U, such that U is given by (8) on the square $[e, 1]^2$. Now, according to Theorem 2.4, since $T_U = \min$, and S_U is continuous, U is local internal uninorm, i.e., $U(x, y) \in \{x, y\}$ for all $(x, y) \in A(e)$.

  According to Theorem 17 from [8], there is a decreasing function $g : [0, 1] \to [0, 1]$ with a fixed point e, which is Id-symmetrical, such that U is given by (2) in $A(e)$ and it is commutative on the set of points $(x, g(x))$ for which holds $x = g^2(x)$. Analogously to the case when U is an idempotent uninorm, there can be proved that function g is given by (10). Since the nilpotent t-conorm S^* (isomorphic to S_L) is on the square $[c, 1]^2$, again according to Theorem 17 from [8], holds $g(1) = e$, and therefore uninorm U on the set $A(e)$ is minimum.

(⇐) On the other hand, if the observed uninorm and semi-t-operator are of the forms (8) and (9), respectively, or of the forms (6) and (7), respectively, the (CDl) condition can be proved as in Theorem 3.4. □

Example 3.8. *Let* $g(x) = \begin{cases} 1 & x \in [0, \frac{2}{3}) \\ \frac{2}{3} & x \in [\frac{2}{3}, 1) \\ \frac{1}{3} & x = 1 \end{cases}$ *be an associated function of a uninorm. An example of operations from Theorem 3.7 (i) is the following*

$$U(x, y) = \begin{cases} \max(x, y) & \text{if } (x, y) \in [\frac{2}{3}, 1]^2 \text{ or } x > \frac{1}{3},\ y = 1 \text{ or } y > \frac{1}{3},\ x = 1, \\ 1 \text{ or } \frac{1}{3} & \text{if } (x, y) \in \{(1, \frac{1}{3}), (\frac{1}{3}, 1)\}, \\ \min(x, y) & \text{otherwise}, \end{cases}$$

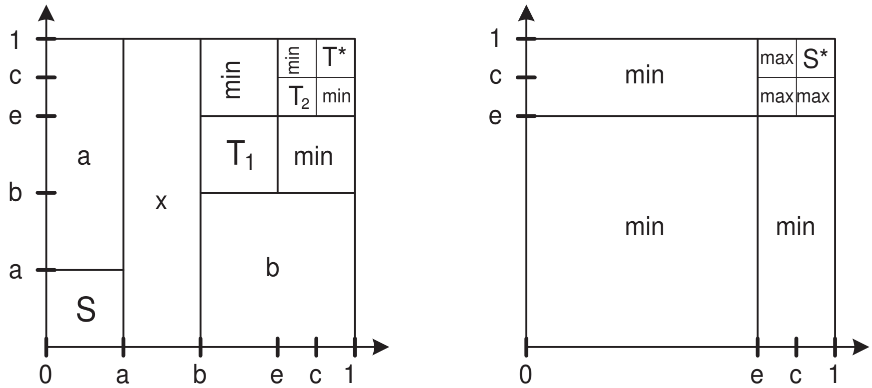

Figure 2. Semi-t-operator (9) and uninorm (8) from Theorem 3.7 (ii).

and the corresponding F is given by

$$F(x,y)=\begin{cases}\max(x,y) & \text{if } (x,y)\in[0,\frac{1}{4}]^2,\\ 6xy-3x-3y+2, & \text{if } (x,y)\in[\frac{1}{2},\frac{2}{3}]^2,\\ 3xy-2x-2y+2, & \text{if } (x,y)\in[\frac{2}{3},1]^2,\\ \frac{1}{4} & \text{if } x\le\frac{1}{4}\le y,\\ \frac{1}{2} & \text{if } y\le\frac{1}{2}\le x,\\ x & \text{if } \frac{1}{4}\le x\le\frac{1}{2},\\ \min(x,y) & \text{otherwise.}\end{cases}$$

The t-norm and t-conorm used in this construction are T_P and S_M. It should be stressed that different associated functions will provide different pairs of conditionally distributive operations.

Remark 3.9. *The following can be concluded from Theorem 3.7.*

- *As solutions of (CDl) only idempotent uninorms, with associated function g such that $g(1)\in[0,e]$, are obtained in case (i). If $g(1)=e$, then $U=U_e^{min}$, and the case (i) of Theorem 25 from [14] is obtained. If $g(1)\in[0,e)$, new idempotent uninorms as solution of (CDl) are obtained. This is the main advantage of this theorem with respect to the corresponding one from [16], where the only one idempotent uninorm was obtained as solution of (CDl).*
- *The case (ii) coincides with the case (ii) of Theorem 25 from [14], where (CDl) has been solved with a starting assumption that a uninorm U is from the class $\mathcal{U}_{\min}$ with continuous underlying t-norm and t-conorm. Therefore, (CDl) is strong condition.*

3.2. *Case II: Conditional distributivity from the right over a conjunctive uninorm*

Similar results hold for the (CDr) condition and they are given by the following theorems. Since the proves are analogous to their counterparts from Case I, they are now omitted.

Lemma 3.10. *Let $F\in\mathcal{F}_{a,b}$, $b<a$, be a semi-t-operator, and U be a conjunctive uninorm.*

(i) If F and U satisfy (CDr) and $a<1$, then $e>a$.

(ii) If F and U satisfy (CDr) and $a=1$, then $e\ge b$.

Theorem 3.11. *Let $F\in\mathcal{F}_{a,b}$, $b<a=1$, be a semi-t-operator, and U be a conjunctive uninorm. F is conditionally distributive over U from the right if and only if $e\ge b$ and U is an idempotent uninorm given by*

$$U(x,y)=\begin{cases}\max(x,y) & \text{if } (x,y)\in[e,1]^2,\\ \min(x,y) & \text{if } (x,y)\in[0,e]^2\cup[0,b)\times(e,1)\cup(e,1)\times[0,b),\\ 0 & \text{if } (x,y)\in\{(0,1),(1,0)\}.\end{cases} \tag{11}$$

Remark 3.12. *As in Theorem 3.4, the structure of uninorm (11) is not determined in full. Again, this does not have any effect on the claim itself and the idempotency is proved. Thus,* $U(x,y) \in \{x,y\}$ *for* $(x,y) \in [b,e)\times(e,1]\cup(e,1]\times[b,e)$.

Theorem 3.13. *Let* $F \in \mathcal{F}_{a,b}$, $b < a < 1$, *be a continuous semi-t-operator, and* U *be a conjunctive uninorm with a continuous underlying t-norm and t-conorm.* F *is conditionally distributive over* U *from the right if and only if* $e > a$ *and exactly one of the following cases is fulfilled:*

(i) U *is an idempotent uninorm given by (6), and* F *is given by*

$$F(x,y) = \begin{cases} bS\left(\frac{x}{b}, \frac{y}{b}\right) & \textit{if } (x,y) \in [0,b]^2, \\ a + (e-a)T_1\left(\frac{x-a}{e-a}, \frac{y-a}{e-a}\right) & \textit{if } (x,y) \in [a,e]^2, \\ e + (1-e)T\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \textit{if } (x,y) \in [e,1]^2, \\ a & \textit{if } x \le a \le y, \\ b & \textit{if } y \le b \le x, \\ y & \textit{if } b \le y \le a, \\ \min(x,y) & \textit{otherwise}, \end{cases} \tag{12}$$

(ii) there is $c \in [e,1)$ *such that* U *is given by (8) and* F *is given by*

$$F(x,y) = \begin{cases} bS\left(\frac{x}{b}, \frac{y}{b}\right) & \textit{if } (x,y) \in [0,b]^2, \\ a + (e-a)T_1\left(\frac{x-a}{e-a}, \frac{y-a}{e-a}\right) & \textit{if } (x,y) \in [a,e]^2, \\ e + (c-e)T_2\left(\frac{x-e}{c-e}, \frac{y-e}{c-e}\right) & \textit{if } (x,y) \in [e,c]^2, \\ c + (1-c)T^*\left(\frac{x-c}{1-c}, \frac{y-c}{1-c}\right) & \textit{if } (x,y) \in [c,1]^2, \\ a & \textit{if } x \le a \le y, \\ b & \textit{if } y \le b \le x, \\ y & \textit{if } b \le y \le a, \\ \min(x,y) & \textit{otherwise}, \end{cases} \tag{13}$$

where S *is a continuous t-conorm,* T, T_1, T_2 *are continuous t-norms, and* T^* *is a t-norm isomorphic to* T_P.

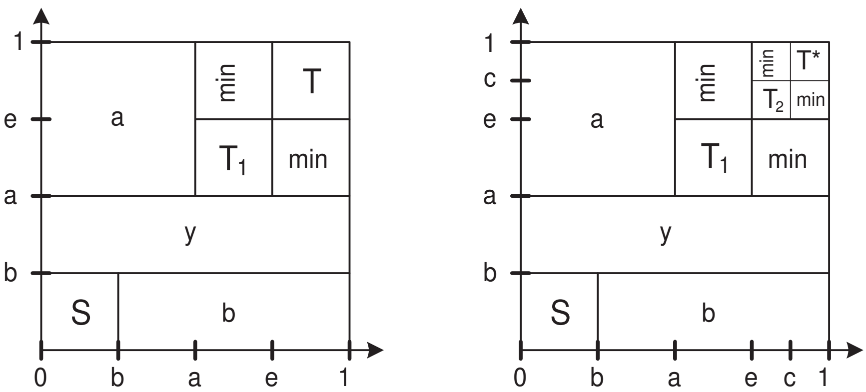

Figure 3. Semi-t-operators of forms (12) and (13) from Theorem 3.13.

Remark 3.14. *As seen in this section, the results for (CDr) are very similar to ones for (CDl). It should be emphasized that (CDl) is considered for* $a < b$ *since a semi-t-operator* F *has the right neutral element* 0 *in the interval* $[0,b]$, *and* *conditional distributivity from the left causes the idempotency of the uninorm U on the square* $[0, b]^2$*. In this manner the necessary and sufficient condition is obtained (see Theorem 3.1, Theorem 3.4 and Theorem 3.7). Analogously, (CDr) is considered when* $a > b$*.*

If (CDr) is observed for $a < b$, or (CDl) for $b < a$, only partial results can be obtained. This is illustrated by the following lemma.

Lemma 3.15. *(i) Let* $F \in \mathcal{F}_{a,b},\ a < b = 1$*, be a semi-t-operator and U be a conjunctive uninorm. If F is conditionally distributive over U from the right then U is idempotent on the square* $[0, a]^2$*.*

(ii) Let $F \in \mathcal{F}_{a,b},\ b < a = 1$*, be a semi-t-operator and U be a conjunctive uninorm. If F is conditionally distributive over U from the left then U is idempotent on the square* $[0, b]^2$*.*

Proof : (i) Let $F \in \mathcal{F}_{a,b},\ a < b = 1$, be a semi-t-operator and U be a conjunctive uninorm such that satisfy (CDr). For $x \in [0, a],\ y = z = 0$ from (CDr) follows

$$x = F(0, x) = F(U(0, 0), x) = U(F(0, x), F(0, x)) = U(x, x).$$

Therefore, U is idempotent on the square $[0, a]^2$.

Proof for claim (ii) is analogous. □

Contrary to the Theorem 3.4, in the previous lemma (i) we have obtain idempotency of the uninorm U only on the square $[0, a]^2$, i.e., only necessary condition is obtained.

4. Conclusion

The main topic of this paper is distributivity equations on the restricted domain between semi-t-operators and conjunctive uninorms. Results from the third section of this paper upgrade the corresponding ones from [14, 19], and continue the research from [16]. The connection with some of current research on this topic is given by the following overview.

(i) Paper [14] considers conditional distributivity from the left (right) of continuous semi-t-operator with respect to uninorm from the class $U_{\min}$ with continuous underlying t-norm and t-conorm. Therefore, results presented here extend research from [14].

(ii) Paper [19] deals with conditional distributivity of continuous nullnorm, i.e., continuous semi-t-operator such that $a = b$, with respect to conjunctive uninorm with continuous underlying t-norm and t-conorm. Thus, results presented here also extend research from [19].

(iii) Papers [5, 29] consider distributivity laws on the whole domain (continuity not required). In that case only idempotent uninorms are solutions. Results presented here show that when distributivity laws hold on the restricted domain, with assumption of continuity for F and U, both idempotent uninorms and uninorms that are non idempotent can be obtained as solutions for (CDl) and (CDr) (see Theorem 3.7 and Theorem 3.13).

(iv) The main difference between [16] and presented paper is that lemmas on relation between neutral element e of the conjunctive uninorm U and parameters a, b of the semi-t-operator F hold without assumption of continuity for F and U. Consequently, some results presented here hold also without assumption of continuity for F and U (see Theorem 3.4 and Theorem 3.11). Also, Theorem 3.7 and Theorem 3.13 provide a larger variety of idempotent uninorms as solutions for (CDl) and (CDr). Additionally, the results from [24] were crucial for the research from [16]. Based on [24], disjunctive (conjunctive) uninorms with continuous underlying t-norms and t-conorms are locally internal on the boundary, i.e., $U(0, y) \in \{0, y\}$ ($U(1, y) \in \{1, y\}$) for all $y \in [0, 1]$, and, for that reason, two directions of research had to be considered in [16]: $U(0, y) = y$ and $U(0, y) = 0,\ y \in (e, 1)$. The notion of locally internal uninorms on the boundary is not needed for the research presented in this paper.

In the forthcoming work analogous study to the one given in this paper will be done when operator F is a general aggregation operation with an absorbing element. Also, since pairs of aggregation operations that satisfying this relaxed distributivity law play an important role in utility theory (see [10, 13]), investigations will be directed in this research area as well.

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