



Operator Roots of Polynomials: Iso-Symmetric Operators

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Abstract. Given Hilbert space operators $A_i, B_i, i = 1, 2$, and X such that A_1 commutes with A_2 and B_1 commutes with B_2 , and integers $m, n \geq 1$, we say that the pairs of operators (B_1, A_1) and (B_2, A_2) are left- $(X, (m, n))$ -symmetric, denoted $((B_1, A_1), (B_2, A_2)) \in \text{left} - (X, (m, n)) - \text{symmetric}$, if

$$\sum_{j=0}^m \sum_{k=0}^n (-1)^{j+k} \binom{m}{j} \binom{n}{k} B_1^{m-j} B_2^{n-k} X A_2^{n-k} A_1^j = 0.$$

An important class of left- $(X, (m, n))$ -symmetric operators is obtained upon choosing $B_1 = B_2 = A_1^* = A_2^* = A^*$ and $X = I$: such operators have been called (m, n) -isosymmetric, and a study of the spectral picture and maximal invariant subspaces of (m, n) -isosymmetric operators has been carried out by Stankus [23]. Using what are essentially algebraic arguments involving elementary operators, we prove results on stability under perturbations by commuting nilpotents and products of commuting left- $(X, (m, n))$ -symmetric operators. It is seen that $(X, (m, n))$ -isosymmetric Drazin invertible operators A have a particularly interesting structure.

1. Introduction

Let $B(\mathcal{H})$ denote the algebra of operators, i.e. bounded linear transformations, on an infinite dimensional complex Hilbert space \mathcal{H} into itself. Let \mathbb{C} denote the complex plane, \mathbb{C}^d the product of d copies of \mathbb{C} for some integer $d \geq 1$, \bar{z} the conjugate of $z \in \mathbb{C}$ and $\mathbf{z} = (z_1, z_2, \dots, z_d) \in \mathbb{C}^d$. For a given polynomial P in \mathbb{C}^d and a d -tuple \mathbf{A} of commuting operators in $B(\mathcal{H})^d$, \mathbf{A} is a hereditary root of P if $P(\mathbf{A}) = 0$. Hereditary roots have attracted the attention of a number of researchers in the recent past. Two particular operator classes of hereditary roots which have been studied extensively are those of m -symmetric (also called m -selfadjoint in the literature) and m -isometric operators, where $A \in B(\mathcal{H})$ is m -symmetric (for some integer $m \geq 1$) if

$$\sum_{j=0}^m (-1)^j \binom{m}{j} A^{*(m-j)} A^j = 0$$

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and $A \in B(\mathcal{H})$ is m -isometric if

$$\sum_{j=0}^m (-1)^j \binom{m}{j} A^{*(m-j)} A^{m-j} = 0.$$

It is clear that A is m -symmetric if and only if it arises as a solution of $P(z) = (\bar{z} - z)^m = 0$, and A is m -isometric if and only if it arises as a solution of $P(z) = (\bar{z}z - 1)^m = 0$. The class of m -symmetric operators was introduced by Helton [20] (albeit not as operator solutions of the polynomial equation $(\bar{z} - z)^m = 0$), who showed that an operator A is 2-Jordan (i.e., $A = T + N$ for some self-adjoint T and a 2-nilpotent N commuting with T) if and only if A and A^* are 3-symmetric. McCullough and Rodman [21] in their consideration of algebraic and spectral properties of m -symmetric operators proved that if an $A \in B(\mathcal{H})$ is self-adjoint and an $N \in B(\mathcal{H})$ is an n -nilpotent which commutes with A , then $A + N$ is $(2n - 1)$ -symmetric. These operators have since been considered by many other authors, amongst them Stankus [23] and Trieu Le [22]

The class of m -isometric operators was introduced by Agler and studied in a series of papers by Agler and Stankus [1–3]; properties of m -isometric operators, amongst them the spectral picture, strict m -isometries, perturbation by commuting nilpotents and the product of m -isometries, have since been studied by a large number of authors, amongst them Sid Ahmed [4], Bayart [5], Bermudez *et al* [7–9], Botelho and Jamison [6], Duggal [11–13], Gu [17, 18] and Gu and Stankus [19].

$A \in B(\mathcal{H})$ is an (m, n) -isosymmetry for some integers $m, n \geq 1$ if

$$\begin{aligned} & \sum_{j=0}^m (-1)^j \binom{m}{j} A^{*(m-j)} \left(\sum_{k=0}^n (-1)^k \binom{n}{k} A^{*(n-k)} A^k \right) A^{m-j} \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} A^{*(n-k)} \left(\sum_{j=0}^m (-1)^j \binom{m}{j} A^{*(m-j)} A^{m-j} \right) A^k \\ &= 0; \end{aligned}$$

(m, n) -isosymmetric operators arise as the hereditary roots of the polynomial $(\bar{z}z - 1)^m (\bar{z} - z)^n = 0$, and a study, amongst other properties, of the spectrum, resolvent inequalities and maximal invariant subspaces of these operators has been carried out by Stankus [23]. In this paper we study a generalisation of (m, n) -isosymmetric operators, but from the point of view of elementary operators. The problem that we consider is that of the permanence of this generalised isosymmetric property under commuting products and perturbation by commuting nilpotents.

For $A, B \in B(\mathcal{H})$, let L_A and $R_B \in B(B(\mathcal{H}))$ denote respectively the operators

$$L_A(X) = AX \text{ and } R_B(X) = XB$$

of left multiplication by A and right multiplication by B . We say that the operator A is left (X, m) -invertible by B , denoted $(B, A) \in \text{left-}(X, m)$ -invertible, for some operator $X \in B(\mathcal{H})$ if

$$\Delta_{B,A}^m(X) = (L_B R_A - I)^m(X) = \sum_{j=0}^m (-1)^j \binom{m}{j} B^{m-j} X A^{m-j} = 0,$$

and that the operator B is an (X, n) -symmetry of A , denoted $(B, A) \in (X, n)$ -symmetry, for some $X \in B(\mathcal{H})$ if

$$\delta_{B,A}^n(X) = (L_B - R_A)^n(X) = \sum_{j=0}^n (-1)^j \binom{n}{j} B^{(n-j)} X A^j = 0.$$

It is clear from these definitions that $\Delta_{B,A}^m(I)$ defines the class of left- m -invertible operators A of [16], $\Delta_{A^*,A}^m(I)$ defines the class of m -isometric operators A , $\delta_{A^*,A}^n(I)$ defines the class of n -symmetric (equivalently, n -selfadjoint) operators, and an operator $A \in B(\mathcal{H})$ is (m, n) -isosymmetric if and only if

$$\Delta_{A^*,A}^m(\delta_{A^*,A}^n(I)) = \delta_{A^*,A}^n(\Delta_{A^*,A}^m(I)) = 0.$$

Let $[A, B] = AB - BA$ denote the commutator of $A, B \in B(\mathcal{H})$. Given operators $A_i, B_i, X \in B(\mathcal{H})$, $i = 1, 2$ and positive integers m and n , such that

$$[A_1, A_2] = [B_1, B_2] = 0,$$

we say in the following that the pairs of operators (B_1, A_1) and (B_2, A_2) are left- $(X, (m, n))$ -symmetric, denoted

$$((B_1, A_1), (B_2, A_2)) \in \text{left} - (X, (m, n)) - \text{symmetric}$$

if

$$\begin{aligned} & \Delta_{B_1, A_1}^m(\delta_{B_2, A_2}^n(X)) = (L_{B_1}R_{A_1} - I)^m((L_{B_2} - R_{A_2})^n(X)) \\ &= \sum_{j=0}^m \sum_{k=0}^n (-1)^{j+k} \binom{m}{j} \binom{n}{k} L_{B_1}^{m-j} L_{B_2}^{n-k} R_{A_2}^k R_{A_1}^{m-j}(X) \\ &= (L_{B_2} - R_{A_2})^n((L_{B_1}R_{A_1} - I)^m(X)) = \delta_{B_2, A_2}^n(\Delta_{B_1, A_1}^m(X)) \\ &= \sum_{k=0}^n \sum_{j=0}^m (-1)^{j+k} \binom{m}{j} \binom{n}{k} L_{B_2}^{n-k} L_{B_1}^{m-j} R_{A_1}^k R_{A_2}^{m-j}(X) \\ &= 0. \end{aligned}$$

Products, and perturbation by commuting nilpotents, of left- $(X, (m, n))$ -symmetric pairs of operators behave in a manner very similar to that of m -isometric and n -symmetric operators. We prove:

Theorem 1.1. *If $A_i, B_i, S_i, T_i, X \in B(\mathcal{H})$, $i = 1, 2$, are such that*

- (i) $[A_1, A_2] = [B_1, B_2] = [A_i, T_i] = [B_i, S_i] = 0$,
- (ii) $((B_1, A_1), (B_2, A_2)) \in \text{left} - (X, (m_1, n_1)) - \text{symmetric}$,
- (iii) $((S_1, T_1), (B_2, A_2)) \in \text{left} - (X, (r_1, n_2)) - \text{symmetric}$,
- (iv) $((B_1, A_1), (S_2, T_2)) \in \text{left} - (X, (m_2, s_1)) - \text{symmetric}$, and
- (v) $((S_1, T_1), (S_2, T_2)) \in \text{left} - (X, (r_2, s_2)) - \text{symmetric}$,

then

$$((S_1B_1, T_1A_1), (S_2B_2, T_2A_2)) \in \text{left} - (X, (m + r - 1, n + s - 1)) - \text{symmetric},$$

where $m = \max(m_1, m_2)$, $n = \max(n_1, n_2)$, $r = \max(r_1, r_2)$ and $s = \max(s_1, s_2)$.

Theorem 1.2. *If $A_i, B_i, M_i, N_i, X \in B(\mathcal{H})$, $i = 1, 2$, are such that*

- (i) $M_i^{m_i} = N_i^{n_i} = 0$, m_i and n_i some positive integers ($i = 1, 2$),
- (ii) $[A_1, A_2] = [B_1, B_2] = [M_1, M_2] = [N_1, N_2] = [A_i, M_i] = [B_i, N_i] = 0$ ($i = 1, 2$) and
- (iii) $((B_1, A_1), (B_2, A_2)) \in \text{left} - (X, (m, n)) - \text{symmetric}$,

then

$$((B_1 + N_1, A_1 + M_1), (B_2 + N_2, A_2 + M_2)) \in \text{left} - (X, (m + m_1 + n_1 - 2, n + m_2 + n_2 - 2)) - \text{symmetric}.$$

An operator $A \in B(\mathcal{H})$ is Drazin invertible, with Drazin inverse A_d , if

$$[A_d, A] = 0, A_d^2A = A_d, A^{p+1}A_d = A^p$$

for some integer $p \geq 1$. (The least integer p for which this holds is then called the Drazin index of A .) No Drazin invertible operator $A \in B(\mathcal{H})$ can be m -isometric (equivalently, left- m -invertible by its adjoint) [15]: there may however exist operators $X \in B(\mathcal{H})$ such that A is left- (X, m) -invertible by A^* . We prove:

Theorem 1.3. Let $A \in B(\mathcal{H})$ be a Drazin invertible operator, with Drazin index p and Drazin inverse A_d . Let $X \in B(\mathcal{H})$, and let m, n be some positive integers.

(i) If $A \in (X, (m, n))$ -isosymmetric, then

$$\Delta_{A_d^*, A}^n \left(\Delta_{A^*, A_d}^m (X) \right) = 0 = \delta_{A_d^*, A}^m \left(\delta_{A_d^*, A_d}^n (X) \right);$$

(ii) if $((A_d^*, A), (A^*, A)) \in \text{left-}(X, (m, n))$ -symmetric, then $\Delta_{A_d^*, A}^n \left(\Delta_{A^*, A_d}^m (X) \right) = 0$;

(iii) if $((A^*, A), (A_d^*, A)) \in \text{left-}(X, (m, n))$ -symmetric, then

$$\delta_{A^*, A_d}^n \left(\delta_{A_d^*, A}^m (X) \right) = 0.$$

We prove Theorem 1.1, and most of our complementary results, in Section 2, tensor products are considered in Section 3, Section 4 is devoted to the proof of Theorem 1.2, and we prove Theorem 1.3 in Section 5.

2. Complementary results, Proof of Theorem 1.1.

We start this section by proving some complementary results. Throughout the following $A_i, B_i, i = 1, 2$, and X will denote operators in $B(\mathcal{H})$, and m, n, t will denote positive integers.

Lemma 2.1. [15, Proposition 2.1] If $(B, A) \in \{(X, n) - \text{symmetric}\} \vee \{\text{left} - (X, m) - \text{invertible}\}$,

(i) then $(B, A) \in \{(X, t) - \text{symmetric}\} \vee (\text{resp.})\{\text{left} - (X, t) - \text{invertible}\}$ for all integers $t \geq n, (\text{resp.})m$;

(ii) and A, B are invertible, then $(B^{-1}, A^{-1}) \in \{(X, n) - \text{symmetric}\} \vee (\text{resp.})\{\text{left} - (X, m) - \text{invertible}\}$.

The following lemma is an easy consequence of Lemma 2.1 (ii).

Lemma 2.2. If A_i, B_i are invertible and $((B_1, A_1), (B_2, A_2)) \in \text{left-}(X, (m, n))$ -symmetric, then $((B_1^{-1}, A_1^{-1}), (B_2^{-1}, A_2^{-1})) \in \text{left-}(X, (m, n))$ -symmetric.

Trivially, $[L_B, R_A] = 0$ for all operators $A, B \in B(\mathcal{H})$. Hence, for every integer $k \geq 1$,

$$\begin{aligned} (L_B^k - R_A^k)^m &= (L_{B^k} - R_{A^k})^m = (L_B - R_A)^m P_{m(k-1)}(L_B, R_A) \\ &= P_{m(k-1)}(L_B, R_A)(L_B - R_A)^m \end{aligned}$$

and

$$\begin{aligned} (L_B^k R_A^k - I)^m &= (L_{B^k} R_{A^k} - I)^m = (L_B R_A - I)^m Q_{m(k-1)}(L_B, R_A) \\ &= Q_{m(k-1)}(L_B, R_A)(L_B R_A - I)^m \end{aligned}$$

for some polynomials P and Q of degree $m(k - 1)$. We have:

Lemma 2.3. [15] If $(B, A) \in \{(X, n) - \text{symmetric}\} \vee \{\text{left} - (X, m) - \text{invertible}\}$, then $(B^k, A^k) \in \{(X, n) - \text{symmetric}\} \vee (\text{resp.})\{\text{left} - (X, m) - \text{invertible}\}$ for every positive integer k .

Considering operators A_i and $B_i, i = 1, 2$, such that $[A_1, A_2] = [B_1, B_2] = 0$, one has

$$[L_{B_i}, R_{A_j}] = [L_{B_1}, L_{B_2}] = [R_{A_1}, R_{A_2}] = 0; i, j = 1, 2.$$

Hence

$$\Delta_{B_1, A_1}^m \left(\delta_{B_2, A_2}^n (X) \right) = \delta_{B_2, A_2}^n \left(\Delta_{B_1, A_1}^m (X) \right).$$

We have:

Lemma 2.4. If $[A_1, A_2] = [B_1, B_2] = 0$, and either $(B_1, A_1) \in \text{left-}(X, m)$ -invertible or $(B_2, A_2) \in (X, n)$ -symmetric, then $\Delta_{B_1, A_1}^m (\delta_{B_2, A_2}^n (X)) = \delta_{B_2, A_2}^n (\Delta_{B_1, A_1}^m (X)) = 0$.

Lemma 2.5. If $[A_1, A_2] = [B_1, B_2] = 0$ and $((B_1, A_1), (B_2, A_2)) \in \text{left-}(X, (m, n))$ -symmetric, then $((B_1, A_1), (B_2, A_2)) \in \{ \text{left-}(X, (m_1, n))$ -symmetric $\} \wedge \{ \text{left-}(X, (m, n_1))$ -symmetric $\} \wedge \{ \text{left-}(X, (m_1, n_1))$ -symmetric $\}$ for all integers $m_1 \geq m$ and $n_1 \geq n$.

Proof. The proof follows since

$$\begin{aligned} \Delta_{B_1, A_1}^{m_1} (\delta_{B_2, A_2}^n) &= \Delta_{B_1, A_1}^{m_1 - m} (\Delta_{A_1, B_1}^m (\delta_{B_2, A_2}^n)), \\ \Delta_{B_1, A_1}^m (\delta_{B_2, A_2}^{n_1}) &= \Delta_{B_1, A_1}^m (\delta_{B_2, A_2}^{n_1 - n} \delta_{B_2, A_2}^n) = \delta_{B_2, A_2}^{n_1 - n} (\Delta_{B_1, A_1}^m \delta_{B_2, A_2}^n) \end{aligned}$$

and

$$\Delta_{B_1, A_1}^{m_1} (\delta_{B_2, A_2}^{n_1}) = \Delta_{B_1, A_1}^{m_1 - m} \delta_{B_2, A_2}^{n_1 - n} (\Delta_{B_1, A_1}^m \delta_{B_2, A_2}^n).$$

□

Proposition 2.6. Let $A_i, B_i, S, T, X \in B(\mathcal{H})$, $i = 1, 2$, be such that

$$[A_1, A_2] = [B_1, B_2] = [A_i, T] = [B_i, S] = 0,$$

and let (a), (b), (c), (d) and (e) be the hypotheses:

- (a) $((B_1, A_1), (B_2, A_2)) \in \text{left-}(X, (m, n))$ -symmetric;
- (b) $(B_1, A_1) \in \text{left-}(X, m)$ -invertible;
- (c) $(B_2, A_2) \in \text{left-}(X, n)$ -symmetric;
- (d) $(S, T) \in \text{left-}(X, t)$ -invertible;
- (e) $(S, T) \in \text{left-}(X, t)$ -symmetric.

(i) If either of the hypotheses (b) or (a)∧(e) or (c)∧(e) is satisfied, then

$$((B_1, A_1), (SB_2, TA_2)) \in \text{left-}(X, (m, n + t - 1))\text{-symmetric}.$$

(ii) If either of the hypotheses (c) or (a)∧(d) or (b)∧(d) is satisfied, then

$$((SB_1, TA_1), (B_2, A_2)) \in \text{left-}(X, (m + t - 1, n))\text{-symmetric}.$$

Proof. The hypotheses $[S, B_2] = [T, A_2] = 0$ imply

$$\begin{aligned} \delta_{SB_2, TA_2}^{n+t-1} &= (L_S L_{B_2} - R_T R_{A_2})^{n+t-1} \\ &= \{L_S (L_{B_2} - R_{A_2}) + (L_S - R_T) R_{A_2}\}^{n+t-1} \\ &= \sum_{j=0}^{n+t-1} \binom{n+t-1}{j} L_S^{n+t-1-j} R_{A_2}^j \delta_{B_2, A_2}^{n+t-1-j} \delta_{S, T}^j \\ &= \sum_{j=0}^{n+t-1} \binom{n+t-1}{j} L_S^{n+t-1-j} R_{A_2}^j \delta_{S, T}^j \delta_{B_2, A_2}^{n+t-1-j} \end{aligned}$$

and the hypotheses $[S, B_1] = [T, A_1] = 0$ imply

$$\begin{aligned} \Delta_{SB_1, TA_1}^{m+t-1} &= (L_S L_{B_1} R_T R_{A_1} - I)^{m+t-1} \\ &= \{L_S R_T (L_{B_1} R_{A_1} - I) + (L_S R_T - I)\}^{m+t-1} \\ &= \sum_{k=0}^{m+t-1} \binom{m+t-1}{k} (L_S R_T)^{m+t-1-k} \Delta_{B_1, A_1}^{m+t-1-k} \Delta_{S, T}^k \\ &= \sum_{k=0}^{m+t-1} \binom{m+t-1}{k} (L_S R_T)^{m+t-1-k} \Delta_{S, T}^k \Delta_{B_1, A_1}^{m+t-1-k}. \end{aligned}$$

This, in view of the commutativity hypotheses on A_i, B_i, S and T ($i = 1, 2$), implies

$$\begin{aligned} & \Delta_{B_1, A_1}^m (\delta_{SB_2, TA_2}^{n+t-1}(X)) \\ &= \sum_{j=0}^{n+t-1} \binom{n+t-1}{j} L_S^{n+t-1-j} R_{A_2}^j \delta_{S, T}^j \delta_{B_2, A_2}^{n+t-1-j} (\Delta_{B_1, A_1}^m (X)) \\ &= \sum_{j=0}^{t-1} \binom{n+t-1}{j} L_S^{n+t-1-j} R_{A_2}^j \delta_{S, T}^j (\Delta_{B_1, A_1}^m \delta_{B_2, A_2}^{n+t-1-j}(X)) \\ &\quad + \sum_{j=t}^{n+t-1} \binom{n+t-1}{j} L_S^{n+t-1-j} R_{A_2}^j \Delta_{B_1, A_1}^m \delta_{B_2, A_2}^{n+t-1-j} (\delta_{S, T}^j(X)) \\ &= \sum_{j=0}^{t-1} \binom{n+t-1}{j} L_S^{n+t-1-j} R_{A_2}^j \delta_{S, T}^j \Delta_{B_1, A_1}^m (\delta_{B_2, A_2}^{n+t-1-j}(X)) \\ &\quad + \sum_{j=t}^{n+t-1} \binom{n+t-1}{j} L_S^{n+t-1-j} R_{A_2}^j \Delta_{B_1, A_1}^m \delta_{B_2, A_2}^{n+t-1-j} (\delta_{S, T}^j(X)) \end{aligned}$$

and

$$\begin{aligned} & \Delta_{SB_1, TA_1}^{m+t-1} (\delta_{B_2, A_2}^n(X)) \\ &= \sum_{k=0}^{m+t-1} \binom{m+t-1}{k} (L_S R_T)^{m+t-1-k} \Delta_{S, T}^k \Delta_{B_1, A_1}^{m+t-1-k} (\delta_{B_2, A_2}^n(X)) \\ &= \sum_{k=0}^{t-1} \binom{m+t-1}{k} (L_S R_T)^{m+t-1-k} \Delta_{S, T}^k (\Delta_{B_1, A_1}^{m+t-1-k} \delta_{B_2, A_2}^n(X)) \\ &\quad + \sum_{k=t}^{m+t-1} \binom{m+t-1}{k} (L_S R_T)^{m+t-1-k} \Delta_{B_1, A_1}^{m+t-1-k} (\Delta_{B_1, A_1}^k \delta_{B_2, A_2}^n(X)) \\ &= \sum_{k=0}^{t-1} \binom{m+t-1}{k} (L_S R_T)^{m+t-1-k} \Delta_{S, T}^k (\delta_{B_2, A_2}^n \Delta_{B_1, A_1}^{m+t-1-k}(X)) \\ &\quad + \sum_{k=t}^{m+t-1} \binom{m+t-1}{k} (L_S R_T)^{m+t-1-k} \Delta_{B_1, A_1}^{m+t-1-k} \delta_{B_2, A_2}^n (\Delta_{S, T}^k(X)). \end{aligned}$$

Recall now that

$$\Delta_{B_1, A_1}^m \delta_{B_2, A_2}^n(X) = 0 \implies \Delta_{B_1, A_1}^m \delta_{B_2, A_2}^{n_1}(X) = 0 \text{ for all } n_1 \geq n$$

and

$$\delta_{S, T}^t(X) = 0 \implies \delta_{S, T}^{t_1}(X) = 0 \text{ for all } t_1 \geq t$$

(see Lemmas 2.5 and 2.1). Hence, if either of the hypotheses (b), or, $(a) \wedge (e)$, or, $(c) \wedge (e)$ is satisfied, then

$$((B_1, A_1), (SB_2, TA_2)) \in \text{left} - (X, (m, n + t - 1)) - \text{symmetric}.$$

Again, since

$$\Delta_{B_1, A_1}^m (\delta_{B_2, A_2}^n(X)) = 0 \implies \Delta_{B_1, A_1}^{m_1} (\delta_{B_2, A_2}^n(X)) = 0 \text{ for all } m_1 \geq m,$$

if either of the hypotheses (c), or, $(a) \wedge (d)$, or, $(b) \wedge (d)$ is satisfied, then

$$((SB_1, TA_1), (B_2, A_2)) \in \text{left} - (X, (m + t - 1, n)) - \text{symmetric}.$$

□

A particularly interesting case of Proposition 2.6 is obtained upon choosing

$$A_1 = A_2 = A, B_1 = B_2 = A^*, S_1 = S_2 = B^* \text{ and } T_1 = T_2 = B.$$

We have:

Corollary 2.7. *If $[A, B] = [A, B^*] = 0$, $A \in (m, n)$ -isosymmetric and B is both left- r -invertible and s -symmetric, then*

$$AB \in \text{left} - (m + r - 1, n + s - 1) - \text{symmetric}.$$

A generalization of Proposition 2.6 is obtained upon replacing the pair of operators (S, T) by the pairs (S_i, T_i) , $i = 1, 2$. Observe that the hypothesis

$$(S_1, T_1) \in \text{left} - (X, t) - \text{invertible} \implies \delta_{E,F}^j (\Delta_{S_1, T_1}^t(X)) = \Delta_{E,F}^k (\Delta_{S_1, T_1}^t(X)) = 0$$

for all $1 \leq j, k$ and $(E, F) = (S_2, T_2) \vee (B_i, A_i)$, $i = 1, 2$, and the hypothesis

$$(S_2, T_2) \in \text{left} - (X, s) - \text{symmetric} \implies \delta_{E,F}^j (\delta_{S_2, T_2}^s(X)) = \Delta_{E,F}^k (\delta_{S_2, T_2}^s(X)) = 0$$

for all $1 \leq j, k$ and $(E, F) = (S_1, T_1) \vee (B_i, A_i)$, $i = 1, 2$.

Corollary 2.8. *If $[A_i, T_j] = [B_i, S_j] = [S_1, S_2] = [B_1, B_2] = [A_1, A_2] = 0$, $1 \leq i, j \leq 2$,*

(i) $((B_1, A_1), (B_2, A_2)) \in \text{left} - (X, (m, n))$ -symmetric;

(ii) $(S_1, T_1) \in \text{left} - (X, r)$ -invertible; and

(iii) $(S_2, T_2) \in (X, s)$ -symmetric,

then

$$((B_1 S_1, A_1 T_1), (B_2 S_2, A_2 T_2)) \in \text{left} - (X, (m + r - 1, n + s - 1)) - \text{symmetric}.$$

Proof. In view of the commutativity hypotheses, Proposition 2.6 implies that if (i) and (ii) are satisfied, then

(iv) $((B_1 S_1, A_1 T_1), (B_2, A_2)) \in \text{left} - (X, (m + r - 1, n))$ -symmetric;

another application of Proposition 2.6, this time since (iv) and (iii) are satisfied, now implies the result. \square

Theorem 1.1, which we now prove, is a generalization of Corollary 2.8.

Proof of Theorem 1.1. Defining the positive integers m, n, r and s as in the statement of the theorem, the commutativity hypotheses imply

$$\begin{aligned} \delta_{S_2 B_2, T_2 A_2}^{n+s-1} &= \{L_{S_2} (L_{B_2} - R_{A_2}) + (L_{S_2} - R_{T_2}) R_{A_2}\}^{n+s-1} \\ &= \sum_{k=0}^{n+s-1} \binom{n+s-1}{k} L_{S_2}^{n+s-1-k} R_{A_2}^k \delta_{B_2, A_2}^{n+s-1-k} \delta_{S_2, T_2}^k; \end{aligned}$$

$$\begin{aligned} \Delta_{S_1 B_1, T_1 A_1}^{m+r-1} &= \{L_{S_1} (L_{B_1} L_{A_1} - I) R_{T_1} + (L_{S_1} R_{T_1} - I)\}^{m+r-1} \\ &= \sum_{j=0}^{m+r-1} \binom{m+r-1}{j} (L_{S_1} R_{T_1})^{m+r-1-j} \Delta_{B_1, A_1}^{m+r-1-j} \Delta_{S_1, T_1}^j; \end{aligned}$$

and

$$\begin{aligned} \Delta_{S_1 B_1, T_1 A_1}^{m+r-1} (\delta_{S_2 B_2, T_2 A_2}^{n+s-1}(X)) &= \left\{ \sum_{j=0}^{m+r-1} \sum_{k=0}^{n+s-1} \binom{m+r-1}{j} \binom{n+s-1}{k} \right\} \times \\ &\quad \times (L_{S_1} R_{T_1})^{m+r-1-j} L_{S_2}^{n+s-1-k} R_{A_2}^k \Delta_{B_1, A_1}^{m+r-1-j} \Delta_{S_1, T_1}^j \delta_{B_2, A_2}^{n+s-1-k} \delta_{S_2, T_2}^k (X) \end{aligned}$$

where the commutativity hypotheses guarantee the commutativity of all the operator entries within the curly brackets. To complete the proof we observe now that:

- if $j \geq r$ and $k \geq s$, then $\Delta_{S_1, T_1}^j \left(\delta_{S_2, T_2}^k(X) \right) = 0$;
- if $j \leq r - 1$ and $k \geq s$, then $\Delta_{B_1, A_1}^{m+r-1-j} \left(\delta_{S_2, T_2}^k(X) \right) = 0$;
- if $j \geq r$ and $k \leq s - 1$, then $\Delta_{S_1, T_1}^j \left(\delta_{B_2, A_2}^{n+s-1-k}(X) \right) = 0$
- and (finally)
- if $j \leq r - 1$ and $k \leq s - 1$, then $\Delta_{B_1, A_1}^{m+r-1-j} \left(\delta_{B_2, A_2}^{n+s-1-k}(X) \right) = 0$.

3. Tensor Products.

Let $\mathcal{H} \otimes \mathcal{H}$, endowed with a reasonable uniform cross norm, denote the completion of the algebraic tensor product of \mathcal{H} with itself, and let $A \otimes B$ denote the tensor product operator defined by $A, B \in B(\mathcal{H})$. Theorem 1.1 has applications to tensor products.

Consider operators $A_i, B_i, i = 1, 2$. Since

$$\begin{aligned} & \Delta_{B_1 \otimes I, A_1 \otimes I}^m \left(\delta_{B_2 \otimes I, A_2 \otimes I}^n \right) \\ &= \sum_{j=0}^m \sum_{k=0}^n (-1)^{j+k} \binom{m}{j} \binom{n}{k} (L_{B_1 \otimes I} R_{A_1 \otimes I})^{m-j} (L_{B_2 \otimes I} R_{A_2 \otimes I})^k \\ &= \sum_{j=0}^m \sum_{k=0}^n (-1)^{j+k} \binom{m}{j} \binom{n}{k} (L_{B_1} R_{A_1})^{m-j} (L_{B_2} R_{A_2})^k, \end{aligned}$$

$$\Delta_{B_1 \otimes I, A_1 \otimes I}^m \left(\delta_{B_2 \otimes I, A_2 \otimes I}^n(X \otimes X) \right) = \Delta_{B_1, A_1}^m \left(\delta_{B_2, A_2}^n(X) \right) \otimes X.$$

Hence, if $((B_1, A_1), (B_2, A_2)) \in \text{left-}(X, (m, n))\text{-symmetric}$, then

$$\Delta_{B_1 \otimes I, A_1 \otimes I}^m \left(\delta_{B_2 \otimes I, A_2 \otimes I}^n(X \otimes X) \right) = 0$$

(and, arguing similarly,

$$\Delta_{I \otimes B_1, I \otimes A_1}^m \left(\delta_{I \otimes B_2, I \otimes A_2}^n(X \otimes X) \right) = 0).$$

Again, if $\Delta_{P_1, Q_1}^r(X) = 0$ (resp., $\delta_{P_2, Q_2}^s(X) = 0$), then $\Delta_{I \otimes P_1, I \otimes Q_1}^r(X \otimes X) = 0$ (resp., $\delta_{I \otimes P_2, I \otimes Q_2}^s(X \otimes X) = 0$).

Corollary 3.1. Let $E_i, F_i, P_i, Q_i \in B(\mathcal{H}), i = 1, 2$, be such that $[E_1, E_2] = [F_1, F_2] = [P_1, Q_1] = [P_2, Q_2] = 0$ and

$$\Delta_{E_1, F_1}^m \left(\delta_{E_2, F_2}^n(X) \right) = \Delta_{P_1, Q_1}^r(X) = \delta_{P_2, Q_2}^s(X) = 0.$$

Then

$$((E_1 \otimes P_1, F_1 \otimes Q_1), (E_2 \otimes P_2, F_2 \otimes Q_2)) \in \text{left-}(X \otimes X, (m + r - 1, n + s - 1))\text{-symmetric}.$$

Proof. The hypothesis $\Delta_{P_1, Q_1}^r(X) = \delta_{P_2, Q_2}^s(X) = 0$ implies $\Delta_{I \otimes P_1, I \otimes Q_1}^r \left(\delta_{E_2 \otimes I, F_2 \otimes I}^n(X \otimes X) \right) = \delta_{I \otimes P_2, I \otimes Q_2}^s \left(\Delta_{E_1 \otimes I, F_1 \otimes I}^m(X \otimes X) \right) = \Delta_{I \otimes P_1, I \otimes Q_1}^r \left(\delta_{I \otimes P_2, I \otimes Q_2}^s(X \otimes X) \right) = 0$. If we let

$$B_i = E_i \otimes I, A_i = F_i \otimes I, S_i = I \otimes P_i \text{ and } T_i = I \otimes Q_i, i = 1, 2,$$

then the hypotheses of Theorem 1.1 are satisfied (with $X \otimes X$ playing the role of X). Hence, since

$$S_i B_i = E_i \otimes P_i \text{ and } T_i A_i = F_i \otimes Q_i, i = 1, 2,$$

the proof follows. \square

For commuting n -isometries $S, T \in B(\mathcal{H})$, Corollary 3.1 take the form:

Corollary 3.2. *If $[S, T] = 0$ and*

$$\Delta_{S^*, S}^n(I) = \delta_{T^*, T}^n(I) = 0,$$

then

$$\Delta_{S^* \otimes T^*, S \otimes T}^{2n-1}(\delta_{S^* \otimes T^*, S \otimes T}^{2n-1}(I \otimes I)) = 0.$$

4. Perturbation by commuting nilpotents.

It is well known, see for example [22], that if $d_{B,A} = \delta_{B,A} \vee \Delta_{B,A}$, $d_{B,A}^m(I) = 0$ and N is an n -nilpotent which commutes with A , then $d_{B,A+N}^{m+n-1}(I) = 0$. This extends to A, B such that $d_{B,A}^m(X) = 0$ for some $X \in B(\mathcal{H})$, as the following argument shows:

$$\begin{aligned} \Delta_{B,A+N}^{m+n-1}(X) &= \{(L_B R_A - I) + L_B R_N\}^{m+n-1}(X) \\ &= \left\{ \sum_{j=0}^{m+n-1} \binom{m+n-1}{j} (L_B R_N)^j \Delta_{B,A}^{m+n-1-j} \right\}(X) \\ &= 0, \end{aligned}$$

since $R_N^j = 0$ for all $j \geq n$ and $\Delta_{B,A}^{m+n-1-j}(X) = 0$ for all $j \leq n-1$ ($\implies m+n-1-j \geq m$); again

$$\begin{aligned} \delta_{B,A+N}^{m+n-1}(X) &= \{(L_B - R_A) - R_N\}^{m+n-1}(X) \\ &= \left\{ \sum_{j=0}^{m+n-1} (-1)^j \binom{m+n-1}{j} R_N^j \delta_{B,A}^{m+n-1-j} \right\}(X) \\ &= 0. \end{aligned}$$

for $R_N^j = 0$ for all $j \geq n$ and $\delta_{B,A}^{m+n-1-j}(X) = 0$ for all $j \leq n-1$.

This argument extends to perturbation by commuting nilpotents of operators $((B_1, A_1), (B_2, A_2)) \in \text{left} - (X, (m, n)) - \text{symmetric}$.

Proof of Theorem 1.2. The commutativity hypothesis $[A_1, A_2] = [B_1, B_2] = 0$ implies

$$\delta_{B_2, A_2}^n(\Delta_{B_1, A_1}^m(X)) = \Delta_{B_1, A_1}^m(\delta_{B_2, A_2}^n(X)) = 0.$$

Set

$$\delta_{B_2, A_2}^n(X) = Y.$$

Then $\Delta_{B_1, A_1}^m(Y) = 0$, and it follows from the argument above that if $[A_1, M_1] = 0$ and $M_1^{m_1} = 0$, then

$$\Delta_{B_1, A_1 + M_1}^{m+m_1-1}(Y) = 0 \iff \Delta_{A_1^* + M_1^*, B_1^*}^{m+m_1-1}(Y^*) = 0,$$

and hence if $[B_1, N_1] = 0, N_1^{n_1} = 0$, then

$$\Delta_{A_1^* + M_1^*, B_1^* + N_1^*}^{m+m_1+n_1-2}(Y^*) = 0 \iff \Delta_{B_1 + N_1, A_1 + M_1}^{m+m_1+n_1-2}(Y) = 0.$$

Let now $\Delta_{B_1+N_1, A_1+M_1}^{m+m_1+n_1-1}(X) = Z$. Then, upon assuming the full force of the commutativity hypotheses, we have

$$\begin{aligned} \delta_{B_2, A_2}^n(Z) = 0 &\implies \delta_{B_2, A_2+M_2}^{n+m_2-1}(Z) = 0 \iff \delta_{A_2^*+M_2^*, B_2^*}^{n+m_2-1}(Z^*) = 0 \\ &\implies \delta_{A_2^*+M_2^*, B_2^*+N_2^*}^{n+m_2+n_2-2}(Z^*) = 0 \iff \delta_{B_2+N_2, A_2+M_2}^{n+m_2+n_2-2}(Z) = 0 \\ &\iff \delta_{B_2+N_2, A_2+M_2}^{n+m_2+n_2-2}(\Delta_{B_1+N_1, A_1+M_1}^{n+m_2+n_2-2}(X)) \\ &= \Delta_{B_1+N_1, A_1+M_1}^{n+m_2+n_2-2}(\delta_{B_2+N_2, A_2+M_2}^{n+m_2+n_2-2}(X)) = 0. \end{aligned}$$

We leave it to the reader to verify that if one drops the hypotheses $[A_1, A_2] = [B_1, B_2] = [M_1, M_2] = [N_1, N_2] = 0$ in the statement of Theorem 1.2, then

$$\delta_{B_2, A_2}^n(\Delta_{B_1, A_1}^m(X)) = 0 \implies \delta_{B_2+N_2, A_2+M_2}^{n+m_2+n_2-2}(\Delta_{B_1+N_1, A_1+M_1}^{n+m_2+n_2-2}(X)) = 0$$

only. For $(X, (m, n))$ -isosymmetric operators A , Theorem 1.2 implies

Corollary 4.1. *If $[A, N] = 0 = N^{n_1}$ and $A \in (X, (m, n))$ -isosymmetric, then $A + N \in (X, (m + 2n_1 - 2, n + 2n_1 - 2))$ -isosymmetric.*

5. Application to Drazin invertible operators.

Let $T \in B(\mathcal{H})$ be a Drazin invertible operator. Then there exists a decomposition

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$$

of \mathcal{H} and a decomposition

$$T = T_1 \oplus T_2 (= T|_{\mathcal{H}_1} \oplus T|_{\mathcal{H}_2})$$

of T such that T_1 is invertible and $T_2^p = 0$ for some integer $p \geq 1$. The Drazin inverse T_d of T has a representation

$$T_d = T_1^{-1} \oplus 0 \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$$

[10, Theorem 2.23]. In the following we use this representation of Drazin invertible operators to prove Theorem 1.3.

Proof of Theorem 1.3 Let $X \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$ have the matrix representation $[X_{i,j}]_{i,j=1}^2$. Then

$$\begin{aligned} (1) \quad 0 &= \Delta_{T^*, T}^m(\delta_{T^*, T}^n(X)) = \delta_{T^*, T}^n(\Delta_{T^*, T}^m(X)) \\ &= \sum_{l=0}^n \sum_{k=0}^m (-1)^{l+k} \binom{n}{l} \binom{m}{k} \{T^{*(n-l+m-k)} X T^{m-k+l}\} \\ &= \sum_{l=0}^n \sum_{k=0}^m (-1)^{l+k} \binom{n}{l} \binom{m}{k} [T_i^{*(n-l+m-k)} X_{ij} T_j^{m-k+l}]_{i,j=1}^2; \end{aligned}$$

$$\begin{aligned} (2) \quad 0 &= \Delta_{T_d^*, T}^m(\delta_{T_d^*, T}^n(X)) = \delta_{T_d^*, T}^n(\Delta_{T_d^*, T}^m(X)) \\ &= \sum_{l=0}^n \sum_{k=0}^m (-1)^{l+k} \binom{n}{l} \binom{m}{k} \begin{pmatrix} T_1^{*(n-l+m+k)} X_{11} T_1^{m-k+l} & T_1^{*(n-l+m+k)} X_{12} T_2^{m-k+l} \\ 0 & 0 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned}
 0 &= \Delta_{T^*, T}^m \left(\delta_{T_d^*, T}^n(X) \right) = \delta_{T_d^*, T}^n \left(\Delta_{T^*, T}^m(X) \right) \\
 (3) \quad &= \sum_{l=0}^n \sum_{k=0}^m (-1)^{l+k} \binom{n}{l} \binom{m}{k} \begin{pmatrix} T_1^{*(m-k-n+l)} X_{11} T_1^{m-k+l} & T_1^{*(m-k-n+l)} X_{12} T_2^{m-k+l} \\ 0 & 0 \end{pmatrix}.
 \end{aligned}$$

We prove that $X_{12} = 0$. Letting s denote either of $n - l$ and $-n + l$, we have

$$\begin{aligned}
 &\sum_{l=0}^n \sum_{k=0}^m (-1)^{l+k} \binom{n}{l} \binom{m}{k} T_i^{*-m+k+s} X_{12} T_2^{m-k+l} = 0 \\
 \iff &\sum_{l=0}^n (-1)^l \binom{n}{l} T_1^{*s} \left\{ \sum_{k=0}^{m-1} (-1)^k \binom{m}{k} T_1^{*-m+k} X_{12} T_2^{m-k} + X_{12} \right\} T_2^l = 0 \\
 \implies &\left\{ \left(\sum_{l=1}^n (-1)^l \binom{n}{l} \sum_{k=0}^{m-1} (-1)^k \binom{m}{k} T_1^{*s-m+k} X_{12} T_2^{m-k+l} + T_1^{*s} X_{12} T_2^l \right) \right. \\
 &\quad \left. + \sum_{k=0}^{m-1} (-1)^k \binom{m}{k} T_1^{*s-m+k} X_{12} T_2^{m-k} + T_1^{*s} X_{12} \right\} T_2^{p-1} = 0 \\
 \iff &T_1^{*s} X_{12} T_2^{p-1} = 0 \iff X_{12} T_2^{p-1} = 0.
 \end{aligned}$$

Repeating this argument next by multiplying the expression within the curly brackets by T_2^{p-2} on the right, and then so on, it is seen that

$$X_{12} T_2^r = 0 \text{ for all } r = 1, 2, \dots.$$

Consequently, it follows (from the expression within the curly brackets) that $X_{12} = 0$. A similar argument shows that $X_{21} = 0$ (in (1)).

(i) Equality (1) implies $T_1 \in (X_{11}, (m, n))$ -isosymmetric, and this (since T_1 is invertible) implies by Lemma 2.1 that $T_1^{-1} \in (m, n)$ -isosymmetric, i.e.,

$$\begin{aligned}
 &\sum_{l=0}^n \sum_{k=0}^m (-1)^{l+k} \binom{n}{l} \binom{m}{k} \{ T_1^{*(n-l+m-k)} X_{11} T_1^{-m+k-l} \} = 0 \\
 \iff &\sum_{l=0}^n (-1)^l \binom{n}{l} T_1^{*(n-l)} \left\{ \sum_{k=0}^m (-1)^k \binom{m}{k} T_1^{*(m-k)} X_{11} T_1^{-(m-k)} \right\} T_1^{-l} = 0 \\
 \iff &\sum_{l=0}^n (-1)^l \binom{n}{l} T_1^{*(n-l)} \left\{ \sum_{k=0}^m (-1)^k \binom{m}{k} T_1^{*(m-k)} X_{11} T_1^{-(m-k)} \right\} T_1^{n-l} = 0 \\
 \iff &\Delta_{T_1^{-1}, T_1}^n \left(\Delta_{T_1^{-1}, T_1^{-1}}^m(X_{11}) \right) = 0 \\
 \iff &\Delta_{T_d^*, T}^n \left(\Delta_{T_d^*, T_d}^m(X) \right) = 0.
 \end{aligned}$$

Again, $T_1^{-1} \in (X_{11}, (m, n))$ -isosymmetric if and only if

$$\begin{aligned} & \sum_{k=0}^m (-1)^k \binom{m}{k} T_1^{*(-m+k)} \left\{ \sum_{l=0}^n (-1)^l \binom{n}{l} T_1^{*(n-l)} X_{11} T_1^{-l} \right\} T_1^{-(m-k)} = 0 \\ \iff & \sum_{k=0}^m (-1)^k \binom{m}{k} T_1^{*(-m+k)} \left\{ \sum_{l=0}^n (-1)^l \binom{n}{l} T_1^{*(n-l)} X_{11} T_1^{-l} \right\} T_1^k = 0 \\ \iff & \delta_{T_1^{-1}, T_1}^m \left(\delta_{T_1^{-1}, T_1}^n (X_{11}) \right) = 0 \\ \iff & \delta_{T_d^*, T}^m \left(\delta_{T_d^*, T_d}^n (X) \right) = 0. \end{aligned}$$

(ii) Since $X_{12} = 0$, and

$$\begin{aligned} & \sum_{l=0}^n \sum_{k=0}^m (-1)^{l+k} \binom{n}{l} \binom{m}{k} \left\{ T_1^{*(-m+k+n-l)} X_{11} T_1^{m-k+l} \right\} = 0 \\ \iff & \sum_{l=0}^n (-1)^l \binom{n}{l} T_1^{*(n-l)} \left\{ \sum_{k=0}^m (-1)^k \binom{m}{k} T_1^{*m-k} X_{11} T_1^{-(m-k)} \right\} T_1^{-l} = 0 \\ \iff & \sum_{l=0}^n (-1)^l \binom{n}{l} T_1^{*(n-l)} \left\{ \sum_{k=0}^m (-1)^k \binom{m}{k} T_1^{*m-k} X_{11} T_1^{-(m-k)} \right\} T_1^{n-l} = 0 \\ \iff & \Delta_{T_1^{-1}, T_1}^n \left(\Delta_{T_1^{-1}, T_1}^m (X) \right) = 0, \text{ we have that} \end{aligned}$$

$$\Delta_{T_d^*, T}^n \left(\Delta_{T_d^*, T_d}^m (X) \right) = 0.$$

(iii) In this case

$$\begin{aligned} & \sum_{l=0}^n \sum_{k=0}^m (-1)^{l+k} \binom{n}{l} \binom{m}{k} \left\{ T_1^{*(m-k-n+l)} X_{11} T_1^{m-k+l} \right\} = 0 \\ \iff & \sum_{l=0}^n \sum_{k=0}^m (-1)^{l+k} \binom{n}{l} \binom{m}{k} \left\{ T_1^{*(m-k-n+l)} X_{11} T_1^{-(m-k+l)} \right\} = 0 \\ \iff & \sum_{l=0}^n (-1)^l \binom{n}{l} T_1^{*(n-l)} \left\{ \sum_{k=0}^m (-1)^k \binom{m}{k} T_1^{*(m-k)} X_{11} T_1^{-(m-k)} \right\} T_1^{-l} = 0 \\ \iff & \sum_{l=0}^n (-1)^l \binom{n}{l} T_1^{*(n-l)} \left\{ \sum_{k=0}^m (-1)^k \binom{m}{k} T_1^{*(m-k)} X_{11} T_1^k \right\} T_1^{-l} = 0 \\ \iff & \delta_{T_1^{-1}, T_1}^n \left(\delta_{T_1^{-1}, T_1}^m (X_{11}) \right) = 0 \\ \iff & \delta_{T_d^*, T_d}^n \left(\delta_{T_d^*, T}^m (X) \right) = 0. \end{aligned}$$

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