Filomat 36:13 (2022), 4553–4562
https://doi.org/10.2298/FIL2213553A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

New Inertial-Based Spectral Projection Method for Solving System of Nonlinear Equations with Convex Constraints

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Abstract. In this paper, a new spectral projection method for solving nonlinear system of equations with convex constraints is proposed based on inertial effect. The inertial technique is integrated into the new proposed search direction with the aim of enhancing the numerical performance. Interestingly, the convergence result of the new method is established based on the assumption that the underlying function is pseudomonotone. This assumption is weaker than monotonicity which is used in many existing methods to prove the convergence. The new method is suitable for large scale problems as well as nonsmooth problems. Numerical experiments presented validate the efficiency of the new method which also outperforms some existing methods in the literature.

1. Introduction

Consider the following system

$$\Omega(z) = 0, \tag{1}$$

2020 *Mathematics Subject Classification*. 90C30; 90C06; 90C56
Keywords. iterative algorithm, line search technique, derivative-free method, nonlinear problem, pseudomonotone
Received: 02 July 2021; Accepted: 07 March 2022
Communicated by Predrag Stanimirović
Corresponding author: Poom Kumam
The authors acknowledge the financial support provided by the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), KMUTT and Center under Computational and Applied Science for Smart Innovation research Cluster (CLASSIC), Faculty of Science, KMUTT. The first author, Aliyu Muhammed Awwal, would like to thank the Postdoctoral Fellowship from King Mongkut's University of Technology Thonburi (KMUTT), Thailand. The authors acknowledge the financial support provided by the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), KMUTT. Moreover, this research was funded by National Science, Research and Innovation Fund (NSRF), King Mongkut's University of Technology North Bangkok with Contract no. KMUTNB-FF-65-13.
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where $z \in \mathbb{R}^n$, $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. The system (1) is considered very crucial problem as it appears in numerous area of applications in science and engineering such as signal processing and image deblurring problems [1–6].
Iterative methods for handling problem (1) fall into different categories and each uses the following iterative formula

$$z_{k+1} = z_k + \theta_k p_k, \quad k = 0, 1, 2, \ldots, \tag{2}$$

where z_k and z_{k+1} are the current and next iterates, respectively. The parameter $\theta_k > 0$ is known as step length which is usually calculated using suitable line search techniques. The nature of the formula for computing the search direction $p_k \in \mathbb{R}^n$ is what categorizes the iterative methods.
For problem (1) with relatively small dimension and the function Ω being smooth, methods such as the Newton method, quasi-Newton methods, Levenberg-Marquardt method and their variants are considered important due to their locally rapid convergence rates [7–9]. If problem (1) is nonsmooth and its dimension is relatively large, iterative methods that avoid computation and storage of matrices are considered more efficient. Some of such iterative methods imitate the behaviour of the *spectral gradient method*.

The spectral gradient iterative methods were originally developed to deal with general unconstrained optimization problems,

$$\min\{f(z) \in \mathbb{R} : z \in \mathbb{R}^n\}.$$

Barzilai and Borwein (BB) [10] is considered the earliest spectral gradient method for solving unconstrained optimization problems where the search direction is taken as the steepest descent direction and the step length is calculated using either of the following formula

$$r_k^{BB1} = \frac{\|s_{k-1}\|^2}{y_{k-1}^T s_{k-1}} \quad \text{or} \quad r_k^{BB2} = \frac{y_{k-1}^T s_{k-1}}{\|y_{k-1}\|^2}, \tag{3}$$

where $s_{k-1} = z_k - z_{k-1}$ and $y_{k-1} = \nabla f(z_k) - \nabla f(z_{k-1})$. However, a notable disadvantage of the BB parameters r_k^{BB1} and r_k^{BB2} is that either could generate negative values for nonconvex objective functions. This motivated Dai et al. [11] to come up with another spectral gradient method where the step length is calculated as the geometric mean of r_k^{BB1} and r_k^{BB2}, i.e.,

$$r_k^{GBB} = \frac{\|s_{k-1}\|}{\|y_{k-1}\|}. \tag{4}$$

Due to the simplicity as well as nice convergence properties of the spectral gradient methods, some researchers have imitated the approach and developed spectral methods for solving problems in the form of (1) [12, 13]. For instance, by replacing $\nabla f(z)$ with $\Omega(z)$ in r_k^{BB1}, Zhang and Zhou [14] presented a nice spectral method for solving (1) where the search direction is given as $p_k = -\widehat{r}_k^{BB1}\Omega(z_k)$, and

$$\widehat{r}_k^{BB1} = \frac{\|s_{k-1}\|^2}{\widehat{y}_{k-1}^T s_{k-1}}, \tag{5}$$

$s_{k-1} = z_k - z_{k-1}$ and $\widehat{y}_{k-1} = \Omega(z_k) - \Omega(z_{k-1}) + c s_{k-1}$, $c > 0$. The introduction of the additional term $c s_{k-1}$ in the definition of the vector $\widehat{y}_{k-1}$ yields $\widehat{y}_{k-1}^T s_{k-1} > 0$, for all k, provided the function Ω in problem (1) is monotone. This means for all k, $\widehat{r}_k^{BB1}$ will produce positive values and this is one of the most interesting thing about the Zang and Zho method. They proposed a new line search technique for calculating the step length θ_k which can be viewed as a modification of the line search in [15]. By incorporating the projection formula of Solodov and Svaiter [15] into their algorithm and assuming that Ω is monotone and Lipscitzian, they showed the method is globally convergent. Following the same approach, Yu et al. [16] considered the modification of problem (1) by taking the domain of z, as a nonempty closed and convex set i.e. $Q \subset \mathbb{R}^n$, and subsequently used the Zhang and Zhou [14] method to solve it. Similarly, Awwal et al. [17] considered computing the search direction as the convex combination of r_k^{BB1} and r_k^{GBB} with $y_{k-1} = \widehat{y}_{k-1}$ and proposed a hybrid spectral projection method for solving problem (1) with convex constraints. Numerical experiments presented showed that the hybrid method has better numerical performance than the method in [16]. For more on modified spectral projection methods, reader may refer to [18, 19].

2. Motivation

Let $\{\beta_k\}$ be the sequence of nonnegative numbers and consider the inertial step

$$w_k = z_k - \beta_k(z_k - z_{k-1}),$$

which is commonly integrated into iterative algorithm so as to enhance its numerical performance, where z_k and z_{k-1} are two given initial points. The first known inertial based method was presented by Polyak [20] to solve a smooth convex minimization problem. Recently, Awwal et al. presented two derivative-free inertial based methods where the search directions were defined by incorporating the inertial step into r_k^{BB1} and r_k^{BB2} with $y_{k-1} = \widehat{y}_{k-1}$. The methods are efficient and work well. However, the two parameters r_k^{BB1} and r_k^{BB2} are defined using the differences between the iterate z_k and the inertial step w_k together with the differences of their images. In this paper, we propose another inertial-based spectral method where the search direction is defined using the difference between previous inertial step and its update as well as their images.

Furthermore, as we have mentioned in the preceding section, Zhang and Zhou [14] used $\widehat{y}_{k-1}$ instead of y_{k-1} so as to take the advantage of the monotonicity assumption on Ω and guarantee positive values for the spectral parameter throughout the iteration process. However, in their numerical experiments, they showed that the smaller the positive constant, c, the better the numerical performance. Therefore, in this paper, our propose method will define its spectral parameter without the third term in the vector $\widehat{y}_{k-1}$, i.e. $c = 0$.

Moreover, all the spectral methods for solving problem (1) discussed above established their convergence with the assumption that the function Ω is monotone, i.e.

$$(\Omega(z) - \Omega(\widehat{z}))^T(z - \widehat{z}) \geq 0, \quad \text{for all } \ z, \widehat{z} \in \mathbb{R}^n.$$

Interestingly, in this paper, we establish the convergence result of the proposed method based on a weaker assumption. That is, the function Ω is assumed to pseudomonotonicity, i.e.,

$$\Omega(\widehat{z})^T(z - \widehat{z}) \geq 0 \implies \Omega(z)^T(z - \widehat{z}) \geq 0, \quad \forall z, \widehat{z} \in \mathbb{R}^n.$$

The rest of this paper is segmented as follows. The algorithm of propose method and its convergence analysis are described in the next section while the numerical experiment is presented in Section 4. Finally some concluding remarks will be given in Section 5.

3. Algorithm (NISPM) and its Global Convergence Results

As we are set to present the algorithm of the new method, these crucial assumptions will be helpful.

Assumption 3.1.

- *The constraint set $Q \subset \mathbb{R}^n$ is convex, closed and nonempty.*
- *The function Ω is pseudomonotone.*
- *The function Ω is Lipschitz continuous.*
- *The solution set of problem* (1) *is nonempty.*

Algorithm 1: New **I**nertial-based **S**pectral **P**rojection **M**ethod (NISPM)

Input : Given initial points $z_{-1},\ z_0 \in Q$, $0 < \ell < 2$, $\sigma, \rho \in (0,1)$, $\zeta > 0$, $\beta_k \in [0,1)$ and $Tol > 0$.
Step 0: Set $k = 0$, calculate $p_0 := -\Omega(z_0)$ and $w_0 := z_0 + \beta_0(z_0 - z_{-1})$.
Step 1: If $\|\Omega(z_k)\| \leq Tol$, stop, otherwise execute *Step 2*.
Step 2: Set $v_k := z_k + \theta_k p_k$ where $\theta_k := \zeta\rho^i$ and i is the least non-negative integer for which

$$-\Omega(v_k)^T p_k \geq \sigma\zeta\rho^i \|p_k\|^2 \min\{1,\ \|\Omega(v_k)\|^{1/t}\},\ t \geq 1. \tag{6}$$

Step 3: If $\|\Omega(v_k)\| = 0$, stop. Else compute

$$z_{k+1} := \Gamma_Q\left[z_k - \ell\frac{\Omega(v_k)^T(z_k - v_k)}{\|\Omega(v_k)\|^2}\Omega(v_k)\right]. \tag{7}$$

Step 4: Update the inertial step: $w_{k+1} := z_{k+1} + \beta_k(z_{k+1} - z_k)$.
Step 5: Set $k := k + 1$, update the search direction and repeat the process from *Step 1*,

$$p_k := -\mu_k\Omega(z_k), \tag{8}$$

$$\mu_k := \min\left\{\frac{\|s_{k-1}\|}{\|\gamma_{k-1}\|} + \frac{\|s_{k-1}\|^2}{s_{k-1}^T\gamma_{k-1}} - \frac{s_{k-1}^T\gamma_{k-1}}{\|\gamma_{k-1}\|^2},\ \mu_{\max}\right\},\ \ 0 << \mu_{\max} << +\infty, \tag{9}$$

$s_{k-1} := w_k - w_{k-1}$ and $\gamma_{k-1} := \Omega(w_k) - \Omega(w_{k-1})$.

Remark 3.2. *The projection operator* $\Gamma_Q(z)$ *in* Step 3 *of Algorithm 1 is defined as* $\Gamma_Q(z) := \arg\min\{\|z - \widehat{z}\| : \widehat{z} \in Q\}$ *and satisfies the inequality*

$$\|\Gamma_Q(z) - \widehat{z}\| \leq \|z - \widehat{z}\|,\ \textit{for all}\ \widehat{z} \in Q. \tag{10}$$

Remark 3.3. *Now, observe that if* $\gamma_{k-1} = \widehat{y}_{k-1}$, $\frac{\|s_{k-1}\|}{\|\gamma_{k-1}\|} = \frac{s_{k-1}^T\gamma_{k-1}}{\|\gamma_{k-1}\|^2}$ *and* $\beta_k = 0$, $\forall\, k$, *then the search direction* (8) *reduces to that of [14].*
By Cauchy Schwarz inequality, it holds that $\frac{\|s_{k-1}\|^2}{s_{k-1}^T\gamma_{k-1}} \geq \frac{\|s_{k-1}\|}{\|\gamma_{k-1}\|} \geq \frac{s_{k-1}^T\gamma_{k-1}}{\|\gamma_{k-1}\|^2}$. *This further means* $\frac{\|s_{k-1}\|^2}{s_{k-1}^T\gamma_{k-1}} - \frac{s_{k-1}^T\gamma_{k-1}}{\|\gamma_{k-1}\|^2} \geq 0$. *Therefore, by the Lipschitz continuity of the function* Ω, *we have*

$$\frac{\|s_{k-1}\|}{\|\gamma_{k-1}\|} + \frac{\|s_{k-1}\|^2}{s_{k-1}^T\gamma_{k-1}} - \frac{s_{k-1}^T\gamma_{k-1}}{\|\gamma_{k-1}\|^2} \geq \frac{\|s_{k-1}\|}{\|\gamma_{k-1}\|} \geq \frac{1}{L}. \tag{11}$$

By (9) *and* (11), *we have*

$$\frac{1}{L} \leq \mu_k \leq \mu_{\max}. \tag{12}$$

Hence, the search direction p_k *satisfies the followings*

$$\Omega(z_k)^T p_k \leq -\frac{1}{L}\|\Omega(z_k)\|^2,\ \textit{and} \tag{13}$$

$$\|p_k\| \leq \mu_{\max}\|\Omega(z_k)\|. \tag{14}$$

Remark 3.4. *We adopt the line search in* Step 3 *of Algorithm 1 from [21]. The line search* (6) *has been shown to contain the line search strategies in [14, 22, 23] as special cases. Therefore, since the line search* (6) *has been shown to be well-defined, then combining with* (12), *we conclude that Algorithm 1 is well-defined.*

Lemma 3.5. *Suppose that the function* Ω *is pseudomonotone and the sequence of iterates* $\{z_k\}$ *is generated by Algorithm 1, then the followings hold*

$$\lim_{k\to\infty} \theta_k\|p_k\| = 0, \quad \text{and} \tag{15}$$

$$\theta_k \geq \min\left\{1, \frac{\rho\|\Omega(z_k)\|^2}{(L+\sigma)Lq_2^2}\right\}. \tag{16}$$

Proof. ***Claim 1:*** *There exists some constant* q_1 *such that* $\|\Omega(z_k)\| \leq q_1, \ \forall\, k \geq 0$.
If $\widehat{z} \in Q$ is the solution of problem (1) then it is clear that $\Omega(\widehat{z})^T(v_k - \widehat{z}) \geq 0$. By the pseudomonotonicity assumption on the function Ω, then it holds that $\Omega(v_k)^T(v_k - \widehat{z}) \geq 0$. This further yields

$$\begin{aligned}\Omega(v_k)^T(z_k - \widehat{z}) &= \Omega(v_k)^T(z_k - v_k + v_k - \widehat{z})\\ &= \Omega(v_k)^T(z_k - v_k) + \Omega(v_k)^T(v_k - \widehat{z})\\ &\geq \Omega(v_k)^T(z_k - v_k).\end{aligned} \tag{17}$$

Now, since $0 < \ell < 2$, then by (7) and (10), we have

$$\begin{aligned}\|z_{k+1} - \widehat{z}\|^2 &= \left\|\Omega_Q\left[z_k - \ell\frac{\Omega(v_k)^T(z_k - v_k)}{\|\Omega(v_k)\|^2}\Omega(v_k)\right] - \widehat{z}\right\|\\ &\leq \left\|(z_k - \widehat{z}) - \ell\frac{\Omega(v_k)^T(z_k - v_k)}{\|\Omega(v_k)\|^2}\Omega(v_k)\right\|^2\\ &= \|z_k - \widehat{z}\|^2 - 2\ell\frac{\Omega(v_k)^T(z_k - v_k)}{\|\Omega(v_k)\|^2}\Omega(v_k)^T(z_k - \widehat{z}) + \ell^2\frac{[\Omega(v_k)^T(z_k - v_k)]^2}{\|\Omega(v_k)\|^2}\\ &\leq \|z_k - \widehat{z}\|^2 - 2\ell\frac{\Omega(v_k)^T(z_k - v_k)}{\|\Omega(v_k)\|^2}\Omega(v_k)^T(z_k - v_k) + \ell^2\frac{[\Omega(v_k)^T(z_k - v_k)]^2}{\|\Omega(v_k)\|^2}\\ &= \|z_k - \widehat{z}\|^2 - \ell(2-\ell)\frac{[\Omega(v_k)^T(z_k - v_k)]^2}{\|\Omega(v_k)\|^2} \qquad (18)\\ &\leq \|z_k - \widehat{z}\|^2. \qquad (19)\end{aligned}$$

The inequality (19) means that $\{\|z_k - \widehat{z}\|\}$ is a decreasing sequence and therefore $\{z_k\}$ is bounded and since Ω is Lipschitz continuous, then the *Claim 1* holds.

Claim 2: *The sequence* $\{p_k\}$ *generated by Algorithm 1 is bounded.*
Indeed, by (14) and *Claim 1*, it is clear that

$$\|p_k\| \leq q_2, \tag{20}$$

where $q_2 := \mu_{\max} q_1$ hence the *Claim 2* is true.

Now, consider the vector v_k defined in *Step 2* of Algorithm 1, then by *Claim 1* and *Claim 2* and the fact that the function Ω is Lipschitz continuous, we can find some constant $q_3 > 0$ such that

$$\|\Omega(v_k)\| \leq q_3, \ \forall\, k \geq 0. \tag{21}$$

Next, consider the line search (6). If $\min\left\{1, \|\Omega(v_k)\|^{1/t}\right\} = \|\Omega(v_k)\|^{1/t}$, then combining the inequalities (6), (18) and (21), we have

$$\sigma^2\theta_k^4\|p_k\|^4 \leq \frac{q_3^{2-2/t}}{\ell(2-\ell)}\left(\|z_k - \widehat{z}\|^2 - \|z_{k+1} - \widehat{z}\|^2\right). \tag{22}$$

On the other hand, if $\min\left\{1, \|\Omega(v_k)\|^{1/t}\right\} = 1$, then (22) reduces to

$$\sigma^2\theta_k^4\|p_k\|^4 \leq \frac{q_3^2}{\ell(2-\ell)}\left(\|z_k - \widehat{z}\|^2 - \|z_{k+1} - \widehat{z}\|^2\right). \tag{23}$$

From (18), it is clear that the $\lim_{k\to\infty}\|z_k - \widehat{z}\|$ exists and therefore since $\sigma > 0$ and $0 < \ell < 2$, then taking limit on both sides of either of (22) or (23) yields (15).

Finally, by the definition of the line search in *Step 3* of Algorithm 1, if $\theta_k \neq \zeta$, then $\theta_k^{'} = \theta_k\rho^{-1}$ will not satisfy (6), that is,

$$\begin{aligned}-\Omega(z_k + \theta_k\rho^{-1}p_k)^T p_k &< \sigma\theta_k\rho^{-1}\|p_k\|^2 \min\{1, \|\Omega(z_k + \theta_k\rho^{-1}p_k)\|\}\\ &\leq \sigma\theta_k\rho^{-1}\|p_k\|^2,\end{aligned} \tag{24}$$

where the last inequality follows the fact that $\min\{a, b\} \leq a,\ \ a, b \geq 0$. The inequality (24) can be rearranged as

$$\Omega(z_k + \theta_k\rho^{-1}p_k)^T p_k + \sigma\theta_k\rho^{-1}\|p_k\|^2 > 0. \tag{25}$$

Now applying Cauchy-Schwarz inequality on (13) and using (20) and (25), we have

$$\begin{aligned}\frac{1}{L}\|\Omega(z_k)\|^2 &\leq -\Omega(z_k)^T p_k\\ &\leq -\Omega(z_k)^T p_k + \Omega(z_k + \theta_k\rho^{-1}p_k)^T p_k + \sigma\theta_k\rho^{-1}\|p_k\|^2\\ &= (\Omega(z_k + \theta_k\rho^{-1}p_k) - \Omega(z_k))^T p_k + \sigma\theta_k\rho^{-1}\|p_k\|^2\\ &\leq L\theta_k\rho^{-1}\|p_k\|^2 + \sigma\theta_k\rho^{-1}\|p_k\|^2\\ &= \theta_k\Big[L\rho^{-1} + \sigma\rho^{-1}\Big]q_2^2.\end{aligned}$$

Hence, making θ_k the subject of the relation yields $\theta_k \geq \min\left\{1, \frac{\rho\|\Omega(z_k)\|^2}{(L+\sigma)Lq_2^2}\right\}$ and hence the proof. □

Theorem 3.6. *Assume that the iterates $\{z_k\}$ is generated by Algorithm 1 such that Assumption 3.1 holds, then*

$$\liminf_{k\to\infty}\|\Omega(z_k)\| = 0. \tag{26}$$

Proof. If the conclusion (26) is false, then there exists some constant, say q_4,

$$\|\Omega(z_k)\| \geq q_4, \text{ for all } k \geq 0. \tag{27}$$

Now, (16) and (27) gives

$$\theta_k \geq \min\left\{1, \frac{\rho q_4^2}{(L+\sigma)Lq_2^2}\right\}. \tag{28}$$

Also, combining (15) and (28) yields

$$\lim_{k\to\infty}\|p_k\| = 0. \tag{29}$$

Moreover, applying Cauchy-Schwarz inequality on (13) and using (27) gives

$$\|\Omega(z_k)\|\|p_k\| \geq \frac{1}{L}\|\Omega_k\|^2. \tag{30}$$

Combining (27) and (30) further yields

$$\|p_k\| \geq \frac{q_4}{L}. \tag{31}$$

It is clear that (29) and (31) yield contradiction and therefore the conclusion (26) must hold. □

4. Numerical Experiments on a Collection Test Problems

This section is devoted to discussing the efficiency together with the numerical performance of the new method. This will be done by implementing Algorithm 1 (NISPM) to solve some test problems taken from the literature. The performance of the new method on the test problems is compared with two methods taken from the literature in order to demonstrate its efficiency. Do to similar characteristics in some sense, these two recently published methods are chosen for the comparison:

(i) "A Modified Spectral Gradient Projection Method for Solving Non-linear Monotone Equations with Convex Constraints and Its Application" (MSGP) [18], and

(ii) "Inertial-Based Derivative-Free Method for System of Monotone Nonlinear Equations and Application" (DAIS1) [21].

The algorithms of the three methods in this discussion are coded in MATLAB R2019b where the new NISPM is executed with the parameters: $\ell = 1.99$, $\sigma = 0.0001$, $\rho = 0.5$, $\zeta = 1$, $\beta_k = 1/(k+1)^2$, $t = 2$ and $\mu_{\max} = 10^{30}$. The parameters used for MSGP and DAIS1 are obtained from [18, 21], respectively. The device used for the experiments is a PC with intel Core(TM) i5-8250u processor with 4 GB of RAM and CPU 1.60 GHZ. The terminating criteria for each algorithm is set as $\|\Omega(z_k)\| \leq 10^{-6}$. Failure is declared if the number of iterations surpasses 1000 and the stopping criteria is yet to be satisfied.

In the course of the experiment, a collection of ten (10) test problems (see, Appendix 5.1) are solved with their dimensions varies as 10000, 30000, 50000, 80000, 100000. The initial points (thirteen (13) of them) used are given in Table 1. This yields a total of six hundred and fifty (650) test problems solved in this experiment. The metrics considered in assessing the performance of the three algorithms are (i) number of iterations (#iter) and (ii) number of function evaluations (#fval). Moreover, for each test problem considered, $\|\Omega(\widehat{z})\|$ (denoted by Norm) is reported in order to determine the degree of accuracy for which an algorithm obtains the solution of a particular problem.

The #iter, #fval and Norm recorded by each algorithm have been compiled and can be accessed through the following link `https://github.com/aliyumagsu/NISPM_Numerical_Results`. Looking through the tables in the link provided above, it can be seen that NISPM obtained the solutions of more than half of the test problems with higher degree of accuracy. Furthermore, it is pleasing to see that the new NISPM attained the solutions of most of the test problems with minimum #iter and #fval compared to MSGP and DAIS1. It can be seen from Table 7 in the link provided above that MSGP and DAIS1 failed to obtain the solution of Problem 5.7 with the thirteenth initial point. The summary of the information, with respect to #iter and #fval, reported in the link provided above are graphically presented in Figures 1 and 2. These two figures were prepared in accordance with Dolan and Moré performance profile [24]. Interestingly, the two figures revealed that the new NISPM outperforms MSGP and DAIS1 as the curve with respect to NISPM stays above those with respect to MSGP and DAIS1. This establishes the efficiency of the MSGP.

Table 1: Initial Points

S/No.	z_0	z_{-1}
1	$(\frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \dots, \frac{1}{10})^T$,	$(\frac{1}{100}, \frac{1}{100}, \frac{1}{100}, \dots, \frac{1}{100})^T$,
2	$(\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots, \frac{1}{2^n})^T$	$(\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots, \frac{1}{2^n})^T$
3	$(2, 2, 2, \dots, 2)^T$	$(1, 1, 1, \dots, 1)^T$
4	$(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n})^T$	$(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n})^T$
5	$(1 - \frac{1}{n}, 1 - \frac{2}{n}, 1 - \frac{3}{n}, \dots, 0)^T$	$(1 - \frac{1}{n}, 1 - \frac{2}{n}, 1 - \frac{3}{n}, \dots, 0)^T$
6	$(0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n})^T$	$(0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n})^T$
7	$(\frac{n-1}{n}, \frac{n-2}{n}, \frac{n-3}{n}, \dots, 0)^T$	$(\frac{n-1}{n}, \frac{n-2}{n}, \frac{n-3}{n}, \dots, 0)^T$
8	$(\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, 1)^T$	$(\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, 1)^T$
9	rand(0, 1)	rand(0, 1)
10	$(1.5, 1.5, 1.5, \dots, 1.5)^T$	$(1, 1, 1, \dots, 1)^T$
11	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})^T$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})^T$
12	$5\min(ih,\ 1 - ih)$	$5\min(ih,\ 1 - ih),\ 1 \leq i \leq n\ , h = 1/(n+1)$
13	$(-1)^i\left(\frac{i}{i+3}\right),\ 1 \leq i \leq n$	$(-1)^i\left(\frac{i}{i+3}\right),\ 1 \leq i \leq n$

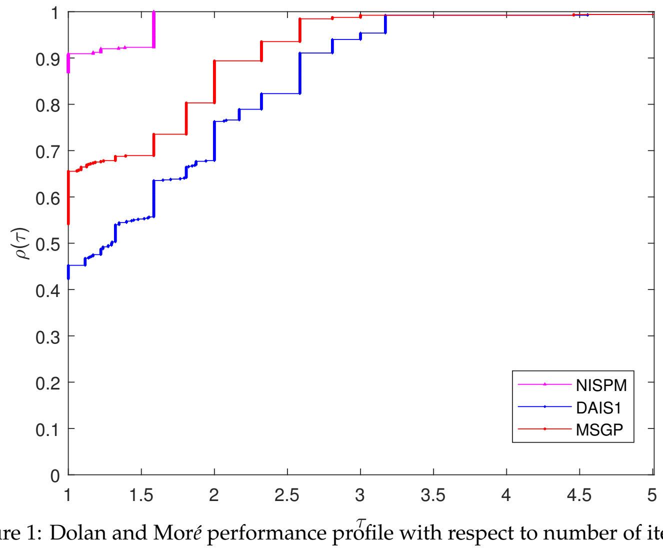

Figure 1: Dolan and Moré performance profile with respect to number of iterations

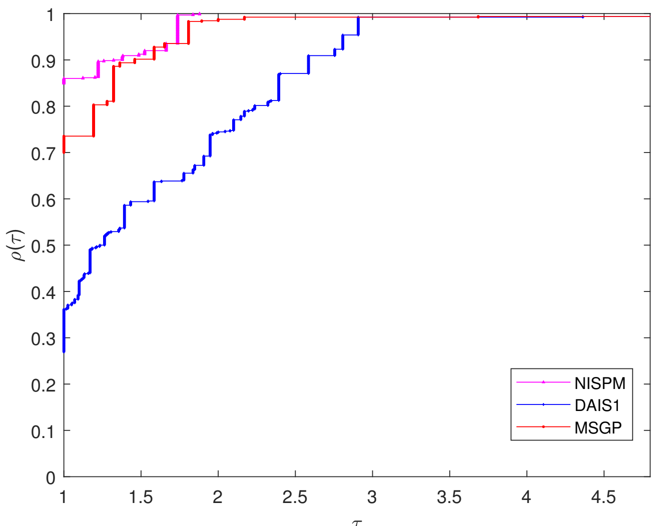

Figure 2: Dolan and Moré performance profile with respect to number of function evaluation

5. Conclusions

A new spectral projection method based on inertial effect has been developed for solving nonlinear system of equations with convex constraints. The details about the new method and how it works have been given is Section 3. With the assumption that the function in Problem (1) is Lipschitzian and pseudomontone, the theoretical convergence result of new methods has been discussed. By implementing the new method to solve some test problems and comparing the performance with two existing methods [18, 21], it was revealed that the new method works well and is more efficient than its competitors. Future research include using the idea in this paper to develop two-step iterative algorithms as presented in [19]. Furthermore, it will be interesting to incorporate the idea in this paper into the methods proposed in [25, 26].

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5.1. Appendix A

We use the following monotone nonlinear equation for the second experiments where $\Omega(z) = (\Omega_1(z), \dots, \Omega_n(z))^T$ and $z = (z_1, \dots, z_n)^T$.

Problem 5.1. *[27]*

$$\begin{aligned}\Omega_1(z) &= e^{z_1} - 1\\ \Omega_i(z) &= e^{z_i} + z_{i-1} - 1,\end{aligned} \quad i = 2, 3, \dots, n, \quad where \ \ Q = \mathbb{R}^n_+.$$

Problem 5.2. *[27]*

$$\Omega_i(z_i) = \log(z_i + 1) - \frac{z_i}{n}, \ \ i = 1, 2, \dots, n, \ \ where \ \ Q = \left\{ z \in \mathbb{R}^n : \sum_{i=1}^{n} z_i \le n, \ z_i > -1, \ i = 1, 2, \dots, n \right\}.$$

Problem 5.3. *[28]*

$$\Omega_i(z) = 2z_i - \sin|z_i|, \ \ i = 1, 2, \dots, n, \ \ where \ \ Q = \mathbb{R}^n_+.$$

Problem 5.4. *[16]*

$$\Omega_i(z) = e^{z_i} - 1, \ \ i = 1, 2, \dots, n, \ \ where \ \ Q = \mathbb{R}^n_+.$$

Problem 5.5. *[29]*

$$\begin{aligned}\Omega_1(z) &= z_1 - \exp\left(\cos\left(\frac{z_1 + z_2}{n+1}\right)\right)\\ \Omega_i(z) &= z_i - \exp\left(\cos\left(\frac{z_{i-1} + z_i + z_{i+1}}{n+1}\right)\right), \ \ 2 \le i \le n-1, \ \ where \ \ Q = \mathbb{R}^n_+.\\ \Omega_n(z) &= z_n - \exp\left(\cos\left(\frac{z_{n-1} + z_n}{n+1}\right)\right),\end{aligned}$$

Problem 5.6. *[30]*

$$\Omega_i(z) = z_i - \sin(|z_i - 1|), \ \ i = 1, 2, \dots, n, \ \ where \ \ Q = \left\{ z \in \mathbb{R}^n : \sum_{i=1}^{n} z_i \le n, \ z_i \ge -1, \ i = 1, 2, \dots, n \right\}.$$

Problem 5.7. *[31]*

$$\Omega_i(z) = e^{z_i^2} + \frac{3}{2}\sin(2z_i) - 1, \ \ i = 1, 2, \dots, n, \ \ where \ \ Q = \mathbb{R}^n_+.$$

Problem 5.8. *[14]*

$$\begin{aligned}\Omega_1(z) &= z_1 + \sin(z_1) - 1\\ \Omega_i(z) &= -z_{i-1} + 2z_i + \sin(z_i) - 1, \ i = 2, \dots, n-1, \ \ where \ \ Q = \mathbb{R}^n_+.\\ \Omega_n(z) &= z_n + \sin(z_n) - 1,\end{aligned}$$

Problem 5.9. *[31]*

$$\Omega_i(z) = \frac{i}{n}e^{z_i} - 1, \ \ i = 1, 2, \dots, n, \ \ where \ \ Q = \mathbb{R}^n_+.$$

Problem 5.10. *[5]*

$$\Omega_i(x) = \cos(z_i) + z_i - 1, \ i = 1, 2, \dots, n, \ \ where \ \ Q = \mathbb{R}^n_+.$$