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Sharp Hölder Continuous Behaviour of Solutions to Vector Network Equilibrium Problems with a Polyhedral Ordering Cone

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Abstract. In this paper, we establish some new results for Hölder continuity of solutions to vector variational inequalities which model vector network equilibrium problems with a polyhedral ordering cone under parametric perturbations. Especially, our approach of studying Hölder continuous behaviour is employed by the properties of the regularized gap function based on the ordering cone generated by a matrix.

1. Introduction

The network equilibrium model was proposed by Wardrop [38] for a transportation network. This model has played a vital role in the traffic network planning and to optimize the traffic control. Based on vector-valued cost functions or multicriteria consideration, many variant kinds of network equilibrium models have been studied, see e.g., [8, 9, 27, 30, 39] and the references therein.

Besides, to reformulate the variational inequality into an equivalent optimization problem, Auslender [3] introduced the notion of gap functions. However, in general, the gap functions considered in [3] are not differentiable. To overcome this disadvantage, Fukushima [11], Yamashita and Fukushima [42] proposed the notion of regularized gap functions for variational inequalities. The regularized gap function is an effective approach to establish the upper estimate of solutions (i.e., error bounds) for problems related to optimization, see e.g., [1, 17, 19, 21–26, 36] and the references therein. Recently, it is also applied to investigate the well-posedness for variational inequalities, see e.g., [29, 37].

On the other hand, the investigate of linear inequalities has led to considering a special class of polyhedral cones. The theory of polyhedral cones associated to the matrices is studied extensively, see e.g., [6, 7, 12, 35, 43]. Very recently, using the partial order provided by a polyhedral cone, Gutiérrez et al. [13, 14] and Hai et al. [15] characterized some properties of exact and approximate efficient solutions of a class of multiobjective optimization problems and vector equilibrium problems. They also showed that the characteristics of solutions established based on the ordering cone generated by some matrix are attractive from a computational point of view. Latest, Hung et al. [18] considered a new class of regularized gap functions and error bounds for vector equilibrium problems with a polyhedral ordering cone. Especially, Hung et al. introduced a new class of vector network equilibrium problems with the partial order provided by a polyhedral cone as a real-world application to illustrate our main results in [18].

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The Hölder continuity has long been recognized as an important feature of the stability analysis of solution mappings for perturbed problems related to optimization. It can calculate the error between the perturbed solution sets and the exact solution sets of the concerning problems. Recently, the Hölder continuity of solution mappings for various equilibrium problems, variational inequalities and optimization problems, etc. has been of considerable interest, e.g. see [2, 4, 5, 16, 28, 41] and the references therein. In particular, Mansour and Scrimali [31] investigated the Hölder continuity for scalar variational inequalities of under parametric perturbations which model traffic network equilibrium problems with elastic travel demand. Very recently, Hung et al. [20] established the Hölder continuity of the solution mapping for a class of parametric variational-hemivariational inequalities via the properties of the regularized gap function. However, to the best of our knowledge, up to now, there does not exist any work concerning the study Hölder continuous behaviour of solutions for vector network equilibrium problems with a polyhedral ordering cone under parametric perturbations.

The main purpose of this paper is to further investigate a model of vector network equilibrium problems with partial order introduced by a polyhedral cone under parametric perturbations. We introduce a formulation of this model by the parametric vector variational inequality with a polyhedral ordering cone. By employing some useful properties of the regularized gap function for this model, we derive the Hölder continuous behaviour of the solution mapping to the concerning problem under some suitable conditions.

2. Preliminaries

Throughout the paper, let $\mathbb{R}^p$ be the p-dimensional Euclidean space and $\mathbb{R}^p_+ = \{(\rho_1, ..., \rho_p) \in \mathbb{R}^p : \rho_i \geq 0, \forall i = 1, ..., p\}$. For any two vectors $\rho = (\rho_1, ..., \rho_p)^\top$ and $\varrho = (\varrho_1, ..., \varrho_p)^\top$, $\rho, \varrho \in \mathbb{R}^p$, we define the relationships: (i) $\rho \leq \varrho$ if and only if $\rho_i \leq \varrho_i$ for all $i \in \{1, ..., p\}$; (ii) $\rho < \varrho$ if and only if $\rho_i < \varrho_i$ for all $i \in \{1, ..., p\}$.

A nonempty set $\mathbf{G} \subset \mathbb{R}^p$ is a *cone* if $\lambda x \in \mathbf{G}$ for all $x \in \mathbf{G}$ and $\lambda \geq 0$. A cone $\mathbf{G}$ is said to be *pointed* if $\mathbf{G} \cap -\mathbf{G} = \{\mathbf{0}\}$, where $\mathbf{0} = (0, ..., 0)^\top \in \mathbb{R}^p$. As usual, a *hyperplane* in $\mathbb{R}^p$ is a set associated with some $(\rho, b) \in \mathbb{R}^p \times \mathbb{R}$, $\rho \neq \mathbf{0}$, and defined as $\{x \in \mathbb{R}^p : \langle \rho, x \rangle = b\}$. The *closed half-space* of $\mathbb{R}^p$ is a set associated with some $(\rho, b) \in \mathbb{R}^p \times \mathbb{R}$, $\rho \neq \mathbf{0}$, and defined as $\{x \in \mathbb{R}^p : \langle \rho, x \rangle \leq b\}$. A set $P \subset \mathbb{R}^p$ is said to be a *polyhedral set* if it can be expressed as the intersection of a finite family of closed half-spaces or hyperplanes.

Proposition 2.1. (see [33]) The following statements are equivalent for a set $\mathbf{G} \subset \mathbb{R}^m$:

(i) $\mathbf{G}$ is a polyhedral cone;
(ii) $\mathbf{G}$ has a representation of the form

$$\mathbf{G} = \{x \in \mathbb{R}^m : \langle \rho_i, x \rangle \leq 0,\ \forall i = 1, ..., p\},$$

for some positive integer p and some $\rho_i \in \mathbb{R}^m$, $i = 1, ..., p$.

Denote the set of all real matrices with p rows and m columns by $\mathbb{R}^{p \times m}$.

Definition 2.2. (see [10]) Let $A \in \mathbb{R}^{p \times m}$. Then

$$\mathbf{G}_A = \{x \in \mathbb{R}^m : Ax \geq \mathbf{0}\}, \tag{1}$$

which is called a *cone generated by* A.

The cone $\mathbf{G}_A$ is polyhedral, and so it is also convex and closed.

Proposition 2.3. (see [32], Proposition 4 and Proposition 5) Let $A \in \mathbb{R}^{p \times m}$. Then

(i) The cone $\mathbf{G}_A$ defined by (1) is pointed if and only if $\text{rank}(A) = m$ ($p \geq m$).
(ii) If the matrix A has no zero rows, then $\text{int}(\mathbf{G}_A) = \{x \in \mathbb{R}^m : Ax > \mathbf{0}\}$.

Lemma 2.4. (see [40], Lemma 1) Let $A \in \mathbb{R}^{p \times m}$. If $\mathbf{G}_A = \{\mathbf{0}\}$, then $\text{rank}(A) = m$ and $p > m$.

Let $A \in \mathbb{R}^{p\times m}$ be a given matrix. The *mapping defined by the matrix A* is also denoted by A, where $A : \mathbb{R}^m \to \mathbb{R}^p$ defined by $x \mapsto Ax$ (or $A(x)$) is a bounded linear mapping.

Proposition 2.5. (see [34], Proposition 4.1) *Let A be a mapping defined by a matrix $A \in \mathbb{R}^{p\times m}$. Assume that the set $\{x \in \mathbb{R}^m : Ax \geq \mathbf{0}\}$ is a pointed cone, or, equivalently, that* $\mathrm{rank}(A) = m$ *and $p \geq m$. Then, the following statements hold:*

(i) *the mapping A is injective,*
(ii) *the image of the set $\{x \in \mathbb{R}^m : Ax \geq \mathbf{0}\}$ under the mapping A is a convex cone included in $\mathbb{R}^p_+$,*
(iii) *if $p = m$, then the image of the space $\mathbb{R}^m$ under the mapping A is $\mathbb{R}^p$ and the image of the cone $\{x \in \mathbb{R}^m : Ax \geq \mathbf{0}\}$ is $\mathbb{R}^p_+$,*
(iv) *if $p > m$, then the image of the space $\mathbb{R}^m$ under the mapping A is a proper subset of $\mathbb{R}^p$ and the image of the cone $\{x \in \mathbb{R}^m : Ax \geq \mathbf{0}\}$ is a proper subset of $\mathbb{R}^p_+$.*

We now revisit a formulation of vector network equilibrium problems with partial order provided by a polyhedral cone generated by some matrix considered in [18].

Consider a transportation network $\mathbf{M} = (\mathcal{N}, \mathcal{E})$, where $\mathcal{N}$ and $\mathcal{E}$ denote the set of nodes and directed arcs, respectively. Let Ω denote the set of origin-destination (O-D) pairs and $\mathcal{P}_\omega$, $\omega \in \Omega$ denotes the set of available paths joining O-D pair w. Let $d = (d_\omega)_{\omega\in\Omega}$ denote the demand vector, where d_ω denotes the demand of traffic flow on O-D pair ω. For a given path $k \in \mathcal{P}_\omega$, let f_k denote the traffic flow on this path and $\mathbf{f} = (f_1, f_2, ..., f_N)^\top \in \mathbb{R}^N$, where $N = \sum_{\omega\in\Omega} |\mathcal{P}_\omega|$ being $|\cdot|$ the cardinality of $\mathcal{P}_\omega$. The path flow vector $\mathbf{f}$ induces a flow z_e on each arc $e \in \mathcal{E}$ given by

$$z_e = \sum_{\omega\in\Omega} \sum_{k\in\mathcal{P}_\omega} \delta_{ek} f_k,$$

where $[\delta_{ek}] \in \mathbb{R}^{\nu\times N}$ $(\nu = |\mathcal{E}|)$ is the arc path incidence matrix with

$$\delta_{ek} = \begin{cases} 1 & \text{if arc } e \text{ belongs to path } k; \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{z} = (z_1, z_2, ..., z_\nu)^\top \in \mathbb{R}^\nu$ be the vector of arc flow. We say that a path flow $\mathbf{f}$ satisfies demands if $\sum_{k\in\mathcal{P}_\omega} f_k = d_\omega$ for all $\omega \in \Omega$. A flow $\mathbf{f} \geq \mathbf{0}$ satisfying the demand is called a feasible flow. Let $\mathbf{K}^+ = \{\mathbf{f} \in \mathbb{R}^N : \mathbf{f} \geq \mathbf{0}\} \subset \mathbb{R}^N$ be a compact and convex set and

$$\mathbf{H} = \left\{ \mathbf{f} \in \mathbf{K}^+ : \sum_{k\in\mathcal{P}_\omega} f_k = d_\omega,\ \forall\omega \in \Omega \right\}.$$

Assume that $\mathbf{H} \neq \emptyset$. It is easy to check that the set $\mathbf{H}$ is compact and convex. Let $\mathbf{c}_e \colon \mathbb{R}^\nu \to \mathbb{R}^m$ be a vector-valued cost function for arc e which is in general a function of all the arc flows. We assume that $\mathbf{c}_e(\mathbf{z}) = (\mathbf{c}_e^1(\mathbf{z}), \mathbf{c}_e^2(\mathbf{z}), ..., \mathbf{c}_e^m(\mathbf{z}))^\top \in \mathbb{R}^m$, $\mathbf{z} \in \mathbb{R}^\nu$. Let $\mathcal{T}_k \colon \mathbb{R}^N \to \mathbb{R}^m$ be a vector-valued cost function along the path k. For each $\omega \in \Omega$ and $k \in \mathcal{P}_\omega$, the vector cost $\mathcal{T}_k$ is assumed to be the sum of all the arc cost of the flow f_k through arcs, which belong to the path k, i.e.,

$$\mathcal{T}_k(\mathbf{f}) = \sum_{e\in\mathcal{E}} \delta_{ek} \mathbf{c}_e(\mathbf{z}) = \begin{pmatrix} \sum_{e\in\mathcal{E}} \delta_{ek} \mathbf{c}_e^1(\mathbf{z}) \\ \sum_{e\in\mathcal{E}} \delta_{ek} \mathbf{c}_e^2(\mathbf{z}) \\ \vdots \\ \sum_{e\in\mathcal{E}} \delta_{ek} \mathbf{c}_e^m(\mathbf{z}) \end{pmatrix}.$$

For each $\omega \in \Omega$, $k \in \mathcal{P}_\omega$, $j \in \{1, 2, ..., m\}$, $\mathbf{z} \in \mathbb{R}^\nu$ and $\mathbf{f} \in \mathbf{H}$, let

$$\mathcal{T}_k^j(\mathbf{f}) = \sum_{e\in\mathcal{E}} \delta_{ek} \mathbf{c}_e^j(\mathbf{z}) \text{ and } \mathcal{T}^j(\mathbf{f}) = (\mathcal{T}_1^j(\mathbf{f}), \mathcal{T}_2^j(\mathbf{f}), ..., \mathcal{T}_N^j(\mathbf{f}))^\top \in \mathbb{R}^N.$$

Then, for each $\mathbf{f} \in \mathbf{H}$, let

$$\mathcal{T}(\mathbf{f}) = (\mathcal{T}^1(\mathbf{f}), \mathcal{T}^2(\mathbf{f}), ..., \mathcal{T}^m(\mathbf{f}))^\top = (\mathcal{T}_1(\mathbf{f}), \mathcal{T}_2(\mathbf{f}), ..., \mathcal{T}_N(\mathbf{f})) \in \mathbb{R}^{m\times N},$$

that is,

$$\mathcal{T}(\mathbf{f}) = \begin{pmatrix} \mathcal{T}_1^1(\mathbf{f}) & \mathcal{T}_2^1(\mathbf{f}) & \cdots & \mathcal{T}_N^1(\mathbf{f}) \\ \mathcal{T}_1^2(\mathbf{f}) & \mathcal{T}_2^2(\mathbf{f}) & \cdots & \mathcal{T}_N^2(\mathbf{f}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{T}_1^m(\mathbf{f}) & \mathcal{T}_2^m(\mathbf{f}) & \cdots & \mathcal{T}_N^m(\mathbf{f}) \end{pmatrix}.$$

Let $A = (a_{ij}) \in \mathbb{R}^{p\times m}$ such that $p \geq m$ and $\text{rank}(A) = m$, and $\mathbf{G}_A$ be the polyhedral cone defined by $\mathbf{G}_A = \{x \in \mathbb{R}^m : Ax \geq \mathbf{0}\}$ such that $\mathbf{G}_A$ has non-empty interior. A flow $\mathbf{f} \in \mathbf{H}$ is said to be in $\mathbf{G}_A$*-equilibrium* if for all $\omega \in \Omega, k \in \mathcal{P}_\omega, l \in \mathcal{P}_\omega$,

$$\mathcal{T}_k(\mathbf{f}) - \mathcal{T}_l(\mathbf{f}) \in \text{int}(\mathbf{G}_A) \Longrightarrow f_k = 0. \tag{2}$$

Let $p = m$. If A is the identity matrix of size m, then $\mathbf{G}_A = \{x \in \mathbb{R}^m : Ax \geq \mathbf{0}\} = \mathbb{R}^m_+$. We get that (2) becomes

$$\mathcal{T}_k(\mathbf{f}) - \mathcal{T}_l(\mathbf{f}) \in \text{int}(\mathbb{R}^m_+) \Longrightarrow f_k = 0.$$

Then the flow $\mathbf{f}$ is in *weak vector equilibrium*, see [8, Definition 3.2].

Proposition 2.6. (see [18], Proposition 4.1) *The path flow* $\mathbf{f}^* \in \mathbf{H}$ *is in* $\mathbf{G}_A$*-equilibrium if* $\mathbf{f}^*$ *solves the vector variational inequality* (*for short,* $\text{VVI}(\mathbf{H}, \mathcal{T}, \mathbf{G}_A)$) :

$$\langle \mathcal{T}(\mathbf{f}^*), \mathbf{h} - \mathbf{f}^* \rangle \notin -\text{int}(\mathbf{G}_A), \quad \forall \mathbf{h} \in \mathbf{H}.$$

3. The mathematical model and related assumptions

In this section, we introduce the perturbed problem of $\text{VVI}(\mathbf{H}, \mathcal{T}, \mathbf{G}_A)$ by the parameters. Moreover, some hypotheses on the data of the perturbe problem are imposed to establish the main results in the next section.

In the rest of paper, unless otherwise specified, let $(\Lambda, \|\cdot\|_\Lambda)$ and $(\Gamma, \|\cdot\|_\Gamma)$ be finite dimensional spaces. Let $d\colon \Lambda \to \mathbb{R}^\nu_+$ be the travel demand. The set of feasible flows is the set-valued map $\mathbf{H}\colon \Lambda \rightrightarrows \mathbb{R}^N$ defined by

$$\mathbf{H}(\lambda) = \{\mathbf{f} \in \mathbf{K}^+ : B\mathbf{f} = d(\lambda)\},$$

where $B = (\phi_{\omega k})_{\omega\in\Omega, k\in\mathcal{P}_\omega}$ is the link-route incidence matrix O-D pairs-paths whose typical entry $\phi_{\omega k}$ is 1 if path k connects the pair ω and 0 otherwise. The conservation condition $B\mathbf{f} = d(\lambda)$ means that flows and hence travelers are not lost or generated in the network. For each $\lambda \in \Lambda$, assume that $\mathbf{H}(\lambda) \neq \emptyset$. Then, $\mathbf{H}(\lambda)$ is a compact and convex set (see [31, p.177]).

Let $\mathbf{c}_e\colon \mathbb{R}^\nu \times \Gamma \to \mathbb{R}^m$ be a vector-valued cost function for arc e. The function $\mathbf{c}_e$ is defined by $\mathbf{c}_e(\mathbf{z}, \gamma) = (\mathbf{c}_e^1(\mathbf{z}, \gamma), \mathbf{c}_e^2(\mathbf{z}, \gamma), ..., \mathbf{c}_e^m(\mathbf{z}, \gamma))^\top \in \mathbb{R}^m$, $\mathbf{z} \in \mathbb{R}^\nu$, $\gamma \in \Gamma$. For each $\gamma \in \Gamma$, $\omega \in \Omega$, $k \in \mathcal{P}_\omega$, $j \in \{1, 2, ..., m\}$, $\mathbf{z} \in \mathbb{R}^\nu$ and $\mathbf{f} \in \mathbf{H}$, let

$$\mathcal{T}_k(\mathbf{f}, \gamma) = \sum_{e\in\mathcal{E}} \delta_{ek}\mathbf{c}_e(\mathbf{z}, \gamma).$$

$$\mathcal{T}_k^j(\mathbf{f}, \gamma) = \sum_{e\in\mathcal{E}} \delta_{ek}\mathbf{c}_e^j(\mathbf{z}, \gamma) \text{ and } \mathcal{T}^j(\mathbf{f}, \gamma) = (\mathcal{T}_1^j(\mathbf{f}, \gamma), \mathcal{T}_2^j(\mathbf{f}, \gamma), ..., \mathcal{T}_N^j(\mathbf{f}, \gamma))^\top \in \mathbb{R}^N.$$

Then, for each $\mathbf{f} \in \mathbf{H}$ and $\gamma \in \Gamma$,

$$\mathcal{T}(\mathbf{f}, \gamma) = (\mathcal{T}^1(\mathbf{f}, \gamma), \mathcal{T}^2(\mathbf{f}, \gamma), ..., \mathcal{T}^m(\mathbf{f}, \gamma))^\top = (\mathcal{T}_1(\mathbf{f}, \gamma), \mathcal{T}_2(\mathbf{f}, \gamma), ..., \mathcal{T}_N(\mathbf{f}, \gamma)) \in \mathbb{R}^{m\times N},$$

that is,

$$\mathcal{T}(\mathbf{f},\gamma)=\begin{pmatrix} \mathcal{T}_1^1(\mathbf{f},\gamma) & \mathcal{T}_2^1(\mathbf{f},\gamma) & \cdots & \mathcal{T}_N^1(\mathbf{f},\gamma)\\ \mathcal{T}_1^2(\mathbf{f},\gamma) & \mathcal{T}_2^2(\mathbf{f},\gamma) & \cdots & \mathcal{T}_N^2(\mathbf{f},\gamma)\\ \vdots & \vdots & \ddots & \vdots\\ \mathcal{T}_1^m(\mathbf{f},\gamma) & \mathcal{T}_2^m(\mathbf{f},\gamma) & \cdots & \mathcal{T}_N^m(\mathbf{f},\gamma) \end{pmatrix}.$$

For given $\lambda \in \Lambda$ and $\gamma \in \Gamma$, the perturbed problem of VVI($\mathbf{H},\mathcal{T},\mathbf{G}_A$) can be stated as follows:
$\mathbf{VQVIP}_{\lambda,\gamma}(\mathbf{H},\mathcal{T},\mathbf{G}_A)$: Find $\mathbf{f}^*_{\lambda,\gamma} \in \mathbf{H}(\lambda)$ such that

$$\langle \mathcal{T}(\mathbf{f}^*_{\lambda,\gamma},\gamma), \mathbf{h}-\mathbf{f}^*_{\lambda,\gamma}\rangle \notin -\text{int}(\mathbf{G}_A), \quad \forall \mathbf{h}\in\mathbf{H}(\lambda).$$

We always assume that the solution set of the problem $\text{VQVIP}_{\lambda,\gamma}(\mathbf{H},\mathcal{T},\mathbf{G}_A)$, $\Phi(\lambda,\gamma)$, is nonempty.
Next, we recall the notion of Hölder continuity of a set-valued mapping.

Definition 3.1 (Classical notion). Let $\mathbf{H}\colon \Lambda \rightrightarrows \mathbb{R}^m$ be a set-valued mapping. $\mathbf{H}$ is said to be *l.α-Hölder continuous* on $V \subset \Lambda$, for some $l > 0$ and $\alpha > 0$, if for any $\lambda_1,\lambda_2 \in V$,

$$\mathbf{H}(\lambda_1) \subset \mathbf{H}(\lambda_2) + l\|\lambda_1-\lambda_2\|^\alpha_\Lambda \mathbb{B}_m, \tag{3}$$

where $\mathbb{B}_m$ indicates the closed unit ball of $\mathbb{R}^m$.
If $\mathbf{H}$ is single-valued mapping, then (3) is equivalent to

$$\|\mathbf{H}(\lambda_1)-\mathbf{H}(\lambda_2)\| \le l\|\lambda_1-\lambda_2\|^\alpha_\Lambda.$$

Let $\lambda\in\Lambda$. If V is a neighborhood of λ, then Condition (3) also states that $\mathbf{H}$ is locally Hölder continuous at λ.

For each $\widetilde{\lambda}\in\Lambda$ and $\widetilde{\gamma}\in\Gamma$ fixed, let $\mathcal{N}(\tilde{\lambda})$ and $\mathcal{N}(\tilde{\gamma})$ be neighborhoods of $\tilde{\lambda}\in\Lambda$ and $\tilde{\gamma}\in\Gamma$, respectively. Now, we impose the following hypotheses on the data of the problem $\text{VQVIP}_{\lambda,\gamma}(\mathbf{H},\mathcal{T},\mathbf{G}_A)$:

($\mathcal{H}_d$): $d\colon \Lambda \to \mathbb{R}^\nu_+$ is $l_d.\eta$-Hölder continuous on $\mathcal{N}(\widetilde{\lambda})$, i.e., there exist $l_d > 0$ and $\eta > 0$,

$$\|d(\lambda_1)-d(\lambda_2)\|_{\mathbb{R}^\nu} \le l_d\|\lambda_1-\lambda_2\|^\eta_\Lambda, \quad \forall \lambda_1,\lambda_2\in\mathcal{N}(\widetilde{\lambda}).$$

($\mathcal{H}_\mathcal{T}$)$_1$: $\bigcap_{i=1}^p \left\{\mathbf{f}\in\mathbf{H}(\lambda) : \sum\limits_{j=1}^m a_{ij}\left(\sum\limits_{\omega\in\Omega}\sum\limits_{k\in\mathcal{P}_\omega}(h_k-f_k)\mathcal{T}^j_k(\mathbf{f},\gamma)\right) \ge 0, \forall \mathbf{h}\in\mathbf{H}(\lambda)\right\} \neq \emptyset$;

($\mathcal{H}_\mathcal{T}$)$_2$: There exists $\sigma > 0$ if, for all $(\mathbf{f},\mathbf{h}) \in \mathbf{K}^+\times\mathbf{K}^+$,

$$\sum_{\omega\in\Omega}\sum_{k\in\mathcal{P}_\omega}\left[(h_k-f_k)\mathcal{T}_k(\mathbf{f},\gamma)+(f_k-h_k)\mathcal{T}_k(\mathbf{h},\gamma)\right]+\sigma\|\mathbf{f}-\mathbf{h}\|^2 e \in -\mathbf{G}_A, \quad \forall\gamma\in\mathcal{N}(\widetilde{\gamma}),$$

where $e=(1,1,...,1)^\top\in\mathbb{R}^m$;

($\mathcal{H}_\mathcal{T}$)$_3$: For each $j\in\{1,...,m\}$, for some $b_{\mathcal{T}^j} > 0$, for all $\mathbf{f}\in\mathbf{K}^+$ one has

$$\left\|\mathcal{T}^j(\mathbf{f},\gamma)\right\| \le b_{\mathcal{T}^j}, \quad \forall\gamma\in\mathcal{N}(\widetilde{\gamma});$$

($\mathcal{H}_\mathcal{T}$)$_4$: For each $j\in\{1,...,m\}$, for some $l_{\mathcal{T}^j},\widetilde{l}_{\mathcal{T}^j} > 0$ and $\theta > 0$, for all $\mathbf{f}_1,\mathbf{f}_2\in\mathbf{K}^+$,

$$\left\|\mathcal{T}^j(\mathbf{f}_2,\gamma_2)-\mathcal{T}^j(\mathbf{f}_1,\gamma_1)\right\| \le l_{\mathcal{T}^j}\|\mathbf{f}_1-\mathbf{f}_2\| + \widetilde{l}_{\mathcal{T}^j}\left\|\gamma_1-\gamma_2\right\|^\theta_\Gamma, \quad \forall\gamma_1,\gamma_2\in\mathcal{N}(\widetilde{\gamma});$$

($\mathcal{H}_\mathbf{R}$): $\mathbf{R} : \mathbf{K}^+\times\mathbf{K}^+ \to \mathbb{R}_+$ is a continuously differentiable function, which satisfies the following property with the associated constants $\beta \ge 2\alpha > 0$:

$$\alpha\|\mathbf{f}-\mathbf{h}\|^2 \le \mathbf{R}(\mathbf{f},\mathbf{h}) \le (\beta-\alpha)\|\mathbf{f}-\mathbf{h}\|^2, \quad \forall \mathbf{f},\mathbf{h}\in\mathbf{K}^+.$$

($\mathcal{H}_{\mathbf{H}}$): $\mathbf{H}\colon \Lambda \rightrightarrows \mathbb{R}^m$ is such that for each $\lambda \in \mathcal{N}(\tilde{\lambda})$, there exists $b_{\mathbf{H}} > 0$, one has

$$\|\mathbf{f}\| \leq b_{\mathbf{H}}, \quad \forall \mathbf{f} \in \mathbf{H}(\lambda).$$

Applying [31, Lemma 1 and Proposition 1], we obtain the following lemma:

Lemma 3.2. *Assume that* ($\mathcal{H}_d$) *holds. Then, there exists $v = v(B) > 0$ such that*

$$\mathbf{H}(\lambda_1) \subset \mathbf{H}(\lambda_2) + vl_d \|\lambda_1 - \lambda_2\|_{\Lambda}^{\eta} \mathbb{B}_m, \quad \forall \lambda_1, \lambda_2 \in \mathcal{N}(\widetilde{\lambda}),$$

that is, $\mathbf{H}$ *is* $(vl_d).\eta$*-Hölder continuous on* $\mathcal{N}(\widetilde{\lambda})$.

4. Main results

In the rest of paper, let $(\tilde{\lambda}, \tilde{\gamma}) \in \Lambda \times \Gamma$ be fixed. In this section, we mainly provide the Hölder continuity of the solution mapping $\Phi(\cdot, \cdot)$ to $\mathrm{VQVIP}_{\lambda,\gamma}(\mathbf{H}, \mathcal{T}, \mathbf{G}_A)$ around the considered point $(\tilde{\lambda}, \tilde{\gamma})$.

Let $\lambda \in \Lambda, \gamma \in \Gamma$ and $\mu > 0$ be arbitrarily given. We now consider the following function $\Upsilon_\mu^{\mathbf{R}}\colon \mathbf{K}^+ \times \Lambda \times \Gamma \to \mathbb{R}$ defined by

$$\Upsilon_\mu^{\mathbf{R}}(\mathbf{f}, \lambda, \gamma) = \sup_{\mathbf{h} \in \mathbf{H}(\lambda)} \left(-\max_{1 \leq i \leq p} \left\{ \sum_{j=1}^{m} a_{ij} \langle \mathcal{T}^j(\mathbf{f}, \gamma), \mathbf{h} - \mathbf{f} \rangle \right\} - \mu \mathbf{R}(\mathbf{f}, \mathbf{h}) \right), \tag{4}$$

where the function $\mathbf{R}\colon \mathbf{K}^+ \times \mathbf{K}^+ \to \mathbb{R}_+$ satisfies the condition ($\mathcal{H}_{\mathbf{R}}$).

Proposition 4.1. *Suppose that the assumption* ($\mathcal{H}_{\mathbf{R}}$) *holds. Then, for each $\mu > 0$, the function $\Upsilon_\mu^{\mathbf{R}}\colon \mathbf{K}^+ \to \mathbb{R}$ defined by* (4) *is a regularized gap function for* $\mathrm{VQVIP}_{\lambda,\gamma}(\mathbf{H}, \mathcal{T}, \mathbf{G}_A)$*, i.e., $\Upsilon_\mu^{\mathbf{R}}$ satisfies the following properties:*

(i) $\Upsilon_\mu^{\mathbf{R}}(\mathbf{f}, \lambda, \gamma) \geq 0$ *for all* $\mathbf{f} \in \mathbf{H}(\lambda)$.

(ii) $\mathbf{f}^*_{\lambda,\gamma} \in \mathbf{H}(\lambda)$ *is such that* $\Upsilon_\mu^{\mathbf{R}}(\mathbf{f}^*_{\lambda,\gamma}, \lambda, \gamma) = 0$ *if and only if* $\mathbf{f}^*_{\lambda,\gamma} \in \Phi(\lambda, \gamma)$*, i.e.,* $\mathbf{f}^*_{\lambda,\gamma}$ *is a solution to* $\mathrm{VQVIP}_{\lambda,\gamma}(\mathbf{H}, \mathcal{T}, \mathbf{G}_A)$.

Proof. The proof is followed from [18, Theorem 4.1]. □

Remark 4.2. For each $\lambda \in \Lambda$, $\gamma \in \Gamma$ and $\mu > 0$, by Proposition 4.1, the close relationship between the regularized gap function $\Upsilon_\mu^{\mathbf{R}}$ and the solution mapping $\Phi(\cdot, \cdot)$ is illustrated as follows:

$$\Phi(\lambda, \gamma) = \left\{ \mathbf{f}_{\lambda,\gamma} \in \mathbf{H}(\lambda) : \Upsilon_\mu^{\mathbf{R}}(\mathbf{f}_{\lambda,\gamma}, \lambda, \gamma) = 0 \right\}.$$

The following resutl gives an upper bound for the problem $\mathrm{VQVIP}_{\lambda,\gamma}(\mathbf{H}, \mathcal{T}, \mathbf{G}_A)$ based on the regularized gap function $\Upsilon_\mu^{\mathbf{R}}$.

Proposition 4.3. *Let $\mathbf{f}^*_{\lambda,\gamma}$ be a solution of the problem* $\mathrm{VQVIP}_{\lambda,\gamma}(\mathbf{H}, \mathcal{T}, \mathbf{G}_A)$*. Assume that the hypotheses* $(\mathcal{H}_{\mathcal{T}})_1$, $(\mathcal{H}_{\mathcal{T}})_2$ *and* ($\mathcal{H}_{\mathbf{R}}$) *hold. If $\mu > 0$ is such that*

$$\min_{1 \leq i \leq p} \left\{ \sum_{k=1}^{m} a_{ik} \right\} \sigma - \mu(\beta - \alpha) > 0,$$

then, for each $\mathbf{f} \in \mathbf{H}(\lambda)$, it holds

$$\left\| \mathbf{f} - \mathbf{f}^*_{\lambda,\gamma} \right\| \leq \sqrt{\frac{\Upsilon_\mu^{\mathbf{R}}(\mathbf{f}, \lambda, \gamma)}{\min_{1 \leq i \leq p} \left\{ \sum_{j=1}^{m} a_{ij} \right\} \sigma - \mu(\beta - \alpha)}}. \tag{5}$$

Proof. Let $\mathbf{f}^*_{\lambda,\gamma}$ be a solution of the problem $\mathrm{VQVIP}_{\lambda,\gamma}(\mathbf{H},\mathcal{T},\mathbf{G}_A)$. For each $\mathbf{f}\in\mathbf{H}(\lambda)$ and $\mu>0$ fixed, since $\mathbf{f}^*_{\lambda,\gamma}\in\mathbf{H}(\lambda)$, it follows from the definition of $\Upsilon^{\mathbf{R}}_{\mu}$ that

$$\Upsilon^{\mathbf{R}}_{\mu}(\mathbf{f},\lambda,\gamma)\geq-\max_{1\leq i\leq p}\left\{\sum_{j=1}^{m}a_{ij}\langle\mathcal{T}^j(\mathbf{f},\gamma),\mathbf{f}^*_{\lambda,\gamma}-\mathbf{f}\rangle\right\}-\mu\mathbf{R}(\mathbf{f},\mathbf{f}^*_{\lambda,\gamma}).$$

Under the hypotheses $(\mathcal{H}_{\mathcal{T}})_1$, $(\mathcal{H}_{\mathcal{T}})_2$ and $(\mathcal{H}_{\mathbf{R}})$, using the same method as in the proof of [18, Theorem 4.1], we obtain an upper bound for the problem $\mathrm{VQVIP}_{\lambda,\gamma}(\mathbf{H},\mathcal{T},\mathbf{G}_A)$ in the inequality (5). □

Next, we derive the following Hölder property of the regularized gap function $\Upsilon^{\mathbf{R}}_{\mu}$ which will be used to study the Hölder continuity of the solution mapping $\Phi(\cdot,\cdot)$.

Proposition 4.4. *Let $\mathcal{N}(\tilde{\lambda})$ and $\mathcal{N}(\tilde{\gamma})$ be neighborhoods of $\tilde{\lambda}\in\Lambda$ and $\tilde{\gamma}\in\Gamma$, respectively. Assume that the hypotheses $(\mathcal{H}_d)$, $(\mathcal{H}_{\mathcal{T}})_3$, $(\mathcal{H}_{\mathcal{T}})_4$, $(\mathcal{H}_{\mathbf{R}})$ and $(\mathcal{H}_{\mathbf{H}})$ hold. Then for each $\mu>0$, for any $(\mathbf{f}_1,\lambda_1,\gamma_1),(\mathbf{f}_2,\lambda_2,\gamma_2)\in\mathbf{H}(\mathcal{N}(\tilde{\lambda}))\times\mathcal{N}(\tilde{\lambda})\times\mathcal{N}(\tilde{\gamma})$, one has*

$$|\Upsilon^{\mathbf{R}}_{\mu}(\mathbf{f}_1,\lambda_1,\gamma_1)-\Upsilon^{\mathbf{R}}_{\mu}(\mathbf{f}_2,\lambda_2,\gamma_2)|\leq l_{\Upsilon^{\mathbf{R}}_{\mu}}(\|\mathbf{f}_1-\mathbf{f}_2\|+\|\lambda_1-\lambda_2\|^{\eta}_{\Lambda}+\|\gamma_1-\gamma_2\|^{\theta}_{\Gamma}),\tag{6}$$

where

$$\begin{aligned}
l_{\Upsilon^{\mathbf{R}}_{\mu}}&=\max\left\{l_{\mathbf{f}},l_{\lambda},l_{\gamma}\right\},\qquad(7)\\
l_{\mathbf{f}}&=\left[\max_{1\leq i\leq p}\left\{\sum_{j=1}^{m}|a_{ij}|\,(2b_{\mathbf{H}}l_{\mathcal{T}^j}+b_{\mathcal{T}^j})\right\}+4b_{\mathbf{H}}\mu(\beta-\alpha)\right],\\
l_{\lambda}&=\left[\max_{1\leq i\leq p}\left\{\sum_{j=1}^{m}|a_{ij}|\,(b_{\mathcal{T}^j}\upsilon l_d)\right\}+4b_{\mathbf{H}}\mu(\beta-\alpha)\upsilon l_d\right],\\
l_{\gamma}&=\max_{1\leq i\leq p}\left\{\sum_{j=1}^{m}|a_{ij}|2b_{\mathbf{H}}\widetilde{l_{\mathcal{T}^j}}\right\}.
\end{aligned}$$

Proof. Let $(\lambda_1,\gamma_1),(\lambda_2,\gamma_2)\in\mathcal{N}(\tilde{\lambda})\times\mathcal{N}(\tilde{\gamma})$ and $(\mathbf{f}_1,\mathbf{f}_2)\in\mathbf{H}(\lambda_1)\times\mathbf{H}(\lambda_2)$ be fixed. By the definition of the regularized gap function $\Upsilon^{\mathbf{R}}_{\mu}$ in (4), we obtain the following assertion: for any $\varepsilon>0$, there exists $\mathbf{h}_{\varepsilon}\in\mathbf{H}(\lambda_1)$ such that

$$\Upsilon^{\mathbf{R}}_{\mu}(\mathbf{f}_1,\lambda_1,\gamma_1)\leq-\max_{1\leq i\leq p}\left\{\sum_{j=1}^{m}a_{ij}\langle\mathcal{T}^j(\mathbf{f}_1,\gamma_1),\mathbf{h}_{\varepsilon}-\mathbf{f}_1\rangle\right\}-\mu\mathbf{R}(\mathbf{f}_1,\mathbf{h}_{\varepsilon})+\varepsilon.\tag{8}$$

By the assumption $(\mathcal{H}_d)$, it follows from Lemma 3.2 that there exist constants $\upsilon,l_d>0$ and $\eta>0$ such that

$$\mathbf{H}(\lambda_1)\subset\mathbf{H}(\lambda_2)+\upsilon l_d\,\|\lambda_1-\lambda_2\|^{\eta}_{\Lambda}\,\mathbb{B}_m.\tag{9}$$

This implies that there exists $\mathbf{h}_2\in\mathbf{H}(\lambda_2)$ such that

$$\|\mathbf{h}_{\varepsilon}-\mathbf{h}_2\|\leq\upsilon l_d\,\|\lambda_1-\lambda_2\|^{\eta}_{\Lambda}.\tag{10}$$

Moreover, we also obtain

$$\Upsilon^{\mathbf{R}}_{\mu}(\mathbf{f}_2,\lambda_2,\gamma_2)\geq-\max_{1\leq i\leq p}\left\{\sum_{j=1}^{m}a_{ij}\langle\mathcal{T}^j(\mathbf{f}_2,\gamma_2),\mathbf{h}_2-\mathbf{f}_2\rangle\right\}-\mu\mathbf{R}(\mathbf{f}_2,\mathbf{h}_2)\tag{11}$$

Using the condition $(\mathcal{H}_{\mathbf{R}})$, one has

$$\begin{aligned}\mathbf{R}(\mathbf{f}_2,\mathbf{h}_2)-\mathbf{R}(\mathbf{f}_1,\mathbf{h}_\varepsilon) &\le (\beta-\alpha)\|\mathbf{f}_2-\mathbf{h}_2\|^2-\alpha\|\mathbf{f}_1-\mathbf{h}_\varepsilon\|^2\\ &\le (\beta-\alpha)(\|\mathbf{f}_2-\mathbf{h}_2\|^2-\|\mathbf{f}_1-\mathbf{h}_\varepsilon\|^2). \end{aligned} \tag{12}$$

From (8), (11) and (12), we get

$$\begin{aligned}&\Upsilon^{\mathbf{R}}_\mu(\mathbf{f}_1,\lambda_1,\gamma_1)-\Upsilon^{\mathbf{R}}_\mu(\mathbf{f}_2,\lambda_2,\gamma_2)\\
&\le \max_{1\le i\le p}\left\{\sum_{j=1}^m a_{ij}\langle\mathcal{T}^j(\mathbf{f}_2,\gamma_2),\mathbf{h}_2-\mathbf{f}_2\rangle\right\}-\max_{1\le i\le p}\left\{\sum_{j=1}^m a_{ij}\langle\mathcal{T}^j(\mathbf{f}_1,\gamma_1),\mathbf{h}_\varepsilon-\mathbf{f}_1\rangle\right\}\\
&\quad+\mu\left(\mathbf{R}(\mathbf{f}_2,\mathbf{h}_2)-\mathbf{R}(\mathbf{f}_1,\mathbf{h}_\varepsilon)\right)+\varepsilon\\
&\le \max_{1\le i\le p}\left\{\sum_{j=1}^m a_{ij}\left(\langle\mathcal{T}^j(\mathbf{f}_2,\gamma_2)-\mathcal{T}^j(\mathbf{f}_1,\gamma_1),\mathbf{h}_\varepsilon-\mathbf{f}_1\rangle+\langle\mathcal{T}^j(\mathbf{f}_2,\gamma_2),\mathbf{f}_1-\mathbf{f}_2+\mathbf{h}_2-\mathbf{h}_\varepsilon\rangle\right)\right\}\\
&\quad+\mu(\beta-\alpha)(\|\mathbf{f}_2-\mathbf{h}_2\|^2-\|\mathbf{f}_1-\mathbf{h}_\varepsilon\|^2)+\varepsilon\\
&\le \max_{1\le i\le p}\left\{\sum_{j=1}^m |a_{ij}|\left\|\mathcal{T}^j(\mathbf{f}_2,\gamma_2)-\mathcal{T}^j(\mathbf{f}_1,\gamma_1)\right\|(\|\mathbf{h}_\varepsilon\|+\|\mathbf{f}_1\|)\right\}\\
&\quad+\max_{1\le i\le p}\left\{\sum_{j=1}^m |a_{ij}|\left\|\mathcal{T}^j(\mathbf{f}_2,\gamma_2)\right\|(\|\mathbf{f}_1-\mathbf{f}_2\|+\|\mathbf{h}_2-\mathbf{h}_\varepsilon\|)\right\}\\
&\quad+\mu(\beta-\alpha)\left(\|\mathbf{f}_2\|+\|\mathbf{h}_2\|+\|\mathbf{f}_1\|+\|\mathbf{h}_\varepsilon\|\right)(\|\mathbf{f}_1-\mathbf{f}_2\|+\|\mathbf{h}_2-\mathbf{h}_\varepsilon\|)+\varepsilon.\end{aligned} \tag{13}$$

Hence, by the conditions $(\mathcal{H}_d)$, $(\mathcal{H}_\mathcal{T})_3$, $(\mathcal{H}_\mathcal{T})_4$, $(\mathcal{H}_\mathbf{H})$, (10), (13) and the arbitrariness of ε, one has

$$\begin{aligned}&\Upsilon^{\mathbf{R}}_\mu(\mathbf{f}_1,\lambda_1,\gamma_1)-\Upsilon^{\mathbf{R}}_\mu(\mathbf{f}_2,\lambda_2,\gamma_2)\\
&\le \max_{1\le i\le p}\left\{\sum_{j=1}^m |a_{ij}|2b_\mathbf{H}(l_{\mathcal{T}^j}\|\mathbf{f}_1-\mathbf{f}_2\|+\widetilde{l}_{\mathcal{T}^j}\left\|\gamma_1-\gamma_2\right\|^\theta_\Gamma)\right\}\\
&\quad+\max_{1\le i\le p}\left\{\sum_{j=1}^m |a_{ij}|b_{\mathcal{T}^j}(\|\mathbf{f}_1-\mathbf{f}_2\|+vl_d\|\lambda_1-\lambda_2\|^\eta_\Lambda\right\}\\
&\quad+4b_\mathbf{H}\mu(\beta-\alpha)\left(\|\mathbf{f}_1-\mathbf{f}_2\|+vl_d\|\lambda_1-\lambda_2\|^\eta_\Lambda\right)\\
&\le\left[\max_{1\le i\le p}\left\{\sum_{j=1}^m |a_{ij}|(2b_\mathbf{H}l_{\mathcal{T}^j}+b_{\mathcal{T}^j})\right\}+4b_\mathbf{H}\mu(\beta-\alpha)\right]\|\mathbf{f}_1-\mathbf{f}_2\|\\
&\quad+\left[\max_{1\le i\le p}\left\{\sum_{j=1}^m |a_{ij}|(b_{\mathcal{T}^j}vl_d)\right\}+4b_\mathbf{H}\mu(\beta-\alpha)vl_d\right]\|\lambda_1-\lambda_2\|^\eta_\Lambda\\
&\quad+\max_{1\le i\le p}\left\{\sum_{j=1}^m |a_{ij}|2b_\mathbf{H}\widetilde{l}_{\mathcal{T}^j}\right\}\left\|\gamma_1-\gamma_2\right\|^\theta_\Gamma\\
&\le l_{\Upsilon^\mathbf{R}_\mu}(\|\mathbf{f}_1-\mathbf{f}_2\|+\|\lambda_1-\lambda_2\|^\eta_\Lambda+\|\gamma_1-\gamma_2\|^\theta_\Gamma),\end{aligned}$$

where $l_{\Upsilon^\mathbf{R}_\mu}$ is defined by (7).

Thus, it follows from the symmetry between $(\mathbf{f}_1,\lambda_1,\gamma_1)$ and $(\mathbf{f}_2,\lambda_2,\gamma_2)$ that the conclusion of Proposition 4.4 is valid. □

In virtue of the properties of the regularized gap function $\Upsilon^{\mathbf{R}}_{\mu}$, we now derive the Hölder continuity of the solution mapping $\Phi(\cdot,\cdot)$ to the problem $\mathrm{VQVIP}_{\lambda,\gamma}(\mathbf{H},\mathcal{T},\mathbf{G}_A)$ around the point $(\tilde{\lambda},\tilde{\gamma})$.

Theorem 4.5. *Let $\mathcal{N}(\tilde{\lambda})$ and $\mathcal{N}(\tilde{\gamma})$ be neighborhoods of $\tilde{\lambda}\in\Lambda$ and $\tilde{\gamma}\in\Gamma$, respectively. Assume that the hypotheses $(\mathcal{H}_d)$, $(\mathcal{H}_{\mathcal{T}})_1$–$(\mathcal{H}_{\mathcal{T}})_4$, $(\mathcal{H}_{\mathbf{R}})$ and $(\mathcal{H}_{\mathbf{H}})$ hold. Then for $\mu>0$ is such that*

$$\min_{1\le i\le p}\left\{\sum_{k=1}^{m}a_{ik}\right\}\sigma-\mu(\beta-\alpha)>0,$$

for any $(\lambda_1,\gamma_1),(\lambda_2,\gamma_2)\in\mathcal{N}(\tilde{\lambda})\times\mathcal{N}(\tilde{\gamma})$, we have

$$\|\mathbf{f}_{\lambda_1,\gamma_1}-\mathbf{f}_{\lambda_2,\gamma_2}\|\le \upsilon l_d\,\|\lambda_1-\lambda_2\|^{\eta}_{\Lambda}+(\Delta l_{\Upsilon^{\mathbf{R}}_{\mu}})^{\frac{1}{2}}\left((1+\upsilon l_d)^{\frac{1}{2}}\,\|\lambda_1-\lambda_2\|^{\frac{\eta}{2}}_{\Lambda}+\|\gamma_1-\gamma_2\|^{\frac{\theta}{2}}_{\Gamma}\right),\tag{14}$$

where $\mathbf{f}_{\lambda_1,\gamma_1}\in\Phi(\lambda_1,\gamma_1)$, $\mathbf{f}_{\lambda_2,\gamma_2}\in\Phi(\lambda_2,\gamma_2)$, $l_{\Upsilon^{\mathbf{R}}_{\mu}}$ is defined by (7) *and*

$$\Delta=\left[\min_{1\le i\le p}\left\{\sum_{j=1}^{m}a_{ij}\right\}\sigma-\mu(\beta-\alpha)\right]^{-1}.\tag{15}$$

Proof. For each $\mu>0$, $\lambda\in\mathcal{N}(\tilde{\lambda})$, $\gamma\in\mathcal{N}(\tilde{\gamma})$ fixed and $\mathbf{f}_{\lambda,\gamma}\in\Phi(\lambda,\gamma)$. It follows from Proposition 4.3 that for each $\mathbf{f}\in\mathbf{H}(\lambda)$, the following inequality holds:

$$\left\|\mathbf{f}-\mathbf{f}_{\lambda,\gamma}\right\|\le\sqrt{\Delta\Upsilon^{\mathbf{R}}_{\mu}(\mathbf{f},\lambda,\gamma)},\tag{16}$$

where Δ is defined by (15).

Let $(\lambda_1,\gamma_1),(\lambda_2,\gamma_2)\in\mathcal{N}(\tilde{\lambda})\times\mathcal{N}(\tilde{\gamma})$ be fixed and $\mathbf{f}_{\lambda_1,\gamma_1}\in\Phi(\lambda_1,\gamma_1)$, $\mathbf{f}_{\lambda_2,\gamma_2}\in\Phi(\lambda_2,\gamma_2)$. Then we have $\mathbf{f}_{\lambda_1,\gamma_1}\in\mathbf{H}(\lambda_1)$ and so, it follows from (9) that there exists $\mathbf{f}_2\in\mathbf{H}(\lambda_2)$ such that

$$\left\|\mathbf{f}_{\lambda_1,\gamma_1}-\mathbf{f}_2\right\|\le\upsilon l_d\,\|\lambda_1-\lambda_2\|^{\eta}_{\Lambda}.\tag{17}$$

Applying (6), (16), (17) and $\mathbf{f}_{\lambda_1,\gamma_1}\in\Phi(\lambda_1,\gamma_1)$, i.e., $\Upsilon^{\mathbf{R}}_{\mu}(\mathbf{f}_{\lambda_1,\gamma_1},\lambda_1,\gamma_1)=0$, we have

$$\begin{aligned}\|\mathbf{f}_{\lambda_1,\gamma_1}-\mathbf{f}_{\lambda_2,\gamma_2}\|&\le\|\mathbf{f}_{\lambda_1,\gamma_1}-\mathbf{f}_2\|+\|\mathbf{f}_2-\mathbf{f}_{\lambda_2,\gamma_2}\|\\&\le\upsilon l_d\,\|\lambda_1-\lambda_2\|^{\eta}_{\Lambda}+\sqrt{\Delta\Upsilon^{\mathbf{R}}_{\mu}(\mathbf{f}_2,\lambda_2,\gamma_2)}\\&=\upsilon l_d\,\|\lambda_1-\lambda_2\|^{\eta}_{\Lambda}+\sqrt{\Delta(\Upsilon^{\mathbf{R}}_{\mu}(\mathbf{f}_2,\lambda_2,\gamma_2)-\Upsilon^{\mathbf{R}}_{\mu}(\mathbf{f}_{\lambda_1,\gamma_1},\lambda_1,\gamma_1))}\\&\le\upsilon l_d\,\|\lambda_1-\lambda_2\|^{\eta}_{\Lambda}+\sqrt{\Delta l_{\Upsilon^{\mathbf{R}}_{\mu}}(\|\mathbf{f}_2-\mathbf{f}_{\lambda_1,\gamma_1}\|+\|\lambda_1-\lambda_2\|^{\eta}_{\Lambda}+\|\gamma_1-\gamma_2\|^{\theta}_{\Gamma})}\\&\le\upsilon l_d\,\|\lambda_1-\lambda_2\|^{\eta}_{\Lambda}+\sqrt{\Delta l_{\Upsilon^{\mathbf{R}}_{\mu}}((1+\upsilon l_d)\,\|\lambda_1-\lambda_2\|^{\eta}_{\Lambda}+\|\gamma_1-\gamma_2\|^{\theta}_{\Gamma})}\\&\le\upsilon l_d\,\|\lambda_1-\lambda_2\|^{\eta}_{\Lambda}+(\Delta l_{\Upsilon^{\mathbf{R}}_{\mu}})^{\frac{1}{2}}\left((1+\upsilon l_d)^{\frac{1}{2}}\,\|\lambda_1-\lambda_2\|^{\frac{\eta}{2}}_{\Lambda}+\|\gamma_1-\gamma_2\|^{\frac{\theta}{2}}_{\Gamma}\right).\end{aligned}$$

Therefore, the inequality (14) holds. □

Remark 4.6. Theorem 4.5 illustrates the Hölder continuous behaviour of the solution mapping to the problem $\mathrm{VQVIP}_{\lambda,\gamma}(\mathbf{H},\mathcal{T},\mathbf{G}_A)$ based on the partial order provided by a polyhedral cone generated by a matrix. This Hölder continuous behaviour depends on the data of $\mathrm{VQVIP}_{\lambda,\gamma}(\mathbf{H},\mathcal{T},\mathbf{G}_A)$ and the choice of the regularized parameter μ of the gap function $\Upsilon^{\mathbf{R}}_{\mu}$.

5. Conclusions

In this paper, we have introduced a model of vector network equilibrium problems based on a polyhedral ordering cone under parametric perturbations (VQVIP$_{\lambda,\gamma}(\mathbf{H},\mathcal{T},\mathbf{G}_A)$). Then, applying the useful properties of the regularized gap function for this model, we have provided the Hölder continuous behaviour of the solution mapping to the problem VQVIP$_{\lambda,\gamma}(\mathbf{H},\mathcal{T},\mathbf{G}_A)$ under some suitable assumptions. To the best of our knowledge, up to now, there is no paper concerning the Hölder continuity for the problem VQVIP$_{\lambda,\gamma}(\mathbf{H},\mathcal{T},\mathbf{G}_A)$ with partial order provided by a polyhedral cone generated by a matrix. Thus, our main results on the Hölder continuity presented in the paper are new.

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