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SEP Matrices and Solution of Matrix Equations

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Abstract. This paper mainly introduces some properties of several generalized inverses of matrices, especially some equivalent characteristics of generalized inverses of matrices, specifically by constructing some specific matrix equations and discussing whether these matrix equations have solutions in a given set to determine whether a group invertible matrix is some generalized inverse of matrices.

1. Introduction

Throughout this paper, $C^{n\times n}$ stands for the set of all $n \times n$ complex matrices. A^H denotes the conjugate transpose matrix of $A \in C^{n\times n}$. Recall that a matrix $A \in C^{n\times n}$ is said to be group invertible [4] if there exists $X \in C^{n\times n}$ such that

$$AXA = A, XAX = X, AX = XA$$

hold. If such matrix X exists, then it is unique, denoted by $A^\#$, and called the group inverse of A. It is well known that the group inverse of $A \in C^{n\times n}$ exists if and only if $rank(A^2) = rank(A)$ [1].

A matrix $A \in C^{n\times n}$ is said to be Moore-Penrose invertible [5–7, 11] if there exists $X \in C^{n\times n}$ such that

$$AXA = A, XAX = X, (AX)^H = AX, (XA)^H = XA$$

hold. According to [11], such matrix X always exists uniquely, denoted by A^+, and called the Moore-Penrose inverse of A.

A matrix $A \in C^{n\times n}$ is called *EP* [12] if $A^\#$ exists and $A^\# = A^+$; A is called partial isometry (or *PI*) [3] if $A^+ = A^H$; A is said to be normal [2] if $AA^H = A^H A$; A is said to be *SEP* if A is *EP* and *PI*.

In [10], it has studied the generalized inverse of an operator with the aid of specific operator equation. In [15], it has studied the connection between the normal element and the existence of solutions to some equation on rings in a given set.

In [13], it has discussed the necessary and sufficient conditions for the EP element and the existence of solutions to equations on rings in a given set.

Inspired by these, the purpose of this paper is that we collect some new characteristics of *EP* matrices, normal matrices and partial isometry matrices through various matrices equations admitting solutions in a definite set $\rho_A = \{A, A^\#, A^+, A^H, (A^\#)^H, (A^+)^H\}, (A^+)^\#, (A^\#)^+\}$.

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2. Several lemmas

We begin with the following lemma.

Lemma 2.1. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a SEP matrix if and only if $A^+ = (A^\#)^H A^+ A^\#$.*
Proof *$'' \Rightarrow ''$ Assume that A is SEP. Then $A^\# = A^+ = A^H$, which implies $(A^\#)^H A^+ A^\# = AA^+A^\# = A^\# = A^+$. $'' \Leftarrow ''$ If $A^+ = (A^\#)^H A^+ A^\#$, then $A^H A^+ = A^H (A^\#)^H A^+ A^\#$. Noting that $A^H (A^\#)^H A^+ = (AA^\#)^H (A^+AA^+) = (A^+A^2A^\#)^H A^+ = (A^+A)^H A^+ = A^+$. Then $A^H A^+ = A^+ A^\#$. Thus A is SEP by [14, Theorem 1.5.3].* ∎

Observing carefully the proof of Lemma 2.1, we have the following corollary.

Corollary 2.2. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then*
(1) $A^H(A^\#)^H A^+ = A^+ = A^+ A^H (A^\#)^H$;
(2) A is a SEP matrix if and only if $A^\# = (A^\#)^H A^+ A^\#$.

Now we give following lemma which proof is routine.

Lemma 2.3. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then*

(1) $(A^+)^\# = (AA^\#)^H A (AA^\#)^H$;

(2) $(A^\#)^+ = A^+ A^3 A^+$;

(3) $(A^\#)^H A^+ A^\#$ is an EP matrix and $((A^\#)^H A^+ A^\#)^+ = A^+ A^3 A^H A^+ A = (A^\#)^+ A A^H A^+ A$.

Corollary 2.4. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a SEP matrix if and only if $(A^\#)^H A^+ = A^+ A$.*

Proof By Corollary 2.2, A is a *SEP* matrix if and only if $A^\# = (A^\#)^H A^+ A^\#$. By Lemma 2.3, we obtain A is a *SEP* matrix if and only if

$$A^+ A^3 A^+ = A^+ A^3 A^H A^+ A.$$

Multiplying the equality one left by $(A^\#)^H A^+ A^\#$ and again by Corollary 2.2, one has A is a *SEP* matrix if and only if $(A^\#)^H A^+ = A^+ A$. ∎

Lemma 2.1 and Lemma 2.3(3) imply the following corollary.

Corollary 2.5. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a SEP matrix if and only if $A = A^+ A^3 A^H A^+ A = (A^\#)^+ A A^H A^+ A$.*

Also Lemma 2.1, Lemma 2.3(1) and (3) imply the following corollary.

Corollary 2.6. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a SEP matrix if and only if $(AA^\#)^H A (AA^\#)^H = A^+ A^3 A^H A^+ A$.*

3. Consistency of relative equations

According to Lemma 2.1, we can construct the following equation.

$$(A^\#)^H X A^\# = A^+. \tag{3.1}$$

Theorem 3.1. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a SEP matrix if and only if Eq.(3.1) is consistent in $C^{n\times n}$, and the general solution of Eq.(3.1) is given by*

$$X = A^+ + U - AA^+ U A A^+, \quad \text{where} \quad U \in C^{n\times n}. \tag{3.2}$$

Proof $'' \Rightarrow ''$ Assume that A is SEP. Then $A^+ = (A^\#)^H A^+ A^\#$ by Lemma 2.1, this gives

$$(A^\#)^H (A^+ + U - AA^+UAA^+)A^\# = A^+. \tag{3.3}$$

Hence the formula (3.2) is the solution of Eq.(3.1).
Now, let $X = X_0$ be any solution of Eq.(3.1). Then $(A^\#)^H X_0 A^\# = A^+$,
which implies

$$AA^+X_0AA^+ = (AA^+)A^H((A^\#)^H X_0 A^\#)A^2A^+ = AA^+A^HA^+A^2A^+.$$

Noting that A is SEP. Then $AA^+A^HA^+A^2A^+ = AA^\#A^\#A^\#A^2A^\# = A^\# = A^+$, it follows that $AA^+X_0AA^+ = A^+$. Choose $U = X_0$. Then $X_0 = A^+ + U - AA^+UAA^+$. This shows that the Formula (3.2) is the general solution of Eq.(3.1).
$'' \Leftarrow ''$ If the general solution of Eq.(3.1) is given by (3.2), then

$$(A^\#)^H (A^+ + U - AA^+UAA^+)A^\# = A^+,$$

$$e.g.\ \ (A^\#)^H A^+ A^\# = A^+.$$

Hence A is SEP by Lemma 2.1. ∎

Remark 3.2. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is an EP matrix if and only if Eq.*(3.1) *is consistent in $C^{n\times n}$.*

In this case, the general solution of Eq.(3.1) *is given by*

$$X = A^H + U - AA^+UAA^+, \ \ where \ \ U \in C^{n\times n}. \tag{3.4}$$

Proof $'' \Rightarrow ''$ If A is EP, then $(A^\#)^H A^H A^\# = (A^\#)^H A^H A^+ = A^+$ by Corollary 2.2. Hence Eq.(3.1) is consistent.
$'' \Leftarrow ''$ Assume that Eq.(3.1) is consistent, then $A^+ = (A^\#)^H X_0 A^\#$ for some $X_0 \in C^{n\times n}$, this gives $A^+ = A^+A^+A$. Hence A is EP.

The rest can be similarly proved as Theorem 3.1. ∎

Now we construct the following equation.

$$A^+XAA^+(A^\#)^H = A^+. \tag{3.5}$$

Theorem 3.3. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then the general solution of Eq.*(3.4) *is given by* (3.3).
Proof *It is routine.* ∎

Remark 3.2 and Theorm 3.3 imply the following corollary.

Corollary 3.4. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is an EP matrix if and only if Eq.*(3.1) *and Eq.*(3.4) *have the same solution.*

Now we consider the following equation.

$$(AA^\#)^H XAA^+ = A^+. \tag{3.6}$$

Theorem 3.5. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then the general solution of Eq.*(3.5) *is given by* (3.2).

Proof Similar to the proof of Theorem 3.1, we can easy prove it. ∎

Clearly, Theorem 3.3 and Thorem 3.5 lead to the following corollary.

Corollary 3.6. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is is an SEP matrix if and only if Eq.*(3.1) *and Eq.*(3.5) *have the same solution.*

4. The solution of a matrix equation in a given set

According to Lemma 2.1, we can construct the following equation.

$$X = (A^{\#})^H X A^{\#}. \tag{4.1}$$

The following theorem follows from [16, Theorem 2.8].

Theorem 4.1. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a SEP matrix if and only if Eq.*(4.1) *has at least one solution in $\rho_A = \{A, A^{\#}, A^+, A^H, (A^{\#})^H, (A^+)^H, (A^+)^{\#}, (A^{\#})^+\}$.*

It is well known that A is *SEP* if and only if $A^{\#}$ is *SEP*. Hence use $A^{\#}$ to replace A in Eq.(4.1), we have the following equation.

$$X = A^H X A. \tag{4.2}$$

Theorem 4.1 implies the following corollary.

Corollary 4.2. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a SEP matrix if and only if Eq.*(4.2) *has at least one solution in ρ_A.*

Noting that if A is *SEP*, then $A^+ = A^{\#}$. Hence we can change Eq.(4.2) as follows.

$$X + A^+ = A^H X A + A^{\#}. \tag{4.3}$$

Theorem 4.3. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a SEP matrix if and only if Eq.*(4.3) *has at least one solution in ρ_A.*

Proof "$\Rightarrow$" If A is a *SEP* matrix, then $X = A^+$ is a solution because $A^+ = A^{\#} = A^H$.
"$\Leftarrow$" 1) If $X = A$, then $A + A^+ = A^H A A + A^{\#}$, Post-multiplying the equality by A^+A, one yields $A^+ = A^+A^+A$. Hence A is *EP*, which implies $A^+ = A^{\#}$, this gives $A = A^H A A$, it follows that $X = A$ is a solution of Eq.(4.2). Hence A is *SEP* by Corollary 4.2;
2) If $X = A^{\#}$, then $A^{\#} + A^+ = A^H A^{\#} A + A^{\#}$. Post-multiplying the equality by A^+A, one yields $A^+ = A^+A^+A$. Hence A is *EP*, which implies $A^{\#} = A^H A^{\#} A$, Thus $X = A^{\#}$ is a solution of Eq.(4.2). Hence A is *SEP* by Corollary 4.2;
3) If $X = A^+$, then $A^+ + A^+ = A^H A^+ A + A^{\#}$. Pre-multiplying the equality by A^+A, one gets $A^{\#} = A^+AA^{\#}$. Hence A is *EP*, it follows that $A^+ = A^H A^+ A$. Thus A is *SEP* by Corollary 4.2;
4) If $X = A^H$, then $A^H + A^+ = A^H A^H A + A^{\#}$. Similar to the proof of 3), we can show that A is *SEP*;
5) If $X = (A^+)^H$, then $(A^+)^H + A^+ = A^H (A^+)^H A + A^{\#}$. Post-multiplying the equality by A^+A, one obtains $A^+ = A^+A^+A$. Hence A is *EP*, which implies $(A^+)^H = A^H (A^+)^H A$. Thus $X = (A^+)^H$ is a solution of Eq.(4.2). By Corollary 4.2, we have A is *SEP*;
6) If $X = (A^{\#})^H$, then $(A^{\#})^H + A^+ = A^H (A^{\#})^H A + A^{\#}$. Pre-multiplying the equality by A^+A, one yields $A^{\#} = A^+AA^{\#}$. Hence A is *EP*, one obtains $(A^{\#})^H = A^H (A^{\#})^H A$. By Corollary 4.2, we have A is *SEP* because $X = (A^{\#})^H$ is a solution Eq.(4.2);
7) If $X = (A^+)^{\#}$, then $(A^+)^{\#} + A^+ = A^H (A^+)^{\#} A + A^{\#}$. Pre-multiplying the equality by A^+A, one has $A^{\#} = A^+AA^{\#}$. Hence A is *EP*, this leads to $X = (A^+)^{\#} = A$ is a solution. Hence A is *SEP* by 1);
8) If $X = (A^{\#})^+$, then $(A^{\#})^+ + A^+ = A^H (A^{\#})^+ A + A^{\#}$. Pre-multiplying the equality by A^+A and using Lemma 2.3, we have $A^{\#} = A^+AA^{\#}$. Hence A is *EP*. Now $X = (A^{\#})^+ = A$ is a solution. Thus A is *SEP* by 1). ∎

5. The general solution of equations

Now we generalize Eq.(4.1) as follows

$$X = (A^{\#})^H Y A^{\#}. \tag{5.1}$$

Theorem 5.1. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then the general solution of Eq.*(5.1) *is given by*

$$\begin{cases} X = (A^\#)^H P A^\# \\ Y = P + V - AA^+VAA^+, \end{cases} \quad \text{where } P, V \in C^{n\times n}. \tag{5.2}$$

Proof First, we have

$$(A^\#)^H(P + V - AA^+VAA^+)A^\# = (A^\#)^H P A^\#.$$

Hence Formula (5.2) is the solution of Eq.(5.1).

Next, let $\begin{cases} X = X_0 \\ Y = Y_0 \end{cases}$ be a solution of Eq.(5.1). Then $X_0 = (A^\#)^H Y_0 A^\#$.

Choose $P = AA^+A^H X_0 A^2 A^+$, $V = Y_0$.

Then $(A^\#)^H P A^\# = (A^\#)^H AA^+A^H X_0 A^2 A^+ A^\# = (A^\#)^H A^H X_0 AA^\#$
$= (A^\#)^H A^H (A^\#)^H Y_0 A^\# AA^\# = (A^\#)^H Y_0 A^\# = X_0$.

And $Y_0 = AA^+Y_0AA^+ + Y_0 - AA^+Y_0AA^+$
$= AA^+A^H(A^\#)^H Y_0 A^\# A^2 A^+ + Y_0 - AA^+Y_0AA^+$
$= AA^+A^H X_0 A^2 A^+ + Y_0 - AA^+Y_0AA^+ = P + V - AA^+VAA^+$

Hence $\begin{cases} X_0 = (A^\#)^H P A^\# \\ Y_0 = P + V - AA^+VAA^+, \end{cases}$

it follows that the general solution of Eq.(5.1) is given by Formula (5.2). ∎

Theorem 5.2. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is SEP if and only if the general solution of Eq.*(5.1) *is given by*

$$\begin{cases} X = (A^\#)^H P A^H \\ Y = P + V - AA^+VAA^+, \end{cases} \quad \text{where } P, V \in C^{n\times n}. \tag{5.3}$$

Proof " $\Rightarrow$ " If A is *SEP*, then $A^\# = A^H$, which implies Formula (5.3) is same as Formula (5.2). By Theorem 5.1, the general solution of Eq.(5.1) is given by Formula (5.3).

" $\Leftarrow$ " If the general solution of Eq.(5.1) is given by Formula (5.3), then $(A^\#)^H P A^H = (A^\#)^H(P + V - AA^+VAA^+)A^\#$,

that is, $(A^\#)^H P A^H = (A^\#)^H P A^\#$ for all $P \in C^{n\times n}$. Especially, choose $P = A^H$. Then we have $A^H = (AA^\#)^H A^\#$, this gives $A = (A^\#)^H AA^\#$. By Theorem 4.1, we obtain that A is *SEP*. ∎

It is not clear which equation has the general solution given by Formula (5.3), so, we construct the following equation.

$$X = (A^\#)^H Y AA^+ A^H. \tag{5.4}$$

Theorem 5.3. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then the general solution of Formula* (5.4) *is given by*

$$\begin{cases} X = (A^\#)^H P A^H \\ Y = P + V - AA^+VAA^+, \end{cases} \quad \text{where } P, V \in C^{n\times n} \text{ and } PA^+ = PAA^+A^+. \tag{5.5}$$

Proof First, Formula (5.5) is the solution to Eq.(5.4). In fact,
$(A^\#)^H(P + V - AA^+VAA^+)AA^+A^H = (A^\#)^H PAA^+A^H$
$= (A^\#)^H PAA^+A^+AA^H = (A^\#)^H PA^+AA^H = (A^\#)^H P A^H$.

Next, let $\begin{cases} X = X_0 \\ Y = Y_0 \end{cases}$ be the any solution of Eq.(5.4). Then

$$X_0 = (A^\#)^H Y_0 AA^+ A^H.$$

Choose $P = AA^+A^H X_0 (A^\#)^H$, $V = Y_0$.

Then $PAA^+A^+ = AA^+A^H X_0 (A^\#)^H AA^+A^+ = AA^+A^H X_0 (A^\#)^H A^+ = PA^+$;
$(A^\#)^H P A^H = (A^\#)^H AA^+A^H X_0 (A^\#)^H A^H = (A^\#)^H A^H X_0 (A^\#)^H A^H$
$= (A^\#)^H A^H ((A^\#)^H Y_0 AA^+ A^H)(A^\#)^H A^H$

$= (A^{\#})^H Y_0 AA^+ A^H = X_0;$
$Y_0 = AA^+ VAA^+ + Y_0 - AA^+ VAA^+$
$= AA^+ Y_0 AA^+ + Y_0 - AA^+ VAA^+$
$= AA^+ A^H (A^{\#})^H Y_0 AA^+ A^H (A^{\#})^H + Y_0 - AA^+ VAA^+$
$= AA^+ A^H X_0 (A^{\#})^H + Y_0 - AA^+ VAA^+$
$= P + Y_0 - AA^+ VAA^+.$
Hence the general solution of Eq.(5.4) is given by Formula (5.5). ∎

Theorem 5.4. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a SEP matrix if and only if Eq.*(5.1) *and Eq.*(5.4) *have the same solution.*

Proof " ⇒ " If A is *SEP*, then A is *EP* and by Theorem 5.2, the general solution of Eq.(5.1) is given by Formula (5.3). Since A is *EP*, we have $PA^+ = PAA^+A^+$ for each $P \in C^{n\times n}$. So by Theorem 5.3, the general solution of Eq.(5.4) is given by Formula (5.3), this implies Eq.(5.1) and Eq.(5.4) have the same solution.
" ⇐ " If Eq.(5.1) and Eq.(5.4) have the same solution, then, by Theorem 5.1, the general solution of Eq.(5.4) is given by Formula (5.2). Thus we have

$$(A^{\#})^H PA^{\#} = (A^{\#})^H (P + V - AA^+ VAA^+) AA^+ A^H$$

i.e. $(A^{\#})^H PA^{\#} = (A^{\#})^H PAA^+ A^H$ for all $P \in C^{n\times n}$, especially, take $P = A^{\#}A$, then $(A^{\#})^H A^{\#} = (AA^{\#})^H$. It follows that $AA^{\#} = (A^{\#})^H A^{\#} = (AA^{\#})^H$, so A is *EP*. And then by Theorem 5.3, the general solution of Eq.(5.4) is given by (5.5), so by (5.3). This leads to the general solution of Eq.(5.1) is given by (5.3). By Theorem 5.2, A is *SEP*. ∎

6. The solutions of bivariate equations in a given set

Nothing that $A^+ = A^+ A^H (A^{\#})^H$. Hence, by Lemma 2.1, we have A is *SEP* if and only if $A^+ A^H (A^{\#})^H = (A^{\#})^H A^+ A^{\#}$. So we can construct the following equation.

$$XA^H Y = YXA^{\#}. \tag{6.1}$$

Theorem 6.1. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a PI matrix if and only if Eq.*(6.1) *has at least one solution in $\tau_A{}^2 = \{(X, Y)|X, Y \in \tau_A\}$, where $\tau_A = \{A, A^{\#}, A^+, (A^+)^H, (A^{\#})^H\}$.*

Proof " ⇒ " Assume that A is *PI*, Then $A^H = A^+$, this infers $(X, Y) = (A, A^{\#})$ is a solution.
" ⇐ " I) If $Y = A$, then we have the following equation.

$$XA^H A = AXA^{\#}. \tag{6.2}$$

1) If $X = A$, then $AA^H A = A^2 A^{\#} = A$. Hence A is *PI*;
2) If $X = A^{\#}$, then $A^{\#} A^H A = AA^{\#} A^{\#} = A^{\#}$, this gives $A = A^2 A^{\#} = A^2 A^{\#} A^H A = AA^H A$. Hence A is *PI*;
3) If $X = A^+$, then $A^+ A^H A = AA^+ A^{\#} = A^{\#}$, which implies $A^{\#} = A^+ A^H A = A^+ A(A^+ A^H A) = A^+ AA^{\#}$. Hence A is *EP*, it follows that $X = A^+ = A^{\#}$. Thus A is *PI* by 2);
4) If $X = (A^+)^H$, then $(A^+)^H A^H A = A(A^+)^H A^{\#}$. e.g. $A = A(A^+)^H A^{\#}$, it follows from Corollary 2.2 that $A^{\#} A = (A^+)^H A^{\#}$. Hence $A = (A^+)^H A^{\#} A = (A^+)^H$, which infers A is *PI*;
5) If $X = (A^{\#})^H$, then $(A^{\#})^H A^H A = A(A^{\#})^H A^{\#}$, one gets

$$(E_n - AA^+)(A^{\#})^H A^H A = 0,$$

this gives $(E_n - AA^+)A^H = (E_n - AA^+)(A^{\#})^H A^H AA^+ A^H = 0$. Hence A is *EP*, which infers $X = (A^{\#})^H = (A^+)^H$. Thus is A is *PI* by 5);

II) If $Y = A^{\#}$, then we have the following equation.

$$XA^H A^{\#} = A^{\#} XA^{\#}. \tag{6.3}$$

6) If $X = A$, then $AA^HA^\# = A^\#AA^\# = A^\#$. Hence A is PI by 2);
7) If $X = A^\#$, then $A^\#A^HA^\# = A^\#A^\#A^\#$, this give $A = A^2(A^\#A^\#A^\#)A^2 = A^2(A^\#A^HA^\#)A^2 = AA^HA$. Hence A is PI;
8) If $X = A^+$, then $A^+A^HA^\# = A^\#A^+A^\# = (A^\#)^3$. This gives $(E_n - A^+A)(A^\#)^3 = 0$, and $(E_n - A^+A)A = (E_n - A^+A)(A^\#)^3A^4 = 0$. Hence A is EP and so $X = A^+ = A^\#$. Thus A is PI by 7);
9) If $X = (A^+)^H$, then $(A^+)^HA^HA^\# = A^\#(A^+)^HA^\#$, e.g. $A^\# = A^\#(A^+)^HA^\#$.
Hence $A = AA^\#A = AA^\#(A^+)^HA^\#A = (A^+)^H$, it follows that A is PI;
10) If $X = (A^\#)^H$, then $(A^\#)^HA^HA^\# = A^\#(A^\#)^HA^\#$.
One obtains $(E_n-A^+A)A^\#(A^\#)^HA^\# = (E_n-A^+A)(A^\#)^HA^HA^\# = 0$, and $(E_n-A^+A)A^\#(A^\#)^H = (E_n-A^+A)A^\#(A^\#)^HA^\#A^2A^+ = 0$.
Hence $(E_n - A^+A)A^\# = (E_n - A^+A)A^\#(A^\#)^HA^HA^+A = 0$, which implies A is EP. Thus $X = (A^\#)^H = (A^+)^H$, by 9), we have A is PI;

III) If $Y = A^+$, then we have the following equation.

$$XA^HA^+ = A^+XA^\#. \tag{6.4}$$

11) If $X = A$, then $AA^HA^+ = A^+AA^\#$, one has $AA^HA^+(E_n - A^+A) = A^+AA^\#(E_n - A^+A) = 0$. Pre-multiplying the last equality by $(A^\#)^HA^+$, one has $A^+ = A^+A^+A$. Hence A is EP, which implies $Y = A^+ = A^\#$. Thus A is PI by II);
12) If $X = A^\#$, then $A^\#A^HA^+ = A^+A^\#A^\#$, it gives $A^+A^\#A^\#(E_n - AA^+) = 0$, and $AA^\#(E_n - AA^+) = A^3A^+A^\#A^\#(E_n - AA^+) = 0$. Hence A is EP, which infers $Y = A^\#$. Thus A is PI by II);
13) If $X = A^+$, then $A^+A^HA^+ = A^+A^+A^\#$, so $A^+A^HA^+(E_n-A^+A) = 0$, and $A^HA^+(E_n-A^+A) = (A^\#A)^HAA^+A^HA^+(E_n-AA^+) = 0$. Hence $A^+(E_n - A^+A) = (A^\#)^HA^HA^+(E_n - A^+A) = 0$, it gives A is EP. Thus $Y = A^+ = A^\#$ and A is PI by II);
14) If $X = (A^+)^H$, then $(A^+)^HA^HA^+ = A^+(A^+)^HA^\#$, e.g. $AA^+A^+ = A^+(A^+)^HA^\#$, this gives $(E_n-AA^+)A^+(A^+)^HA^\# = 0$, and by Corollary 2.2 $(E_n - AA^+)A^+(A^+)^H = (E_n - AA^+)A^+(A^+)^HA^\#A = 0$.
Hence $(E_n - AA^+)A^+ = (E_n - AA^+)A^+(A^+)^HA^H = 0$, which implies A is EP and so $Y = A^+ = A^\#$. Thus A is PI by II);
15) If $X = (A^\#)^H$, then $(A^\#)^HA^HA^+ = A^+(A^\#)^HA^\#$, e.g. $A^+ = A^+(A^\#)^HA^\#$.
Hence A is EP and so $X = (A^\#)^H = (A^+)^H$. By 14), we have A is PI;

IV) If $Y = (A^+)^H$, then we have the following equation

$$XA^H(A^+)^H = (A^+)^HXA^\#.$$

that is,

$$XA^+A = (A^+)^HXA^\#. \tag{6.5}$$

16) If $X = A$, then $A = AA^+A = (A^+)^HAA^\# = (A^+)^H$. Hence A is PI;
17) If $X = A^\#$, then $A^\# = A^\#A^+A = (A^+)^HA^\#A^\#$, this gives $A = A^\#A^2 = (A^+)^HA^\#A^\#A^2 = (A^+)^H$. Hence A is PI;
18) If $X = A^+$, then $A^+A^+A = (A^+)^HA^+A^\#$, so $(E_n-AA^+)A^+A^+A = 0$ and $(E_n-AA^+)A^+ = (E_n-AA^+)A^+A^+A(AA^\#)^H = 0$. Hence A is EP, which implies $X = A^+ = A^\#$. Thus A is PI by 17);
19) If $X = (A^+)^H$, then $(A^+)^H = (A^+)^HA^+A = (A^+)^H(A^+)^HA^\#$, it follows $(A^+)^HA = (A^+)^H(A^+)^HA^\#A = (A^+)^H(A^+)^H$.
Hence $A^+A^2 = A^H(A^+)^HA = A^H(A^+)^H(A^+)^H = A^+A(A^+)^H$ and $A^2 = A(A^+)^H$. Thus A is PI;
20) If $X = (A^\#)^H$, then $(A^\#)^HA^+A = (A^+)^H(A^\#)^HA^\#$, this gives $(E_n - AA^+)(A^\#)^HA^+A = 0$, and $(E_n - AA^+)A^+ = (E_n - AA^+)(A^\#)^HA^+AA^HA^+ = 0$. Hence A is EP and so $X = (A^\#)^H = (A^+)^H$. Thus A is PI by 19);

V) If $Y = (A^\#)^H$, then we have the following equation.

$$XA^H(A^\#)^H = (A^\#)^HXA^\#. \tag{6.6}$$

21) If $X = A$, then $AA^H(A^\#)^H = (A^\#)^H AA^\#$, this gives

$$(E_n - AA^+)(A^\#)^H AA^\# = 0.$$

It follows that

$$(E_n - AA^+)(A^\#)^H = (E_n - AA^+)(A^\#)^H AA^\# A^2 A^+ = 0.$$

Hence A is EP, which implies $Y = (A^\#)^H = (A^+)^H$. By IV), we have A is PI;
22) If $X = A^\#$, then $A^\# A^H (A^\#)^H = (A^\#)^H A^\# A^\#$, it gives

$$A^\# A^H (A^\#)^H (E_n - A^+ A) = 0.$$

Pre-multiplying the last equality by $A^+A^+A^2$, one has $A^+(E_n - A^+A) = 0$. Hence A is EP and so $Y = (A^\#)^H = (A^+)^H$. By IV), A is PI;
23) If $X = A^+$, then $A^+A^H(A^\#)^H = (A^\#)^H A^+ A^\#$, e.g. $A^+ = (A^\#)^H A^+ A^\#$, one yields

$$A^+ = (A^\#)^H A^+ A^\# A^+ A = A^+ A^+ A.$$

Hence A is EP, this infers $Y = (A^+)^H$. Thus A is PI by IV);
24) If $X = (A^+)^H$, then $(A^+)^H A^H (A^\#)^H = (A^\#)^H (A^+)^H A^\#$.
Applying the involution to the equality, one has $A^\# AA^+ = (A^\#)^H A^+ A^\#$, this gives

$$A^\# AA^+ (E_n - A^+ A) = 0,$$

and

$$A^+(E_n - A^+A) = A^+ AA^\# AA^+ (E_n - A^+A) = 0.$$

Hence A is EP and so $Y = (A^+)^H$. Thus A is PI by IV);
25) If $X = (A^\#)^H$, then $(A^\#)^H = (A^\#)^H A^H (A^\#)^H = (A^\#)^H (A^\#)^H A^\#$, it follows that

$$A^+ = A^+ A^H (A^\#)^H = A^+ A^H (A^\#)^H (A^\#)^H A^\# = A^+ (A^\#)^H A^\#,$$

so $A^+(E_n - A^+A) = A^+(A^\#)^H A^\# (E_n - A^+A) = 0$, Hence A is EP, it follows that $A^+A = A^+(A^\#)^H A^\# A = A^+(A^+)^H A^\# A = A^+(A^+)^H$.Thus A is PI. ∎

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