Filomat 36:13 (2022), 4599–4615
https://doi.org/10.2298/FIL2213599B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A New High-Order Accurate Difference Scheme for the Benjamin-Bona-Mahoney-Burgers (BBMB) Equation

Khedidja Bayarassou[a], Khaled Omrani[b]

[a]*École Supérieure des Sciences et de la Technologie de Hammam Sousse, Université de Sousse, Tunisia.*
[b]*Laboratoire : Physique, Mathématique, Modélisation Quantique et Conception Mécanique, Institut Préparatoire aux Etudes d'Ingénieurs d'El Manar, Tunis 2092, Université de Tunis El Manar, Tunisia.*

Abstract. In this article, a high-order linearized difference scheme is presented for the periodic initial value problem of the Benjamin-Bona-Mahoney-Burgers (BBMB) equation. It is proved that the proposed scheme is uniquely solvable and unconditionally convergent, with convergence order of $O(h^4 + k^2)$ in the L^∞-norm. An application on the regularised long wave is thoroughly studied numerically. Furthermore, interaction of solitary waves with different amplitudes is shown. The three invariants of the motion are evaluated to determine the conservation properties of the system. Numerical experiments including the comparisons with other numerical methods are reported to demonstrate the accuracy and efficiency of our difference scheme and to confirm the theoretical analysis.

1. Introduction

Recent work emphasizes on the improvement of accuracy in many application of fluid mechanics [1, 2] and other areas of aero-acoustics [3]. In this paper, we treat the nonlinear Benjamin-Bona-Mahony-Burgers (BBMB) equation

$$u_t - \mu u_{xxt} - \alpha u_{xx} + u_x + \beta u u_x = 0, \tag{1}$$

which describes propagation of surface water waves in a channel [4]. In (1) the nonlinearity, the dissipative and the dispersive coefficients characterized by β, α and μ, respectively. For $\mu = 0$, (1) is the Burgers' equation which describes wave propagation in acoustics and hydrodynamics. For $\alpha = 0$, (1) is the BBM (or the Regularized Long Wave (RLW)) equation which incorporates the dispersive effects. Several numerical solution have been acheived in literature for the Burgers and the BBM equation by different methods, noting finite difference method and finite element method.
The propagation of acoustic waves needs to be accurately simulated over long time periods and far distances. Many different methods have been used to estimate BBMB equation. In [5, 6] the Benjamin-Bona-Mahony-Burgers (BBMB) and Generalized BBMB equations are solved by meshless methods. Fourth-order conservative compact difference scheme for the generalized BBM (GRLW) equation, the generalized symmetric regularized long-wave (SRLW) equation and generalized Rosenau-RLW equation are discussed

2020 *Mathematics Subject Classification*. 65M06, 65M12, 65M15.
Keywords. BBMB equation, high order linearized difference scheme, solvability, convergence, solitary waves.
Received: 11 September 2021; Revised: 02 June 2022; Accepted: 01 December 2022
Communicated by Marko Petković
Email addresses: bayarassoukhedidja@gmail.com (Khedidja Bayarassou), Khaled.Omrani@issatso.rnu.tn (Khaled Omrani)

respectively by Li in [7]-[10].
Throughout this article, we discuss the periodic boundary value problem for the BBMB equation, thus we seek a real-valued function $u(x,t)$, that satisfies:

$$u_t - u_{xxt} - \alpha u_{xx} + u_x + \beta u u_x = 0, \quad x \in (x_l, x_r), \quad t \in [0,T], \tag{2}$$

$$u(x,0) = u_0(x), \quad x \in (x_l, x_r), \tag{3}$$

$$u(x,t) = u(x + (x_r - x_l), t), \quad 0 < t \leq T, \tag{4}$$

where β is real constant while α is positive constant, we assume that the initial condition $u_0(x)$ is sufficiently smooth as required by the error analysis.
The key aspect of this paper is to use the fourth-order accurate difference scheme for the BBMB equation, and prove that the difference scheme is unconditionally stable and convergent with convergence order of $O(h^4 + k^2)$ in the discrete L^∞-norm.
This paper is organized as follows. In section 2, a high-order linearized difference scheme is derived. In section 3, the discrete dissipation law of the difference scheme and a priori estimates are also discussed. Section 4 is devoted to the solvability of the linearized difference scheme. The convergence and stability are proved in section 5. In the last section, some numerical examples are presented to prove the theoretical results.
Throughout this article, C denotes a generic positive constant which is independent of the discretization parameters h and k, but may have different values at different places.

2. Construction of linearized difference scheme

In this section, we propose a three-level linearized difference scheme for BBMB equation (2)-(4). For convenience, the following notations are used. For a positive integer N, let time-step $k = \frac{T}{N}$, $t_n = nk$, $n = 0, 1, \cdots, N$. Let space-step $h = \frac{x_r - x_l}{J}$, $x_j = x_l + jh$, $j = 0, \cdots, J$. Denote $Q_T = [x_l, x_r] \times [0, T]$ and

$$\mathbb{R}^J_{per} = \{V = (V_j)_{j\in\mathbb{Z}} \mid V_j \in \mathbb{R} \text{ and } V_{j+J} = V_j, \ j \in \mathbb{Z}\}.$$

For a function $V^n \in \mathbb{R}^J_{per}$, define the difference operators as:

$$(V^n_j)_x = \frac{V^n_{j+1} - V^n_j}{h}, \quad (V^n_j)_{\bar{x}} = \frac{V^n_j - V^n_{j-1}}{h}, \quad (V^n_j)_{\hat{x}} = \frac{V^n_{j+1} - V^n_{j-1}}{2h},$$

$$\bar{V}^n_j = \frac{V^{n+1}_j + V^{n-1}_j}{2}, \quad V^{n+\frac{1}{2}}_j = \frac{V^{n+1}_j + V^n_j}{2}, \quad (V^n_j)_{\hat{t}} = \frac{V^{n+1}_j - V^{n-1}_j}{2k}, \quad (V^n_j)_t = \frac{V^{n+1}_j - V^n_j}{k}.$$

For any function $V^n, W^n \in \mathbb{R}^J_{per}$, we introduce the discrete L^2 inner product in $\mathbb{R}^J_{per}$ as:

$$\langle V^n, W^n \rangle = h \sum_{j=1}^{J} V^n_j W^n_j.$$

The discrete L^2-norm $\|V^n\|$, the discrete semi-norm $\|V^n_x\|$, $\|V^n_{\hat{x}}\|$ and L^∞-norm are defined respectively as follows:

$$\|V^n\| = \sqrt{h \sum_{j=1}^{J} (V^n_j)^2}, \qquad \|V^n_x\| = \sqrt{h \sum_{j=1}^{J} [(V^n_j)_x]^2},$$

$$\|V^n_{\hat{x}}\| = \sqrt{h \sum_{j=1}^{J} [(V^n_j)_{\hat{x}}]^2}, \qquad \|V^n\|_\infty = \max_{1 \leqslant j \leqslant J} |V^n_j|.$$

Denote $H^m_{per}(\Omega)$ the periodic Sobolev space of order m.
To construct a high-order linearized difference scheme for solving BBMB equation, the following formulas should be introduced

$$\begin{aligned}
&\frac{1}{6}\Big[u^{'}(x_{j+1}) + 4u^{'}(x_j) + u^{'}(x_{j-1})\Big] = \frac{1}{2h}\Big[u(x_{j+1}) - u(x_{j-1})\Big] + O(h^4), \quad 1 \le j \le J,\\
&\frac{1}{12}\Big[u^{''}(x_{j+1}) + 10u^{''}(x_j) + u^{''}(x_{j-1})\Big] = \frac{1}{h^2}\Big[u(x_{j+1}) - 2u(x_j) + u(x_{j-1})\Big] + O(h^4), \quad 1 \le j \le J.
\end{aligned} \tag{5}$$

By virtue of (5), let

$$\begin{aligned}
A_1 U^n_j &= U^n_j + \frac{h^2}{6}(U^n_j)_{x\bar{x}} = \frac{1}{6}(U^n_{j-1} + 4U^n_j + U^n_{j+1}), \quad 1 \le j \le J,\\
A_2 U^n_j &= U^n_j + \frac{h^2}{12}(U^n_j)_{x\bar{x}} = \frac{1}{12}(U^n_{j-1} + 10U^n_j + U^n_{j+1}), \quad 1 \le j \le J.
\end{aligned}$$

We introduce the new functions y, z, v, w and ϕ, the equation (2) is reduced to an equivalent second-order system of differential equations as:

$$y_t = z + v - \alpha w, \tag{6}$$
$$v = u_x, \tag{7}$$
$$w = u_{xx}, \tag{8}$$
$$z = \beta\phi, \tag{9}$$
$$\phi = \frac{1}{2}(u^2)_x, \tag{10}$$
$$y = -u + w. \tag{11}$$

Based on notations above, we construct the following linearized difference scheme for solving the system (6)-(11)

$$(Y^n_j)_{\bar{t}} = \bar{Z}^n_j + \bar{V}^n_j - \alpha\bar{W}^n_j, \tag{12}$$
$$A_1\bar{V}^n_j = (\bar{U}^n_j)_{\hat{x}}, \tag{13}$$
$$A_2\bar{W}^n_j = (\bar{U}^n_j)_{x\bar{x}}, \tag{14}$$
$$A_1\bar{Z}^n_j = \beta\phi(U^n_j, \bar{U}^n_j), \tag{15}$$
$$\phi(U^n_j, \bar{U}^n_j) = \frac{1}{3}\Big((\bar{U}^n_j)_{\hat{x}} U^n_j + (U^n_j \bar{U}^n_j)_{\hat{x}}\Big), \tag{16}$$
$$Y^n_j = -U^n_j + W^n_j. \tag{17}$$

It follows from (12)-(17) that

$$A_1(U^n_j)_{\bar{t}} - A_1A_2^{-1}(U^n_j)_{x\bar{x}\bar{t}} - \alpha A_1A_2^{-1}(\bar{U}^n_j)_{x\bar{x}} + (\bar{U}^n_j)_{\hat{x}} + \beta\phi(U^n_j, \bar{U}^n_j) = 0. \tag{18}$$

Hence, by operating A_1^{-1} on the both sides of the above equality, we obtain

$$(U^n_j)_{\bar{t}} - A_2^{-1}(U^n_j)_{x\bar{x}\bar{t}} - \alpha A_2^{-1}(\bar{U}^n_j)_{x\bar{x}} + A_1^{-1}(\bar{U}^n_j)_{\hat{x}} + \beta A_1^{-1}\phi(U^n_j, \bar{U}^n_j) = 0. \tag{19}$$

We introduce now the vector and matrix notations as

$$\bar{\mathbf{U}}^n = (\bar{U}^n_1, \bar{U}^n_2, \cdots, \bar{U}^n_J)^T,$$

$$\mathbf{\Phi}(\mathbf{U}^n, \bar{\mathbf{U}}^n) = \Big(\phi(U^n_1, \bar{U}^n_1), \phi(U^n_2, \bar{U}^n_2), \cdots, \phi(U^n_J, \bar{U}^n_J)\Big)^T,$$

$$\mathbf{M}_1 = \frac{1}{6}\begin{pmatrix} 4 & 1 & 0 & \cdots & 1 \\ 1 & 4 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 4 & 1 \\ 1 & \cdots & 0 & 1 & 4 \end{pmatrix}_{J\times J}, \qquad \mathbf{M}_2 = \frac{1}{12}\begin{pmatrix} 10 & 1 & 0 & \cdots & 1 \\ 1 & 10 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 10 & 1 \\ 1 & \cdots & 0 & 1 & 10 \end{pmatrix}_{J\times J}$$

Where $\mathbf{M}_1$ and $\mathbf{M}_2$ the matrix form of A_1 and A_2 operators, respectively.
Note that $\mathbf{M}_1$ and $\mathbf{M}_2$ are two real symmetric positive definite matrices, there exist two real symmetric positive definite matrices $\mathbf{H}_1$ and $\mathbf{H}_2$, such that $\mathbf{H}_1 = \mathbf{M}_1^{-1}$, $\mathbf{H}_2 = \mathbf{M}_2^{-1}$. So, the difference scheme (19) can be rewritten as the following matrix form:

$$\mathbf{U}^n_{\hat{t}} - \mathbf{H}_2\mathbf{U}^n_{x\bar{x}\bar{t}} - \alpha\mathbf{H}_2\bar{\mathbf{U}}^n_{x\bar{x}} + \mathbf{H}_1\bar{\mathbf{U}}^n_{\hat{x}} + \beta\mathbf{H}_1\mathbf{\Phi}(\mathbf{U}^n, \bar{\mathbf{U}}^n) = 0, \quad n = 1, \cdots, N-1, \tag{20}$$

$$U^n_j = U^n_{j+J}, \quad j = 1, \cdots, J, \quad n = 0, \cdots, N, \tag{21}$$

$$U^0_j = u_0(x_j), \quad j = 1, \cdots, J. \tag{22}$$

As a scheme is a three level method, we can get $\mathbf{U}^1$ by the following two level nonlinear difference scheme:

$$\mathbf{U}^0_t - \mathbf{H}_2\mathbf{U}^0_{x\bar{x}t} - \alpha\mathbf{H}_2\mathbf{U}^{\frac{1}{2}}_{x\bar{x}} + \mathbf{H}_1\mathbf{U}^{\frac{1}{2}}_{\hat{x}} + \beta\mathbf{H}_1\mathbf{\Psi}(\mathbf{U}^{\frac{1}{2}}, \mathbf{U}^{\frac{1}{2}}) = 0, \tag{23}$$

where

$$\mathbf{\Psi}(\mathbf{U}^{\frac{1}{2}}, \mathbf{U}^{\frac{1}{2}}) = \Big(\psi(U^{\frac{1}{2}}_1, U^{\frac{1}{2}}_1), \psi(U^{\frac{1}{2}}_2, U^{\frac{1}{2}}_2), \cdots, \psi(U^{\frac{1}{2}}_J, U^{\frac{1}{2}}_J)\Big)^T,$$

with

$$\psi(U^{\frac{1}{2}}_j, U^{\frac{1}{2}}_j) = \frac{1}{3}\Big((U^{\frac{1}{2}}_j)_{\hat{x}}U^{\frac{1}{2}}_j + [(U^{\frac{1}{2}}_j)^2]_{\hat{x}}\Big), \qquad j = 1, \cdots, J.$$

In view of difference properties and the periodic boundary condition we obtain the following Lemmas

2.1. Some useful lemmas

Lemma 2.1 ([11]). *For any real-value symmetric positive definite matrix* $\mathbf{H}$ *and a periodic grid function* $\mathbf{U}^n$*, there is*

$$\langle \mathbf{H}\mathbf{U}^n_{x\bar{x}}, \mathbf{U}^n \rangle = -\|\mathbf{Q}\mathbf{U}^n_x\|^2,$$

where $\mathbf{Q} = Chol(\mathbf{H})$*, the Cholesky factorization.*

Lemma 2.2. *For* $\mathbf{H}_1 = \mathbf{Q}_1^T\mathbf{Q}_1$ *and* $\mathbf{H}_2 = \mathbf{Q}_2^T\mathbf{Q}_2$*, where* $\mathbf{Q}_1$ *and* $\mathbf{Q}_2$ *are two real upper triangular matrices, then*

$$\|\mathbf{U}^n\|^2 \leqslant \langle \mathbf{H}_1\mathbf{U}^n, \mathbf{U}^n \rangle = \|\mathbf{Q}_1\mathbf{U}^n\|^2 \leqslant 3\|\mathbf{U}^n\|^2, \tag{24}$$

$$\|\mathbf{U}^n\|^2 \leqslant \langle \mathbf{H}_2\mathbf{U}^n, \mathbf{U}^n \rangle = \|\mathbf{Q}_2\mathbf{U}^n\|^2 \leqslant \frac{3}{2}\|\mathbf{U}^n\|^2. \tag{25}$$

Proof. The eigenvalues of the matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ are respectively

$$\lambda_{\mathbf{M}_1,j} = \frac{1}{3}\Big(2 + cos(\frac{2\pi j}{J})\Big), \qquad \lambda_{\mathbf{M}_2,j} = \frac{1}{6}\Big(5 + cos(\frac{2\pi j}{J})\Big), \quad j = 1, \ldots, J.$$

Therefore

$$\frac{1}{3} \leqslant \lambda_{\mathbf{M}_1,j} \leqslant 1, \quad \frac{2}{3} \leqslant \lambda_{\mathbf{M}_2,j} \leqslant 1, \quad j = 1, \ldots, J.$$

Thus

$$1 \leqslant \lambda_{\mathbf{H}_1,j} \leqslant 3, \quad 1 \leqslant \lambda_{\mathbf{H}_2,j} \leqslant \frac{3}{2}, \quad j = 1, \ldots, J. \tag{26}$$

Noticing that

$$\langle \mathbf{H}_1\mathbf{U}^n, \mathbf{U}^n\rangle = \langle \mathbf{Q}_1\mathbf{U}^n, \mathbf{Q}_1\mathbf{U}^n\rangle = \|\mathbf{Q}_1\mathbf{U}^n\|^2,$$
$$\langle \mathbf{H}_2\mathbf{U}^n, \mathbf{U}^n\rangle = \langle \mathbf{Q}_2\mathbf{U}^n, \mathbf{Q}_2\mathbf{U}^n\rangle = \|\mathbf{Q}_2\mathbf{U}^n\|^2.$$

Therefore, from (26), the proof holds.

Lemma 2.3. *For* $\mathbf{U}^n \in \mathbb{R}^J_{per}$*, there is*

$$\langle \mathbf{H}_1\bar{\mathbf{U}}^n_{\hat{x}}, \bar{\mathbf{U}}^n\rangle = 0, \tag{27}$$
$$\langle \mathbf{H}_1\mathbf{\Phi}(\mathbf{U}^n, \bar{\mathbf{U}}^n), \bar{\mathbf{U}}^n\rangle = 0. \tag{28}$$

Proof. For any grid functions $\mathbf{U}^n, \mathbf{V}^n \in \mathbb{R}^J_{per}$, we have

$$\langle \mathbf{H}_1\bar{\mathbf{U}}^n_{\hat{x}}, \mathbf{V}^n\rangle = \langle \bar{\mathbf{U}}^n_{\hat{x}}, \mathbf{H}_1\mathbf{V}^n\rangle = -\langle \bar{\mathbf{U}}^n, \mathbf{H}_1\mathbf{V}^n_{\hat{x}}\rangle.$$

Obviously, we have in particular

$$\langle \mathbf{H}_1\bar{\mathbf{U}}^n_{\hat{x}}, \bar{\mathbf{U}}^n\rangle = 0.$$

For $\mathbf{U}^n \in \mathbb{R}^J_{per}$, we have

$$\begin{aligned}\langle \mathbf{H}_1\mathbf{\Phi}(\mathbf{U}^n, \bar{\mathbf{U}}^n), \bar{\mathbf{U}}^n\rangle &= \frac{1}{3}\langle \mathbf{H}_1\bar{\mathbf{U}}^n_{\hat{x}}\mathbf{U}^n, \bar{\mathbf{U}}^n\rangle + \frac{1}{3}\langle \mathbf{H}_1(\mathbf{U}^n\bar{\mathbf{U}}^n)_{\hat{x}}, \bar{\mathbf{U}}^n\rangle\\ &= \frac{1}{3}\langle \mathbf{Q}_1\bar{\mathbf{U}}^n_{\hat{x}}, \mathbf{Q}_1(\mathbf{U}^n\bar{\mathbf{U}}^n)\rangle - \frac{1}{3}\langle \mathbf{Q}_1(\mathbf{U}^n\bar{\mathbf{U}}^n), \mathbf{Q}_1\bar{\mathbf{U}}^n_{\hat{x}}\rangle\\ &= 0.\end{aligned}$$

Remark. Similarly, we can prove:

$$\langle \mathbf{H}_1\mathbf{U}^{\frac{1}{2}}_{\hat{x}}, \mathbf{U}^{\frac{1}{2}}\rangle = 0, \tag{29}$$
$$\langle \mathbf{H}_1\mathbf{\Psi}(\mathbf{U}^{\frac{1}{2}}, \mathbf{U}^{\frac{1}{2}}), \mathbf{U}^{\frac{1}{2}}\rangle = 0. \tag{30}$$

3. Analysis of the linearized difference scheme

3.1. Discrete dissipative law

Bellow, we cite the dissipation energy of the linearized difference scheme.

Theorem 3.1. *The difference scheme (20)-(23) satisfies:*

$$\varepsilon^n \leqslant \varepsilon^{n-1} \leqslant \cdots \leqslant \varepsilon^0, \tag{31}$$

where

$$\varepsilon^n = \frac{1}{2}(\|\mathbf{U}^{n+1}\|^2 + \|\mathbf{U}^n\|^2) + \frac{1}{2}(\|\mathbf{Q}_2\mathbf{U}^{n+1}_x\|^2 + \|\mathbf{Q}_2\mathbf{U}^n_x\|^2).$$

Proof. Computing the inner product of (20) with $2\bar{\mathbf{U}}^n$ and using Lemma 2.1, we obtain

$$\begin{aligned}&\frac{1}{2k}(\|\mathbf{U}^{n+1}\|^2 - \|\mathbf{U}^{n-1}\|^2) + \frac{1}{2k}(\|\mathbf{Q}_2\mathbf{U}^{n+1}_x\|^2 - \|\mathbf{Q}_2\mathbf{U}^{n-1}_x\|^2) + 2\langle \mathbf{H}_1\bar{\mathbf{U}}^n_{\hat{x}}, \bar{\mathbf{U}}^n\rangle\\ &\qquad + 2\alpha\|\mathbf{Q}_2\bar{\mathbf{U}}^n_x\|^2 + 2\langle \mathbf{H}_1\mathbf{\Phi}(\mathbf{U}^n, \bar{\mathbf{U}}^n), \bar{\mathbf{U}}^n\rangle = 0.\end{aligned} \tag{32}$$

It follows from (27) and (28) that

$$\frac{1}{2k}(\|\mathbf{U}^{n+1}\|^2 - \|\mathbf{U}^{n-1}\|^2) + \frac{1}{2k}(\|\mathbf{Q}_2\mathbf{U}^{n+1}_x\|^2 - \|\mathbf{Q}_2\mathbf{U}^{n-1}_x\|^2) = -2\alpha\|\mathbf{Q}_2\bar{\mathbf{U}}^n_x\|^2.$$

As $\alpha > 0$, we have

$$\frac{1}{2}(\|\mathbf{U}^{n+1}\|^2 - \|\mathbf{U}^{n-1}\|^2) + \frac{1}{2}(\|\mathbf{Q}_2\mathbf{U}_x^{n+1}\|^2 - \|\mathbf{Q}_2\mathbf{U}_x^{n-1}\|^2) \leqslant 0.$$

Thus

$$\varepsilon^n \leqslant \varepsilon^{n-1} \leqslant \cdots \leqslant \varepsilon^0. \tag{33}$$

Taking now the inner product of (23) with $\mathbf{U}^{\frac{1}{2}}$ and using Lemma 2.1, (29) and (30), yields to

$$\frac{1}{2k}(\|\mathbf{U}^1\|^2 - \|\mathbf{U}^0\|^2) + \frac{1}{2k}(\|\mathbf{Q}_2\mathbf{U}_x^1\|^2 - \|\mathbf{Q}_2\mathbf{U}_x^0\|^2) = -\alpha\|\mathbf{Q}_2\mathbf{U}_x^{\frac{1}{2}}\|^2 \leqslant 0.$$

Therefore

$$\frac{1}{2}(\|\mathbf{U}^1\|^2 + \|\mathbf{Q}_2\mathbf{U}_x^1\|^2) - \frac{1}{2}(\|\mathbf{U}^0\|^2 + \|\mathbf{Q}_2\mathbf{U}_x^0\|^2) \leqslant 0. \tag{34}$$

Thus (33) can be written

$$\varepsilon^n \leqslant \cdots \leqslant \varepsilon^0 = \|\mathbf{U}^0\|^2 + \|\mathbf{Q}_2\mathbf{U}_x^0\|^2.$$

This completes the proof.

3.2. *A priori estimates*

We give the following a priori estimates.

Lemma 3.2 (Discrete Sobolev's inequality [12]). *There exist two constants C_1 and C_2 such that:*

$$\|\mathbf{U}^n\|_\infty \leqslant C_1\|\mathbf{U}^n\| + C_2\|\mathbf{U}_x^n\|. \tag{35}$$

Theorem 3.3. *Assume that $u_0 \in H^1_{per}(\Omega)$. The solution of the difference scheme (20)-(23) satisfies a priori estimates as follows:*

$$\|\mathbf{U}^n\| \leqslant C, \qquad \|\mathbf{U}_x^n\| \leqslant C, \qquad \|\mathbf{U}^n\|_\infty \leqslant C, \tag{36}$$

where C is a positive constant independent of both h and k.

Proof. Using (24), we find

$$\|\mathbf{U}^n\|^2 + \|\mathbf{U}_x^n\|^2 \leqslant \|\mathbf{U}^n\|^2 + \|\mathbf{Q}_2\mathbf{U}_x^n\|^2 \leqslant 2\varepsilon^n = \|\mathbf{U}^{n+1}\|^2 + \|\mathbf{Q}_2\mathbf{U}_x^{n+1}\|^2 + \|\mathbf{U}^n\|^2 + \|\mathbf{Q}_2\mathbf{U}_x^n\|^2.$$

It follows from Theorem 3.1 and (24) that

$$\|\mathbf{U}^n\|^2 + \|\mathbf{U}_x^n\|^2 \leqslant 2\varepsilon^0 = 2(\|\mathbf{U}^0\|^2 + \|\mathbf{Q}_2\mathbf{U}_x^0\|^2) \leqslant C(\|\mathbf{U}^0\|^2 + \|\mathbf{U}_x^0\|^2).$$

Hence $u_0 \in H^1_{per}(\Omega)$, we obtain

$$\|\mathbf{U}^n\|^2 + \|\mathbf{U}_x^n\|^2 \leqslant C.$$

Thus

$$\|\mathbf{U}^n\| \leqslant C, \qquad \|\mathbf{U}_x^n\| \leqslant C.$$

It follows from Lemma 3.2 that

$$\|\mathbf{U}^n\|_\infty \leqslant C.$$

This completes the proof.

4. Solvability of the linearized difference scheme

Theorem 4.1. *The linearized difference scheme (20)-(23) is uniquely solvable.*

Proof. By mathematical induction, it is obvious that $\mathbf{U}^0$ is uniquely solvable from (22), $\mathbf{U}^1$ can be uniquely determined by a fourth order method (23). Now, we assume that $\mathbf{U}^0, \cdots, \mathbf{U}^n$, $(n \leqslant N-1)$ are uniquely solvable. It follows from (20) that

$$\frac{1}{2k}\mathbf{U}^{n+1} - \frac{1}{2k}\mathbf{H}_2\mathbf{U}^{n+1}_{x\bar{x}} - \alpha\mathbf{H}_2\mathbf{U}^{n+1}_{x\bar{x}} + \mathbf{H}_1\mathbf{U}^{n+1}_{\hat{x}} + \beta\mathbf{H}_1\mathbf{\Phi}(\mathbf{U}^n, \mathbf{U}^{n+1}) = 0. \tag{37}$$

Computing the inner product of (37) with $\mathbf{U}^{n+1}$ and applying Lemma 2.1 and Lemma 2.3 , we find

$$\|\mathbf{U}^{n+1}\|^2 + \|\mathbf{Q}_2\mathbf{U}^{n+1}_x\|^2 \leqslant 0. \tag{38}$$

This yields

$$\mathbf{U}^{n+1} = 0.$$

That is, the system (37) determines $\mathbf{U}^{n+1}$ uniquely. This completes the proof.

5. Convergence and stability

In this section, we will prove the convergence of the linearized difference scheme.

Lemma 5.1 (Discrete Gronwall inequality [13]). *Assume $\{G^n / n \geqslant 0\}$ is non-negative sequences and satisfies*

$$G^0 \leqslant A, \qquad G^n \leqslant A + Bk\sum_{i=0}^{n-1} G^i, \qquad n = 1, 2, \cdots,$$

where A and B are non negative constants. Then G satisfies:

$$G^n \leqslant Ae^{Bnk}, \qquad n = 0, 1, 2, \cdots.$$

Lemma 5.2 ([14]). *For any discrete function $\mathbf{U}^n \in \mathbb{R}^J_{per}$, we have*

$$\|\mathbf{U}^n_{\hat{x}}\| \leqslant \|\mathbf{U}^n_x\|.$$

Lemma 5.3. *For any grid function $\mathbf{U}^n \in \mathbb{R}^J_{per}$, we have*

$$\|\mathbf{H}_1\mathbf{U}^n_{\hat{x}}\| \leqslant C\|\mathbf{U}^n_x\|.$$

Proof. It follows from Lemma 6 and the definition of the matrix $\mathbf{H}_1$ that

$$\|\mathbf{H}_1\mathbf{U}^n_{\hat{x}}\| \leq \|\mathbf{H}_1\|_\infty \cdot \|\mathbf{U}^n_{\hat{x}}\| \leq C\|\mathbf{U}^n_x\|.$$

Theorem 5.4. *Suppose that the solution of problem (2)-(4) $u(x,t) \in C^{6,3}(Q_T)$. Then the solution of the difference scheme (20)-(23) converges to the solution of the problem (2)-(4) in the discrete L^∞-norm and the rate of convergence is $O(h^4 + k^2)$ when h and k are small, i.e.,*

$$\|u^n - U^n\|_\infty \leqslant C(h^4 + k^2), \qquad 0 \leqslant n \leqslant N,$$

where C is a positive constant independent of k and h.

Proof. Let $\mathbf{r}^n = (r_1^n, \cdots, r_J^n)^T \in \mathbb{R}^J_{per}$ be the consistency error of linearized difference scheme (20)-(23)

$$\mathbf{r}^n = \mathbf{u}^n_{\hat{t}} - \mathbf{H}_2\mathbf{u}^n_{x\bar{x}\hat{t}} - \alpha\mathbf{H}_2\bar{\mathbf{u}}^n_{x\bar{x}} + \mathbf{H}_1\bar{\mathbf{u}}^n_{\hat{x}} + \beta\mathbf{H}_1\mathbf{\Phi}(\mathbf{u}^n, \bar{\mathbf{u}}^n), \qquad n = 1, \cdots, N-1, \tag{39}$$

$$\mathbf{r}^0 = \mathbf{u}^0_t - \mathbf{H}_2\mathbf{u}^0_{x\bar{x}t} - \alpha\mathbf{H}_2\mathbf{u}^{\frac{1}{2}}_{x\bar{x}} + \mathbf{H}_1\mathbf{u}^{\frac{1}{2}}_{\hat{x}} + \beta\mathbf{H}_1\mathbf{\Psi}(\mathbf{u}^{\frac{1}{2}}, \mathbf{u}^{\frac{1}{2}}), \tag{40}$$

$$u^n_j = u^n_{j+J}, \quad j = 1, \cdots, J, \quad n = 1, \cdots, N, \tag{41}$$

$$u^0_j = u_0(x_j), \quad j = 1, \cdots, J. \tag{42}$$

According to Taylor's expansion, it follows that

$$\max_{j,n} |\mathbf{r}^n_j| \leqslant C(h^4 + k^2), \quad j = 1, \cdots, J, \quad n = 0, \cdots, N. \tag{43}$$

Letting $\mathbf{e}^n = \mathbf{u}^n - \mathbf{U}^n$ and substracting (39)-(42) from (20)-(23), we obtain

$$\mathbf{r}^n = \mathbf{e}^n_{\hat{t}} - \mathbf{H}_2\mathbf{e}^n_{x\bar{x}\hat{t}} - \alpha\mathbf{H}_2\bar{\mathbf{e}}^n_{x\bar{x}} + \mathbf{H}_1\bar{\mathbf{e}}^n_{\hat{x}} + \beta(\mathbf{H}_1\mathbf{\Phi}(\mathbf{u}^n, \bar{\mathbf{u}}^n) - \mathbf{H}_1\mathbf{\Phi}(\mathbf{U}^n, \bar{\mathbf{U}}^n)), \quad n = 1, \cdots, N-1, \tag{44}$$

$$\mathbf{r}^0 = \mathbf{e}^0_t - \mathbf{H}_2\mathbf{e}^0_{x\bar{x}t} - \alpha\mathbf{H}_2\mathbf{e}^{\frac{1}{2}}_{x\bar{x}} + \mathbf{H}_1\mathbf{e}^{\frac{1}{2}}_{\hat{x}} + \beta(\mathbf{H}_1\mathbf{\Psi}(\mathbf{u}^{\frac{1}{2}}, \mathbf{u}^{\frac{1}{2}}) - \mathbf{H}_1\mathbf{\Psi}(\mathbf{U}^{\frac{1}{2}}, \mathbf{U}^{\frac{1}{2}})), \tag{45}$$

$$e^n_j = e^n_{j+J}, \quad j = 1, \cdots, J, \quad n = 0, \cdots, N-1, \tag{46}$$

$$e^0_j = 0, \quad j = 1, \cdots, J. \tag{47}$$

We will prove that

$$\|\mathbf{e}^n\|_\infty \leqslant C(h^4 + k^2), \qquad n = 0, \cdots, N.$$

Taking the inner product of (44) with $2\bar{\mathbf{e}}^n$ and using Lemma 2.1 and (27), we have

$$\|\mathbf{e}^n\|^2_{\hat{t}} + \|\mathbf{Q}_2\mathbf{e}^n_x\|^2_{\hat{t}} + 2\alpha\|\mathbf{Q}_2\bar{\mathbf{e}}^n_x\|^2 + 2\beta\langle\mathbf{H}_1\mathbf{\Phi}(\mathbf{u}^n, \bar{\mathbf{u}}^n) - \mathbf{H}_1\mathbf{\Phi}(\mathbf{U}^n, \bar{\mathbf{U}}^n), \bar{\mathbf{e}}^n\rangle = 2\langle\mathbf{r}^n, \bar{\mathbf{e}}^n\rangle. \tag{48}$$

Noting that

$$3\langle\mathbf{H}_1\mathbf{\Phi}(\mathbf{u}^n, \bar{\mathbf{u}}^n) - \mathbf{H}_1\mathbf{\Phi}(\mathbf{U}^n, \bar{\mathbf{U}}^n), \bar{\mathbf{e}}^n\rangle = \langle\mathbf{H}_1\bar{\mathbf{e}}^n_{\hat{x}}\mathbf{U}^n, \bar{\mathbf{e}}^n\rangle + \langle\mathbf{H}_1\bar{\mathbf{u}}^n_{\hat{x}}\mathbf{e}^n, \bar{\mathbf{e}}^n\rangle - \langle\mathbf{u}^n\bar{\mathbf{e}}^n, \mathbf{H}_1\bar{\mathbf{e}}^n_{\hat{x}}\rangle - \langle\mathbf{e}^n\bar{\mathbf{U}}^n, \mathbf{H}_1\bar{\mathbf{e}}^n_{\hat{x}}\rangle.$$

It follows from the regularity assumption of the solution u, Theorem 3.3, Lemma 5.2 and Lemma 5.3 that

$$\langle\mathbf{H}_1\mathbf{\Phi}(\mathbf{u}^n, \bar{\mathbf{u}}^n) - \mathbf{H}_1\mathbf{\Phi}(\mathbf{U}^n, \bar{\mathbf{U}}^n), \bar{\mathbf{e}}^n\rangle \leqslant C(\|\mathbf{e}^{n+1}\|^2 + \|\mathbf{e}^n\|^2 + \|\mathbf{e}^{n-1}\|^2 + \|\mathbf{e}^{n+1}_x\|^2 + \|\mathbf{e}^{n-1}_x\|^2). \tag{49}$$

Using (24), we obtain

$$\begin{aligned}\langle\mathbf{H}_1\mathbf{\Phi}(\mathbf{u}^n, \bar{\mathbf{u}}^n) - \mathbf{H}_1\mathbf{\Phi}(\mathbf{U}^n, \bar{\mathbf{U}}^n), \bar{\mathbf{e}}^n\rangle \leqslant & C\Big(\|\mathbf{e}^{n+1}\|^2 + \|\mathbf{e}^n\|^2 + \|\mathbf{e}^{n-1}\|^2 + \|\mathbf{Q}_2\mathbf{e}^{n-1}_x\|^2 \\ & + \|\mathbf{Q}_2\mathbf{e}^n_x\|^2 + \|\mathbf{Q}_2\mathbf{e}^{n+1}_x\|^2\Big).\end{aligned} \tag{50}$$

Substituting (50) into (48), we obtain

$$\begin{aligned}\frac{1}{2k}(\|\mathbf{e}^{n+1}\|^2 - \|\mathbf{e}^{n-1}\|^2) + \frac{1}{2k}(\|\mathbf{Q}_2\mathbf{e}^{n+1}_x\|^2 - \|\mathbf{Q}_2\mathbf{e}^{n-1}_x\|^2) \leqslant \|\mathbf{r}^n\|^2 + C\Big[\|\mathbf{e}^{n+1}\|^2 + \|\mathbf{e}^{n-1}\|^2 \\ + \|\mathbf{e}^n\|^2 + \|\mathbf{Q}_2\mathbf{e}^{n+1}_x\|^2 + \|\mathbf{Q}_2\mathbf{e}^n_x\|^2 + \|\mathbf{Q}_2\mathbf{e}^{n-1}_x\|^2\Big].\end{aligned} \tag{51}$$

Let $A^n = \frac{1}{2}(\|\mathbf{e}^{n+1}\|^2 + \|\mathbf{e}^n\|^2) + \frac{1}{2}(\|\mathbf{Q}_2\mathbf{e}^{n+1}_x\|^2 + \|\mathbf{Q}_2\mathbf{e}^n_x\|^2)$. Therefore, (51) can be written as follows

$$A^n - A^{n-1} \leq k\|\mathbf{r}^n\|^2 + Ck(A^n + A^{n-1}).$$

Summing up the above inequality from 1 to n, we obtain

$$A^n \leqslant A^0 + k\sum_{\ell=1}^{n}\|\mathbf{r}^\ell\|^2 + Ck\sum_{\ell=0}^{n}A^\ell. \tag{52}$$

Using (43), we have

$$k\sum_{\ell=1}^{n}\|\mathbf{r}^{\ell}\|^2 \leqslant nk\max_{1\leqslant\ell\leqslant n}\|\mathbf{r}^{\ell}\|^2 \leqslant CT(h^4+k^2)^2. \tag{53}$$

Since $e_j^0 = 0$, it is easy to know from (45) and (43) that

$$A^0 \leqslant C(h^4+k^2)^2. \tag{54}$$

Substituting (53)-(54) into (52), we obtain

$$A^n \leqslant C(h^4+k^2)^2 + Ck\sum_{\ell=0}^{n}A^{\ell}.$$

Hence,

$$(1-Ck)A^n \leqslant C(h^4+k^2)^2 + Ck\sum_{\ell=0}^{n-1}A^{\ell}.$$

For k sufficiently small such that $(1-Ck)>0$, we have

$$A^n \leqslant C(h^4+k^2)^2 + Ck\sum_{\ell=0}^{n-1}A^{\ell}.$$

According to Lemma 5.1, we have

$$A^n \leqslant C(h^4+k^2)^2 e^{CT} \leqslant C(h^4+k^2)^2.$$

Consequently, we arrive at

$$\|\mathbf{e}^n\| \leqslant C(h^4+k^2), \qquad \|\mathbf{Q}_2\mathbf{e}_x^n\| \leqslant C(h^4+k^2). \tag{55}$$

It follows from (24) that

$$\|\mathbf{e}^n\| \leqslant C(h^4+k^2), \qquad \|\mathbf{e}_x^n\| \leqslant C(h^4+k^2). \tag{56}$$

Applying Lemma 3.2, we obtain

$$\|\mathbf{e}^n\|_\infty \leqslant C(h^4+k^2), \qquad n=0,\cdots,N.$$

This completes the proof.
Similarly, we can prove the stability of the difference solution.

Theorem 5.5. *Under the condition of Theorem 5.4, the solution of the difference scheme (20)-(23) is stable for initial data by the* $\|.\|_\infty$ *norm.*

6. Numerical Experiments

In this section, we apply the proposed numerical scheme to solve the BBMB equation and test their numerical accurate in order to validate our theoretical results that have been presented above. A comparison of our scheme with other existing studies are made. We show also some result about the BBM equation. All computations were obtained by using Matlab.

6.1. Dissipation of Energy

Example 1

We consider the following periodic initial value problem of BBMB equation

$$u_t - u_{xxt} - u_{xx} + u_x + uu_x = 0, \quad x \in [0,1], \quad t \in [0,1], \tag{57}$$

$$u(x,0) = \sin(2\pi x), \quad x \in [0,1]. \tag{58}$$

We plot the dissipation of energy with $h = k = 1/50$. Figure 1 confirms the result found in Theorem 3.1.

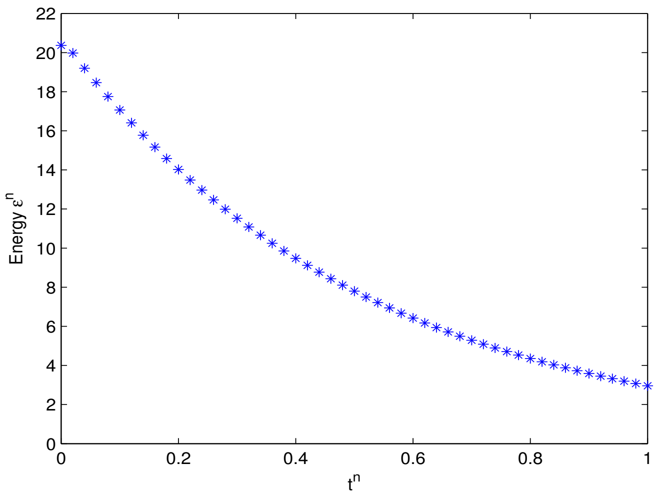

Figure 1: Profil of the discrete energy.

Example 2

To further illustrate the dissipation of energy, we consider the equation (2) with $\alpha = 1$ and $\beta = 12$ as follows

$$u_t - u_{xxt} - u_{xx} + u_x + 12uu_x = 0, \quad x \in [-1,1], \quad t \in [0,1], \tag{59}$$

we adopt the exact solution mentioned in [15]. Accordingly, we take

$$u(x,0) = \frac{-23}{120} - \frac{1}{5}\tanh(x) + \frac{1}{10}\tanh^2(x), \quad x \in [-1,1]. \tag{60}$$

We compute the numerical solution of the problem (59)-(60) by the difference scheme (20)-(23) with $h = \frac{2}{J}$ and $k = \frac{1}{N}$. In Table 1, we give some numerical values of the discrete energy at various times t^n for $k = 0.02$ and $h = 0.02$.

t^n	$\mathcal{E}^n$
0.04	10.718041129435278
0.1	9.682853076162946
0.2	8.199629744154645
0.5	5.106264138262407
0.6	4.401404773312493
1	2.563745956306620

Table 1: Values of discrete energy $\mathcal{E}^n$ at different time t^n.

From Table 1, we can see the dissipation of energy of the numerical solution for (59)-(60). This also supports the result found in Theorem 3.1.

6.2. Error estimates and order of convergence

Consider the following periodic initial value problem of the BBMB equation

$$u_t - u_{xxt} - u_{xx} + u_x + uu_x = f(x,t), \quad x \in [0,1], \quad t \in [0,1], \tag{61}$$
$$u(x,0) = \sin(2\pi x), \quad x \in [0,1], \tag{62}$$

where

$$f(x,t) = \exp(-t)[2\pi\cos(2\pi x) - \sin(2\pi x) + \pi\exp(-t)\sin(4\pi x)].$$

The exact solution is

$$u(x,t) = \exp(-t)\sin(2\pi x).$$

We represent the numerical solution of the BBMB equation for $h = 1/100$ and $k = 1/2$.
Figure 2 shows the behavior of the BBMB equation obtained from $t = 0$ to $t = 5$. Obviously, we can see the accuracy of the numerical solutions which indicates that the method is well suited and reaches a balance with the solution of the BBMB equation.

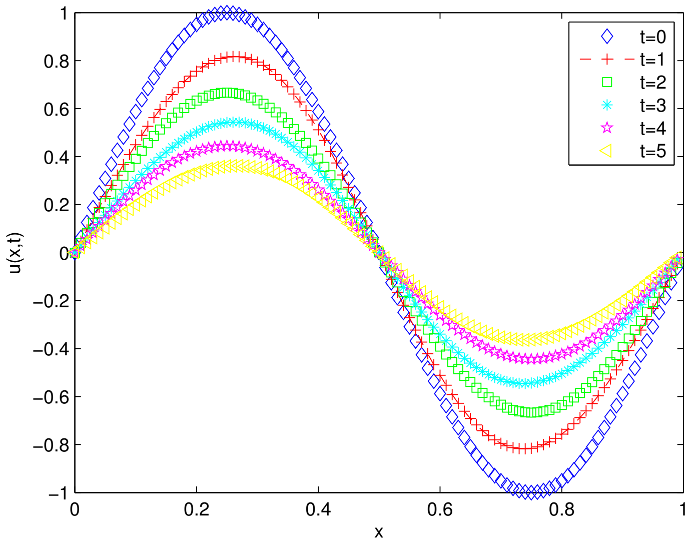

Figure 2: The numerical solution obtained by the difference scheme.

In order to compare the numerical scheme more qualitatively, we compute the maximum norm error of the numerical solution

$$e_\infty(h,k) = \max_{0\leqslant n\leqslant N} \|u^n - U^n\|_\infty.$$

The convergence order in temporal and spatial directions are defined as

$$Order_k = log_2\Big(\frac{e_\infty(h,2k)}{e_\infty(h,k)}\Big), \qquad Order_h = log_2\Big(\frac{e_\infty(2h,k)}{e_\infty(h,k)}\Big),$$

when k and h are sufficiently small, respectively.
Tables 2 and 3 present some maximum norm errors and the corresponding convergence orders of our difference scheme. We can see that the results obtained confirm the theoretical order of convergence found in Theorem 5.4.

k	$e_\infty(h,k)$	$Order_k$
1/8	1.9772×10^{-03}	–
1/16	4.8806×10^{-04}	2.0183
1/32	1.2130×10^{-04}	2.0084
1/64	3.0239×10^{-05}	2.0042

Table 2: The maximum norm errors and temporal convergence order with various k when $h = 0.01$.

h	$e_\infty(h,k)$	$Order_h$
1/8	2.5564×10^{-04}	–
1/16	1.3626×10^{-05}	4.2296
1/32	8.4018×10^{-07}	4.0196
1/64	5.1847×10^{-08}	4.0184

Table 3: The maximum norm errors and spatial convergence order with various h and $k = h^2$.

6.3. Comparison with the scheme proposed in [16]

We conduct a comparison between our difference scheme and the numerical method in [16] with fixed k and various h in Table 4. In this Table, it is clear that the results obtained by the linearized difference scheme is more accurate and robust.

h	Present scheme	CPU time	Scheme [16]	CPU time
1/10	9.6398×10^{-05}	$1.369051s$	1.0584×10^{-02}	$4.313302s$
1/20	5.3935×10^{-06}	$2.238243s$	2.7591×10^{-03}	$6.812266s$
1/40	3.7614×10^{-07}	$3.226216s$	7.0556×10^{-04}	$8.787010s$
1/80	2.2318×10^{-08}	$4.150066s$	1.7845×10^{-04}	$10.056571s$

Table 4: Comparison of error estimates in the maximum norm with $k = 0.001$.

6.4. A comparison through BBM Equation ($\alpha = 0$ in (1))

In this section, we have studied the BBM equation where three problem will be shown: motion of single solitary wave, the interaction of two positive solitary waves and the undular bore.

6.4.1. Motion of single solitary wave

We consider the BBM equation with the boundary conditions $u \to 0$ as $x \to \pm\infty$

$$u_t - u_{xxt} + u_x + uu_x = 0, \quad x \in [-40, 60], \quad t \in [0, T], \tag{63}$$

and the initial data is

$$u(x,0) = 3c\, sech^2\Big(k_0[x - x_0]\Big),$$

the exact solution is

$$u(x,t) = 3c\, sech^2\Big(k_0[x - vt - x_0]\Big),$$

with amplitude $3c$ where c, x_0 are arbitrary constants, $v = 1 + c$ is the wave velocity and $k_0 = \frac{1}{2}\sqrt{\frac{c}{(1+c)}}$.
To allow comparison with the previous method, parameters are taken as $c = 0.1,\ x_0 = 0$.
We discretize the problem (63) by the following finite difference scheme:

$$\begin{aligned}
&\mathbf{U}^n_{\hat{t}} - \mathbf{H}_2\mathbf{U}^n_{x\bar{x}\hat{t}} + \mathbf{H}_1\bar{\mathbf{U}}^n_{\hat{x}} + \beta\mathbf{H}_1\mathbf{\Phi}(\mathbf{U}^n, \bar{\mathbf{U}}^n) = 0, \quad n = 1, \cdots, N-1,\\
&\mathbf{U}^0_j = u_0(x_j), \quad j = 1, \ldots, J.
\end{aligned}$$

We examined our results by calculating the following three conservative laws: mass, momentum, and energy which can be expressed as:

$$I_1 = \int_a^b u dx \simeq h \sum_{j=1}^{J} U_j^n,$$

$$I_2 = \int_a^b (u^2 + \mu u_x^2) dx \simeq h \sum_{j=1}^{J} \Big[(U_j^n)^2 + \mu (U_j^n)_{\hat{x}}^2 \Big],$$

$$I_3 = \int_a^b (3u^2 + u^3) dx \simeq h \sum_{j=1}^{J} \Big[3(U_j^n)^2 + (U_j^n)^3 \Big].$$

In Table 5, we display the invariants and we treat the error estimates in the maximum norm for $k = 0.001,\ h = 0.5$, it shows that the results of our scheme is more accurate. A comparison of invariants obtained by the present method for $h = k = 0.2$ and some existing results ([18] for $h = k = 0.2$, [19] for $h = 0.125, k = 0.1$ and [20] for $h = 3, k = 0.01$) is listed in Table 6 at time $t = 20$.

Time	method	I_1	I_2	I_3	L^∞
0		3.97992	0.81038	2.57901	−
1	Present method	3.97993	0.81038	2.57901	3.41×10^{-07}
	[17]	3.97993	0.81046	2.57901	4.57×10^{-07}
2	Present method	3.97994	0.81038	2.57901	6.96×10^{-07}
	[17]	3.97994	0.81046	2.57901	3.79×10^{-05}
3	Present method	3.97994	0.81038	2.57901	1.05×10^{-06}
	[17]	3.97994	0.81046	2.57901	3.80×10^{-05}
4	Present method	3.97994	0.81038	2.57901	1.41×10^{-06}
	[17]	3.97995	0.81046	2.57901	3.80×10^{-05}

Table 5: Comparison of error estimates in the maximum norm and the conservative laws.

method	I_1	I_2	I_3
Analytical	3.979950	0.810462	2.579007
Present method	3.979951	0.810459	2.579006
[18]	3.979942	0.810462	2.579006
[19]	3.98203	0.810467	2.57302
[20]	3.990464	0.823457	2.673990

Table 6: Comparison of invariants for $t = 20$.

Figure 3 and 4 represent the profiles of single solitary waves at $T = 0$, $T = 25$ and $h = 0.5,\ k = 0.1$, for $c = 0.1$ and $c = 0.03$.

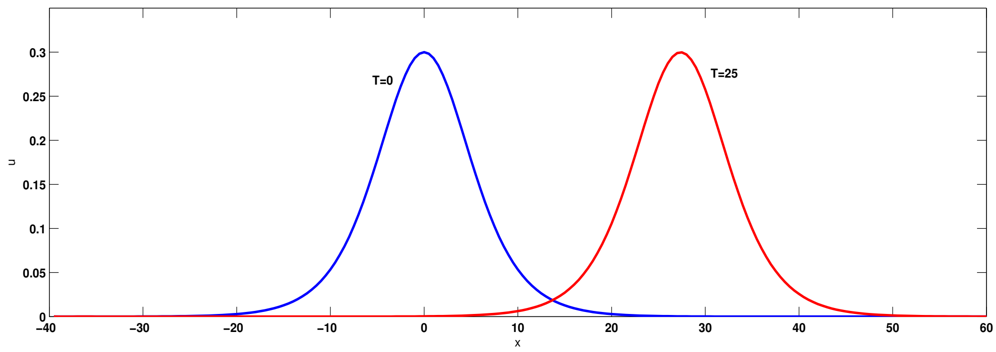

Figure 3: Profile of single solitary wave for $c = 0.1$.

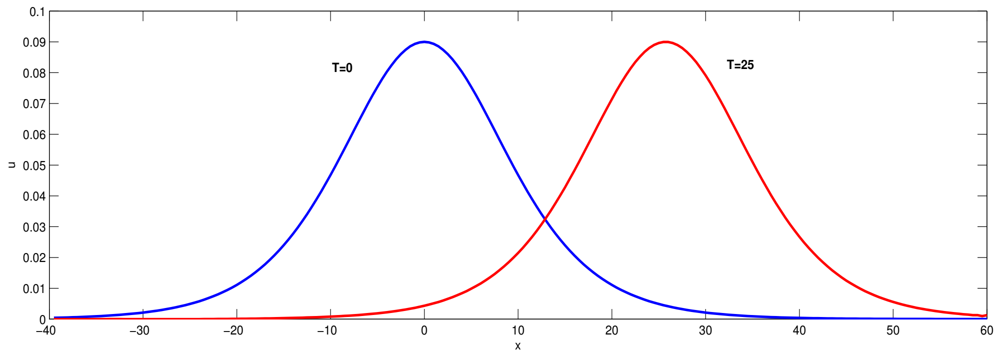

Figure 4: Profile of single solitary wave for $c = 0.03$.

6.4.2. Interaction of two positive solitary waves

In this part, we will investigate the interaction of two positive solitary waves having different amplitudes. We consider the BBM equation with initial conditions given by

$$u(x,0) = \sum_{i=1}^{2} 3c_i\, sech^2\Big(k_i(x - x_i)\Big), \tag{64}$$

where $c_i = \frac{4k_i^2}{1-4k_i^2}$, c_i and x_i are constants, $i = 1, 2$, and with the boundary conditions $u \to 0$ as $x \to \pm\infty$. We choose parameters as $\mu = 1$, $k_1 = 0.4$, $k_2 = 0.3$, $x_1 = 15$, $x_2 = 35$, $h = 0.3$ and $k = 0.1$ with interval $[0, 120]$. Figure 5 displays the profile of interaction of two positive solitary waves. As is well known, solitary waves with smaller amplitudes have a less velocity than another of larger amplitudes. It is appeared from Figure 5 that the larger wave goes up the smaller wave and passes it at t = 25.

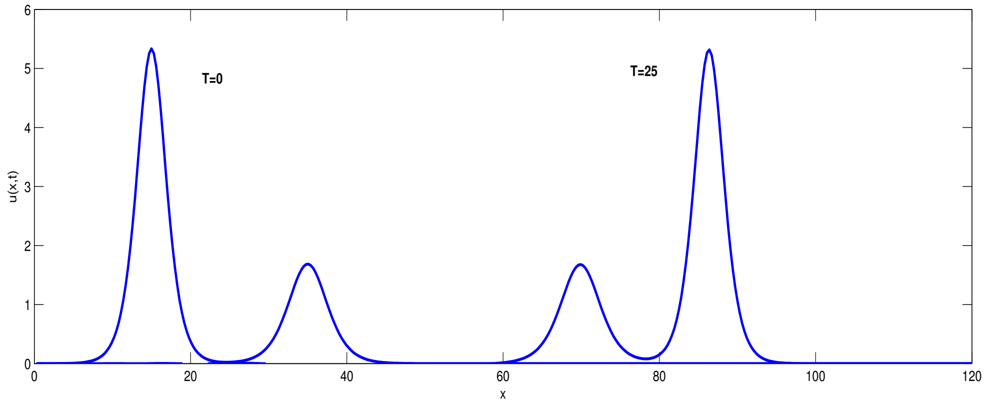

Figure 5: Profil of interaction of two solitary waves.

Table 7 shows a comparison of invariants for the interaction of two positive solitary waves obtained for $h = 0.3$ and $k = 0.1$ at different time, from present method and the method in [18].

Time	method	I_1	I_2	I_3
0		37.916482	120.479974	744.081208
2	Present method	37.925772	120.480658	744.074345
	[18]	37.916850	120.515270	743.998856
4	Present method	37.928355	120.480931	744.064772
	[18]	37.916972	120.513172	743.956686
6	Present method	37.929586	120.480995	744.044281
	[18]	37.917095	120.511737	743.917027

Table 7: Invariants for the interaction of two solitary waves.

6.4.3. *The undular bore*

We consider the equation (63) with the boundary conditions $u \to 0$ as $x \to \infty$ and $u \to c_0$ as $x \to -\infty$ and the initial condition defined by [21]

$$u(x,0) = \frac{c_0}{2}\Big[1 - tanh(\frac{x - x_0}{d})\Big], \qquad x \in [-36, 300],$$

where $u(x,0)$ represents the elevation of the water surface, d represents the slope between the still water and deeper water and c_0 represents the magnitude of the change in water level. We take the parameters to have the following values: $\beta = 1.5,\ \mu = 1/6,\ c_0 = 0.1\ and\ x_0 = 0$.
The behavior of the wave at $t = 100$ for the slope $d = 2$ and $d = 5$ have been presented in Figure 6 and 7, respectively. The numbre of undulations formed increases with time and decreases with the increase of d from $d = 2$ to $d = 5$.

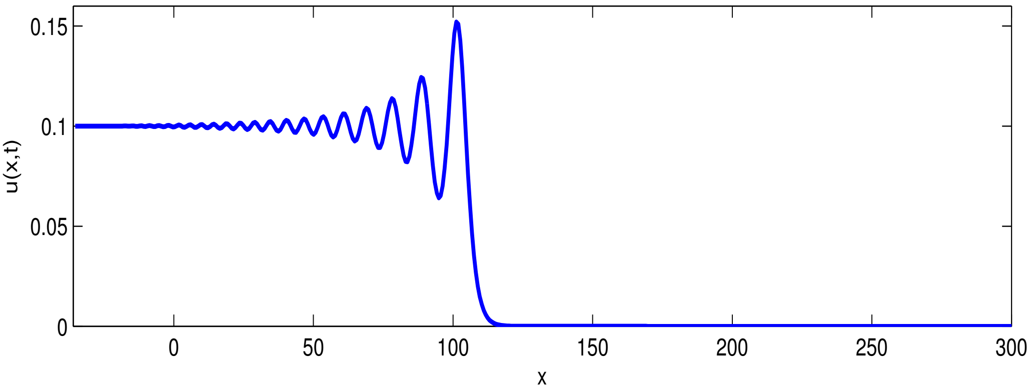

Figure 6: The undular graph for $d = 2,\ J = 480,\ T = 100$.

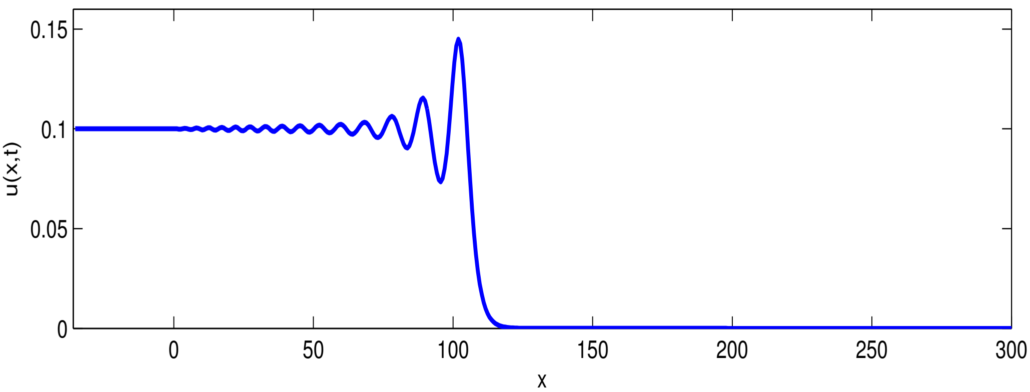

Figure 7: The undular graph for $d = 5,\ J = 480,\ T = 100$.

7. Conclusion

In this article, we construct a high-order linearized difference scheme for the BBMB equation. The solvability of the difference scheme is shown. The proposed scheme is without any restrictions on the grid ratio, convergent at fourth order in space and second order in time. Further, we have compared our difference scheme with earlier published results and it was shown that our linearized scheme is more accurate, efficient and robust. As a particular case, we have studied the BBM equation where the invariants I_1, I_2 and I_3 were explored for single solitary wave and interaction of two positive solitary waves, also the undular bore development was maintained.

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