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β-matrices and β-tensors

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Abstract. In this manuscript we introduce the class of β-matrices, which gives a new sufficient condition for the positivity of the determinant. However, we show that nonnegative β-matrices are not necessarily P-matrices. For column stochastic matrices, the property of being a β-matrix is weaker than strict diagonal dominance. We extend β-matrices to tensors and call them β-tensors. Although they are not in general P-tensors, we prove that nonnegative β-tensors of odd order are P-tensors

1. Introduction

By the Levy-Desplanques theorem (see Corollary 5.6.17 of [4]), strictly diagonally dominant matrices with positive diagonal entries provide an example of matrices with positive determinant. In fact, they are also P-*matrices*, that is, all their principal minors are positive. A B-*matrix* is a matrix with positive row sums and such that each off-diagonal entry is less than the corresponding row sum. B-matrices form another class of P-matrices (see [8]) that is, in general, far from diagonally dominant matrices. In this paper, we introduce a new class of matrices with positive determinant (called β-matrices) that is also, in general, far from diagonal dominance. We call them β-matrices and we also show that they are not necessarily P-matrices. For column stochastic matrices, the property of being a β-matrix is weaker than strict diagonal dominance.

Strictly diagonally dominant matrices and B-matrices and their generalizations (see [6]) have been extended to tensors (see [7], [9]). We also extend β-matrices to β-tensors and we prove that nonnegative β-tensors of odd order are P-tensors.

The paper is organized as follows. Section 2 introduces β-matrices with their properties, examples and counterexamples. In particular, we prove that a β-matrix has always a positive determinant. Their relationship with other classes of matrices is also analyzed. Section 3 is devoted to β-tensors. We analyze their relationship with other classes of tensors and some associated decompositions. We prove that nonnegative β-tensors of odd order are P-tensors.

We finish the introduction with some basic definitions and notations. A real matrix $A = (a_{ij})_{1\leq i,j\leq n}$ is a *Z-matrix* if all its off-diagonal entries are nonpositive, i.e., $a_{ij} \leq 0$ for $i \neq j$. If all its entries are nonnegative, then A is called *nonnegative* and it is denoted by $A \geq 0$. We say that a matrix $A = (a_{ij})_{1\leq i,j\leq n}$ is *strictly diagonally dominant* (by rows) if $|a_{ii}| > \sum_{i\neq j} |a_{ij}|$ and that it is *diagonally dominant* (by rows) if $|a_{ii}| \geq \sum_{i\neq j} |a_{ij}|$.

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Finally, we say that A is (*strictly*) *diagonally dominant by columns* if A^T is (strictly) diagonally dominant by rows.

2. β-matrices

We start this section by introducing the class of β-matrices.

Definition 2.1. *Let $A = (a_{ij})_{1\le i,j\le n}$ be a square real matrix with $n > 2$ such that, for all $j = 1, \dots, n$, $C_j := \sum_{i=1}^n a_{ij} \neq 0$, and let $\tilde{a}_{ij} := \frac{a_{ij}}{C_j}$ for all i, j and*

$$s_i := \min_{1\le j\le n}\{\tilde{a}_{ij}\}, \quad i = 1, \dots, n. \tag{1}$$

We say that A is a β-matrix if, for all $j = 1, \dots, n$, $C_j > 0$ and

$$\tilde{a}_{ii} > s_i > \frac{\left(\sum_{k\neq i}\tilde{a}_{ik}\right) - \tilde{a}_{ii}}{n-2}, \quad i = 1, \dots, n. \tag{2}$$

The following theorem shows that a β-matrix has always positive determinant.

Theorem 2.2. *If A is a β-matrix, then* $\det A > 0$.

Proof. If we define the matrix $\tilde{A} = (\tilde{a}_{ij})_{1\le i,j\le n}$ and the diagonal matrix $D := \mathrm{diag}\{C_1, \dots, C_n\}$, observe that $A = \tilde{A}D$ and so it is sufficient to prove that $\det\tilde{A} > 0$ because D has positive diagonal entries. The matrix $\tilde{A}$ satisfies $\tilde{A}^Te = e$, where $e = (1, \dots, 1)^T$. Therefore, 1 is an eigenvalue of $\tilde{A}$. Since A is real, its complex non-real eigenvalues occur in conjugate pairs, whose product is positive. Since $\det\tilde{A}$ is the product of its complex non-real eigenvalues and the real ones, it is sufficient to see that, if $\lambda \neq 1$ is a real eigenvalue of $\tilde{A}$, then $\lambda > 0$.

If $s = (s_1, \dots, s_n)^T$, we can write

$$\tilde{A} = \tilde{A}^+ + C, \tag{3}$$

where $\tilde{A}^+ = (\tilde{a}_{ij} - s_i)_{1\le i,j\le n}$ for all i, j and $C := se^T$. By (2), $\tilde{A}^+$ has positive diagonal entries and, for all $i = 1, \dots, n$,

$$\sum_{k\neq i}(\tilde{a}_{ik} - s_i) < \tilde{a}_{ii} - s_i$$

because

$$\sum_{k\neq i}\tilde{a}_{ik} - (n-2)s_i < \tilde{a}_{ii}.$$

Thus, $\tilde{A}^+$ is a strictly diagonally dominant matrix with positive diagonal entries. Then, by applying the Gerschgorin circles by rows to $\tilde{A}^+$, we deduce that the real eigenvalues of $\tilde{A}^+$ are positive.

Since $\lambda(\neq 1)$ is a real eigenvalue of $\tilde{A}$, there exists an eigenvector $x(\neq 0)$ such that $\tilde{A}x = \lambda x$. Trasposing both parts of this equality, we have that $\lambda x^T = x^T\tilde{A}^T$ and multiplying by e, we get

$$\lambda x^Te = x^T\tilde{A}^Te = x^Te$$

and so, $(\lambda - 1)(x^Te) = 0$, which implies that $x^Te = 0$ and so $e^Tx = 0$. Hence, by (3), we deduce that

$$\tilde{A}^+x = (\tilde{A} - C)x = \tilde{A}x - se^Tx = \tilde{A}x = \lambda x$$

and λ is also an eigenvalue of $\tilde{A}^+$, and so positive, which proves the result. □

Remark 2.3. *Let us notice that Theorem 2.2 still holds if we extend Definition 2.1 to the case $n = 2$ by modifying condition* (2)*. In fact, for $n = 2$,* (2) *can be replaced by $\tilde{a}_{ii} > s_i$ for $i = 1, 2$. Following the argumentation given in the proof of Theorem 2.2, we see that, when $n = 2$, this new condition implies that the matrix $\tilde{A}^+$ in* (3) *is a diagonal matrix with positive diagonal entries. Hence, it has positive determinant.*

With some sign restrictions, let us see some relations of β-matrices with linear complementarity problems. Let us recall that, given an $n \times n$ real matrix A and $q \in \mathbf{R}^n$, the *linear complementarity problem*, denoted by LCP(A, q) consists of finding, if possible, vectors $x \in \mathbf{R}^n$ satisfying

$$Ax + q \geq 0, \quad x \geq 0, \quad x^T(Ax + q) = 0,$$

where the inequalities are entry wise. It is well known that A is a P-matrix if and only if the LCP(A, q) has a unique solution x^* for any $q \in \mathbf{R}^n$. Let us also recall that an $n \times n$ real matrix A is called a *Q-matrix* if LCP(A, q) has a solution for any $q \in \mathbf{R}^n$ (see [1]).

Proposition 2.4. *Let $A = (a_{ij})_{1\leq i,j\leq n}$ be a β-matrix. Then the following properties hold.*

i) *If A is a Z-matrix, then it is strictly diagonally dominant by columns with positive diagonal entries and so it is a P-matrix.*
ii) *If A is nonnegative, then it has positive diagonal entries and so it is a Q-matrix.*

Proof. (i) If a Z-matrix is also a β-matrix, then it is strictly diagonally dominant by columns with positive diagonal entries because it has positive column sums. It is well known that a strictly diagonally dominant matrix with positive diagonal entries is a P-matrix.

(ii) If A is a nonnegative β-matrix, all entries $\tilde{a}_{ij}$ are also nonnegative and then (1) and (2) imply that $\tilde{a}_{ii} > s_i \geq 0$ for all i. Then the positivity of all column sums C_i also implies that A has positive diagonal entries. Now the fact that A is a Q-matrix follows from Theorem (3.10) of Chapter 10 of [1] because it is a nonnegative matrix with positive diagonal entries. □

However, as the following example shows, not all β-matrices are Q-matrices.

Example 2.5. *Let us consider the matrix*

$$A = \begin{pmatrix} 10 & 3 & 3 \\ -4 & 1 & -2 \\ -1 & -1 & 1 \end{pmatrix}.$$

We can see that A is a β-matrix since it has positive column sums and $\tilde{A}$ satisfies (2)*. However, this example does not satisfy the hypotheses of Proposition 2.4 i) or ii). In fact, we now show that it is not a Q-matrix because the LCP(A, q) does not have a solution for $q = (0, -1, -1)^T$. A feasible solution $x = (x_1, x_2, x_3)$ should verify that $Ax + q \geq 0$, i.e.,*

$$\begin{cases} 10x_1 + 3x_3 + 3x_3 \geq 0, \\ -1 - 4x_1 + x_2 - 2x_3 \geq 0, \\ -1 - x_1 - x_2 + x_3 \geq 0, \end{cases}$$

with $x_1, x_2, x_3 \geq 0$. The first inequality holds for any nonnegative value of the variables. However, the second and third inequalities are incompatible. If $-1 - 4x_1 + x_2 - 2x_3$ and $-1 - x_1 - x_2 + x_3$ are nonnegative, its sum should be also nonnegative. But $-2 - 5x_1 - x_3 \ngeq 0$ for any $x_1, x_3 \geq 0$, and hence, the LCP(A, q) does not have a solution and A is not a Q-matrix.

Observe that the matrix A of Example 2.5 also shows that the transpose of a β-matrix is not necessarily a β-matrix because A^T has columns with negative sums.

The following remark shows that, for matrices A stochastic by columns (that is, $A \geq 0$ and $A^T e = e$), the concept of β-matrix is weaker than strict diagonal dominance by rows.

Remark 2.6. *Let $n > 2$ and let $A = (a_{ij})_{1\le i,j\le n}$ be a matrix stochastic by columns. Then $C_j = 1$ for all $j = 1,\dots,n$ and so $\tilde{a}_{ij} = a_{ij}$ for all i, j. So, a matrix A stochastic by columns is a β-matrix if and only if the following condition holds:*

$$a_{ii} > s_i' > \frac{\left(\sum_{k\neq i} a_{ik}\right) - a_{ii}}{n-2}, \quad s_i' := \min_{1\le j\le n}\{a_{ij}\}, \quad i = 1,\dots,n. \tag{4}$$

Observe also that, if a matrix stochastic by columns A is also strictly diagonally dominant by rows, then A is a β-matrix because (4) clearly holds:

$$a_{ii} > s_i' \ge 0 > \frac{\left(\sum_{k\neq i} a_{ik}\right) - a_{ii}}{n-2}, \quad i = 1,\dots,n.$$

The next remark shows that, in general, we cannot replace in Theorem 2.2 the condition (2) of Definition 2.1 by the condition (4).

Remark 2.7. *A matrix $A = (a_{ij})_{1\le i,j\le n}$ with positive column sums and satisfying (4) can have nonpositive determinant. In fact, take $\varepsilon > 0$ and*

$$A = \begin{pmatrix} 2+\varepsilon & 2 & 0 \\ 2 & 3+\varepsilon & 3 \\ 0 & 1 & 2+\varepsilon \end{pmatrix}.$$

Then $\det A = (2+\varepsilon)(\varepsilon^2 + 5\varepsilon - 1) < 0$ for ε small enough. However, A has positive column sums and satisfies (4): $2+\varepsilon > 0 > -\varepsilon$, $3+\varepsilon > 2 > 2-\varepsilon$ and $2+\varepsilon > 0 > -1-\varepsilon$.

The following example shows that, in spite of having positive determinant, nonnegative β-matrices are not necessarily P-matrices.

Example 2.8. *Let us consider the following matrix*

$$C := \begin{pmatrix} 3+\varepsilon & 2 & 0 & 1 \\ 0 & 2+\varepsilon & 2 & 0 \\ 2 & 2 & 3+\varepsilon & 3 \\ 1 & 0 & 1 & 2+\varepsilon \end{pmatrix}. \tag{5}$$

We can see that C is a β-matrix. The column sums are positive, $C_j = 6+\varepsilon > 0$ for $j = 1,\dots,4$, and the matrix $\tilde{C}$ given by Definition 2.1 satisfies (2) for $i = 1, 2, 3, 4$. However, C is not a P-matrix. As it can be seen in Remark 2.7, the principal minor using indices $2, 3$ and 4 is given by $\det A = (2+\varepsilon)(\varepsilon^2 + 5\varepsilon - 1)$ and it takes negative values for ε small enough.

Observe that the previous example also shows that the property of being a β-matrix is not inherited by principal submatrices. In fact, C is a β-matrix and its principal submatrix A is not a β-matrix (take into account Remark 2.7 and Theorem 2.2).

The following examples show nonsymmetric and symmetric β-matrices that are far from being strictly diagonally dominant matrices and from being B-matrices, which are other classes of matrices with positive determinant.

Example 2.9. *Let us first consider the $n \times n$ ($n > 2$) matrix A:*

$$A = \begin{pmatrix} n+\varepsilon & 1 & \cdots & \cdots & \cdots & 1 & n \\ n & \ddots & \ddots & & & & 1 \\ 1 & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & \cdots & 1 & n & n+\varepsilon \end{pmatrix}, \quad \varepsilon > 0.$$

Observe that A is not strictly diagonally dominant and that it is not a B-matrix because $n > \frac{3n-2+\varepsilon}{n}$. The matrix A has positive column sums and, if we construct the matrix $\tilde{A}$ given by Definition 2.1, we can check that (2) holds:

$$\frac{n+\varepsilon}{3n-2+\varepsilon} > \frac{1}{3n-2+\varepsilon} > \frac{2(n-1)-(n+\varepsilon)}{(3n-2+\varepsilon)(n-2)} = \frac{n-2+\varepsilon}{(3n-2+\varepsilon)(n-2)}.$$

Then A is a β-matrix and, by Theorem 2.2, $\det A > 0$.

The next matrix B is very close to the previous matrix A, although B is symmetric. The $n \times n$ ($n > 2$ even) symmetric matrix B has also $n+\varepsilon$ on the main diagonal, it has $n, 1, n, 1, \ldots, n, 1, n$ on the line below (and above) the main diagonal, and 1's elsewhere. Observe again that B is not strictly diagonally dominant and that it is not a B-matrix because $n > \frac{3n-2+\varepsilon}{n}$. The matrix B also satisfies Definition 2.1, and so B is also a β-matrix and, by Theorem 2.2, $\det B > 0$.

3. β-tensors

A real mth order n-dimensional tensor $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$ is a multi-array of real entries $a_{i_1\cdots i_m} \in \mathbb{R}$, where $i_k \in N := \{1, \ldots, n\}$ for $k = 1, \ldots, m$. We call the set of entries $a_{ii_2\cdots i_m}$ the i-th row of $\mathcal{A}$ for $i, i_2, \ldots, i_m \in N$. A tensor $\mathcal{A}$ is called *diagonally dominant* if

$$|a_{i\cdots i}| \geq \sum_{i_2,\ldots,i_m \neq (i,\ldots,i)}^{n} |a_{ii_2\cdots i_m}|, \ i \in N. \tag{6}$$

If (6) holds strictly, then $\mathcal{A}$ is called *strictly diagonally dominant*.

We say that $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$ is *nonnegative* if $a_{i_1\cdots i_m} \geq 0$ for all $i_1, \ldots, i_m \in N$ and that $\mathcal{A}$ is a *Z-tensor* if all its off-diagonal entries are nonpositive. Let us now introduce the important concept of P-tensor and some previous notations. Let us first recall that, given an m-th order tensor $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$ and $x \in \mathbb{R}^n$, then $\mathcal{A}x^{m-1} \in \mathbb{R}^n$ is given by

$$\left(\mathcal{A}x^{m-1}\right)_i := \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m}, \quad \text{for each } i = 1, \ldots, n.$$

Given an index $i_k \in N$ with $k \in \{1, \ldots, m\}$, let us define the i_kth mode-k sum of $\mathcal{A}$ (see [2]), $r(\mathcal{A}, i_k, k)$, as

$$r(\mathcal{A}, i_k, k) = \sum_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_m=1}^{n} a_{i_1\cdots i_k\cdots i_m}. \tag{7}$$

This sum will play the role of the row sums of the matrix whenever $k = 1$ and the role of the column sums for a given $j \in \{2, \ldots, m\}$. We are also interested in the case where the tensor is diagonally dominant with respect to this index j. In this case, we say that the tensor $\mathcal{A}$ is *strictly k-diagonally dominant* if

$$|a_{i\cdots i}| > \sum_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_m \neq (i,\ldots,i)}^{n} |a_{i_1\cdots i\cdots i_m}|, \; i \in N. \tag{8}$$

Definition 3.1. *(see [3] or page 192 of [9]) A tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is called a P-tensor if for each nonzero $x \in \mathbb{R}^n$ there exists an index $i \in N$ such that*

$$x_i^{m-1}(\mathcal{A}x^{m-1})_i > 0. \tag{9}$$

For the case of tensors of order 2, a P-tensor coincides with a P-matrix (see page 338 of [3]). We now consider an extension of the definition of β-matrices to the higher order case. This definition will give us a sufficient condition to identify nonnegative odd order P-tensors.

Definition 3.2. *Given $m > 2$ and $k \in \{2,\ldots,m\}$, let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$ be a real tensor such that for all $j = 1,\ldots,n$*

$$C_j := r(\mathcal{A}, j, k) = \sum_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_m=1}^{n} a_{i_1\cdots j\cdots i_m} \neq 0, \tag{10}$$

let $\tilde{a}_{ii_2\cdots i_m} = \frac{a_{ii_2\cdots i_m}}{C_{i_2}\cdots C_{i_m}}$ for all $i, i_2, \ldots, i_m$ and

$$s_i = \min_{i_2,\ldots,i_m}\{\tilde{a}_{ii_2\cdots i_m}\} \quad for\ i = 1,\ldots,n. \tag{11}$$

We say that $\mathcal{A}$ is a β-tensor (for the index k) if, for all $i = 1,\ldots,n$, $C_i > 0$ and

$$\tilde{a}_{i\cdots i} > s_i > \frac{\sum_{i_2,\ldots,i_m \neq (i,\ldots,i)} \tilde{a}_{ii_2\cdots i_m} - \tilde{a}_{i\cdots i}}{n^{m-1} - 2}. \tag{12}$$

As it has been the case with structured matrices and the linear complementarity problem, structured tensors and its application to the *tensor complementarity problem* have received a lot of attention recently. Given a tensor $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$ and a vector $q \in \mathbf{R}^n$, the tensor complementarity problem, denoted by TCP($\mathcal{A}$,q), consists of finding a vector $x \in \mathbf{R}^n$ such that

$$x \geq 0, \; \mathcal{A}x^{m-1} + q \geq 0, \quad x^T(\mathcal{A}x^{m-1} + q) = 0.$$

We say that $\mathcal{A}$ is a *Q-tensor* if the TCP($\mathcal{A}$,q) has a solution for all $q \in \mathbf{R}^n$.

Proposition 3.3. *Let $\mathcal{A}$ be a β-tensor for an index $k \in \{2,\ldots,m\}$. Then the following properties hold:*

i) *If $\mathcal{A}$ is a Z-tensor, then it is strictly k-diagonally dominant with positive diagonal entries.*
ii) *If $\mathcal{A}$ is nonnegative, then it has positive diagonal entries and so it is a Q-tensor.*

Proof. i) If a β-tensor is also a Z-tensor, it is strictly k-diagonally dominant with positive diagonal entries because its mode-k sums (10) are positive.

ii) If A is a nonnegative β-tensor, formula (11) implies that $\tilde{a}_{i\cdots i} > s_i \geq 0$ for all $i \in N$. Moreover, since its mode-k sums (10) are positive, $\mathcal{A}$ has positive diagonal entries. Hence, $\mathcal{A}$ is a nonnegative tensor with positive diagonal entries and it is a Q-tensor by Theorem 3.2 of [5]. □

Let us now introduce the *Yang-Yang transformation*, first used in [10]. Given n nonzero real numbers $d_1,\ldots,d_n$, we define the tensor

$$\mathcal{T} = (t_{i_1\cdots i_m}) = Y(\mathcal{A}, d_1,\ldots,d_n),$$

whose entries are given by

$$t_{i_1\cdots i_m} = (d_{i_1})^{-(m-1)} d_{i_2}\cdots d_{i_m} a_{i_1\cdots i_m}$$

for any $i_j \in N$, $j = 1, \ldots, m$. Given a β-tensor $\mathcal{A}$, let us define

$$\hat{\mathcal{A}} := Y(\mathcal{A}, 1/C_1, \ldots, 1/C_n), \tag{13}$$

where C_j are the sums defined in (10) for $j = 1, \ldots, n$. We are going to see that, when $\mathcal{A}$ is a β-tensor, $\hat{\mathcal{A}}$ can be decomposed as the sum of a strictly diagonally dominant tensor and a rank-one tensor.

Proposition 3.4. *Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ be a β-tensor and let $\hat{\mathcal{A}}$ be the tensor given by* (13)*. Then*

$$\hat{\mathcal{A}} = \mathcal{B} + C, \tag{14}$$

where $\mathcal{B}$ is a strictly diagonally dominant tensor with positive diagonal entries and C is a rank-one tensor.

Proof. Let us first define the tensor $C := (c_{i_1 \cdots i_m})$ such that $c_{i_1 \cdots i_m} = s_{i_1} C_{i_1}^{m-1}$, where s_{i_1} and C_{i_1} are given by formulas (10) and (11). Then we consider the tensor $\mathcal{B} := \mathcal{A} - C$. Let us check that $\mathcal{B}$ is strictly diagonally dominant with positive diagonal entries. For $i = 1, \ldots, n$,

$$\sum_{i_2, \ldots, i_m \neq (i, \ldots, i)}^{n} |\tilde{a}_{i i_2 \cdots i_m} C_i^{m-1} - s_i C_i^{m-1}| = \sum_{i_2, \ldots, i_m \neq (i, \ldots, i)}^{n} (\tilde{a}_{i i_2 \cdots i_m} - s_i) C_i^{m-1}.$$

Then we need to prove the following inequality

$$\sum_{i_2, \ldots, i_m \neq (i, \ldots, i)}^{n} (\tilde{a}_{i i_2 \cdots i_m} - s_i) C_i^{m-1} < a_{i \cdots i} - s_i C_i^{m-1}, \tag{15}$$

or analogously,

$$\sum_{i_2, \ldots, i_m \neq (i, \ldots, i)}^{n} (\tilde{a}_{i i_2 \cdots i_m} - s_i) < \tilde{a}_{i \cdots i} - s_i.$$

After some computations we can rewrite (15) as

$$\sum_{i_2, \ldots, i_m \neq (i, \ldots, i)}^{n} \tilde{a}_{i i_2 \cdots i_m} - (n^{m-1} - 2) s_i < \tilde{a}_{i \cdots i},$$

which holds because of (12). Hence, $\mathcal{B}$ is strictly diagonally dominant with positive diagonal entries. □

Now we analyze the relationship of nonnegative β-tensors with P-tensors. By Example 2.8 we know that β-tensors of even order are not necessarily P-tensors. As a consequence of the decomposition (14), in the proof of the following result we are going to deduce that $\hat{\mathcal{A}}$ is a P-tensor whenever it is a nonnegative tensor of odd order. Then, because of the nice properties of the Yang-Yang transformation, we can conclude that $\mathcal{A}$ is also a P-tensor, and so, nonnegative β-tensors of odd order are always P-tensors.

Theorem 3.5. *Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ be a nonnegative β-tensor of odd order m. Then $\mathcal{A}$ is a P-tensor.*

Proof. Given $x \neq 0 \in \mathbb{R}^n$, let us consider the decomposition (14) of $\hat{\mathcal{A}}$. We have that $s_i \geq 0$ and that $(Cx^{m-1})_i = s_i C_i^{m-1}(x_1 + \ldots + x_n)^{m-1} \geq 0$ for all $i \in N$ because $\mathcal{A}$ is nonnegative. Hence, $x_i^{m-1} s_i C_i^{m-1}(x_1 + \ldots + x_n)^{m-1} \geq 0$ for $i \in N$. Since $\mathcal{B}$ is a strictly diagonally dominant tensor with positive diagonal entries, it is a P-tensor by Corollary 3.2 of [3]. So there exists an index $i \in N$ such that $x_i^{m-1}(\mathcal{B}x^{m-1})_i > 0$. Hence, for that index we deduce that

$$x_i^{m-1}(\hat{\mathcal{A}}x^{m-1})_i = x_i^{m-1}(\mathcal{B}x^{m-1})_i + x_i^{m-1}(Cx^{m-1})_i > 0,$$

and so $\hat{\mathcal{A}}$ is a P-tensor.

Given a nonzero vector x, let us now check that $\mathcal{A}$ is a P-tensor. Given an index $j \in N$, because of the relationship between $\mathcal{A}$ and $\hat{\mathcal{A}}$ we see that

$$
\begin{aligned}
(\mathcal{A}x^{m-1})_j &= \sum_{i_2,\ldots,i_m=1}^{n} a_{ji_2\cdots i_m} x_{i_2}\cdots x_{i_m} \\
&= \frac{1}{C_j^{(m-1)}} \sum_{i_2,\ldots,i_m=1}^{n} C_j^{(m-1)} \frac{a_{ji_2\cdots i_m}}{C_{i_2}\cdots C_{i_m}} C_{i_2}x_{i_2}\cdots C_{i_m}x_{i_m} \\
&= \frac{1}{C_j^{(m-1)}} (\hat{\mathcal{A}}y^{m-1})_j,
\end{aligned}
$$

where $y = (C_1x_1, \ldots, C_nx_n)$. We have that $y \neq 0$ because $C_j > 0$ for all $j \in N$. Then, since $\hat{\mathcal{A}}$ is a P-tensor, we deduce that there exists and index $i \in N$ such that $y_i^{m-1}(\hat{\mathcal{A}}y^{m-1})_i > 0$. Hence, using again that $C_i > 0$ we conclude that

$$
\frac{y_i^{m-1}}{C_i^{m-1}} \cdot \frac{1}{C_i^{m-1}} (\hat{\mathcal{A}}y^{m-1})_i = x_i^{m-1}(\mathcal{A}x^{m-1})_i > 0,
$$

and so, that $\mathcal{A}$ is a P-tensor. □

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