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From Fuzzy Scalar Henstock to Fuzzy Scalar McShane Integrability

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Abstract. We introduce the notion of the scalar fuzzy McShane and Henstock integrals for fuzzy number valued functions and we discuss their relationship and we give a fuzzy scalar version of a Gordon theorem [24].

1. Introduction

The McShane integral as it was described in [22, 24, 43] is a Riemann-type integral using "gauge-limit". It is equivalent to the Lebesgue integral for real functions. The Dunford, Pettis and Bochner integrals are generalisations of the Lebesgue integral to Banach space-valued functions. The McShane integral of a vector-valued functions and its relationship to the Bochner integral, Pettis integral were discussed in [19, 22, 23, 32, 43]. An interesting convergence theorem for the McShane integral was proved by D. H. Fremlin and J. Mendoza in [22]. In general, McShane integrability lies strictly between Bochner and Pettis integrability, but McShane and Pettis integrability are equivalent for functions taking values in separable Banach spaces; see, [20, 22, 24].

There is a great deal of literature on Bochner and Pettis integration for the space set-valued functions (see El Amri and Hess [3] or Hess an Ziat [26] for further references) of several types that have shown to be a useful tool in many branches of mathematics such as mathematical economy, control theory, differential inclusions, convex analysis and optimisation.

At the end of last century Ziat ([46, 47]) and El Amri and Hess [3] presented several criteria for a set-valued functions having as its values convex weakly compact subsets of a Banach space, to be Pettis integrable. In [15] Di Piazza and Musial studied the natural generalization of Pettis integral of a set-valued functions obtained by replacing the Lebesgue integrability of the support functions by their Kurzweil-Henstock integrability (they call such an integral Kurzweil-Henstock-Pettis). There it is proved that the Kurzweil-Henstock-Pettis integral is in some way a translation of the Pettis one. The same Authors in [16] deal with the Henstock and McShane integrals of set-valued functions. Such integrals are generalisations, by means the notion of the Hausdorff distance, of the definitions of Henstock and McShane integrals for vector valued functions. There it is also presented a characterisation of Henstock integrable set-valued

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functions with convex compact values, similar to that in [15]: each Henstock integrable set-valued function is the sum one of its Henstock integrable selections and of a McShane integrable set-valued function. There is also proved that if the multifunctions are compact convex valued and the target Banach space is separable, then the Pettis and the McShane integrals coincide, as in case of functions taking their values in a separable Banach space ([22]).

Park in [33] introduced the Pettis integral for fuzzy mappings in separable Banach spaces. In [18] Di Piazza and Marraffa continued the study of the Pettis integral for fuzzy mappings in Banach spaces not necessary separable. The important result of this paper is the following decomposition theorem:
$\Gamma : [0,1] \longrightarrow \mathcal{F}(X)$ (where $\mathcal{F}(X)$) is the generalized fuzzy number space associated to the Banach space X) is a Pettis integrable fuzzy mapping if and only if Γ can be represented as $\Gamma(t) = G(t) + f(t)$, where $G : [0,1] \longrightarrow \mathcal{F}(X)$ is a Pettis integrable fuzzy mapping whose support functions are nonnegative and f is a Pettis integrable fuzzy mapping generated by a Pettis integrable selection of Γ. For decomposition results see also [6], [7], [8], [17].

In the present work, we introduce the notion of the *fuzzy scalar McShane and Henstock integrals* for fuzzy-number-valued functions and discuss their relationship. More precisely, we seek to determine when the fuzzy scalar Henstock and the fuzzy scalar McShane integrals are equivalent. Given this goal, the following result due to Gordon (Theorem 9.13 [24]) dealing with the real-valued McShane and Henstock integrals will guide our investigation: the two integrals are equal for bounded functions. This result will play a vital role in our proof of main result (Theorem 4.1).

2. Preliminaries

Let X be a real Banach space, whose norm is denoted by $\|.\|$ and whose closed unit ball is denoted by $\overline{B}_X$ and let X^* be the topological dual of X. The closed unit ball of X^* is denoted by $\overline{B}_{X^*}$. By w, we denote the weak topology of X, and w^* the weak topology of X^*. $cwk(X)$ is the family of all nonempty weakly compact convex subsets of X endowed with the Hausdorff distance

$$d_H(A,B) := \max\{\sup_{x\in A}\inf_{y\in B}\|x-y\|, \sup_{y\in B}\inf_{x\in A}\|x-y\|\}$$

and the operations

$$A+B = \{x+y : \ x\in A, \ y\in B\}, \qquad kA = \{kx : \ x\in A\},$$

for A and B in $cwk(X)$. The space $cwk(X)$ endowed with the Hausdorff distance is a complete metric space. For every $A \in cwk(X)$ the support function of A is denoted by $\delta^*(.,A)$ and defined by $\delta^*(x^*,A) = \sup\{\langle x^*, y\rangle : y \in A\}$, for each $x^* \in X^*$. Clearly the map $x^* \to \delta^*(x^*,A)$ is sublinear on X^* and $-\delta^*(-x^*,A) = \inf\{\langle x^*, y\rangle : y \in A\}$, for each $x^* \in X^*$. The distance functional is a mapping $d : X \times \mathcal{P}(X) \to \mathbb{R}^+$ such that $d(x,A) := \inf\{\|x-a\|, a \in A\}=\sup_{x^*\in B_{X^*}}[\langle x^*, x\rangle - \delta^*(x^*,A)]$.

According to Hormander's formula ([1, 11], for A and B non empty members of $cwk(X)$ we have the equality

$$d_H(A,B) = \sup_{x^*\in\overline{B}_{X^*}} |\delta^*(x^*,A) - \delta^*(x^*,B)| = \sup_{x\in X} |d(x,A) - d(x,B)|.$$

Moreover for $A \in cwk(X)$ we define

$$\|A\| = d_H(A,\{0\}) = \sup_{x\in A}\|x\|.$$

On $cwk(X)$ we will consider the following convergence given in [5]: a sequence (C_n) in $cwk(X)$ scalarly converges (Shortly S-converges) to $C \in cwk(X)$ if the following condition is satisfied:
$\forall x^* \in X^*$, $\lim_{n\to\infty} \delta^*(x^*,C_n) = \delta^*(x^*,C)$. A sequence (C_n) in $cwk(X)$ converges in the Hausdorff topology to

$C \in cwk(X)$ if the following condition is satisfied: $\lim_{n\to\infty} d_H(C_n, C) = 0$.

Throughout this paper $[0,1]$ is the unit interval of the real line equipped with the usual topology and the Lebesgue measure λ. The family of Lebesgue measurable subsets of $[0,1]$ is denoted by $\mathcal{L}$. We say that a subset $\mathcal{H}$ of Lebesgue-integrable functions defined on $[0,1]$ is uniformly integrable ([32]) if

$$\lim_{a\to\infty} \sup_{h\in\mathcal{H}} \int_{\{t\in[0,1]:|h(t)|\geq a\}} |h|\, d\lambda = 0.$$

It is well known ([32]) that $\mathcal{H}$ is uniformly integrable if and only if it is *bounded* for the L_1-norm (i.e. $\sup_{h\in\mathcal{H}} \int_0^1 |h|\, d\lambda$ is finite) and ***equi-continuous***, i.e.

$$\lim_{\lambda(A)\to 0} \sup_{h\in\mathcal{H}} \int_A |h|\, d\lambda = 0.$$

A function $f : [0,1] \to X$ is said to be *scalarly measurable* (resp. *scalarly integrable*, alias *Dunford integrable*) if for every $x^* \in X^*$, the real-valued function $\langle x^*, f\rangle$ is measurable (resp. Lebesgue integrable). If $f : [0,1] \to X$ is a scalarly integrable function, then for each $E \in \mathcal{L}$, there is $x_E^{**} \in X^{**}$ such that

$$\langle x^*, x_E^{**}\rangle = \int_E \langle x^*, f\rangle\, d\lambda.$$

In the case that $x_E^{**} \in X$ for all $E \in \mathcal{L}$, then f is called *Pettis integrable*, or simply *$\mathcal{P}$-integrable* and we set $(\mathcal{P}) \int_E f\, d\lambda$ instead of x_E^{**} to denote the *Pettis integral* of f over E. If $f : [0,1] \to X$ is a Pettis integrable function, then the set $\{\langle x^*, f\rangle : x^* \in B_{X^*}\}$ is relatively weakly compact in L_1 for the weak topology of L_1 ([13], Theorem II. 3.8) (see also [30]); equivalently it is *uniformly integrable* ([13], Theorem III. 2.15). For an extensive study of Banach space-valued Pettis integral, the reader is referred to Musiał ([30]).

A *partial McShane partition* is a finite collection $\{(I_i, t_i) : 1 \leq i \leq k\}$, where $I_1, ..., I_k$ are non-overlapping subintervals of $[0,1]$ and t_i is a point of $[0,1]$ for each $1 \leq i \leq k$. If the union of all the elements I_i of the partition equals $[0,1]$, then it is a *McShane partition* of $[0,1]$. A *gauge* on $[0,1]$ is a function $\delta : [0,1] \to]0, +\infty[$. For a given a gauge δ on $[0,1]$, we say that a McShane partition $\{(I_i, t_i) : 1 \leq i \leq k\}$ is *subordinate* to δ if $I_i \subset]t_i - \delta(t_i), t_i + \delta(t_i)[$ for every $1 \leq i \leq k$. Let $f : [0,1] \to X$ be a function. We set

$$\sigma(f, P) := \sum_{i=1}^{k} \lambda(I_i) f(t_i),$$

for each McShane partition $P := \{(I_i, t_i) : 1 \leq i \leq k\}$. A sequence (P_n) of McShane partitions of [0,1] is said to be *adapted* to a sequence of gauges (δ_n) on $[0,1]$ if P_n is subordinate to δ_n for each $n \geq 1$

• A function $f : [0,1] \to X$ is *McShane integrable* on $[0,1]$, with *McShane integral* $\varpi \in X$, if for every $\varepsilon > 0$ there is a gauge δ on $[0,1]$ such that

$$\|\sigma(f, P) - \varpi\| < \varepsilon,$$

for every McShane partition P of $[0,1]$ subordinate to δ. We set $\varpi := (\mathcal{M}) \int_0^1 f\, d\lambda$.

Remark 2.1. *Recall that the partitions employed in this definition can be replaced with measurable partitions of* $[0,1]$ *(that is, a finite collection* $\{(E_i, t_i) : 1 \leq i \leq k\}$*, where* $E_1, ..., E_k$ *is a finite disjoint cover of [0,1] by measurable sets and points* $t_1, ..., t_k \in [0,1]$*) ([24], [28], [36]. See also [22], [21], [9], [8], [7]) for a more general setting.*

It is known [22] that if a function $f : [0,1] \to X$ is McShane integrable on $[0,1]$, then it is Pettis integrable on $[0,1]$ and the corresponding integrals are equals, but the converse does not hold in general (see [14, 22, 37].

Nevertheless, for some classes of Banach spaces these two notions coincides: this happens for separable spaces ([20, 22, 23], super-reflexive spaces, $c_0(I)$ (for any non-empty set I) [19], $L^1_{\mathbb{R}}(\nu)$ (for any probability measure ν) [37] and subspaces of a *Hilbert generated Banach space* [14]. More recently, R. Deville and J. Rodriguez [14] have proved the coincidence of the McShane and Pettis integrals for functions taking values in a subspace of a *Hilbert generated Banach space*, thus generalizing all previously mentioned results on such coincidence.

In particular, a real-valued function is Lebesgue integrable on $[0,1]$ if and only if it is McShane integrable on $[0,1]$ and the corresponding integrals are equal in both cases (see [24]).

A *partial Henstock partition* (resp. *Henstock partition* of $[0,1]$) is a partial McShane partition (resp. McShane partition of $[0,1]$) $\{(I_i,t_i) : 1 \le i \le k\}$ such that t_i is a point of I_i for each $1 \le i \le k$.

• A function $f : [0,1] \to X$ is *Henstock integrable*, with *Henstock integral* $\varpi \in X$, if for every $\varepsilon > 0$ there is a gauge δ on $[0,1]$ such that

$$\|\sigma(f,P) - \varpi\| < \varepsilon,$$

for every Henstock partition P of $[0,1]$ subordinate to δ. We set $\varpi := (\mathcal{H}) \int_0^1 f\,d\lambda$.

In case when X is the real line, f is called *Kurzweil-Henstock integrable*, or simply $\mathcal{KH}$-integrable.

Remark 2.2. *It is interesting to observe the following sequential formulation of the preceding definitions.*

A function $f : [0,1] \to X$ is McShane integrable (resp. Henstock integrable), with integral ϖ, if and only if there is a sequence of gauges (δ_n) on $[0,1]$ such that

$$\lim_{n\to\infty} \sup_{P\in\Pi_{\mathcal{M}}(\delta_n)} \|\sigma(f,P) - \varpi\| = 0$$

$$(\textit{resp.} \lim_{n\to\infty} \sup_{P\in\Pi_{\mathcal{H}}(\delta_n)} \|\sigma(f,P) - \varpi\| = 0),$$

where $\Pi_{\mathcal{M}}(\delta_n)$ (resp. $(\Pi_{\mathcal{H}}(\delta_n))$ denotes the collection of all McShane (resp. Henstock) partitions of $[0,1]$ subordinate to δ_n.

Equivalently

$$\lim_{n\to\infty} \|\sigma(f,P_n) - \varpi\| = 0$$

for every sequence P_n of McShane (resp. Henstock) partitions of $[0,1]$ adapted to (δ_n)

The following results concerning the Lebesgue, Kurzweil-Henstock and McShane integral for real-valued function play a vital role in our paper in which we present a fuzzy scalar version of Theorem 2.4 (Theorem 9.13 [24]).

Theorem 2.3. *(Theorem 10.11 [24]).*
Let $f : [0,1] \longrightarrow \mathbb{R}$ be a real-valued function. Then f is McShane integrable if and only if it is Lebesgue integrable. We have then

$$(\mathcal{M}) \int_0^1 f\,d\lambda = \int_0^1 f\,d\lambda.$$

Theorem 2.4. *(Theorem 9.13 [24]).*
Let $f : [0,1] \longrightarrow \mathbb{R}$ be a bounded real-valued function. Then f is McShane integrable if and only if it is $\mathcal{KH}$-integrable. We have then

$$(\mathcal{M}) \int_I f\,d\lambda = (\mathcal{KH}) \int_I f\,d\lambda,$$

for every closed subinterval $I \subset [0,1]$.

Lemma 2.5 (*Saks Henstock*). *(**Lemma 9.11, [24]**). Let $f : [0,1] \to \mathbb{R}$ be a $\mathcal{KH}$-integrable function on $[0,1]$ and $\varepsilon > 0$. Let δ be a gauge of $[0,1]$ such that*

$$|\sigma(f,P) - (\mathcal{KH})\int_0^1 f\,d\lambda| \le \varepsilon,$$

for every Henstock partition P of $[0,1]$ subordinate to δ. Then

$$|\sum_{i=1}^{m}[\lambda(I_i)f(t_i) - (\mathcal{KH})\int_{I_i} f\,d\lambda]| \le \varepsilon,$$

for every partial Henstock partition $\{(I_i,t_i) : i = 1,...,m\}$ of $[0,1]$ subordinate to δ.

We recall now some definitions concerning set-valued functions.

A function $f : [0,1] \longrightarrow X$ is called selection of a set-valued function $F : [0,1] \longrightarrow cwk(X)$ if, for every $t \in [0,1]$, on has $f(t) \in F(t)$. By $S^0(F)$ (resp. $S_{\mathcal{P}}(F)$) we denote the family of all measurable (resp. Pettis integrable) selections.
A set-valued functions $F : [0,1] \longrightarrow cwk(X)$ is said to be scalarly measurable (resp. scalarly integrable) if for every $x^* \in X^*$, the real valued function $\delta^*(x^*, F)$ is measurable (resp. Lebesgue integrable)(see [25]).

Definition 2.6. *(Definition 3.1, [3]). A set-valued function $F : [0,1] \longrightarrow cwk(X)$ is said to be Pettis integrable on $[0,1]$ if F is scalarly integrable on $[0,1]$ and for each $E \in \mathcal{L}$ there exists a set $W_E \in cwk(X)$ such that for each $x^* \in X^*$, we have*

$$\delta^*(x^*, W_E) = \int_E \delta^*(x^*, F)\,d\lambda.$$

Then we set $W_E = (\mathcal{P})\int_E F\,d\lambda$, for each $E \in \mathcal{L}$.

Given $F : [0,1] \longrightarrow cwk(X)$ and a partition $P = \{(I_i,t_i) : 1 \le i \le k\}$ in $[0,1]$ we set

$$\sigma(F,P) := \sum_{i=1}^{k} \lambda(I_i)F(t_i)$$

Definition 2.7. *([16]). A set-valued function $F : [0,1] \longrightarrow cwk(X)$ is said to be McShane (resp. Henstock) integrable on $[0,1]$ if there exists a nonempty set $W \in cwk(X)$ with the following property: for every $\varepsilon > 0$ there is a gauge δ on $[0,1]$ such that for each McShane (resp. Henstock) partition P of $[0,1]$, we have*

$$d_H(\sigma(F,P), W) < \varepsilon.$$

Notation: $W = (\mathcal{M})\int_0^1 F\,d\lambda$ (resp. $W = (\mathcal{H})\int_0^1 F\,d\lambda$).

Remark 2.8. *A set-valued function $F : [0,1] \longrightarrow cwk(X)$ McShane integrable (resp. Henstock integrable) on $[0,1]$, with integral $W \in cwk(X)$, if and only if there is a sequence of gauges (δ_n) on $[0,1]$ such that*

$$\lim_{n\to\infty} \sup_{P\in\Pi_{\mathcal{M}}(\delta_n)} d_H(\sigma(F,P), W) = 0$$

$$(resp.\ \lim_{n\to\infty} \sup_{P\in\Pi_{\mathcal{H}}(\delta_n)} d_H(\sigma(F,P), W) = 0),$$

where $\Pi_{\mathcal{M}}(\delta_n)$ (resp. $(\Pi_{\mathcal{H}}(\delta_n))$ denotes the collection of all McShane (resp. Henstock) partitions of $[0,1]$ subordinate to δ_n.

Equivalently

$$\lim_{n\to\infty} d_H(\sigma(F,P_n), W) = 0$$

for every sequence P_n of McShane (resp. Henstock) partitions of $[0,1]$ adapted to (δ_n).

3. Fuzzy McShane and Fuzzy Henstock integrals

Let $u : X \to [0,1]$. We set $L_r[u] := \{x \in X : u(x) \geq r\}$, for $r \in]0,1]$ the r-level set of u and $supp[u] := cl\{x \in X : u(x) > 0\} = cl(\cup_{r\in]0,1]} L_r[u])$; u is called a generalized fuzzy number, as in [44, 45], (fuzzy number as in [27]) on X if, for each $r \in]0,1]$, $L_r[u] \in cwk(X)$.
Let $\mathcal{F}(X)$ denote the generalized fuzzy number space. We define $\theta : X \to [0,1]$ as follows:

$$\theta(x) = 1_{\{0\}}(x) = \begin{cases} 1 \ if \ x = 0 \\ 0 \ if \ x \neq 0 \end{cases}$$

Then $\theta \in \mathcal{F}(X)$ and θ is called the null element of $\mathcal{F}(X)$.
In the sequel we will use the following representation theorem (see [27]).

Theorem 3.1. *If $u \in \mathcal{F}(X)$, then*
(1) $L_r[u] \in cwk(X)$, $supp[u] \in cwk(X)$, for all $r \in]0,1]$.
(2) $L_{r_2}[u] \subset L_{r_1}[u]$, for $0 < r_1 \leq r_2 \leq 1$.
(3) If r_k is a nondecreasing sequence converging to $r > 0$, then

$$L_r[u] = \bigcap_{k\geq 1} L_{r_k}[u].$$

Conversely, if $\{A_r : r \in]0,1]\}$ is a family of subsets of X satisfying (1), (2) *and* (3), *then there exists a unique $u \in \mathcal{F}(X)$ such that $L_r[u] = A_r$ for every $r \in]0,1]$.*

A linear structure in $\mathcal{F}(X)$ is defined by the operations:

$$(u + v)(x) = \sup_{y+z=x} \min\{u(y), v(z)\}$$

$$(ku)(x) = \begin{cases} u(\frac{x}{k}) \ if \ k \neq 0 \\ 1_{\{0\}}(x) \ if \ k = 0 \end{cases}$$

where $u, v \in \mathcal{F}(X)$ and $k \in \mathbb{R}$ (see [33, 34]).

Remark 3.2. *We can define the above two operations as follows (see [29]: p. 585):*

$$(u + v)(x) = \sup\{r \in]0,1] : \ x \in L_r[u] + L_r[v]\} \ and$$

$$(ku)(x) = \sup\{r \in]0,1] : \ x \in kL_r[u]\}.$$

In the next theorem we list basic properties of the generalized fuzzy numbers will be needed in this paper. There are borrowed from [27, 44, 45].

Theorem 3.3. *Let $u \in \mathcal{F}(X)$. Then*
(1) u is normal fuzzy set, i.e, there exists $x_0 \in X$, such that $u(x_0) = 1$.
(2) u is a convex fuzzy set (or quasiconcave), i.e, $u(tx + (1-t)y) \geq \min\{u(x), u(y)\}$ for any $x, \ y \in X$, $t \in [0,1]$.
(3) u is upper semi-continuous, i.e, for each $r \in]0,1]$, the r-level set $L_r[u]$ is closed subset of X.
(4) $supp[u] := cl\{x \in X : u(x) > 0\}$ is compact.
(5) $L_r[u + v] = L_r[u] + L_r[v]$, for every $u, \ v \in \mathcal{F}(X)$ and $r \in]0,1]$.
(6)$L_r[ku] = kL_r[u]$, for every $u \in \mathcal{F}(X)$, $k \in \mathbb{R}$ and $r \in]0,1]$.

A fuzzy-number valued function $\Gamma : [0,1] \longrightarrow \mathcal{F}(X)$ is said to be measurable if for each $r \in]0,1]$, the set-valued $L_r[\Gamma]$ is measurable. From now $\Gamma : [0,1] \longrightarrow \mathcal{F}(X)$ is a measurable fuzzy-number valued function.

Define $d_\infty : \mathcal{F}(X) \times \mathcal{F}(X) \longrightarrow \mathbb{R}^+$ by the equality:

$$d_\infty(u,v) = \sup_{r \in]0,1]} d_H(L_r[u], L_r[v]).$$

We start with the definition of fuzzy Pettis, Henstock and McShane integrals.

Definition 3.4. *(Definition 3.2, [33]). A measurable fuzzy-number-valued function $\Gamma : [0,1] \longrightarrow \mathcal{F}(X)$ is said to be fuzzy Pettis integrable on $[0,1]$ if and only if for each $E \in \mathcal{L}$, there exists a generalized fuzzy number $u_E \in \mathcal{F}(X)$ such that for any $r \in]0,1]$ and for any $x^* \in X^*$ we have*

$$\delta^*(x^*, L_r[u_E]) = \int_E \delta^*(x^*, L_r[\Gamma])\, d\lambda.$$

In this case $u_E := (\mathcal{FP}) \int_E \Gamma\, d\lambda$ is called the fuzzy Pettis integral of Γ over E.

For the properties of fuzzy Pettis integral, the reader is referred to [18, 33].

A fuzzy-number-valued function $\Gamma : [0,1] \longrightarrow \mathcal{F}(X)$ is said to be scalarly (resp. scalarly Henstock-Kurzweil) integrable on $[0,1]$ if for all $r \in]0,1]$ the set-valued function $L_r[\Gamma] : [0,1] \longrightarrow cwk(X)$ is scalarly (resp. scalarly Henstock-Kurzweil) integrable.

Definition 3.5. *A fuzzy-number-valued function $\Gamma : [0,1] \longrightarrow \mathcal{F}(X)$ is said to be weakly fuzzy McShane (resp. Henstock) integrable on $[0,1]$ if and only if for any $r \in]0,1]$ the set-valued function $L_r[\Gamma] : [0,1] \longrightarrow cwk(X)$ is McShane (resp. Henstock) integrable and there exists a generalized fuzzy number $u \in \mathcal{F}(X)$ such that for any $r \in]0,1]$ and for any $x^* \in X^*$ we have*

$$\delta^*(x^*, L_r[u]) = \int_0^1 \delta^*(x^*, L_r[\Gamma])\, d\lambda.$$

or

$$\delta^*(x^*, L_r[u]) = (\mathcal{KH}) \int_0^1 \delta^*(x^*, L_r[\Gamma])\, d\lambda,$$

respectively. In this case $u := (\mathcal{WFM}) \int_E \Gamma\, d\lambda$ (resp. $u := (\mathcal{WFH}) \int_E \Gamma\, d\lambda$) is called the weak fuzzy McShane (weak fuzzy Henstock) integral of Γ over $[0,1]$. (see [6, 31].

Given $\Gamma : [0,1] \longrightarrow \mathcal{F}(X)$ a fuzzy-number-valued function and a partition $P = \{(I_i, t_i) : 1 \leq i \leq k\}$ in $[0,1]$ we set

$$\sigma(\Gamma, P) := \sum_{i=1}^{k} \lambda(I_i)\Gamma(t_i)$$

Definition 3.6. *A fuzzy-number-valued function $\Gamma : [0,1] \longrightarrow \mathcal{F}(X)$ is said to be fuzzy McShane (resp. Henstock) integrable on $[0,1]$ if there exists a fuzzy number $u \in \mathcal{F}(X)$ such that for every $\varepsilon > 0$ if there is a gauge δ on $[0,1]$ such that for every McShane (resp. Henstock) partition of $[0,1]$ subordinate to δ, we have*

$$d_\infty(\sigma(\Gamma, P), u) < \varepsilon.$$

Notation: $u = (\mathcal{FM}) \int_0^1 \Gamma\, d\lambda$ (resp. $u = (\mathcal{FH}) \int_0^1 \Gamma\, d\lambda$). (See [6, 12, 31].

Remark 3.7. *For sake of comparison with the fuzzy scalar McShane (resp. Henstock) integrability, it is interesting to observe the following sequential formulation of the preceding definition.*

A fuzzy-number-valued function $\Gamma : [0,1] \longrightarrow \mathcal{F}(X)$ *is fuzzy McShane (resp. Henstock) integrable on* $[0,1]$ *with integral* $u \in \mathcal{F}(X)$ *if there exists a sequence* (δ_n) *of gauges on* $[0,1]$ *such that,*

$$\lim_{n\to+\infty} \sup_{P\in\Pi_{\mathcal{M}}(\delta_n)} d_\infty(\sigma(\Gamma,P),u) = 0,$$

$$resp.\ \lim_{n\to+\infty} \sup_{P\in\Pi_{\mathcal{H}}(\delta_n)} d_\infty(\sigma(\Gamma,P),u) = 0,$$

where $\Pi_{\mathcal{M}}(\delta_n)$ *(resp.* $\Pi_{\mathcal{H}}(\delta_n)$*) denotes the collection of all McShane (resp. Henstock) partitions of* $[0,1]$ *subordinate to* δ_n*.*
Equivalently, for each sequence (P_n) *of McShane (resp. Henstock) partitions of* $[0,1]$ *adapted to* (δ_n)*, we have*

$$\lim_{n\to+\infty} d_\infty(\sigma(\Gamma,P_n),u) = 0.$$

Now we define our new notion of fuzzy McShane and Henstock integrability namely fuzzy scalar McShane (resp. Henstock) integrability:

Definition 3.8. *A fuzzy-number-valued function* $\Gamma : [0,1] \longrightarrow \mathcal{F}(X)$ *is said to be fuzzy scalar McShane integrable (*$\mathcal{FSM}$*-integrable for short) on* $[0,1]$ *if there exists a fuzzy number* $u \in \mathcal{F}(X)$*) such that, there is a sequence* (δ_n) *of gauges on* $[0,1]$ *such that, for all* $r \in]0,1]$ *and for each sequence* (P_n) *of McShane partitions adapted to* (δ_n)*, we have*

$$\lim_{n\to+\infty} \delta^*(x^*, L_r[\sigma(\Gamma,P_n)]) = \delta^*(x^*, L_r[u]) \text{ for all } x^* \in X^*.$$

Notation: $u = (\mathcal{SFM})\int_0^1 \Gamma\, d\lambda$.

Definition 3.9. *A fuzzy-number-valued function* $\Gamma : [0,1] \longrightarrow \mathcal{F}(X)$ *is said to be fuzzy scalar Henstock integrable (*$\mathcal{FSH}$*-integrable for short) on* $[0,1]$ *if there exists a fuzzy number* $u \in \mathcal{F}(X)$*) such that, there is a sequence* (δ_n) *of gauges on* $[0,1]$ *such that, for all* $r \in]0,1]$ *and for each sequence* (P_n) *of Henstock partitions adapted to* (δ_n)*, we have*

$$\lim_{n\to+\infty} \delta^*(x^*, L_r[\sigma(\Gamma,P_n)]) = \delta^*(x^*, L_r[u]) \text{ for all } x^* \in X^*.$$

Notation: $u = (\mathcal{SFH})\int_0^1 \Gamma\, d\lambda$.

Proposition 3.10. *If a fuzzy-number-valued function* $\Gamma : [0,1] \longrightarrow \mathcal{F}(X)$ *is* $\mathcal{FSM}$*-integrable on* $[0,1]$*, then it is* $\mathcal{FSH}$*-integrable on* $[0,1]$*. We have then*

$$(\mathcal{SFM})\int_0^1 \Gamma\, d\lambda = (\mathcal{SFH})\int_0^1 \Gamma\, d\lambda.$$

Proposition 3.11. *If a fuzzy-number-valued function* $\Gamma : [0,1] \longrightarrow \mathcal{F}(X)$ *is* $\mathcal{FSM}$*-integrable (resp.* $\mathcal{FSH}$*-integrable) integrable on* $[0,1]$*, then it is* $\mathcal{WFM}$*-integrable (resp.* $\mathcal{WFH}$*-integrable) on* $[0,1]$*. We have then*

$$(\mathcal{SFM})\int_0^1 \Gamma\, d\lambda = (\mathcal{WFM})\int_0^1 \Gamma\, d\lambda$$

or

$$(\mathcal{SFH})\int_0^1 \Gamma\, d\lambda = (\mathcal{WFH})\int_0^1 \Gamma\, d\lambda.$$

respectively.

Proof. As consequence of Definition 3.2, Remark 2.3, Hormander's equality, and ([11]: Proposition III.35). ■

Proposition 3.12. *If a fuzzy-number-valued function* $\Gamma : [0,1] \longrightarrow \mathcal{F}(X)$ *is fuzzy McShane (resp. Henstock) integrable on* $[0,1]$*, then it is* $\mathcal{FSM}$*-integrable (resp.* $\mathcal{FSH}$*-integrable) on* $[0,1]$ *and the two integrals are respectively equal.*

Proof. As consequence of Remark 3.7 and Hormander's equality. ■

The next theorem provides the linearity properties of the fuzzy scalar McShane integral.

Theorem 3.13. *Let* Γ, G*:* $[0,1] \to \mathcal{F}(X)$ *be two fuzzy number valued functions.*

(1) If Γ *and* G *are* $\mathcal{FSM}$*-integrable on* $[0,1]$*, then* $\Gamma + G$ *is* $\mathcal{FSM}$*-integrable on* $[0,1]$ *and*

$$(\mathcal{FSM}) \int_0^1 (\Gamma + G)\, d\lambda = (\mathcal{FSM}) \int_0^1 \Gamma\, d\lambda + (\mathcal{FSM}) \int_0^1 G\, d\lambda.$$

(2) If Γ *is* $\mathcal{SFM}$*-integrable on* $[0,1]$ *and if* α *is a real nonnegative number, then* $\alpha\Gamma$ *is* $\mathcal{FSM}$*-integrable on* $[0,1]$ *and*

$$(\mathcal{FSM}) \int_0^1 \alpha\Gamma\, d\lambda = \alpha(\mathcal{FSM}) \int_0^1 \Gamma\, d\lambda.$$

(3) If Γ *is* $\mathcal{FSM}$*-integrable on* $[0,1]$ *and if* $\Gamma = G$ λ*-a.e., then the function* G *is* $\mathcal{FSM}$*-integrable on* $[0,1]$ *and*

$$(\mathcal{FSM}) \int_0^1 G\, d\lambda = (\mathcal{FSM}) \int_0^1 \Gamma\, d\lambda.$$

Remark 3.14. *Proceeding analogously, Theorem 3.13 remains valid in the context of* $\mathcal{FSH}$*-integrals.*

Proposition 3.15. *A fuzzy-number-valued function* $\Gamma : [0,1] \longrightarrow \mathcal{F}(X)$ *is* $\mathcal{FSH}$*-integrable on* $[0,1]$*, with fuzzy scalar Henstock integral* u*, if and only if there is a sequence* (δ_n) *of gauges on* $[0,1]$ *such that*

$$\lim_{n\to+\infty} \sup_{P\in\Pi_{\mathcal{H}}(\delta_n)} |\delta^*(x^*, L_r[\sigma(\Gamma, P)]) - \delta^*(x^*, L_r[u])| = 0 \text{ for all } x^* \in X^*,$$

for all $r \in]0,1]$*, where* $\Pi_{\mathcal{H}}(\delta_n)$ *denotes the collection of all Henstock partitions subordinate to* δ_n *.*

Proof. The "if" part is trivial. To prove the "only if" part let (δ_n) be as mentioned in Definition 3.8 and set

$$a_n := \sup_{P\in\Pi_{\mathcal{H}}(\delta_n)} |\delta^*(x^*, L_r[\sigma(\Gamma, P)]) - \delta^*(x^*, L_r[u])| \qquad (n \geq 1).$$

For each $n \geq 1$ select $P_n \in \Pi_{\mathcal{H}}(\delta_n)$ such that

$$\begin{aligned} |\delta^*(x^*, L_r[\sigma(\Gamma, P_n)]) - \delta^*(x^*, L_r[u])| &\geq a_n - \frac{1}{n} \text{ if } a_n < \infty \\ &\geq 1 \qquad\quad \text{if } a_n = \infty. \end{aligned}$$

Since

$$\lim_{n\to\infty} |\delta^*(x^*, L_r[\sigma(\Gamma, P_n)]) - \delta^*(x^*, L_r[u])| = 0,$$

we must have $a_n < \infty$ except perhaps for a finite number of indices n, and it follows then that $\lim_{n\to\infty} a_n = 0$.
■

Remark 3.16. *Proceeding analogously, Proposition 3.15 remains valid in the context of* $\mathcal{FSM}$*-integrals.*

Proposition 3.17. *Let $\Gamma : [0,1] \longrightarrow \mathcal{F}(X)$ be a fuzzy number $\mathcal{FSH}$-integrable valued function. Then for each $r \in]0,1]$ the function $\delta^*(x^*, L_r[\Gamma])$ is scalarly Henstock integrable and*

$$\delta^*(x^*, L_r[(\mathcal{FSH})\int_0^1 \Gamma\, d\lambda]) = (\mathcal{KH})\int_0^1 \delta^*(x^*, L_r[\Gamma])\, d\lambda,$$

for all $x^* \in X^*$ and $r \in]0,1]$.

Proof. As consequence of Remark 2.2 and Proposition 3.15. ∎

Lemma 3.18 (*Saks scalar fuzzy Henstock lemma*). *Let $\Gamma : [0,1] \longrightarrow \mathcal{F}(X)$ be a fuzzy number $\mathcal{FSH}$-integrable valued function and (δ_n) be a sequence of gauges on $[0,1]$ such that*

$$\lim_{n\to+\infty} \delta^*(x^*, L_r[\sigma(\Gamma, P_n)]) = \delta^*(x^*, L_r[(\mathcal{FSH})\int_0^1 \Gamma\, d\lambda]),$$

for all $x^* \in X^*$, for all $r \in]0,1]$ and for each sequence (P_n) of Henstock partitions adapted to (δ_n). Then

$$\lim_{n\to+\infty} |\sum_{i=1}^{k_n}[\lambda(I_i^n)\delta^*(x^*, L_r\Gamma(t_i^n)]) - (\mathcal{KH})\int_{I_i^n} \delta^*(x^*, L_r[\Gamma])\, d\lambda| = 0,$$

for all $x^* \in X^*$, for all $r \in]0,1]$ and for each sequence $((I_i^n, t_i^n)_{1\le i\le k_n})_{n\ge 1}$ of partial Henstock partitions adapted to (δ_n).

Proof. By Propositions 3.15 and 3.17, there exists a sequence (δ_n) of gauges from $[0,1]$ such that

$$\lim_{n\to+\infty} \sup_{P\in\Pi_{\mathcal{H}(\delta_n)}} |\sigma(\delta^*(x^*, L_r[\Gamma]), P) - (\mathcal{KH})\int_0^1 \delta^*(x^*, L_r[\Gamma])\, d\lambda| = 0.$$

for all $x^* \in X^*$, for all $r \in]0,1]$, where $\Pi_{\mathcal{H}}(\delta_n)$ denotes the collection of all Henstock partitions subordinate to δ_n. Let $x^* \in X^*$, $r \in]0,1]$ and $\varepsilon > 0$. Then there exists $N \ge 1$ (possibly depending on x^*) such that

$$\sup_{P\in\Pi_{\mathcal{H}(\delta_n)}} |\sigma(\delta^*(x^*, L_r[\Gamma]), P) - (\mathcal{KH})\int_0^1 \delta^*(x^*, L_r[\Gamma])\, d\lambda| \le \varepsilon \text{ for all } n \ge N.$$

By application of Lemma 2.5 to the function $\delta^*(x^*, L_r[\Gamma])$ we get

$$\sup_{(I_i,t_i)_{1\le i\le k}\in\Pi^p_{\mathcal{H}}(\delta_n)} |\sum_{i=1}^{k}[\lambda(I_i)\delta^*(x^*, L_r[\Gamma(t_i)]) - (\mathcal{KH})\int_{I_i} \delta^*(x^*, L_r[\Gamma])\, d\lambda]| \le \varepsilon,$$

for all $n \ge N$, where $\Pi^p_{\mathcal{H}}(\delta_n)$ denotes the collection of all partial Henstock partitions subordinate to δ_n. Since this holds for all $n \ge N$ and ε was arbitrary, it follows that

$$\lim_{n\to+\infty} \sup_{(I_i,t_i)_{1\le i\le k}\in\Pi^p_{\mathcal{H}}(\delta_n)} |\sum_{i=1}^{k}[\lambda(I_i)\delta^*(x^*, L_r\Gamma(t_i)]) - (\mathcal{KH})\int_{I_i} \delta^*(x^*, L_r[\Gamma])\, d\lambda]| = 0.$$

∎

4. From Fuzzy scalar Henstock to Fuzzy scalar McShane integrability

In this section we present our principal result in which we give a sufficient condition so that a fuzzy scalar Henstock integrable function is also fuzzy scalar McShane integrable as the following theorem shows.

Theorem 4.1. *Let* $\Gamma : [0,1] \longrightarrow \mathcal{F}(X)$ *be a fuzzy-number valued function on* $[0,1]$, *such that the set-valued function* $supp[\Gamma]$ *is bounded on* $[0,1]$. *Then* Γ *is* $\mathcal{SFM}$*-integrable on* $[0,1]$ *if and only if* $\mathcal{SFH}$*-integrable on* $[0,1]$. *Then we have*

$$(\mathcal{SFM})\int_0^1 \Gamma\, d\lambda = (\mathcal{SFH})\int_0^1 \Gamma\, d\lambda.$$

The proof of Theorem 4.1 involves the following technic lemma:

Lemma 4.2. *Let* $\phi : [0,1] \longrightarrow \mathbb{R}$ *be a function,* $(\delta_n)_{n\geq 1}$ *sequence of gauges on* $[0,1]$ *and* $\varepsilon, \eta > 0$ *such that*

$$\limsup_{n\to+\infty} \sigma(\phi, P_n^H) \leq \eta,$$

for each sequence (P_n^H) *of partial Henstock partitions adapted to* (δ_n). *Then*

$$\limsup_{n\to+\infty} \lambda([0,1]\bigcap \bigcup_{\phi(t)\geq\varepsilon}]t-\delta_n(t), t+\delta_n(t)[) \leq \frac{\eta}{\varepsilon}.$$

Proof. For each $n \geq 1$, choose a compact K_n of $[0,1]$ contained on $[0,1]\bigcap\bigcup_{\phi(t)\geq\varepsilon}]t-\delta_n(t), t+\delta_n(t)[$ such that

$$\lambda([0,1]\bigcap \bigcup_{\phi(t)\geq\varepsilon}]t-\delta_n(t), t+\delta_n(t)[\backslash K_n) \leq \frac{1}{n}.$$

According to Lemma 5 [21], there exists a sequence $(P_n^H)_{n\geq 1} = ((I_i^n, t_i^n)_{1\leq i\leq k_n})_{n\geq 1}$ of partial Henstock partitions adapted to (δ_n) such that

$$\phi(t_i^n) \geq \varepsilon \ \text{ for all } \ 1 \leq i \leq k_n \ \text{ and } \ K_n \subset \bigcup_{i=1}^{k_n} I_i^n,$$

for every $n \geq 1$. Therefore

$$\varepsilon\lambda(K_n) \leq \varepsilon \sum_{i=1}^{k_n} \lambda(I_i^n) \leq \sigma(\phi, P_n^H) \ \text{ for all } \ n \geq 1.$$

By letting $n \to +\infty$ in the following inequality

$$\begin{aligned} & \varepsilon\lambda([0,1]\bigcap \bigcup_{\phi(t)\geq\varepsilon}]t-\delta_n(t), t+\delta_n(t)[) \\ = \ & \varepsilon\lambda([0,1]\bigcap \bigcup_{\phi(t)\geq\varepsilon}]t-\delta_n(t), t+\delta_n(t)[\backslash K_n) + \varepsilon\lambda(K_n) \\ \leq \ & \frac{\varepsilon}{n} + \sigma(\phi, P_n^H) \ \text{ for all } \ n \geq 1 \end{aligned}$$

we get

$$\varepsilon \limsup_{n\to+\infty} \lambda([0,1]\bigcap \bigcup_{\phi(t)\geq\varepsilon}]t-\delta_n(t), t+\delta_n(t)[) \leq \eta.$$

This finish the proof of lemma. ∎

Proof of Theorem 4.1.

If Γ is $\mathcal{FSM}$-integrable on $[0,1]$, then it is $\mathcal{FSH}$-integrable on $[0,1]$ (Proposition 3.10). Conversely, suppose that Γ is $\mathcal{FSH}$-integrable on $[0,1]$ and prove that it is $\mathcal{FSM}$-integrable on $[0,1]$ and

$$(\mathcal{FSM})\int_0^1 \Gamma\, d\lambda = (\mathcal{FSH})\int_0^1 \Gamma\, d\lambda.$$

Let $M := \sup_{t\in[0,1]} \|supp[\Gamma(t)]\|$ and $\varepsilon > 0$. According to Proposition 3.17, Theorem 9.13 [24] and the following inequality

$$\text{(4.1.1)} \qquad \delta^*(x^*, L_r[\Gamma]) \le \delta^*(x^*, supp[\Gamma]) \le M$$

for each $r \in]0,1]$, the set-valued function $L_r[\Gamma]$ is scalarly integrable and

$$\text{(4.1.2)} \qquad \int_I \delta^*(x^*, L_r[\Gamma])\, d\lambda = (\mathcal{KH})\int_I \delta^*(x^*, L_r[\Gamma])\, d\lambda,$$

for all $x^* \in X^*$, $r \in]0,1]$ and for every closed subinterval $I \subset [0,1]$. Next, define the set

$$C = \{\delta^*(x^*, L_r[\Gamma]),\ \ x^* \in \overline{B}_{X^*},\ r \in]0,1]\}.$$

Since the space of Lebesgue integrable functions, provided with the L_1-norm, is separable, we can find a sequence $(\phi_\ell)_{\ell\ge1}$ which is L_1-dense in C. That is

$$\forall \phi \in C\ \exists l \ge 1,\ \int_0^1 |\phi - \phi_l|\, d\lambda \le \varepsilon^2.$$

The functions ϕ_l are McShane integrables, because are Lebesgue integrables (Theorem 10.11 [24]), therefore, for each $l \ge 1$ we can select a sequence $(\delta_{n,l})_{n\ge1}$ of gauges on $[0,1]$ such that, for every sequence P_n^M of McShane partitions of $[0,1]$ adapted to $(\delta_{n,l})_{n\ge1}$, we have

$$\lim_{n\to+\infty} |\sigma(\phi_l, P_n^M) - \int_0^1 \phi_l\, d\lambda| = 0.$$

For each $n \ge 1$ define a gauge Δ_n^1 by

$$\Delta_n^1 := \inf\{\delta_{n,1}, ..., \delta_{n,n}\}.$$

Then, for every sequence P_n^M of McShane partitions of $[0,1]$ adapted to Δ_n^1, we have

$$\text{(4.1.3)} \qquad \lim_{n\to+\infty} |\sigma(\phi_l, P_n^M) - \int_0^1 \phi_l\, d\lambda| = 0 \text{ for all } l \ge 1.$$

Next, as Γ is $\mathcal{FSH}$-integrable on $[0,1]$, we can select a sequence (Δ_n^2) of gauges on $[0,1]$ such that, for every $r \in]0,1]$ and for every sequence P_n^H of Henstock partitions of $[0,1]$ adapted to (Δ_n^2), we have

$$\lim_{n\to+\infty} \delta^*(x^*, L_r[\sigma(\Gamma, P_n^H)]) = \delta^*(x^*, L_r[(\mathcal{FSH})\int_0^1 \Gamma\, d\lambda]),$$

for all $x^* \in X^*$. Define a sequence (Δ_n) of gauges on $[0,1]$ by writing

$$\Delta_n := \inf\{\Delta_n^1, \Delta_n^2\} \ \ n \ge 1.$$

Let $\phi \in C$ be fixed but arbitrary and take $l_0 \ge 1$ such that

$$\text{(4.1.4)} \qquad \int_0^1 |\phi - \phi_{l_0}|\, d\lambda \le \varepsilon^2.$$

We seek to prove that

$$\limsup_{n\to+\infty} \lambda([0,1] \bigcap V_n) \leq \varepsilon,$$

where

$$V_n := \bigcup_{\phi(t)-\phi_{l_0}(t)\geq\varepsilon}]t-\Delta_n(t), t+\Delta_n(t)[, \;\; n \geq 1.$$

Indeed, if $(P_n^H)_{n\geq 1} = ((K_j^n, u_j^n)_{1\leq j\leq s_n})_{n\geq 1}$ be a sequence of partial Henstock partitions adapted to (Δ_n), then according to Lemma 3.18 and (4.1.2), we have

$$\lim_{n\to+\infty} |\sum_{j=1}^{s_n} [\lambda(K_j^n)\delta^*(x^*, L_r[\Gamma(u_j^n)]) - \int_{K_j^n} \delta^*(x^*, L_r[\Gamma])\, d\lambda]| = 0,$$

for all $x^* \in \overline{B}_{X^*}$ and $r \in]0,1]$. In other words

$$\lim_{n\to+\infty} |\sigma(\psi, P_n^H) - \int_{H_n} \psi\, d\lambda| = 0 \;\text{ for all }\; \psi \in C,$$

where $H_n = \bigcup_{j=1}^{s_n} K_j^n$ $n \geq 1$. In particular, we have

$$\text{(4.1.5)} \qquad \lim_{n\to+\infty} |\sigma(\phi - \phi_{l_0}, P_n^H) - \int_{H_n} \phi - \phi_{l_0}\, d\lambda| = 0.$$

According to (4.1.4), (4.1.5) and the following inequality

$$\begin{aligned}\sigma(\phi - \phi_{l_0}, P_n^H) &\leq \int_0^1 |\phi - \phi_{l_0}| d\lambda + |\sigma(\phi - \phi_{l_0}, P_n^H) - \int_{H_n} \phi - \phi_{l_0}\, d\lambda| \\ &\leq \varepsilon^2 + |\sigma(\phi - \phi_{l_0}, P_n^H) - \int_{H_n} \phi - \phi_{l_0}\, d\lambda|,\end{aligned}$$

we get

$$\limsup_{n\to+\infty} \sigma(\phi - \phi_{l_0}, P_n^H) \leq \varepsilon^2.$$

Therefore, we can invoke Lemma 4.1, which yields the inequality

$$\limsup_{n\to+\infty} \lambda([0,1] \bigcap V_n) \leq \varepsilon.$$

Now let $(P_n^M)_{n\geq 1} = ((I_i^n, t_i^n)_{1\leq i\leq k_n})_{n\geq 1}$ be a sequence of McShane partitions on $[0,1]$ adapted to (Δ_n), then by remark that

$$\bigcup_{1\leq i\leq k_n,\; \phi_{l_0}(t_i^n)-\phi(t_i^n)\geq\varepsilon} I_i^n \subset [0,1] \bigcap V_n \;\text{ for all }\; n \geq 1,$$

we get

$$\limsup_{n\to+\infty} \sum_{1\leq i\leq k_n,\; \phi(t_i^n)-\phi_{l_0}(t_i^n)\geq\varepsilon} \lambda(I_i^n) \leq \varepsilon.$$

Similarly,

$$\limsup_{n\to+\infty} \sum_{1\leq i\leq k_n,\; \phi_{l_0}(t_i^n)-\phi(t_i^n)\geq\varepsilon} \lambda(I_i^n) \leq \varepsilon.$$

So

$$\text{(4.1.6)} \qquad \limsup_{n\to+\infty} \sum_{1\leq i\leq k_n,\; |\phi(t_i^n)-\phi_{l_0}(t_i^n)|\geq\varepsilon} \lambda(I_i^n) \leq 2\varepsilon.$$

On the other hand

$$\begin{aligned}\sum_{i=1}^{k_n}\lambda(I_i^n)|\phi(t_i^n)-\phi_{I_0}(t_i^n)| &= \sum_{1\le i\le k_n,\ |\phi(t_i^n)-\phi_{I_0}(t_i^n)|\ge\varepsilon}\lambda(I_i^n)|\phi(t_i^n)-\phi_{I_0}(t_i^n)+\sum_{1\le i\le k_n,\ |\phi(t_i^n)-\phi_{I_0}(t_i^n)|\le\varepsilon}\lambda(I_i^n)|\phi(t_i^n)-\phi_{I_0}(t_i^n)|\\ &\le 2M\sum_{1\le i\le k_n,\ |\phi(t_i^n)-\phi_{I_0}(t_i^n)|\ge\varepsilon}\lambda(I_i^n)+\varepsilon.\end{aligned}$$

because

$$|\phi(t_i^n)-\phi_{I_0}(t_i^n)|\le 2M \text{ for all } 1\le i\le k_n \text{ and } n\ge 1,$$

This together with (4.1.4) implies

$$\begin{aligned}|\sigma(\phi,P_n^M)-\int_0^1\phi\,d\lambda| &\le |\int_0^1\phi\,d\lambda-\int_0^1\phi_{I_0}\,d\lambda|+|\sigma(\phi_{I_0},P_n^M)-\int_0^1\phi_{I_0}\,d\lambda|+|\sigma(\phi,P_n^M)-\sigma(\phi_{I_0},P_n^M)|\\ &\le \varepsilon^2+|\sigma(\phi_{I_0},P_n^M)-\int_0^1\phi_{I_0}\,d\lambda|+2M\sum_{1\le i\le k_n,\ |\phi(t_i^n)-\phi_{I_0}(t_i^n)|\ge\varepsilon}\lambda(I_i^n)+\varepsilon,\end{aligned}$$

for every $n\ge 1$. Therefore, using (4.1.3) and (4.1.6) we get

$$\limsup_{n\to+\infty}|\sigma(\phi,P_n^M)-\int_0^1\phi\,d\lambda|\le\varepsilon^2+\varepsilon+4M\varepsilon.$$

By the arbitrariness of $\varepsilon>0$, we conclude that

$$\lim_{n\to+\infty}\sigma(\phi,P_n^M)=\int_0^1\phi\,d\lambda \quad\text{for all } \phi\in C.$$

Using (4.1.2) and Proposition 3.17 this equality becomes

$$\lim_{n\to+\infty}\delta(x^*,L_r[\sigma(\Gamma,P_n^M)])=\delta^*(x^*,L_r[(\mathcal{FSH})\int_0^1\Gamma\,d\lambda)],$$

for all $x^*\in\overline{B}_{X^*}$ and $r\in]0,1]$. Thus Γ is $\mathcal{FSM}$-integrable on $[0,1]$ and

$$(\mathcal{FSM})\int_0^1\Gamma\,d\lambda=(\mathcal{FSH})\int_0^1\Gamma\,d\lambda.$$

This finish the proof of our theorem. ■

Remark 4.3. *Remark that if $\Gamma(t)=1_{\{f(t)\}}$, where f is a real-valued function, Theorem 4.1 becomes Gordon theorem (Theorem 9.13 [24]) concerning real-valued McShane and $\mathcal{KH}$-integrals.*

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