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Kuelbs-Steadman Spaces with Bounded Variable Exponents

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Abstract. Kuelbs-Steadman spaces are studied within the framework of Henstock-Kurzweil integrable function spaces with bounded variable exponent. We describe a relationship between the Lebesgue spaces with bounded variable exponents and variable Kuelbs-Steadman spaces. The geometrical properties of the spaces are studied. Finally, we discuss the boundedness behaviour of the maximal operator on variable Kuelbs-Steadman spaces.

1. Introduction and preliminaries

Let Ω be a set in $\mathbb{R}^n$ with $|\Omega| > 0$. Lebesgue spaces with variable exponent appeared for the first time in 1931 by W. Orlicz is a generalization of classical L^p spaces, replacing the constant exponent p with an exponent function $p(.)$ consist of all functions f such that $\int_{\mathbb{R}^n} |f(x)|^{p(x)} dx < \infty$. Nakano (see [16, 17]) has conducted extensive research on this space. Various results on maximal, potential and singular operators in variable Lebesgue spaces were obtained in the articles [3, 4, 11, 12, 14, 16, 20, 22]. Sharapudinov in [23] introduced the Lexemberg norm for the lebesgue space and shown the reflexivity. In literature Kuelbs-Steadman space $KS^p(\Omega)$ was introduced by T.L. Gill and W.W. Zachary in 2008 (see [9]). Interesting fact of this space is that it is a Banach space which parallels the standard L^p spaces, but contains as dense compact embeddings. These spaces are of particular interest because they contain the Henstock-Kurzweil integrable functions and the HK-measure, which generalizes the Lebesgue measure . In all section of the article the Lebesgue measure of set or functions are separable. We denote the Lebesgue measure and the characteristic function for a set $A \subset \mathbb{R}^n$ by $\mu(A)$ and $ch(A)$. We denote $\mathcal{P}(\Omega)$ the family of all (measurable) functions $\mathcal{P} : \Omega \to [0, \infty]$. For $p \in \mathcal{P}(\Omega)$, we put $\Omega_1 = \{x \in \Omega : p(x) = 1\}$, $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$, $\Omega_0 = \Omega \setminus (\Omega_1 \cup \Omega_\infty)$ and $p_* = ess \inf_{\Omega_0} p(x)$, $p^* = ess \sup_{\Omega_0} p(x)$ if $|\Omega| > 0$. We assume $p^* < \infty$ in our work. Under this assumption with the assumption of boundedness of $p(.)$; $L^{p(.)}$ gives good behaviour for many fundamental results (see [21] and references therein). Throughout the article $C_0^\infty(\mathbb{R}) = \left\{ f \in C^\infty(\mathbb{R}) : \overline{supp f} \subsetneq \mathbb{R} \right\}$ is the space of bump functions i.e., functions that are both smooth, in the sense of having continuous (strong) derivatives of all orders, and compactly supported. In section 2, we discuss Kuelbs-Steadman space with variable exponent with its fundamental properties and geometrical properties. Under our assumption of $p(x)$, $p(x)$ is not allowed to tend to infinity. In this case of a bounded set Ω, the function $p(x)$ will be supposed to satisfy

$$1 \leq p_0 \leq p(x) \leq p < \infty, \; x \in \Omega \tag{1}$$

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$$|p(x) - p(y)| \leq \frac{A}{\ln\left(\frac{1}{|x-y|}\right)},\ |x-y| \leq \frac{1}{2},\ x, y \in \Omega. \tag{2}$$

When Ω is unbounded that is $p(\infty) = \lim_{|x|\to\infty} p(x)$ and

$$|p(x) - p(y)| \leq \frac{C}{\ln[e + \min(|x|, |y|)]}\ x, y \in \Omega. \tag{3}$$

For $L^{p(.)}$, we can recall the following results with their proof:
Given Ω, $p(x) \in \mathcal{P}(\Omega)$ in short we write $p(.) \in \mathcal{P}(\Omega)$ and a measurable function f, define the modular functional associated with $p(.)$ by

$$\rho(f) = \rho_{p(.)}(f) = \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)} dx + \|f\|_{L^\infty(\Omega_\infty)} \tag{4}$$

Proposition 1.0.1. *Given Ω, $p(.) \in \mathcal{P}(\Omega)$. If $p^* < \infty$, then $f \in L^{p(.)}(\Omega)$ if and only if $\rho(f) = \int_\Omega |f(x)|^{p(x)} dx < \infty$.*

Proof. Since $p^* < \infty$, we can drop the L^∞ term in the modular. If $\rho(f) < \infty$, then $f \in L^{p(.)}$.

Coversely, by [25, Property 5, Proposition 2.7], we have $\rho\left(\frac{f}{\lambda}\right) < \infty$ for some $\lambda > 1$. But then

$$\begin{aligned}\rho(f) &= \int_\Omega \left(\frac{|f(x)\lambda}{\lambda}\right)^{p(x)} d\mu(x)\\ &\leq \lambda^{p^*(\Omega)} \rho\left(\frac{f}{\lambda}\right)\\ &< \infty.\end{aligned}$$

□

Theorem 1.0.2. *$L^{p(.)}(\Omega)$ is a Banach space endowed with a norm*

$$\|f\|_{p(.)} = \inf\left\{\lambda > 0\ \ \rho_p(\frac{f}{\lambda}) \leq 1\right\},\ f \in L^{p(.)}(\Omega). \tag{5}$$

Proof. Let $\{f_k\} \subset L^{p(.)}(\Omega)$ be a Cauchy sequence. Choose k_1 such that $\|f_i - f_j\|_{p(.)} < 2^{-1}$ for $i,\ j \geq k_1$; choose $k_2 > k_1$ such that $\|f_i - f_j\|_{p(.)} < 2^{-2}$ for $i,\ j \geq k_2$, and so on.

This construction yields a subsequence $\{f_{k_j}\}$, $k_{j+1} > k_j$, such that

$$\|f_{k_{j+1}} - f_{k_j}\|_{p(.)} < 2^{-j}.$$

Define the new sequence $\{g_j\}$ by $g_1 = f_{k_1}$ and for $j > 1$, $g_j = f_{k_j} - f_{k_{j-1}}$. Then for all j we get the telescoping sum

$$\sum_{i=1}^{j} g_i = f_{k_j};$$

further, we have that

$$\sum_{j=1}^{\infty} \|g_j\|_{p(.)} \leq \|f_{k_1}\|_{p(.)} + \sum_{j=1}^{\infty} 2^{-j} < \infty.$$

Therefore, by [25, Theorem 2.24], there exists $f \in L^{p(.)}(\Omega)$ such that $f_{k_j} \to f$ in norm.

Finally, by the triangle inequality we have that

$$\|f - f_k\|_{p(.)} \leq \|f - f_{k_j}\|_{p(.)} + \|f_{k_j} - f_k\|_{p(.)};$$

since $\{f_k\}$ is a Cauchy sequence, we can make both terms on the right-hand side as small as desired. Hence, $f_k \to f$ in norm. □

Theorem 1.0.3. *Given Ω, and $p(.) \in \mathcal{P}(\Omega)$. Suppose $p^* < \infty$ for any sequence $(f_n) \subset L^{p(.)}(\Omega)$ and $f \in L^{p(.)}(\Omega)$ then*

$$\|f_n - f\|_{p(.)} \to 0 \Leftrightarrow \rho(f - f_n) \to 0.$$

Proof. Suppose the sequence converges in norm. By [25, Corollary 2.16], for all k sufficiently large,

$$\rho(f - f_k) \le \|f - f_k\|_{p(.)} \le 1,$$

and so $\rho(f - f_k) \to 0$.

To prove the converse, fix $\lambda < 1$. By [25, Proposition 2.10],

$$\rho\Big(\frac{(f - f_k)}{\lambda}\Big) \le \Big(\frac{1}{\lambda}\Big)^{p^*} \rho(f - f_k).$$

Hence, for all k sufficiently large we have that

$$\rho\Big(\frac{f - f_k}{\lambda}\Big) \le 1.$$

Equivalently, for all such k, $\|f - f_k\|_{p(.)} \le \lambda$. Since λ was arbitrary, $f_k \to f$ in norm. □

2. Kuelbs-Steadman spaces with variable exponent

Kuelbs-Steadman spaces with variable exponents are a concept that will be introduced in this section. We recall the construction of $KS^p(\mathbb{R}^n)$ as follows:

Let $\{B_k\}_{k=1}^{\infty}$ is the countable collection of balls in $\mathbb{R}^n$ such that radius $B_r = \Gamma(B_l)$ is of the form 2^{-l}, $l \in \mathbb{N}$, and the centre of B_k is contained in $\mathbb{Q}^n$. Let $\tau = \{t_k\}$ be a non negative real sequence such that $\sum_{k=1}^{\infty} t_k = 1$. Let $\mathcal{E}_k(x)$ be the characteristic function of B_k, so that $\mathcal{E}_k(x)$ is in $L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ for $1 \le p < \infty$. Recalling the space $KS^p(\mathbb{R}^n)$ is the closure of $L^p(\mathbb{R}^n)$ with respect the norm

$$\|f\|_{KS^p} = \Big(\sum_{k=1}^{\infty} t_k \Big| \int_{B_k} \mathcal{E}_k(x) f(x) d\mu(x)\Big|^p\Big)^{\frac{1}{p}}.$$

By defining the weighted l^p space $l^p(\tau)$, we can write this as follows:

$$\|\{\sigma_k\}\|_{l^p(\tau)} = \Big(\sum_{k=1}^{\infty} t_k |\sigma_k|^p\Big)^{\frac{1}{p}}.$$

Then,

$$\begin{aligned}\Big\|f\Big\|_{KS^p} &= \Big\|\Big\{\int_{B_k} f(x)\Big\}\Big\|_{l^p(\tau)} \\ &= \Big\|\Big\{f(B_k)\Big\}\Big\|_{l^p(\tau)}.\end{aligned}$$

To extend this definition to the variable exponent setting, define $p(.) : \mathbb{R}^n \to [1, \infty)$ to be a measurable exponent,

$$p_k = p_{B_k} = \Big(\frac{1}{|B_k|} \int_{B_k} \frac{1}{P(X)} dX\Big)^{-1}.$$

In the other words p_k is the harmonic mean of $p(.)$ on B_k. Define $l^{p_k}(\tau)$ to be the variable exponent sequence space with the norm

$$\|\{\sigma_k\}\|_{l^{p_k}}(\tau) = \inf\Big\{\lambda > 0 \ : \sum_{k=1}^{\infty} t_k \Big(\frac{|\sigma_k|}{\lambda}\Big)^{p_k} \le 1\Big\}.$$

This is a Banach function space and behaves much as other sequence spaces to. We now define $KS^{p(.)}$ to be the completion of $L^{p(.)}$ with respect to the norm

$$\rho_0(f) = \|f\|_{KS^{p(.)}} = \|\{f(B_k)\}\|_{l^{p_k}}(\tau).$$

This norm is well defined as $f \in L^{p(.)}$ implies $\|f\|_{KS^{p(.)}} < \infty$.

Definition 2.0.1. *Given an exponent function $p(.) \in \mathcal{P}(\Omega)$ we define $KS^{p(.)}(\Omega)$ to be the Henstock-Kurzweil integrable function (measurable with compact support) f such that $\rho_0(\frac{f}{\lambda}) < \infty$ for some $\lambda > 0$.*

Proposition 2.0.2. *Given Ω, and $p(.) \in \mathcal{P}(\Omega)$ then:*

1. *For all f, $\rho_0(f) \geq 0$ and $\rho_0(|f|) = \rho_0(f)$.*
2. *$\rho_0(f) = 0$ if and only if $f = 0$ for a.e. $x \in \Omega$.*
3. *If $\rho_0(f) < \infty$ then $f(x) < \infty$ for a.e. $x \in \Omega$.*
4. *ρ_0 is convex given α, $\beta \geq 0$, $\alpha + \beta = 1$*
   $\rho_0(\alpha f + \beta g) \leq \alpha\rho_0(f) + \beta\rho_0(g)$
5. *If $|f(x)| \geq |g(x)|$ a.e. then $\rho_0(f) \geq \rho_0(g)$.*

Proof. For (1) using the definition of $\rho_0(f)$.
To prove (2): Let $\rho_0(f) = 0$ if and only if $f = 0$ a.e. $x \in \Omega$.

$$\begin{aligned}
\rho_0(f) = 0 &\Leftrightarrow \inf\left\{\lambda > 0 : \sum_{k=1}^{\infty} t_k\left(\frac{f(B_k)}{\lambda}\right)^{p_k} \leq 1\right\} = 0 \\
&\Leftrightarrow \sum_{k=1}^{\infty} f_k\left(\frac{f(B_k)}{\lambda}\right)^{p_k} = 0 \\
&\Leftrightarrow f(B_k) = 0 \\
&\Leftrightarrow f = 0 \text{ a.e. } x \in \Omega.
\end{aligned}$$

For (3) using property of L^∞ norm.
For (4) using (3)
For (5) As $|f(x)| \geq |g(x)|$ so, $f(x) \geq g(x)$. Using $f(x) \geq g(x)$ in definition of $\rho_0(f)$, we get the proof. □

Proposition 2.0.3. *Given Ω, $p(.) \in \mathcal{P}(\Omega)$. If $p^* < \infty$, then $f \in KS^{p(.)}(\Omega)$ if and only if $\rho_0(f) < \infty$.*

Theorem 2.0.4. *If $p(.) \in \mathcal{P}(\Omega)$ for Ω then $KS^{p(.)}(\Omega)$ is a vector space.*

Proof. Since the set of all Lebesgue measurable functions is itself a vector space, and since $0 \in KS^{p(.)}(\Omega)$,it will suffice to show that for all $\alpha, \beta \in \mathbb{R}$ not both zero, if $f, g \in KS^{p(.)}(\Omega)$, then $\alpha f + \beta g \in KS^{p(.)}(\Omega)$. Let $\mu = (|\alpha| + |\beta|)\lambda$ then,

$$\begin{aligned}
\rho_0\left(\frac{\alpha f + \beta g}{\mu}\right) &= \rho_0\left(\frac{|\alpha f + \beta g|}{\mu}\right) \\
&\leq \rho_0\left(\frac{|\alpha|}{|\alpha| + |\beta|}\frac{|f|}{\lambda} + \frac{|\beta|}{|\alpha| + |\beta|}\frac{|g|}{\lambda}\right) \\
&\leq \frac{|\alpha|}{|\alpha| + |\beta|}\rho_0(f) + \frac{|\beta|}{|\alpha| + |\beta|}\rho_0(g) \\
&< \infty.
\end{aligned}$$

Therefore, $\alpha f + \beta g \in KS^{p(.)}(\Omega)$. □

On the Kuelbs-Steadman spaces, if $1 \leq p < \infty$, then the norm is gotten directly from the modular:

$$\left\|f\right\|_{KS^p} = \left\{ \ \left\{\sum_{k=1}^{\infty} t_k \left|\int_{\mathbb{R}^n} \mathcal{E}_k(x) f(\mathbf{x}) dx\right|^p\right\}^{1/p}, 1 \leqslant p < \infty.\right.$$

Such a definition obviously fails since we cannot replace the constant exponent $\frac{1}{p}$ outside the integral with the exponent function $\frac{1}{p(.)}$. The solution is a more subtle approach which is similar to that used to define the Luxemburg norm on Orlicz spaces. We define norm of $KS^{p(.)}(\Omega)$ as

$$\|f\|_{p(.)} = \inf\left\{\lambda > 0 \ \sum_{k=1}^{\infty} t_k\left(\frac{f(B_k)}{\lambda}\right)^{p_k} \leq 1\right\}, \ f \in L^{p(.)}(\Omega). \tag{6}$$

Since $KS^{p(.)}(\Omega)$ is the completion of $L^{p(.)}(\Omega)$ so we conclude the following theorem.

Theorem 2.0.5. *For $1 \leq q_0 \leq q(x) \leq q < \infty$, $x \in \Omega$, $L^{q(.)}(\Omega) \subset KS^{p(.)}(\Omega)$ as a continuous dense embeddings.*

Remark 2.0.6. *The statement is very precise from [24, Corollary 2.27] that, given any Ω and $p(.) \in \mathcal{P}(\Omega)$ if $f \in KS^{p(.)}(\Omega)$ then f is locally integrable.*

Theorem 2.0.7. *$KS^{p(.)}(\Omega)$ is a Banach space endowed with a norm*

$$\|f\|_{p(.)} = \inf\left\{\lambda > 0 \ \sum_{k=1}^{\infty} t_k\left(\frac{f(B_k)}{\lambda}\right) \leq 1\right\}, \ f \in L^{p(.)}(\Omega). \tag{7}$$

Proof. We need to prove the following properties :

1. $\|f\|_{p(.)} = 0 \ \Leftrightarrow \ f = 0.$
2. For all $\alpha \in \mathbb{R}$, $\|\alpha f\|_{p(.)} = |\alpha|\|f\|_{p(.)}$
3. For $f, g \in KS^{p(.)}$, $\|f + g\|_{p(.)} \leq \|f\|_{p(.)} + \|g\|_{p(.)}$

For (1) If $f = 0$, then $f(B_k) = 0 < 1$ for all $\lambda > 0$. Hence $\|f\|_{p(.)} = 0$.
Conversely, let $\|f\|_{p(.)} = 0$. Then for all $\lambda > 0$,

$$\|f\|_{p(.)} = \inf\{\lambda > 0 : \ \sum_{k=1}^{\infty} t_k(\frac{f(B_k)}{\lambda})^{p_k} \leq 1\} = 0$$

implies that $f(B_k) = 0$ for $\lambda > 0$. Hence $f = 0$ a.e.
For (2) If $\alpha = 0$ then the condition is true. Let $\alpha \neq 0$

$$\begin{aligned}\|\alpha f\|_{p(.)} &= \inf\left\{\alpha > 0 : \ \sum_{k=1}^{\infty} t_k(\frac{|\alpha| f(B_k)}{\lambda})^{p_k} \leq 1\right\} \\ &= |\alpha| \inf\left\{\frac{\lambda}{|\alpha|} > 0 \ : \ \sum_{k=1}^{\infty} t_k\left(\frac{f(B_k)}{\frac{\lambda}{|\alpha|}}\right) \leq 1\right\} \\ &= |\alpha|\|f\|_{p(.)}.\end{aligned}$$

For (3) let $f, g \in L^{p(.)}$. Now,

$$\begin{aligned}\|f + g\|_{KS^{p(.)}} &= \inf\left\{\lambda > 0 : \ \sum_{k=1}^{\infty} t_k\left(\frac{(f+g)(B_k)}{\lambda}\right)^{p_k} \leq 1\right\} \\ &\leq \inf\left\{\lambda > 0 : \ \sum_{k=1}^{\infty} \left(\frac{f(B_k)}{\lambda}\right)^{p_k} \leq 1\right\} + \inf\{\lambda > 0 : \ \sum_{k=1}^{\infty} \left(\frac{g(B_k)}{\lambda}\right)^{p_k} \leq 1\right\}.\end{aligned}$$

So,

$$\|f+g\|_{p(.)} \le \|f\|_{p(.)} + \|g\|_{p(.)}.$$

The similar technique of the proof of the Theorem 1.0.2 can be used to prove $KS^{p(.)}(\Omega)$ is a Banach space endowed with a norm

$$\|f\|_{p(.)} = \inf\left\{\lambda > 0 \ \sum_{k=1}^{\infty} t_k\left(\frac{f(B_k)}{\lambda}\right) \le 1\right\},\ f \in L^{p(.)}(\Omega). \tag{8}$$

□

Theorem 2.0.8. *Let K be a weakly compact subset of $L^{p(.)}$, then K is a compact subset of $KS^{p(.)}$.*

Proof. Let (f_n) is any weakly convergence in K with limit f, then

$$\rho(f - f_n) \to 0 \ \textit{this gives} \int_{\Omega} |(f - f_n)(x)|^{p(x)} dx \to 0.$$

So, $\sum_{k=1}^{\infty} t_k\left(\frac{f(B_k)}{\lambda}\right)^{p_k} \to 0$ for $\lambda > 0$.
This implies, $\rho_0(f - f_n) \to 0$. Therefore, K is compact subset of $KS^{p(.)}(\Omega)$. □

2.1. Separability of $KS^{p(.)}(\Omega)$

In this subsection, we discuss the separability of $KS^{p(.)}$. We have study few denseness property of $KS^{p(.)}$ for separability as follows:

Lemma 2.1.1. *Given an open set Ω and $p(.) \in \mathcal{P}(\Omega)$. Then the set of Henstock-Kurzweil integrable function which is bounded with compact support with $supp(f) \subset \Omega$ is dense in $KS^{p(.)}$.*

Proof. The set of Henstock-Kurzweil integrable bounded function with compact support is Lebesgue integrable. Using ([24, Theorem 2.72]) *supp*(f) is dense in $KS^{p(.)}(\Omega)$. □

Proposition 2.1.2. *Given an open set Ω and $p(.) \in \mathcal{P}(\Omega)$. If $p^* < \infty$ then the set $C_c(\Omega)$ is dense in $KS^{p(.)}(\Omega)$.*

Proof. Let $f \in KS^{p(.)}(\Omega)$ and fix $\epsilon > 0$, then there exists a function $g \in C_c(\Omega)$ such that

$$\|f - g\|_{p(.)} < \epsilon.$$

Now, using the Lemma 2.10 there exists a bounded function of compact support h, such that

$$\|f - h\|_{p(.)} < \frac{\epsilon}{2}.$$

Let *supp*$(h) \subset B \cap \Omega$ for some open ball B. Since $p^* < \infty$, $C_c(B \cap \Omega)$ is dense in $KS^{p^*}(B \cap \Omega)$ thus there exists $g_0 \in C_c(B \cap \Omega) \subset C_c(\Omega)$. So,

$$\begin{aligned}\|g_0 - g\|_{p(.)} &= \|g_0 - g\|_{p(.)}(B \cap \Omega)\\ &< (1 + |B \cap \Omega|)\|g_0 - g\|_{p^*}(B \cap \Omega)\\ &< \frac{\epsilon}{2}.\end{aligned}$$

□

Corollary 2.1.3. $C_0^{\infty}(\Omega)$ *is dense in* $KS^{p(.)}(\Omega)$.

Theorem 2.1.4. *Given an open set Ω, and $p(.) \in \mathcal{P}(\Omega)$, then $KS^{p(.)}(\Omega)$ is separable if $p^* < \infty$.*

Proof. Let $p^* < \infty$. Then the proof is similar as the Proposition 2.11. Let $\Omega = \cup_{k=1}^{\infty} B_k(0) \cap \Omega$. Since $B_k(0) \cap \Omega$ is open, $KS^{p(.)}(B_k(0) \cap \Omega)$ is separable. So, it contains a countable dense subset. The union of all these sets is a countable set contained in $KS^{p(.)}(\Omega)$ so, this set is dense in $KS^{p(.)}(\Omega)$. □

Remark 2.1.5. *If $p^* = \infty$ that is $|\Omega_\infty| = 0$ then $KS^{p(.)}(\Omega)$ is non separable.*

Theorem 2.1.6. **Holder's type inequality** *Let $p, q, r \in \mathcal{P}(\Omega)$ such that $\frac{1}{r(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}$ for μ−a.e $y \in \Omega$ then,*

$$\rho_{0_{r(.)}}(fg) \le \rho_{0_{p(.)}}(f) + \rho_{0_{q(.)}}(g), \tag{9}$$

$$\rho_{0_{r(.)}}(fg) \le 2\|f\|_{p(.)}\|q\|_{q(.)}. \tag{10}$$

Proof. Let $f \in KS^{p(.)}$ and $q \in KS^{q(.)}$. Since f, g are measurable also, fg is measurable. Now using Young's inequalities by integration over $y \in \Omega$ is the required result of the Theorem (2.1.6).
If $\|f\|_{p(.)} \le 1$ and $\|g\|_{q(.)} \le 1$ then $\rho_{0_{p(.)}} \le 1$ and $\rho_{0_{q(.)}} \le 1$. Using the unit ball property and the Theorem (2.1.6), we get

$$\begin{aligned}\rho_{0_r}(\frac{1}{2}fg) &\le \frac{1}{2}\rho_{0_{r(.)}}(fg)\\ &\le \frac{1}{2}(\rho_{0_{p(.)}}(f) + \rho_{0_{q(.)}}(g))\\ &\le 1.\end{aligned}$$

Hence $\frac{1}{2}\rho_{0_{r(.)}}(fg) \le 1$. Consequently, using $\|f\|_{p(.)} \le 1$ and $\|g\|_{q(.)} \le 1$, gives

$$\rho_{0_{r(.)}}(fg) \le 2\|f\|_{p(.)}\|g\|_{q(.)}.$$

□

2.2. Geometric properties of $KS^{p(.)}$

This part of our article contributes to the study of the geometric characteristics of the spaces $KS^{p(x)}$. We present a characterization of their basic geometric properties, namely reflexivity, uniform convexity.

Definition 2.2.1. *The Banach space $KS^{p(.)}(\Omega)$ is a Banach function space if the following axioms are satisfied:*

1. $f \in KS^{p(.)}$ *if and only if* $\|f\|_{p(.)} < \infty$,
2. $\|f\|_{p(.)} = \||g|\|_{p(.)}$ *for every measurable function on* Ω,
3. $0 \le f_n \to f$ μ*−a.e. implies* $\|f_n\| \to \|f\|$,
4. $\|ch(E)\|_{p(.)} < \infty$ *for every* $E \subset \Omega$ *such that* $\mu(E) < \infty$,
5. *For every $E \subset \Omega$ such that $\mu(E) < \infty$, there exist a constant C_E such that $\int_E \mathcal{E}_k(x)f(x)d\mu(x) \le C_E\|f\|$ for every $f \in KS^{p(.)}$.*

We will define an absolutely continuous norm as follows:

Definition 2.2.2. *$f \in KS^{p(.)}$ has an absolutely continuous norm if for every decreasing sequence $\{D_n\}$ of subsets of Ω satisfying $\mu(D_n) \to 0$ then $\|ch(D_n)\| \to 0$.*

Recalling the Uniformly convex Banach space as follows:

Definition 2.2.3. *[2] A Banach space X is called uniformly convex if for every $\epsilon \in (0, 2]$ there exists a $\delta > 0$ such that*

$$\|\frac{1}{2}(x + y)\| \le 1 - \delta,$$

whenever $x, y \in B_X$, B_X is unit sphere with $\|x - y\| \ge \epsilon$.

Proposition 2.2.4. *Let μ be non atomic and $p^* < \infty$, then $KS^{p(.)}$ has absolutely continuous norm.*

Proof. Let $p^* < \infty$. Let $f \in KS^{p(.)}$ with $\|f\|_{p(.)} = 1$. Assume $\{E_n\}$ is a sequence of sets such that $\mu(E_n) \to 0$. Let $z \in \mathbb{N}$ with $\epsilon > 0$ such that $\|fch(E_n)\| > 1 - \epsilon$. If $\phi = fch(\Omega \setminus E_z)$; $\chi = fch(E_z)$, from the [15, Lemma 2.2] $\int_\Omega |\frac{\phi(x)}{\|\phi\|}|^{p(x)} d\mu(x) = 1$ and $\int_\Omega |\frac{\chi(x)}{\|\chi\|}|^{p(x)} d\mu(x) = 1$. So,

$$\sum_{k=1}^{\infty} | \int_\Omega \mathcal{E}_k(x) \frac{\phi(x)}{\|\phi\|} d\mu(x)|^{p(x)} = 1$$

and $\sum_{k=1}^{\infty} | \int_\Omega \mathcal{E}_k(x) \frac{\chi(x)}{\|\chi\|} d\mu(x)|^{p(x)} = 1$.

Now,

$$\begin{aligned}
\|\chi\|^{p^*} &\leq \left| \sum_{k=1}^{\infty} \nu_k(x) \int_\Omega \mathcal{E}_k(x) \chi(x) d\mu(x) \right|^p \\
&\leq 1 - \sum_{k=1}^{\infty} \nu_k(x) \left| \int_\Omega \mathcal{E}_k(x) \phi(x) d\mu(x) \right|^p \\
&\leq 1 - \|\phi\|^{p^*}.
\end{aligned}$$

Therefore, $\|\chi\| \leq 1 - (1-\epsilon)^{p^*}$. □

Theorem 2.2.5. *Every space $KS^{p(.)}(\Omega)$ is a Banach function space.*

Proof. To prove $KS^{p(.)}(\Omega)$ is a Banach function space, we need to show $f \in KS^{p(.)}$ must satisfy Definition 2.2.1.

$KS^{p(.)}$ satisfies (1), (2) and (4) with very obviously.

For (3), let a sequence f_n with $0 \leq f_n \to f$ (μ -a.e.) implies $\|f_n\| \to \|f\|$. Since $p^* < \infty$, then there exists $g \in KS^{p(.)}(\Omega)$ such that $|f_n(x)| \leq g(x)$ a.e.(using [24, Theorem 6.2]).

For (5) let $E \subset \Omega$ with $\mu(E) < \infty$. Let

$$\mathbb{E}_0 = \{x \in E \cap \Omega_0 \ : |f(x)| < 1 \ \} \tag{11}$$

$$\mathbb{E}_1 = \{x \in E \cap \Omega_0 \ : \ |f(x)| \geq 1\}. \tag{12}$$

Then,

$\frac{1}{\|f\|}\left[\sup \left(\sum_{k=1}^{\infty} t_k(x) \left| \int_\Omega \mathcal{E}_k(x) f(x) d\mu(x) \right| \right) + ess \sup_{E\cap\Omega_\infty} |f(x)| \right]$

$= \sup(\sum_{k=1}^{\infty} t_k(x)| \int_\Omega \frac{\mathcal{E}_k(x)f(x)}{\|f\|} d\mu(x)|) + ess \sup_{E\cap\Omega_\infty} \frac{|f(x)|}{\|f\|}$

$\leq \sup[\sum_{k=1}^{\infty} t_k(x)| \int_{\mathbb{E}_0} \frac{\mathcal{E}_k(x)f(x)}{\|f\|} d\mu(x)| + \sum_{k=1}^{\infty} t_k(x)| \int_{\mathbb{E}_1} \frac{\mathcal{E}_k(x)f(x)}{\|f\|} d\mu(x)|] + ess \sup_{E\cap\Omega_\infty} \frac{|f|}{\|f\|}$

$\leq \mu(E) + 1 = C_E$.

Therefore,

$$\sup \left(\sum_{k=1}^{\infty} t_k(x)| \int_\Omega \mathcal{E}_k(x) f(x) d\mu(x)| \right) \leq C_E \|f\|.$$

So,

$$\left| \int_\Omega \mathcal{E}_k(x) f(x) d\mu(x) \right| \leq C_E \|f\|.$$

This completes the proof. □

Theorem 2.2.6. *$KS^{q(x)}$ is isomorphic to the associated space of $KS^{p(x)}$.*

Proof. In term of Banach function space the Theorem 2.2.5 and the Proposition 2.2.4 gives $KS^{q(x)}$ is isomorphic to the associated space of $KS^{p(x)}$. □

Theorem 2.2.7. *Assume that μ is nonatomic and $p^* < \infty$, then the following are equivalent:*

1. *$KS^{p(.)}$ is reflexive.*
2. *The space $KS^{p(.)}$ and $KS^{q(.)}$ have absolutely continuous norm.*

Proof. Since $KS^{p(.)}$ is Banach function space. Using [1, Corollary 4.4], (1) $\Leftrightarrow$ (2). □

Theorem 2.2.8. *If $1 < p_* \leq p^* < \infty$, then $KS^{p(.)}$ is uniformly convex.*

Proof. The proof follow from a modification of the proof of Clarkson inequalities for l^p norms of [2] and (4) $\Longrightarrow$ (2) of [15, Theorem 3.3]. □

3. Boundedness of maximal operators in $KS^{p(.)}$

In this section, we discuss about boundedness of Maximal operator. If $B(x,r)$ is an arbitrary ball centre at x and radius r, then for $f \in L^1_{loc}(\Omega)$, $\mathcal{M}_{B(x,r)}f = \sum_{k=1}^{\infty} t_k(y)|\fint_{B(x,r)} \mathcal{E}_k(y)f(y)dy|$, where $\fint_{B(x,r)}$ is the mean value integral over $B(x,r)$.

Definition 3.0.1. Maximal operator: *Let*

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\Big(\sum_{k=1}^{\infty} t_k(y)|\int_{B(x,r)\cap\Omega} \mathcal{E}_k(y)f(y)d\mu(y)|\Big)$$

be the maximal operator.

Clearly for any $p(.) \in KS^{p(.)}(\mathbb{R}^n)$ if $f \in KS^{p(.)}(\mathbb{R}^n)$, then $\mathcal{M}f(x)$ is well defined and $\mathcal{M}f(x) < \infty$ is a.e..

Proposition 3.0.2. ([6, Lemma 3.4]) Let p be a bounded exponent on Ω with condition (3) then there exists a constant $C(p) > 0$ such that for all $\|f\|_{p(.)} \leq 1$ then

$$(\mathcal{M}f(x))^{\frac{p(x)}{p_*}} \leq C(p)(\mathcal{M}(|f|^{\frac{p}{p_*}})(x) + 1)\ \forall x \in \Omega. \tag{13}$$

Theorem 3.0.3. *Let Ω be a bounded domain under (2) and (3), the maximal operator $\mathcal{M}$ is bounded in the space $KS^{p(.)}(\Omega)$.*

Proof. Since $\mathcal{M}f$ is a positive homogeneous, i.e $\mathcal{M}(\lambda f) = \|\lambda\|\mathcal{M}_f$. We need to show $\|\mathcal{M}f\|_{p(.)} \leq C(p)\ \forall f$ *with* $\|f\|_{p(.)} \leq 1$. Since in our assumption $p^* < \infty$, then it is sufficient to prove $\rho_0(\mathcal{M}f) \leq C(p)\ \forall \|f\|_{p(.)} \leq 1$. If $f \in KS^{p(.)}(\Omega)$ with $\|f\|_{p(.)} \leq 1$ then $\rho_0(f) \leq 1$. Let $q = \frac{p}{p_*}$, using the Proposition 3.0.2, we get our need $\rho_0(\mathcal{M}f) \leq C(p)$ for all $\|f\|_{p(.)} \leq 1$. □

Theorem 3.0.4. *Let $p(x)$ satisfy condition* (2), (3) *and* (4), *then the maximal operator $\mathcal{M}$ is bounded in the space $KS^{p(.)}(\Omega)$.*

Proof. Condition (4) is a natural analogue of (3) at infinity. So, there must a number p_∞ such that $|x| \to \infty$. This limit holds uniformly in all direction. So proof is just extension of Theorem 3.0.3. □

Let $\mathcal{P}(\mathbb{R}^n)$ denote the set of exponents p with $1 < p_* \leq p^* < \infty$ such that $\mathcal{M}$ is bounded in $KS^{p(.)}(\mathbb{R}^n)$. Clearly $\mathcal{P}(\mathbb{R}^n)$ is closed under some simple operations ([5, Theorem 2.2]). Also if $p \in \mathcal{P}(\mathbb{R}^n)$ and $s \in [1,\infty)$ then

$$\|\mathcal{M}f\|^s_{sp(.)} = C\|f\|^s_{sp(.)}. \tag{14}$$

Hence $sp \in \mathcal{P}(\mathbb{R}^n)$.

Theorem 3.0.5. *For $p^* < \infty$. Let $\mathcal{M}$ is bounded in $KS^{p(.)}$, then $\mathcal{M}$ is bounded in $KS^{\frac{p(.)}{s}}(\mathbb{R}^n)$ for every $s \in [1,\infty)$.*

Proof. Proof is similar as [25, Theorem 3.38]. Hence, we have omitted the proof. □

Theorem 3.0.6. *$\mathcal{M}$ is bounded in $L^{p(.)}$ then $\mathcal{M}$ is bounded on $KS^{p(.)}$.*

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