Filomat 36:13 (2022), 4385–4395
https://doi.org/10.2298/FIL2213385W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Self-Adjoint Perturbations of Left (Right) Weyl Spectrum for Upper Triangular Operator Matrices

Xiufeng Wu[a], Junjie Huang[b], Alatancang Chen[a]

[a]*School of Mathematical Sciences, Inner Mongolia Normal University, Hohhot 010022, China*
[b]*School of Mathematical Sciences, Inner Mongolia University, Hohhot 010021, China*

Abstract. Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space. Given the operators $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{H})$, we define $M_X := \left[\begin{smallmatrix} A & X \\ 0 & B \end{smallmatrix}\right]$ where $X \in \mathcal{S}(\mathcal{H})$ is a self-adjoint operator. In this paper, a necessary and sufficient condition is given for M_X to be a left (right) Weyl operator for some $X \in \mathcal{S}(\mathcal{H})$. Moreover, it is shown that

$$\bigcap_{X\in\mathcal{S}(\mathcal{H})} \sigma_\star(M_X) = \bigcap_{X\in\mathcal{S}(\mathcal{H})\cap\mathrm{Inv}(\mathcal{H})} \sigma_\star(M_X) = \bigcap_{X\in\mathcal{B}(\mathcal{H})} \sigma_\star(M_X) \cup \Delta,$$

where σ_* is the left (right) Weyl spectrum. Finally, we further characterize the perturbation of the left (right) Weyl spectrum for Hamiltonian operators.

1. Introduction

We assume throughout that $\mathcal{H}$ and $\mathcal{K}$ are separable infinite dimensional Hilbert spaces. If T is a bounded linear operator from $\mathcal{H}$ to $\mathcal{K}$, we write $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and, if $\mathcal{H} = \mathcal{K}$, $T \in \mathcal{B}(\mathcal{H})$. By $\mathcal{S}(\mathcal{H})$ denote the subset of $\mathcal{B}(\mathcal{H})$ whose elements are self-adjoint. The identity operator on $\mathcal{H}$ is denoted by $I_{\mathcal{H}}$ and simply by I if the underlying space is clear from the context. Let $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$. Then $\mathcal{N}(T)$, $\mathcal{R}(T)$ and $T^*$are, respectively, used to denote the kernel, the range and the adjoint of T, and we write $n(T) := \dim \mathcal{N}(T)$ and $d(T) := \dim \mathcal{N}(T^*)$.

For $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ with closed range $\mathcal{R}(T)$, T is said to be left Fredholm, if $n(T) < \infty$; while if $d(T) < \infty$, we say T is right Fredholm. If T is both left and right Fredholm, then it is Fredholm. For $T \in \mathcal{B}(\mathcal{H})$, the left (right) essential spectrum and essential spectrum are defined by

$$\sigma_{le}(T)(\sigma_{re}(T)) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not left (right) Fredholm}\},$$
$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}.$$

2020 *Mathematics Subject Classification*. 47A53; 47A55; 47B99
Keywords. upper triangular operator matrix, self-adjoint operator, left (right) Weyl operator, Hamiltonian operator
Received: 25 May 2021; Revised: 03 July 2022; Accepted: 10 July 2022
Communicated by Dragan S. Djordjević
This work is partially supported by the National Natural Science Foundation of China (No. 11901323), the Program for Young Talents of Science and Technology in Universities of Inner Mongolia Autonomous Region (No. NJYT22029), the Natural Science Foundation of Inner Mongolia (No. 2022LHMS01003), the Fundamental research Funds for the Inner Mongolia Normal University (No. 2022JBBJ009).
Email addresses: wuxiufeng68@163.com (Xiufeng Wu), huangjunjie@rocketmail.com (Junjie Huang), alatanca@imu.edu.cn (Alatancang Chen)

If $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is left or right Fredholm, we define the index of T by $\operatorname{ind}(T) := n(T) - d(T)$. Then T is called left Weyl if it is left Fredholm with $\operatorname{ind}(T) \leq 0$, right Weyl if right Fredholm with $\operatorname{ind}(T) \geq 0$, and Weyl if Fredholm with $\operatorname{ind}(T) = 0$. For $T \in \mathcal{B}(\mathcal{H})$, the sets

$$\sigma_{lw}(T)(\sigma_{rw}(T)) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not left (right) Weyl}\},$$
$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$$

are called left (right) Weyl spectrum and Weyl spectrum. For convenience, we define $\rho_\star(T) := \mathbb{C} \setminus \sigma_\star(T)$ in which $\sigma_\star \in \{\sigma_{le}, \sigma_{re}, \sigma_e\}$ and $\rho_\star \in \{\rho_{le}, \rho_{re}, \rho_e\}$.

For given $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, define

$$M_X := \begin{bmatrix} A & X \\ 0 & B \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$$

where $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ is an unknown element. The spectrum and its various subdivisions of M_X are considered in many papers such as [2–5, 7–9, 11–18] and the references therein. In [4] and [5], the perturbations of the left and right Weyl spectra of M_X were, respectively, given by

$$\begin{aligned} \bigcap_{X \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{lw}(M_X) = {} & \sigma_{le}(A) \cup \{\lambda \in \sigma_m(B) : d(A - \lambda) < \infty\} \\ & \cup \{\lambda \in \rho_m(B) : n(B - \lambda) = d(B - \lambda) = \infty, d(A - \lambda) < \infty\} \\ & \cup \{\lambda \in \rho_m(B) : n(A - \lambda) + n(B - \lambda) > d(A - \lambda) + d(B - \lambda)\}, \end{aligned}$$

$$\begin{aligned} \bigcap_{X \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{rw}(M_X) = {} & \sigma_{re}(B) \cup \{\lambda \in \sigma_m(A) : n(B - \lambda) < \infty\} \\ & \cup \{\lambda \in \rho_m(A) : n(A - \lambda) = d(A - \lambda) = \infty, n(B - \lambda) < \infty\} \\ & \cup \{\lambda \in \rho_m(A) : n(A - \lambda) + n(B - \lambda) < d(A - \lambda) + d(B - \lambda)\}. \end{aligned}$$

In [16], the authors proved that

$$\bigcap_{X \in \mathrm{Inv}(\mathcal{K},\mathcal{H})} \sigma_{lw}(M_X) = \bigcap_{X \in \mathcal{B}(\mathcal{K},\mathcal{H})} \sigma_{lw}(M_X) \cup \{\lambda \in \mathbb{C} : B - \lambda \text{ is compact}\},$$

where $\mathrm{Inv}(\mathcal{K}, \mathcal{H})$ denotes the set of all the invertible operators of $\mathcal{B}(\mathcal{K}, \mathcal{H})$. In [9, 18], the authors making use of the single-valued extension property, estimated the defect sets $(\sigma_\star(A) \cup \sigma_\star(B)) \setminus \sigma_\star(M_X)$ and obtained some sufficient conditions for

$$\sigma_\star(M_X) = \sigma_\star(A) \cup \sigma_\star(B),$$

where $\sigma_\star$ runs different spectra.

Let $A \in \mathcal{B}(\mathcal{H})$. Recall that an upper triangular Hamiltonian operator is a block operator matrix of the particular form

$$H_X := \begin{bmatrix} A & X \\ 0 & -A^* \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}),$$

where $X \in \mathcal{S}(\mathcal{H})$. Hamiltonian operators play a fundamental role in algebraic Riccati equations, control theory, elasticity mechanics and other areas. This paper is motivated by the perturbation of left (right) Weyl spectrum for H_X. Note that, for a Hamiltonian operator H_X, $H_X - \lambda$ is not necessary a Hamiltonian operator. Thus, we consider the following more general questions:

Question 1. Is there a self-adjoint operator $X \in \mathcal{S}(\mathcal{H})$ such that M_X is left (right) Weyl, left (right) Browder, left (right) Drazin?

Question 2. $\bigcap_{X \in \mathcal{S}(\mathcal{H})} \sigma_\star(M_X) =?$ where $\sigma_\star$ is any type of spectrum.

In [11, 13, 17], the authors investigated the self-adjoint perturbations of the spectra and Weyl spectra of M_X.

This paper mainly aims to characterize the left (right) Weylness of M_X for some $X \in \mathcal{S}(\mathcal{H})$. A second aim is to describe the following self-adjoint perturbations

$$\bigcap_{X\in\mathcal{S}(\mathcal{H})} \sigma_{lw}(M_X), \qquad \bigcap_{X\in\mathcal{S}(\mathcal{H})} \sigma_{rw}(M_X),$$

and explore the relationship between

$$\bigcap_{X\in\mathcal{B}(\mathcal{H})} \sigma_\star(M_X), \qquad \bigcap_{X\in\mathcal{S}(\mathcal{H})} \sigma_\star(M_X) \qquad \text{and} \bigcap_{X\in\mathcal{S}(\mathcal{H})\cap\mathrm{Inv}(\mathcal{H})} \sigma_\star(M_X),$$

where $\sigma_\star \in \{\sigma_{lw}, \sigma_{rw}\}$. As a byproduct, we also obtain a necessary and sufficient condition such that

$$\sigma_\star(M_X) = \sigma_\star(A) \cup \sigma_\star(B) \quad \text{for every } X \in \mathcal{S}(\mathcal{H})$$

by using the spectral properties of the given diagonal entries $A, B \in \mathcal{B}(\mathcal{H})$. Finally, a third aim is to develop the analogues for Hamiltonian operators, which is actually our original motivation for considering such self-adjoint perturbations.

2. Preliminaries

We begin with some basic lemmas, which are useful for the proofs of the main results of this paper.

Lemma 2.1 (see [1, Remark 1.54]). *Let $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be left (rightt) Fredholm, and let $S \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be a compact operator. Then $T + S$ is a left (right) Fredholm operator with* $\mathrm{ind}(T + S) = \mathrm{ind}(T)$.

Lemma 2.2 (see [6, Lemma 5.8]). *Let $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$. Then T is compact if and only if $\mathcal{R}(T)$ contains no closed infinite dimensional subspaces.*

Lemma 2.3 (see [4, Theorem 2.1]). *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then M_X is a left Weyl operator for some $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ if and only if A is left Fredholm, and one of the following statements is fulfilled:*
(i) $d(A) = \infty$;
(ii) $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ *is a left Weyl operator.*

Lemma 2.4 (see [4, Theorem 2.3]). *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then M_X is a right Weyl operator for some $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ if and only if B is right Fredholm, and one of the following statements is fulfilled:*
(i) $n(B) = \infty$;
(ii) $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ *is a right Weyl operator.*

3. Main results

In this section, we present the main results of this paper and their proofs. First, we establish the left Weylness of M_X.

For a linear subspace $\mathcal{M} \subseteq \mathcal{H}$, $\overline{\mathcal{M}}$ and $\mathcal{M}^\perp$ stand for the closure and the orthogonal complement of $\mathcal{M}$, respectively. Write $T|_\mathcal{M}$ for the restriction of T to $\mathcal{M}$ and $P_\mathcal{M}$ for the orthogonal projection onto $\mathcal{M}$ along $\mathcal{M}^\perp$ when $\mathcal{M}$ is closed.

Theorem 3.1. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then M_X is a left Weyl operator for some $X \in \mathcal{S}(\mathcal{H})$ if and only if A is left Fredholm, and one of the following statements is fulfilled:*
(i) $B|_{\mathcal{R}(A)} : \mathcal{R}(A) \to \mathcal{H}$ *is a left Fredholm operator with* $\mathrm{ind}(B|_{\mathcal{R}(A)}) \le -n(A)$;
(ii) $B|_{\mathcal{R}(A)^\perp} : \mathcal{R}(A)^\perp \to \mathcal{H}$ *is a non-compact operator. In addition, the collection of all $X \in \mathcal{S}(\mathcal{H})$, completing M_X as a left Weyl operator, is further given by*

$$S_{LW}(A,B) = \{X \in \mathcal{S}(\mathcal{H}) : \begin{bmatrix} P_{\mathcal{R}(A)^\perp} X \\ B \end{bmatrix} : \mathcal{H} \to \begin{bmatrix} \mathcal{R}(A)^\perp \\ \mathcal{H} \end{bmatrix} \text{ is left Fredholm with } \mathrm{ind}(\begin{bmatrix} P_{\mathcal{R}(A)^\perp} X \\ B \end{bmatrix}) \le -n(A)\}. \qquad (1)$$

Proof. Let A is left Fredholm. Picking a finite dimensional subspace $\mathcal{M}$ of $\mathcal{H}$ satisfying $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ and $\dim \mathcal{M} = n(A)$. Then, we have

$$M_X = \begin{bmatrix} 0 & A_1 & X_{11} & X_{12} \\ 0 & 0 & X_{21} & X_{22} \\ 0 & 0 & B_{11} & B_{12} \\ 0 & 0 & B_{21} & B_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{N}(A) \\ \mathcal{N}(A)^\perp \\ \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \\ \mathcal{M}^\perp \\ \mathcal{M} \end{bmatrix} \tag{2}$$

for any $X \in \mathcal{S}(\mathcal{H})$, where $A_1 : \mathcal{N}(A)^\perp \to \mathcal{R}(A)$ is invertible and $X_{22}^* = X_{22}$. Hence there exists the invertible operator

$$V := \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & -A_1^{-1}X_{11} & -A_1^{-1}X_{12} \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} : \begin{bmatrix} \mathcal{N}(A) \\ \mathcal{N}(A)^\perp \\ \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(A) \\ \mathcal{N}(A)^\perp \\ \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \end{bmatrix}$$

such that

$$M_X V = \begin{bmatrix} 0 & A_1 & 0 & 0 \\ 0 & 0 & X_{21} & X_{22} \\ 0 & 0 & B_{11} & B_{12} \\ 0 & 0 & B_{21} & B_{22} \end{bmatrix}. \tag{3}$$

Necessity. Assume that M_X be a left Weyl operator for some $X \in \mathcal{S}(\mathcal{H})$. Clearly, A is left Fredholm. From (3) and Lemma 2.1, it follows that

$$\begin{bmatrix} X_{21} & X_{22} \\ B_{11} & B_{12} \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A)^\perp \\ \mathcal{M}^\perp \end{bmatrix} \tag{4}$$

is left Weyl. Thus there exists an invertible operator $W \in \mathcal{B}(\mathcal{M}^\perp, \mathcal{R}(A))$ such that

$$\begin{bmatrix} B_{11}W & B_{12} \\ X_{21}W & X_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} W & 0 \\ 0 & I_{\mathcal{R}(A)^\perp} \end{bmatrix} \tag{5}$$

is left Weyl. Now we consider two cases.

Case 1: B_{12} is a compact operator. By Lemma 2.1, the left Weylness of the operator matrix (5) implies that

$$\begin{bmatrix} B_{11}W & 0 \\ X_{21}W & X_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{M}^\perp \\ \mathcal{R}(A)^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{M}^\perp \\ \mathcal{R}(A)^\perp \end{bmatrix}$$

is left Weyl. By Lemma 2.3, X_{22} is left Fredholm, which together with $X_{22} \in \mathcal{S}(\mathcal{R}(A)^\perp)$ implies the Weylness of X_{22}. It follows that $B_{11}W$ is a left Weyl operator. This implies that

$$\begin{bmatrix} B_{11} \\ 0 \end{bmatrix} : \mathcal{R}(A) \to \begin{bmatrix} \mathcal{M}^\perp \\ \mathcal{M} \end{bmatrix}$$

is left Fredholm and $\operatorname{ind}(\left[\begin{smallmatrix} B_{11} \\ 0 \end{smallmatrix}\right]) \le -n(A)$. From Lemma 2.1, it follows that $\left[\begin{smallmatrix} B_{11} \\ B_{21} \end{smallmatrix}\right]$ is left Fredholm and $\operatorname{ind}(\left[\begin{smallmatrix} B_{11} \\ B_{21} \end{smallmatrix}\right]) \le -n(A)$. The assertion (i) follows from $B|_{\mathcal{R}(A)} = \left[\begin{smallmatrix} B_{11} \\ B_{21} \end{smallmatrix}\right]$ right away.

Case 2: B_{12} is a non-compact operator. Since $\dim \mathcal{M} < \infty$, it follows that

$$\begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix} : \mathcal{R}(A)^\perp \to \begin{bmatrix} \mathcal{M}^\perp \\ \mathcal{M} \end{bmatrix}$$

is non-compact, assertion (ii) is proven.

Sufficiency. Let A is left Fredholm. From assertion (i), we easily see that $B_{11} : \mathcal{R}(A) \to \mathcal{M}^{\perp}$ is a left Weyl operator. If B_{12} is a compact operator, then define by

$$X := \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathcal{R}(A)^{\perp}} \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^{\perp} \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^{\perp} \end{bmatrix}$$

and we verify that M_X is clearly left Weyl.

Now assume that assertion (ii) holds. From the relation (3), we need only show that the operator matrix (4) is left Weyl for some $X_{21} \in \mathcal{B}(\mathcal{R}(A), \mathcal{R}(A)^{\perp})$ and $X_{22} \in \mathcal{S}(\mathcal{R}(A)^{\perp})$ in order to prove the desired result. Define $X_{22} := I_{\mathcal{R}(A)^{\perp}}$. It is easy to see that

$$\begin{bmatrix} B_{12} \\ X_{22} \end{bmatrix} : \mathcal{R}(A)^{\perp} \to \begin{bmatrix} \mathcal{M}^{\perp} \\ \mathcal{R}(A)^{\perp} \end{bmatrix}$$

is left Fredholm and $P_{\mathcal{R}(A)^{\perp}}(\mathcal{N}[X_{22}^* \;\; B_{12}^*])$ contains a closed infinite dimensional subspace $\mathcal{G}$ from Lemma 2.2. We take an orthogonal decomposition $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2$ of $\mathcal{G}$ such that $\mathcal{G}_1$ and $\mathcal{G}_2$ are closed infinite-dimensional subspace of $\mathcal{G}$. Then there exists a right invertible operator $S \in \mathcal{B}(\mathcal{R}(A)^{\perp}, \mathcal{R}(A))$ such that $\mathcal{N}(S)^{\perp} = \mathcal{G}_1$. Since $\mathcal{R}(P_{\mathcal{G}}) \subset \mathcal{R}(B_{12}^*)$, therefore $(B_{12}^*)^{\dagger} P_{\mathcal{G}_1} \in \mathcal{B}(\mathcal{R}(A)^{\perp}, \mathcal{R}(A))$. Define

$$X_{21}^* = S + B_{11}^*(B_{12}^*)^{\dagger} P_{\mathcal{G}_1}. \tag{6}$$

Then the operator matrix

$$\begin{bmatrix} X_{22}^* & B_{12}^* \\ X_{21}^* & B_{11}^* \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A)^{\perp} \\ \mathcal{M}^{\perp} \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A)^{\perp} \\ \mathcal{R}(A) \end{bmatrix} \tag{7}$$

is right Weyl. In fact, let $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathcal{R}([X_{22}^* \;\; B_{12}^*]) \oplus \mathcal{R}(A)$. Since $\mathcal{R}(P_{\mathcal{G}}) \subset \mathcal{R}(B_{12}^*)$, there exist $x_0 \in \mathcal{G}^{\perp}$ (with $\mathcal{R}(A)^{\perp} = \mathcal{G} \oplus \mathcal{G}^{\perp}$) and $y_0 \in \mathcal{M}^{\perp}$ such that $x_0 + B_{12}^* y_0 = u_1$. From the definition of S, it follows that $S\hat{x}_0 = u_2 - B_{11}^* y_0$ for some $\hat{x}_0 \in \mathcal{G}_1$. If we choose $x_1 := x_0 + \hat{x}_0$ and $y_1 := y_0 - (B_{12}^*)^{+}\hat{x}_0$, then we get

$$\begin{bmatrix} X_{22}^* & B_{12}^* \\ X_{21}^* & B_{11}^* \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 + \hat{x}_0 + B_{12}^* y_1 \\ S\hat{x}_0 + B_{11}^*(B_{12}^*)^{\dagger}\hat{x}_0 + B_{11}^* y_1 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

This proves the right Fredholmness of (7). Note that $\mathcal{R}(X_{22}^* P_{\mathcal{G}}) \subset \mathcal{R}(B_{12}^*)$. Then there exists $y_0' \in \mathcal{M}^{\perp}$ such that $x_0' + B_{12}^* y_0' = 0$ for all $x_0' \in \mathcal{G}_2$. The right invertibility of S further implies $S\hat{x}_0' = -B_{11}^* y_0'$ for some $\hat{x}_0' \in \mathcal{G}_1$. Define $x_1 := x_0' + \hat{x}_0'$ and $y_1 := y_0' - (B_{12}^*)^{+}\hat{x}_0'$, then

$$\begin{bmatrix} X_{22}^* & B_{12}^* \\ X_{21}^* & B_{11}^* \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0' + \hat{x}_0' + B_{12}^* y_1 \\ S\hat{x}_0' + B_{11}^*(B_{12}^*)^{\dagger}\hat{x}_0' + B_{11}^* y_1 \end{bmatrix} = 0.$$

The arbitrariness of $x_0' \in \mathcal{G}_2$ results in

$$n(\begin{bmatrix} X_{22}^* & B_{12}^* \\ X_{21}^* & B_{11}^* \end{bmatrix}) = \infty > d(\begin{bmatrix} X_{22}^* & B_{12}^* \\ X_{21}^* & B_{11}^* \end{bmatrix}).$$

Therefore, the operator matrix (4) is left Weyl. Define $X \in \mathcal{S}(\mathcal{H})$

$$X := \begin{bmatrix} 0 & X_{21}^* \\ X_{21} & X_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^{\perp} \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^{\perp} \end{bmatrix}.$$

Then M_X is a left Weyl operator.

From the fact $\begin{bmatrix} X_{21} & X_{22} \end{bmatrix} = P_{\mathcal{R}(A)^{\perp}} X$ and the previous proof, the relation (1) is clearly valid. □

Remark 3.2. In the Theorem above, the assertion (i) holds if and only if BA is a left Fredholm operator with $\operatorname{ind}(BA) \leq 0$, since $\mathcal{R}(B|_{\mathcal{R}(A)}) = \mathcal{R}(BA)$ and $n(BA) = n(A) + n(B|_{\mathcal{R}(A)})$. Furthermore, if $d(A) < \infty$, then we easily obtain that

$$\operatorname{ind}(B|_{\mathcal{R}(A)}) = n(B) - d(A) - d(B). \tag{8}$$

The following is a dual result of Theorem 3.1.

Theorem 3.3. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then M_X is a right Weyl operator for some $X \in \mathcal{S}(\mathcal{H})$ if and only if B is right Fredholm, and one of the following statements is fulfilled:*

(i) $A^* |_{\mathcal{R}(B^*)}: \mathcal{R}(B^*) \to \mathcal{H}$ is a left Fredholm operator and $\operatorname{ind}(A^* |_{\mathcal{R}(B^*)}) \leq -d(B)$;

(ii) $A^* |_{\mathcal{R}(B^*)^\perp}: \mathcal{R}(B^*)^\perp \to \mathcal{H}$ is non-compact operator. In addition, the set of all $X \in \mathcal{S}(\mathcal{H})$, completing M_X as a right Weyl operator, is further given by

$$S_{RW}(A, B) = \{X \in \mathcal{S}(\mathcal{H}) : [A \; X|_{\mathcal{N}(B)}] : \begin{bmatrix} \mathcal{H} \\ \mathcal{N}(B) \end{bmatrix} \to \mathcal{H} \text{ is right Fredholm and } \operatorname{ind}(\begin{bmatrix} P_{\mathcal{R}(A)^\perp} X \\ B \end{bmatrix}) \leq -d(B)\}.$$

Theorem 3.4. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then M_X is a left Weyl operator for some $X \in \mathcal{S}(\mathcal{H}) \cap \operatorname{Inv}(\mathcal{H})$ if and only if M_X is a left Weyl operator for some $X \in \mathcal{S}(\mathcal{H})$.*

Proof. For the proof, we only need to prove the Sufficiency. Let M_X is a left Weyl operator for some $X \in \mathcal{S}(\mathcal{H})$. Again, M_X has the representation (2) for any $X \in \mathcal{S}(\mathcal{H})$. By Theorem 3.1, $B_{11} : \mathcal{R}(A) \to \mathcal{M}^\perp$ is a left Weyl operator and B_{12} is a compact operator, or B_{12} is a non-compact operator. If $B_{11} : \mathcal{R}(A) \to \mathcal{M}^\perp$ is a left Weyl operator and B_{12} is a compact operator, then define by

$$X := \begin{bmatrix} I_{\mathcal{R}(A)} & 0 \\ 0 & I_{\mathcal{R}(A)^\perp} \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \end{bmatrix}$$

and we verify that M_X is clearly left Weyl. If B_{12} is a non-compact operator, define $X \in \mathcal{S}(\mathcal{H}) \cap \operatorname{Inv}(\mathcal{H})$

$$X := \begin{bmatrix} X_{11} & X_{21}^* \\ X_{21} & X_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \end{bmatrix},$$

where $X_{11} = 4\|X_{21}\|^2 I_{\mathcal{R}(A)}$, $X_{22} = I_{\mathcal{R}(A)^\perp}$, and X_{21}^* as in (6). It is easy to see that $X \in \mathcal{S}(\mathcal{H})$. Now we will prove that $X \in \operatorname{Inv}(\mathcal{H})$. Since $X_{11} = 4\|X_{21}\|^2 I_{\mathcal{R}(A)} \in \operatorname{Inv}(\mathcal{R}(A))$, thus the invertible operators $U \in \mathcal{B}(\mathcal{R}(A) \oplus \mathcal{R}(A)^\perp)$ and $V \in \mathcal{B}(\mathcal{R}(A) \oplus \mathcal{R}(A)^\perp)$ given by

$$U := \begin{bmatrix} I_{\mathcal{R}(A)} & 0 \\ -X_{21}X_{11}^{-1} & I_{\mathcal{R}(A)^\perp} \end{bmatrix}, \quad V := \begin{bmatrix} I_{\mathcal{R}(A)} & -X_{11}^{-1}X_{21}^* \\ 0 & I_{\mathcal{R}(A)^\perp} \end{bmatrix}$$

are such that

$$UXV = \begin{bmatrix} X_{11} & 0 \\ 0 & I_{\mathcal{R}(A)^\perp} - X_{21}X_{11}^{-1}X_{21}^* \end{bmatrix}.$$

Note that

$$\|X_{21}X_{11}^{-1}X_{21}^*\| \leq \|X_{21}\|\|X_{11}^{-1}\|\|X_{21}^*\| = \frac{1}{4} < 1,$$

it follows that $I_{\mathcal{R}(A)^\perp} - X_{21}X_{11}^{-1}X_{21}^* \in \operatorname{Inv}(\mathcal{R}(A)^\perp)$. This together with the invertibility of X_{11} implies that $X \in \operatorname{Inv}(\mathcal{H})$. From the proof of the sufficiency of Theorem 3.1, we obtain that M_X is a left Weyl operator. □

The following is a dual result of Theorem 3.4.

Theorem 3.5. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then M_X is a right Weyl operator for some $X \in \mathcal{S}(\mathcal{H}) \cap \operatorname{Inv}(\mathcal{H})$ if and only if M_X is a right Weyl operator for some $X \in \mathcal{S}(\mathcal{H})$.*

As a direct consequence of Theorem 3.1, of Theorem 3.3, one can obtain

Corollary 3.6. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned}
&\bigcap_{X\in\mathcal{S}(\mathcal{H})} \sigma_{lw}(M_X) = \bigcap_{X\in\mathcal{S}(\mathcal{H})\cap \mathrm{Inv}(\mathcal{H})} \sigma_{lw}(M_X) \\
&= \sigma_{le}(A) \cup \{\lambda \in \rho_{le}(A) : (B-\lambda)\mid_{\mathcal{R}(A-\lambda)} \text{ is not left Fredholm}, (B-\lambda)\mid_{\mathcal{R}(A-\lambda)^{\perp}} \text{ is compact}\} \\
&\quad \cup\{\lambda \in \rho_{le}(A) : (B-\lambda)\mid_{\mathcal{R}(A-\lambda)} \text{ is left Fredholm}, (B-\lambda)\mid_{\mathcal{R}(A-\lambda)^{\perp}} \text{ is compact}, \\
&\quad\quad \mathrm{ind}((B-\lambda)\mid_{\mathcal{R}(A-\lambda)}) > -n(A-\lambda)\}, \\
&\bigcap_{X\in\mathcal{S}(\mathcal{H})} \sigma_{rw}(M_X) = \bigcap_{X\in\mathcal{S}(\mathcal{H})\cap \mathrm{Inv}(\mathcal{H})} \sigma_{rw}(M_X) \\
&= \sigma_{re}(B) \cup \{\lambda \in \rho_{re}(B) : (A^*-\overline{\lambda})\mid_{\mathcal{R}(B^*-\overline{\lambda})} \text{ is not left Fredholm}, (A^*-\overline{\lambda})\mid_{\mathcal{R}(B^*-\overline{\lambda})^{\perp}} \text{ is compact}, \} \\
&\quad \cup\{\lambda \in \rho_{re}(B) : (A^*-\overline{\lambda})\mid_{\mathcal{R}(B^*-\overline{\lambda})} \text{ is left Fredholm}, (A^*-\overline{\lambda})\mid_{\mathcal{R}(B^*-\overline{\lambda})^{\perp}} \text{ is compact}, \\
&\quad\quad \mathrm{ind}((A^*-\overline{\lambda})\mid_{\mathcal{R}(B^*-\overline{\lambda})}) > -n(B^*-\overline{\lambda})\}.
\end{aligned}$$

Corollary 3.7. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\bigcap_{X\in\mathcal{S}(\mathcal{H})} \sigma_{lw}(M_X) = \bigcap_{X\in\mathcal{S}(\mathcal{H})\cap \mathrm{Inv}(\mathcal{H})} \sigma_{lw}(M_X) = \bigcap_{X\in\mathcal{B}(\mathcal{H})} \sigma_{lw}(M_X) \cup \Delta,$$

where

$$\begin{aligned}
\Delta = &\{\lambda \in \rho_{le}(A) : d(A-\lambda)=\infty, (B-\lambda)\mid_{\mathcal{R}(A-\lambda)} \text{ is not left Fredholm}, (B-\lambda)\mid_{\mathcal{R}(A-\lambda)^{\perp}} \text{ is compact}\} \\
&\cup\{\lambda \in \rho_{le}(A) : d(A-\lambda)=\infty, (B-\lambda)\mid_{\mathcal{R}(A-\lambda)} \text{ is left Fredholm}, \\
&\quad (B-\lambda)\mid_{\mathcal{R}(A-\lambda)^{\perp}} \text{ is compact}, \ \mathrm{ind}((B-\lambda)\mid_{\mathcal{R}(A-\lambda)}) > -n(A-\lambda)\}.
\end{aligned}$$

In particular,

$$\bigcap_{X\in\mathcal{S}(\mathcal{H})} \sigma_{lw}(M_X) = \bigcap_{X\in\mathcal{B}(\mathcal{H})} \sigma_{lw}(M_X)$$

if and only if $\Delta = \emptyset$.

Proof. For the proof, we need only use Lemma 2.3 and Theorem 3.1 directly. □

Corollary 3.8. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\bigcap_{X\in\mathcal{S}(\mathcal{H})} \sigma_{rw}(M_X) = \bigcap_{X\in\mathcal{S}(\mathcal{H})\cap \mathrm{Inv}(\mathcal{H})} \sigma_{rw}(M_X) = \bigcap_{X\in\mathcal{B}(\mathcal{H})} \sigma_{rw}(M_X) \cup \Delta,$$

where

$$\begin{aligned}
\Delta = &\{\lambda \in \rho_{re}(B) : n(B-\lambda)=\infty, (A^*-\overline{\lambda})\mid_{\mathcal{R}(B^*-\overline{\lambda})} \text{ is not left Fredholm}, (A^*-\overline{\lambda})\mid_{\mathcal{R}(B^*-\overline{\lambda})^{\perp}} \text{ is compact}\} \\
&\cup\{\lambda \in \rho_{re}(B) : n(B-\lambda)=\infty, (A^*-\overline{\lambda})\mid_{\mathcal{R}(B^*-\overline{\lambda})} \text{ is left Fredholm}, \\
&\quad (A^*-\overline{\lambda})\mid_{\mathcal{R}(B^*-\overline{\lambda})^{\perp}} \text{ is compact}, \ \mathrm{ind}((A^*-\overline{\lambda})\mid_{\mathcal{R}(B^*-\overline{\lambda})}) > -d(B-\lambda)\}.
\end{aligned}$$

In particular,

$$\bigcap_{X\in\mathcal{S}(\mathcal{H})} \sigma_{rw}(M_X) = \bigcap_{X\in\mathcal{B}(\mathcal{H})} \sigma_{rw}(M_X)$$

if and only if $\Delta = \emptyset$.

Corollary 3.9. *Let $A, B \in \mathcal{B}(\mathcal{H})$. Then*

$$\sigma_{lw}(A) \cup \sigma_{lw}(B) = \sigma_{lw}(M_X) \cup \bigcup_{k=1}^{4} \Delta_k$$

holds for every $X \in \mathcal{S}(\mathcal{H})$, *where*

$$\begin{aligned}
\Delta_1 &:= \{\lambda \in \rho_{le}(A) \cap \rho_{le}(B) : n(A-\lambda) > d(A-\lambda), n(A-\lambda) + n(B-\lambda) \le d(A-\lambda) + d(B-\lambda)\},\\
\Delta_2 &:= \{\lambda \in \rho_{le}(A) \cap \rho_{le}(B) : n(B-\lambda) > d(B-\lambda), d(A-\lambda) < \infty, n(A-\lambda) + n(B-\lambda) \le d(A-\lambda) + d(B-\lambda)\},\\
\Delta_3 &:= \{\lambda \in \rho_{le}(A) \cap \sigma_{lw}(B) : d(A-\lambda) = \infty, (B-\lambda)\,|_{\mathcal{R}(A-\lambda)} \text{ is left Fredholm}, \operatorname{ind}((B-\lambda)\,|_{\mathcal{R}(A-\lambda)}) \le -n(A-\lambda)\},\\
\Delta_4 &:= \{\lambda \in \rho_{le}(A) \cap \sigma_{lw}(B) : d(A-\lambda) = \infty, (B-\lambda)\,|_{\mathcal{R}(A-\lambda)^\perp} \text{ is non-compact}\}.
\end{aligned}$$

Proof. The inclusion $\sigma_{lw}(M_X) \cup \bigcup_{k=1}^{4} \Delta_k \subseteq \sigma_{lw}(A) \cup \sigma_{lw}(B)$ for every $X \in \mathcal{S}(\mathcal{H})$ is trivial.

We prove here the opposite inclusion. Let $\lambda \in (\sigma_{lw}(A) \cup \sigma_{lw}(B)) \setminus \sigma_{lw}(M_X)$ for some $X \in \mathcal{S}(\mathcal{H})$. Then, it is obvious that $\lambda \in \rho_{le}(A)$. If $\lambda \in \sigma_{lw}(A) \setminus \sigma_{lw}(B)$, then $\lambda \in \rho_{le}(B)$ and $n(A-\lambda) > d(A-\lambda)$. This, together with $\lambda \notin \sigma_{lw}(M_X)$ implies that $n(A-\lambda) + n(B-\lambda) \le d(A-\lambda) + d(B-\lambda)$ from Theorem 3.1 and equation (8). Thus, $\lambda \in \Delta_1$. If $\lambda \in \sigma_{lw}(B)$, then $\lambda \in \Delta_2 \cup \Delta_3 \cup \Delta_4$ from Theorem 3.1. Therefore, $\sigma_{lw}(A) \cup \sigma_{lw}(B) \subseteq \sigma_{lw}(M_X) \cup \bigcup_{k=1}^{4} \Delta_k$. □

Corollary 3.10. *Let* $A, B \in \mathcal{B}(\mathcal{H})$. *Then*

$$\sigma_{lw}(A) \cup \sigma_{lw}(B) = \sigma_{lw}(M_X)$$

holds for every $X \in \mathcal{S}(\mathcal{H})$ *if and only if*

$$\Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4 = \emptyset,$$

where $\Delta_k (k = 1, 2, 3, 4)$ *defined as in the Corollary* 3.9.

Proof. From the proof of Corollary 3.9, we immediately have the desired result. □

The following is a dual result of Corollary 3.9.

Corollary 3.11. *Let* $A, B \in \mathcal{B}(\mathcal{H})$. *Then*

$$\sigma_{rw}(A) \cup \sigma_{rw}(B) = \sigma_{rw}(M_X) \cup \bigcup_{k=1}^{4} \Delta_k$$

holds for every $X \in \mathcal{S}(\mathcal{H})$, *where*

$$\begin{aligned}
\Delta_1 &:= \{\lambda \in \rho_{re}(A) \cap \rho_{re}(B) : d(A-\lambda) > n(A-\lambda), n(B-\lambda) < \infty, d(A-\lambda) + d(B-\lambda) \le n(A-\lambda) + n(B-\lambda)\},\\
\Delta_2 &:= \{\lambda \in \rho_{re}(A) \cap \rho_{re}(B) : d(B-\lambda) > n(B-\lambda), d(A-\lambda) + d(B-\lambda) \le n(A-\lambda) + n(B-\lambda)\},\\
\Delta_3 &:= \{\lambda \in \rho_{re}(B) \cap \sigma_{rw}(A) : n(B-\lambda) = \infty, (A^* - \overline{\lambda})\,|_{\mathcal{R}(B^*-\overline{\lambda})} \text{ is left Fredholm}, \operatorname{ind}((A^* - \overline{\lambda})\,|_{\mathcal{R}(B^*-\overline{\lambda})}) \le -d(B-\lambda)\},\\
\Delta_4 &:= \{\lambda \in \rho_{re}(B) \cap \sigma_{rw}(A) : n(B-\lambda) = \infty, (A^* - \overline{\lambda})\,|_{\mathcal{R}(B^*-\overline{\lambda})^\perp} \text{ is non-compact}\}.
\end{aligned}$$

Corollary 3.12. *Let* $A, B \in \mathcal{B}(\mathcal{H})$. *Then*

$$\sigma_{rw}(A) \cup \sigma_{rw}(B) = \sigma_{rw}(M_X)$$

holds for every $X \in \mathcal{S}(\mathcal{H})$ *if and only if*

$$\Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4 = \emptyset,$$

where $\Delta_k (k = 1, 2, 3, 4)$ *defined as in the Corollary* 3.11.

We end this section by analyzing some special cases of our main results.

Corollary 3.13. *Let* $A, B \in \mathcal{B}(\mathcal{H})$. *If* A *is left Fredholm, then* M_X *is left Weyl for some* $X \in \mathcal{S}(\mathcal{H})$ *if and only if* $P_{\mathcal{M}^\perp} B|_{\mathcal{R}(A)} + P_{\mathcal{M}^\perp} B|_{\mathcal{R}(A)^\perp} F$ *is left Weyl for some* $F \in \mathcal{B}(\mathcal{R}(A), \mathcal{R}(A)^\perp)$, *where* $\mathcal{M}$ *is a finite dimensional subspace of* $\mathcal{H}$ *with* $\dim \mathcal{M} = n(A)$.

Proof. Write $B_1 := B|_{\mathcal{R}(A)}$ and $B_2 := B|_{\mathcal{R}(A)^\perp}$. Let $\mathcal{M}$ be a finite dimensional subspace of $\mathcal{H}$ with $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$ and $\dim \mathcal{M} = n(A)$. Assume that M_X is left Weyl for some $X \in \mathcal{S}(\mathcal{H})$. By Theorem 3.1, we have that $P_{\mathcal{M}^\perp} B_1$ is a left Weyl operator, or that $P_{\mathcal{M}^\perp} B_2$ is a non-compact operator. Note that $\mathcal{R}(A)$ is an infinite dimensional closed subspace of $\mathcal{H}$. Then there exists an invertible operator $U \in \mathcal{B}(\mathcal{M}^\perp, \mathcal{R}(A))$ such that $P_{\mathcal{M}^\perp} B_1 U$ is a left Weyl operator or $P_{\mathcal{M}^\perp} B_2$ is a non-compact operator.

If $P_{\mathcal{M}^\perp} B_2$ is a non-compact operator, then, from the proof of the sufficiency of Theorem 3.1, there exists $F \in \mathcal{B}(\mathcal{R}(A), \mathcal{R}(A)^\perp)$ such that

$$\begin{bmatrix} P_{\mathcal{M}^\perp} B_1 U & P_{\mathcal{M}^\perp} B_2 \\ -FU & I_{\mathcal{R}(A)^\perp} \end{bmatrix} : \begin{bmatrix} \mathcal{M}^\perp \\ \mathcal{R}(A)^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{M}^\perp \\ \mathcal{R}(A)^\perp \end{bmatrix}$$

is left Weyl. Since

$$\begin{bmatrix} P_{\mathcal{M}^\perp} B_1 & P_{\mathcal{M}^\perp} B_2 \\ -F & I_{\mathcal{R}(A)^\perp} \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I_{\mathcal{R}(A)^\perp} \end{bmatrix} = \begin{bmatrix} P_{\mathcal{M}^\perp} B_1 U & P_{\mathcal{M}^\perp} B_2 \\ -FU & I_{\mathcal{R}(A)^\perp} \end{bmatrix},$$

it follows that

$$\begin{bmatrix} P_{\mathcal{M}^\perp} B_1 & P_{\mathcal{M}^\perp} B_2 \\ -F & I_{\mathcal{R}(A)^\perp} \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{M}^\perp \\ \mathcal{R}(A)^\perp \end{bmatrix} \tag{9}$$

is left Weyl. This, together with

$$\begin{aligned} &\begin{bmatrix} I_{M^\perp} & -P_{\mathcal{M}^\perp} B_2 \\ 0 & I_{\mathcal{R}(A)^\perp} \end{bmatrix} \begin{bmatrix} P_{\mathcal{M}^\perp} B_1 & P_{\mathcal{M}^\perp} B_2 \\ -F & I_{\mathcal{R}(A)^\perp} \end{bmatrix} \begin{bmatrix} I_{\mathcal{R}(A)} & 0 \\ F & I_{\mathcal{R}(A)^\perp} \end{bmatrix} \\ = &\begin{bmatrix} P_{\mathcal{M}^\perp} B_1 + P_{\mathcal{M}^\perp} B_2 F & 0 \\ 0 & I \end{bmatrix} \end{aligned} \tag{10}$$

implies that $P_{\mathcal{M}^\perp} B_1 + P_{\mathcal{M}^\perp} B_2 F$ is a left Weyl operator.

If $P_{\mathcal{M}^\perp} B_2$ is a compact operator, then $P_{\mathcal{M}^\perp} B_1$ is a left Weyl operator. Then, for any $F \in \mathcal{B}(\mathcal{R}(A), \mathcal{R}(A)^\perp)$, we see that

$$\begin{bmatrix} P_{\mathcal{M}^\perp} B_1 & 0 \\ -F & I_{\mathcal{R}(A)^\perp} \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{R}(A)^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{M}^\perp \\ \mathcal{R}(A)^\perp \end{bmatrix}$$

is left Weyl. Applying Lemma 2.1, we infer that (9) is left Weyl. By the factorization (10), we conclude that $P_{\mathcal{M}^\perp} B_1 + P_{\mathcal{M}^\perp} B_2 F$ is a left Weyl operator.

Conversely, let $P_{\mathcal{M}^\perp} B_1 + P_{\mathcal{M}^\perp} B_2 F$ is left Weyl for some $F \in \mathcal{B}(\mathcal{R}(A), \mathcal{R}(A)^\perp)$. Then, either B_2 is compact and $P_{\mathcal{M}^\perp} B_1$ is left Weyl or B_2 is a non-compact operator. Note that $\dim \mathcal{M} = n(A) < \infty$. From Theorem 3.1, we see that M_X is left Weyl for some $X \in \mathcal{S}(\mathcal{H})$. □

Corollary 3.14. *Let $A, B \in \mathcal{B}(\mathcal{H})$. If B is right Fredholm, then M_X is right Weyl for some $X \in \mathcal{S}(\mathcal{H})$ if and only if $P_{\mathcal{M}^\perp} A^*|_{\mathcal{R}(B^*)} + P_{\mathcal{M}^\perp} A^*|_{\mathcal{R}(B^*)^\perp} F$ is right Weyl for some $F \in \mathcal{B}(\mathcal{R}(B^*), \mathcal{R}(B^*)^\perp)$, where $\mathcal{M}$ is a finite dimensional subspace of $\mathcal{H}$ with* $\dim \mathcal{M} = d(B)$.

Corollary 3.15. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be given operators with $d(A) < \infty$. Then M_X is left Weyl for some $X \in \mathcal{S}(\mathcal{H})$ if and only if M_X is left Weyl for some $X \in \mathcal{B}(\mathcal{H})$.*

Proof. Let M_X be left Weyl for some $X \in \mathcal{B}(\mathcal{H})$. Then, in combination with $d(A) < \infty$, we obtain that A is left Fredholm, B is left Fredholm and $n(A) + n(B) \leq d(A) + d(B)$. Hence $B|_{\mathcal{R}(A)}$ is Fredholm and $\operatorname{ind}(B|_{\mathcal{R}(A)}) \leq -n(A)$. By Theorem 3.1, M_X is left Weyl for some $X \in \mathcal{S}(\mathcal{H})$. The opposite implication is trivial. □

The following is a dual result of Corollary 3.15.

Corollary 3.16. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be given operators with $n(B) < \infty$. Then M_X is right Weyl for some $X \in \mathcal{S}(\mathcal{H})$ if and only if M_X is right Weyl for some $X \in \mathcal{B}(\mathcal{H})$.*

4. Applications and examples

Let $A \in \mathcal{B}(\mathcal{H})$. We denote H_X by the operator on $\mathcal{H} \oplus \mathcal{H}$ of the form

$$H_X := \begin{bmatrix} A & X \\ 0 & -A^* \end{bmatrix}$$

with $X \in \mathcal{S}(\mathcal{H})$ unknown, which is clearly the so-called Hamiltonian operator. As applications, we now present the analogues of Hamiltonian operators.

Proposition 4.1. *Let $A \in \mathcal{B}(\mathcal{H})$. Then H_X is left Weyl for some $X \in \mathcal{S}(\mathcal{H})$ if and only if A is left Fredholm.*

Proof. Let H_X be left Weyl for some $X \in \mathcal{S}(\mathcal{H})$. By Theorem 3.1, A is left Fredholm. Conversely, if A is left Fredholm, then $\mathcal{R}(-A^*|_{\mathcal{R}(A)}) = \mathcal{R}(A^*)$ is closed and $\text{ind}(-A^*|_{\mathcal{R}(A)}) = -n(A)$. By Theorem 3.1, H_X is left Weyl for some $X \in \mathcal{S}(\mathcal{H})$. □

Proposition 4.2. *Let $A \in \mathcal{B}(\mathcal{H})$. Then H_X is left Weyl for some $X \in \mathcal{S}(\mathcal{H}) \cap \text{Inv}(\mathcal{H})$ if and only if A is left Fredholm.*

Proof. From Theorem 3.4 and Proposition 4.1, the desired result follows right away. □

Similarly, we get the following conclusions.

Proposition 4.3. *Let $A \in \mathcal{B}(\mathcal{H})$. Then H_X is right Weyl for some $X \in \mathcal{S}(\mathcal{H})$ if and only if A is left Fredholm.*

Proposition 4.4. *Let $A \in \mathcal{B}(\mathcal{H})$. Then H_X is right Weyl for some $X \in \mathcal{S}(\mathcal{H}) \cap \text{Inv}(\mathcal{H})$ if and only if A is left Fredholm.*

Proposition 4.5. *Let $A \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned}
&\bigcap_{X\in\mathcal{S}(\mathcal{H})} \sigma_{lw}(H_X) = \bigcap_{X\in\mathcal{S}(\mathcal{H})\cap\text{Inv}(\mathcal{H})} \sigma_{lw}(H_X)\\
&= \sigma_{le}(A) \cup \{\lambda \in \rho_{le}(A) : (-A^* - \lambda)|_{\mathcal{R}(A-\lambda)} \text{ is not left Fredholm}, (-A^* - \lambda)|_{\mathcal{R}(A-\lambda)^\perp} \text{ is compact}\}\\
&\qquad \cup \{\lambda \in \rho_{le}(A) : (-A^* - \lambda)|_{\mathcal{R}(A-\lambda)} \text{ is left Fredholm}, (-A^* - \lambda)|_{\mathcal{R}(A-\lambda)^\perp} \text{ is compact},\\
&\qquad \text{ind}((-A^* - \lambda)|_{\mathcal{R}(A-\lambda)}) > -n(A-\lambda)\},\\
&\bigcap_{X\in\mathcal{S}(\mathcal{H})} \sigma_{rw}(H_X) = \bigcap_{X\in\mathcal{S}(\mathcal{H})\cap\text{Inv}(\mathcal{H})} \sigma_{rw}(H_X)\\
&= \sigma_{re}(-A^*) \cup \{\lambda \in \rho_{re}(-A^*) : (A^* - \overline{\lambda})|_{\mathcal{R}(-A-\overline{\lambda})} \text{ is not left Fredholm}, (A^* - \overline{\lambda})|_{\mathcal{R}(-A-\overline{\lambda})^\perp} \text{ is compact}\}\\
&\qquad \cup \{\lambda \in \rho_{re}(-A^*) : (A^* - \overline{\lambda})|_{\mathcal{R}(-A-\overline{\lambda})} \text{ is left Fredholm}, (A^* - \overline{\lambda})|_{\mathcal{R}(-A-\overline{\lambda})^\perp} \text{ is compact},\\
&\qquad \text{ind}((A^* - \overline{\lambda})|_{\mathcal{R}(-A-\overline{\lambda})}) > -n(-A-\overline{\lambda})\}.
\end{aligned}$$

Proof. Note that $\sigma_{re}(-A^*) = \{\lambda \in \mathbb{C} : -\overline{\lambda} \in \sigma_{le}(A)\}$ and $n(-A^* - \lambda) = d(A + \overline{\lambda})$. By Corollary 3.6, we directly obtain the result. □

Remark 4.6. Unlike the general operator matrix case, $\bigcap_{X\in\mathcal{S}(H)} \sigma_{lw}(H_X)$ and $\bigcap_{X\in\mathcal{S}(H)} \sigma_{rw}(H_X)$ can not be derived from Propositions 4.1 and 4.3, repectively.

We conclude this section with two illustrating examples of the previous results.

Example 4.7. *Let $\mathcal{H} = \mathcal{K} = \ell^2$, and let $A, B \in \mathcal{B}(\ell^2)$ be defined by*

$$\begin{aligned}
Ax &= (0, x_3, 0, x_4, 0, x_5, \cdots),\\
Bx &= (0, x_1, \tfrac{x_2}{2}, x_5, \tfrac{x_6}{6}, x_9, \tfrac{x_{10}}{10}, \cdots)
\end{aligned}$$

for $x = (x_1, x_2, x_3, \cdots) \in \ell^2$. Then we claim that $M_X = \left[\begin{smallmatrix} A & X \\ 0 & B \end{smallmatrix}\right]$ is left Weyl for some $X \in \mathcal{S}(\ell^2)$.

It is esay to see that A is left Fredholm and $B|_{\mathcal{R}(A)^{\perp}}$ is non-compact. By Theorem 3.1, we obtain that $M_X = \begin{bmatrix} A & X \\ 0 & B \end{bmatrix}$ is left Weyl for some $X \in \mathcal{S}(\ell^2)$. In fact, define the self-adjoint operator

$$Xx = (x_1 + x_2, x_1, x_3, x_5, x_5 + x_4, , x_9, x_7, x_{13}, x_9 + x_6, x_{17}, x_{11}, x_{21}, x_{13} + x_8, \cdots)$$

for $x = (x_1, x_2, x_3, \cdots) \in \ell^2$. Then we can check that M_X is closed, $n(M_X) = 2$, $d(M_X) = \infty$, and hence M_X is a left Weyl operator.

Example 4.8. *Let $\mathcal{H} = \mathcal{K} = \ell^2$, and let $A, B \in \mathcal{B}(\ell^2)$ be defined by*

$$Ax = (0, x_2, 0, x_3, 0, x_4, \cdots),$$
$$Bx = (x_1, x_4, \tfrac{x_3}{3} + x_6, x_8, \tfrac{x_5}{5} + x_{10}, x_{12}, \tfrac{x_7}{7} + x_{14}, \cdots)$$

for $x = (x_1, x_2, x_3, \cdots) \in \ell^2$.

Clearly, A is left Fredholm and $B\,|_{\mathcal{R}(A)^{\perp}}$ is compact. Direct calculations show that $B\,|_{\mathcal{R}(A)}$ is left Fredholm and $\mathrm{ind}(B\,|_{\mathcal{R}(A)}) = 0 > -1 = -n(A)$. By Corollary 3.6,

$$0 \in \bigcap_{X \in \mathcal{S}(\mathcal{H})} \sigma_{lw}(M_X).$$

Note that $d(A) = \infty$, it follows from Lemma 2.3 that

$$0 \notin \bigcap_{X \in \mathcal{B}(\mathcal{H})} \sigma_{lw}(M_X).$$

Indeed, if we take the operator by

$$X_0 x = (x_1, 0, x_2, 0, x_3, 0, \cdots)$$

for $x = (x_1, x_2, x_3, \cdots) \in \ell^2$. Then, we immediately see that $0 \notin \sigma_{lw}(M_{X_0})$, and hence

$$0 \notin \bigcap_{X \in \mathcal{B}(\mathcal{H})} \sigma_{lw}(M_X).$$

References

[1] P. Aienia, Fredholm and Local Spectral Theory with Applications to Multipliers, Kluwer Academic Publishers, Dordrecht, 2004.
[2] M. Barraa, M. Boumazgour, On the perturbations of spectra of upper triangular operator matrices, J. Math. Anal. Appl. 347 (2008) 315–322.
[3] C. Benhida, E.H. Zerouali, H. Zguitti, Spectra of upper triangular operator matrices, Proc. Amer. Math. Soc. 133 (2005) 3013–3020.
[4] X.H. Cao, B. Meng, Essential approximate point spectra and Weyl's theorem for operator matrices, J. Math. Anal. Appl. 304 (2005) 759–771.
[5] D.S. Djordjević, Perturbations of spectra of operator matrices, J. Operator Theory 48 (2002) 467–486.
[6] R.G. Douglas, Banach Algebra Technique in Operator Theory (2nd ed.), Springer-Verlag, New York, 2003.
[7] S.V. Djordjević, Y.M. Han, A note on Weyl's theorem for operator matrices, Proc. Amer. Math. Soc. 131 (2002) 2543–2547.
[8] H.K. Du, J. Pan, Perturbation of spectrums of 2×2 operator matrices, Proc. Amer. Math. Soc. 121 (1994) 761–766.
[9] S.V. Djordjević, H. Zguitti, Essential point spectra of operator matrices trough local spectral theory, J. Math. Anal. Appl. 338 (2008) 285–291.
[10] I. Gohberg, S. Goldberg, M.A. Kaashoek, Classes of Linear Operators, Birkhäuser Verlag, Basel, 1990.
[11] G.J. Hai, A. Chen, On the invertibility of upper triangular operator matrices, Linear and Multilinear Algebra 62 (2014) 538–547.
[12] G.J. Hai, D.S. Cvetković-llić, Generalized left and right Weyl spectra of upper triangular operator matrices, Electronic Journal of Linear Algebra, 32 (2017), 41-50.
[13] J.J. Huang, A. Chen, H.Wang, Self-adjoint perturbation of spectra of upper triangular operator matrices, Acta Math. Sinica (Chin. Ser.) 53(6) (2010) 1193–1200.
[14] I.S. Hwang, W.Y. Lee, The boundedness below of 2×2 upper triangular operator matrices, Integr. Equ. Oper. Theory 39 (2001) 267–276.
[15] W.Y. Lee, Weyl spectra of operator matrices, Proc. Amer. Math. Soc. 129 (2001) 131–138.
[16] Y.Li, H.K. Du, The intersection of essential approximate point spectra of operator matrices, J. Math. Anal. Appl. 323 (2006) 1171–1183.
[17] X.F. Wu, J.J. Huang, A. Chen, Self-adjoint perturbations of spectra for upper triangular operator matrices, Linear Algebra Appl. 531 (2017) 1–21.
[18] E.H. Zerouali, H. Zguitti, Perturbation of spectra of operator matrices and local spectral theory, J. Math. Anal. Appl. 324 (2006) 992–1005.