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Generating Functions of Binary Products of (p, q)-Fibonacci-Like Numbers with Odd and Even Certain Numbers and Polynomials

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Abstract. In this paper, we study the symmetric and the generating functions for odd and even terms of the second-order linear recurrence sequences. we introduce a operator in order to derive a new family of generating functions of odd and even terms of Mersenne numbers, Mersenne Lucas numbers, (p, q)-Fibonacci-like numbers, k-Pell polynomials and k-Pell Lucas polynomials. By making use of the operator defined in this paper, we give some new generating functions of the products of (p, q)-Fibonacci-like numbers with odd and even terms of certain numbers and polynomials.

1. Introduction

The incomplete numbers and polynomials and their generalizations have been studied in various research. For example, Djordjević and Srivastava in [6], studied the generalizations of the incomplete Fibonacci and Lucas polynomials. The same authors in [7] calculated the generating functions of the incomplete generalized Jacobsthal and generalized Jacobsthal Lucas numbers. Srivastava et al. in [8] defined the incomplete q-Fibonacci and q-Lucas polynomials and they presented some interesting properties of them.

Djordjević and Srivastava in [5] introduced and investigated some properties and relations involving two sequences of the numbers $\{C_{n,3}(a, b, r) \equiv C_{n,3}\}$ and $\{C_{n,4}(a, b, c, r) \equiv C_{n,4}\}$. In 2020, the authors in [10] defined a new class of q-starlike functions by applying the q-derivative operator. In the next year Frontczak et al. in [19] investigated two special families of series involving the reciprocal central binomial coefficients and Lucas numbers. Also, Raina and Srivastava introduced the explicit hypergeometric representations, generating functions and summation formulas of a new class of numbers associated with the familiar Lucas numbers in [20].

Recently, there are many recursive sequences that have been discussed in the literature. The well-known example of these sequences is Fibonacci-like sequence [4]. For $n \geq 2$, the Fibonacci-like sequence is defined by:

$$S_n = S_{n-1} + S_{n-2},$$

with $S_0 = S_1 = 2$. The associated initial conditions S_0 and S_1 are the sum of the Fibonacci and Lucas sequences respectively, i.e. $S_0 = F_0 + L_0 = 2$ and $S_1 = F_1 + L_1 = 2$. The authors in [21] defined the generalized

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Fibonacci-like sequence namely k-Fibonacci-like sequence and derived its identities like Catalan's, Cassini's and d'Ocagnes identities. Also they established some of the interesting properties of them. For $k \geq 1$ and $n \geq 2$, the k-Fibonacci-like sequence is defined by:

$$S_{k,n} = kS_{k,n-1} + S_{k,n-2},$$

with $S_{k,0} = 2$ and $S_{k,1} = 2k$. Another generalization of Fibonacci-like sequence called (p,q)-Fibonacci-like sequence $\left\{S_{p,q,n}\right\}_{n\in\mathbb{N}}$ which is given by: For p,q positive real numbers and $n \geq 2$,

$$S_{p,q,n} = pS_{p,q,n-1} + qS_{p,q,n-2},$$

with $S_{p,q,0} = 2$ and $S_{p,q,1} = 2p$ (see [3]). It observe that for $p = k$ and $q = 1$, the k-Fibonacci-like sequence is obtained.

In [12], the authors defined and studied the Mersenne Lucas numbers. They gave Binet's formula, generating function and symmetric function of Mersenne Lucas numbers. By using the Binet's formula they obtained some well-known identities such as Catalan's identity, Cassini's identity and d'Ocagne's identity. The Mersenne Lucas numbers of order n denoted by $\{m_n\}_{n\in\mathbb{N}}$ are a numbers of the form $m_n = 2^n + 1$, where n is a nonnegative numbers. This identity is called as the Binet's formula for Mersenne Lucas numbers and it comes from the fact that the Mersenne Lucas numbers can also be defined recursively by:

$$m_{n+1} = 2m_n - 1, \tag{1.1}$$

with the initial conditions $m_0 = 2$ and $m_1 = 3$. Since this recurrence is inhomogeneous, substituting n by $n+1$, we obtain:

$$m_{n+2} = 2m_{n+1} - 1, \tag{1.2}$$

subtracting (1.1) to (1.2), we have that $m_{n+2} = 3m_{n+1} - 2m_n$, other form for the recurrence relation of Mersenne Lucas numbers, with initial conditions $m_0 = 2$ and $m_1 = 3$ (see [12]). The Mersenne numbers denoted by $\{M_n\}_{n\in\mathbb{N}}$ are defined by the same manner but with the initial terms $M_0 = 0$ and $M_1 = 1$ (see [18]). The explicit formulas of Mersenne and Mersenne Lucas numbers are given in [13] as follows:

$$M_n = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^j \binom{n-j-1}{j} 3^{n-2j-1} 2^j \text{ and } m_n = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} 3^{n-2j} 2^j.$$

The k-Pell polynomials $\{P_{k,n}(x)\}_{n\in\mathbb{N}}$ and k-Pell Lucas polynomials $\{Q_{k,n}(x)\}_{n\in\mathbb{N}}$ [17] are defined by the same second-order homogeneous linear recurrence relation but with different initial terms as:

$$P_{k,n+2}(x) = 2xP_{k,n+1}(x) + kP_{k,n}(x),\ P_{k,0}(x) = 0,\ P_{k,1}(x) = 1,$$

$$Q_{k,n+2}(x) = 2xQ_{k,n+1}(x) + kQ_{k,n}(x),\ Q_{k,0}(x) = 2,\ Q_{k,1}(x) = 2x.$$

The Binet's formulas which are also called closed forms of k-Pell and k-Pell Lucas polynomials are given by:

$$P_{k,n}(x) = \frac{e_1^n - e_2^n}{e_1 - e_2},$$
$$Q_{k,n}(x) = e_1^n + e_2^n,$$

with $e_1 = x + \sqrt{x^2+k}$ and $e_2 = x - \sqrt{x^2+k}$. The generating and symmetric functions of $P_{k,n}(x)$ and $Q_{k,n}(x)$ are given by (see [16]):

$$\sum_{n=0}^{\infty} P_{k,n}(x) z^n = \frac{z}{1-2xz-kz^2}, \text{ with } P_{k,n}(x) = S_{n-1}(e_1 + [-e_2]), \tag{1.3}$$

$$\sum_{n=0}^{\infty} Q_{k,n}(x) z^n = \frac{2-2xz}{1-2xz-kz^2}, \text{ with } Q_{k,n}(x) = 2S_n(e_1+[-e_2]) - 2xS_{n-1}(e_1+[-e_2]). \tag{1.4}$$

Next, we give some new properties of k-Pell and k-Pell Lucas polynomials.

Corollary 1.1. *By the Eqs.* (1.3) *and* (1.4), *we get:*

$$Q_{k,n}(x) = 2P_{k,n+1}(x) - 2xP_{k,n}(x).$$

Theorem 1.2. *The new explicit formulas for k-Pell and k-Pell Lucas polynomials are given respectively by:*

$$P_{k,n}(x) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j} (2x)^{n-2j-1} k^j \text{ and } Q_{k,n}(x) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-j} \binom{n-j}{j} (2x)^{n-2j} k^j.$$

Proof. It can be proved easily by using induction method. □

This study proposes to present new class of generating functions for odd and even some special numbers and polynomials by using the symmetric functions. The structure of this paper is arranged in the following way:

- **In Section 2:** We present some preliminary facts and results on the symmetric functions.
- **In Section 3:** We introduce and prove new theorem on generating functions and we present some special cases of them.
- **In Section 4:** By making use the symmetric functions, we derive some new generating functions of odd and even terms of (p,q)-Fibonacci-like numbers, Mersenne and Mersenne Lucas numbers, k-Pell and k-Pell Lucas polynomials.
- **In Section 5:** We investigate the new generating functions of the products of (p,q)-Fibonacci-like numbers with odd and even Mersenne and Mersenne Lucas numbers, k-Pell and k-Pell Lucas polynomials by using the theorem given in the Section 3.

Notation: In the rest of this paper, the (p,q)-Fibonacci-like numbers and k-Fibonacci-like numbers will be denoted by $l_{p,q,n}$ and $l_{k,n}$ instead of $S_{p,q,n}$ and $S_{k,n}$ respectively, because the symmetric function denoted by $S_n(A)$.

2. Preliminary results on symmetric functions

In this section, we present some backgrounds and results about the symmetric functions.

Definition 2.1. *[1] Let A and E be any two alphabets. We define $S_n(A-E)$ by the following form:*

$$\frac{\prod_{e\in E}(1-ez)}{\prod_{a\in A}(1-az)} = \sum_{n=0}^{\infty} S_n(A-E)z^n, \tag{2.1}$$

with the condition $S_n(A-E)=0$ for $n<0$.

Equation (2.1) can be rewritten in the following form:

$$\sum_{n=0}^{\infty} S_n(A-E)z^n = \left(\sum_{n=0}^{\infty} S_n(A)z^n\right) \times \left(\sum_{n=0}^{\infty} S_n(-E)z^n\right),$$

where

$$S_n(A-E) = \sum_{j=0}^{n} S_{n-j}(-E)S_j(A).$$

Definition 2.2. *[9] Given a function f on $\mathbb{R}^n$, the divided difference operator is defined as follows:*

$$\partial_{e_ie_{i+1}}(f) = \frac{f(e_1,\cdots,e_i,e_{i+1},\cdots,e_n) - f(e_1,\cdots,e_{i-1},e_{i+1},e_i,e_{i+2},\cdots,e_n)}{e_i - e_{i+1}}.$$

Definition 2.3. *[14] Let n be positive integer and $E = \{e_1, e_2\}$ are set of given variables. Then, the n^{th} symmetric function $S_n(e_1 + e_2)$ is defined by:*

$$S_n(E) = S_n(e_1 + e_2) = \frac{e_1^{n+1} - e_2^{n+1}}{e_1 - e_2},$$

with

$$S_0(E) = S_0(e_1 + e_2) = 1,\ S_1(E) = S_1(e_1 + e_2) = e_1 + e_2,\ S_2(E) = S_2(e_1 + e_2) = e_1^2 + e_1e_2 + e_2^2, ...$$

Proposition 2.4. *For $n \in \mathbb{N}$, the following identities are verified:*

$$l_{p,q,n} = 2S_n(e_1 + [-e_2]),\ \text{with } e_1 = \frac{p + \sqrt{p^2 + 4q}}{2} \text{ and } e_2 = \frac{p - \sqrt{p^2 + 4q}}{2}.$$
$$M_n = S_{n-1}(e_1 + [-e_2]) \text{ and } m_n = 2S_n(e_1 + [-e_2]) - 3S_{n-1}(e_1 + [-e_2]),\ \text{with } e_1 = 2 \text{ and } e_2 = 1.$$

Definition 2.5. *[2] Given an alphabet $E = \{e_1, e_2\}$, the symmetrizing operator $\delta^j_{e_1e_2}$ is defined by:*

$$\delta^j_{e_1e_2}(f) = \frac{e_1^j f(e_1) - e_2^j f(e_2)}{e_1 - e_2},\ \text{for all } j \in \mathbb{N}_0 := \{\mathbb{N} \cup \{0\}\} = \{0, 1, 2, 3, \cdots\}. \tag{2.2}$$

Remark 2.6. *If $j = 0$, the operator (2.2) gives us:*

$$\delta^0_{e_1e_2}(f) = \frac{f(e_1) - f(e_2)}{e_1 - e_2} = \partial_{e_1e_2}(f).$$

3. New theorem on generating function and some special cases

In this section, by using the symmetrizing operator $\delta^j_{e_1e_2}$ we prove the main theorem of the paper, which combines all previously known results in a unified way, treating them as special cases (see [11, 15]).

Theorem 3.1. *Given two alphabets $A = \{a_1, a_2\}$ and $E = \{e_1, e_2\}$, then we have:*

$$\sum_{n=0}^{\infty} S_n(A) S_{2n+j-1}(E) z^n = \frac{\sum\limits_{n=0}^{\infty} S_n(-A) \frac{e_1^j e_2^{2n} - e_2^j e_1^{2n}}{e_1 - e_2} z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_1^{2n} z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A) e_2^{2n} z^n\right)}. \tag{3.1}$$

Proof. By applying the operator $\delta^j_{e_1e_2}$ to the series $f(e_1z) = \sum\limits_{n=0}^{\infty} S_n(A) e_1^{2n} z^n$, the left-hand side of the formula (3.1) can be written as:

$$\delta^j_{e_1e_2} f(e_1z) = \frac{e_1^j \sum\limits_{n=0}^{\infty} S_n(A) e_1^{2n} z^n - e_2^j \sum\limits_{n=0}^{\infty} S_n(A) e_2^{2n} z^n}{e_1 - e_2}$$
$$= \sum_{n=0}^{\infty} S_n(A) \left(\frac{e_1^{2n+j} - e_2^{2n+j}}{e_1 - e_2}\right) z^n$$

$$=\sum_{n=0}^{\infty} S_n(A)\, S_{2n+j-1}(E)\, z^n.$$

By applying the operator $\delta^{j}_{e_1e_2}$ to the series $f(e_1z)=\frac{1}{\sum\limits_{n=0}^{\infty} S_n(-A)e_1^{2n}z^n}$, the right-hand side of the formula (3.1) can be expressed as:

$$\delta^{j}_{e_1e_2} f(e_1z)=\frac{\frac{e_1^j}{\sum\limits_{n=0}^{\infty} S_n(-A)e_1^{2n}z^n}-\frac{e_2^j}{\sum\limits_{n=0}^{\infty} S_n(-A)e_2^{2n}z^n}}{e_1-e_2}$$

$$=\frac{e_1^j\sum\limits_{n=0}^{\infty} S_n(-A)\, e_2^{2n}z^n-e_2^j\sum\limits_{n=0}^{\infty} S_n(-A)\, e_1^{2n}z^n}{(e_1-e_2)\left(\sum\limits_{n=0}^{\infty} S_n(-A)\, e_1^{2n}z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)\, e_2^{2n}z^n\right)}$$

$$=\frac{\sum\limits_{n=0}^{\infty} S_n(-A)\,\frac{e_1^je_2^{2n}-e_2^je_1^{2n}}{e_1-e_2}z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-A)\, e_1^{2n}z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)\, e_2^{2n}z^n\right)}.$$

Therefore:

$$\sum_{n=0}^{\infty} S_n(A)\, S_{2n+j-1}(E)\, z^n=\frac{\sum\limits_{n=0}^{\infty} S_n(-A)\,\frac{e_1^je_2^{2n}-e_2^je_1^{2n}}{e_1-e_2}z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-A)\, e_1^{2n}z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)\, e_2^{2n}z^n\right)}.$$

Thus, this completes the proof. □

- For $A=\{a_1,a_2\}$, $E=\{e_1,e_2\}$, $j=0$, $j=1$ and $j=2$ in the previous theorem we deduce the following lemmas.

Lemma 3.2. *Given two alphabets $A=\{a_1,a_2\}$ and $E=\{e_1,e_2\}$, then we have:*

$$\sum_{n=0}^{\infty} S_n(A)\, S_{2n-1}(E)\, z^n=\frac{(a_1+a_2)(e_1+e_2)z-a_1a_2(e_1+e_2)\left((e_1+e_2)^2-2e_1e_2\right)z^2}{\left(\sum\limits_{n=0}^{\infty} S_n(-A)\, e_1^{2n}z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)\, e_2^{2n}z^n\right)}. \quad (3.2)$$

Lemma 3.3. *Given two alphabets $A=\{a_1,a_2\}$ and $E=\{e_1,e_2\}$, then we have:*

$$\sum_{n=0}^{\infty} S_n(A)\, S_{2n}(E)\, z^n=\frac{1+e_1e_2(a_1+a_2)z-e_1e_2a_1a_2\left((e_1+e_2)^2-e_1e_2\right)z^2}{\left(\sum\limits_{n=0}^{\infty} S_n(-A)\, e_1^{2n}z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)\, e_2^{2n}z^n\right)}. \quad (3.3)$$

Lemma 3.4. *Given two alphabets $A=\{a_1,a_2\}$ and $E=\{e_1,e_2\}$, then we have:*

$$\sum_{n=0}^{\infty} S_n(A)\, S_{2n+1}(E)\, z^n=\frac{e_1+e_2-a_1a_2e_1^2e_2^2(e_1+e_2)z^2}{\left(\sum\limits_{n=0}^{\infty} S_n(-A)\, e_1^{2n}z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)\, e_2^{2n}z^n\right)}. \quad (3.4)$$

- For $A=\{1,0\}$, $E=\{e_1,e_2\}$, $j=0$, $j=1$ and $j=2$ in the Theorem 3.1 we deduce the following lemmas.

Lemma 3.5. *Given an alphabet $E = \{e_1, e_2\}$, then we have:*

$$\sum_{n=0}^{\infty} S_{2n-1}(E) z^n = \frac{(e_1 + e_2) z}{\left(1 - e_1^2 z\right)\left(1 - e_2^2 z\right)}. \tag{3.5}$$

Lemma 3.6. *Given an alphabet $E = \{e_1, e_2\}$, then we have:*

$$\sum_{n=0}^{\infty} S_{2n}(E) z^n = \frac{1 + e_1 e_2 z}{\left(1 - e_1^2 z\right)\left(1 - e_2^2 z\right)}. \tag{3.6}$$

Lemma 3.7. *Given an alphabet $E = \{e_1, e_2\}$, then we have:*

$$\sum_{n=0}^{\infty} S_{2n+1}(E) z^n = \frac{e_1 + e_2}{\left(1 - e_1^2 z\right)\left(1 - e_2^2 z\right)}. \tag{3.7}$$

4. Generating functions of even and odd certain numbers and polynomials

In this part, we now derive the new generating functions for odd and even terms of (p, q)-Fibonacci-like numbers, Mersenne numbers, Mersenne Lucas numbers, k-Pell polynomials and k-Pell Lucas polynomials.

- For $E = \{e_1, -e_2\}$ in the Lemma 3.5, Lemma 3.6 and Lemma 3.7 we deduce the following relationships:

$$\sum_{n=0}^{\infty} S_{2n-1}(e_1 + [-e_2]) z^n = \frac{(e_1 - e_2) z}{1 - \left((e_1 - e_2)^2 + 2e_1 e_2\right) z + e_1^2 e_2^2 z^2}. \tag{4.1}$$

$$\sum_{n=0}^{\infty} S_{2n}(e_1 + [-e_2]) z^n = \frac{1 - e_1 e_2 z}{1 - \left((e_1 - e_2)^2 + 2e_1 e_2\right) z + e_1^2 e_2^2 z^2}. \tag{4.2}$$

$$\sum_{n=0}^{\infty} S_{2n+1}(e_1 + [-e_2]) z^n = \frac{e_1 - e_2}{1 - \left((e_1 - e_2)^2 + 2e_1 e_2\right) z + e_1^2 e_2^2 z^2}. \tag{4.3}$$

This part consists of three cases.

Case 1. *The substitution of* $\begin{cases} e_1 - e_2 = p \\ e_1 e_2 = q \end{cases}$ *in* (4.2) *and* (4.3), *we obtain:*

$$\sum_{n=0}^{\infty} S_{2n}(e_1 + [-e_2]) z^n = \frac{1 - qz}{1 - (p^2 + 2q) z + q^2 z^2}, \tag{4.4}$$

$$\sum_{n=0}^{\infty} S_{2n+1}(e_1 + [-e_2]) z^n = \frac{p}{1 - (p^2 + 2q) z + q^2 z^2}, \tag{4.5}$$

respectively, and we have the following proposition.

Proposition 4.1. *For $n \in \mathbb{N}$, the new generating functions of even and odd (p, q)-Fibonacci-like numbers are respectively given by:*

$$\sum_{n=0}^{\infty} l_{p,q,2n} z^n = \frac{2 - 2qz}{1 - (p^2 + 2q) z + q^2 z^2}. \tag{4.6}$$

$$\sum_{n=0}^{\infty} l_{p,q,2n+1} z^n = \frac{2p}{1 - (p^2 + 2q) z + q^2 z^2}. \tag{4.7}$$

Proof. we have:

$$l_{p,q,n} = 2S_n\left(e_1 + [-e_2]\right).$$

Then, by using the Eqs. (4.4) and (4.5), we get:

$$\sum_{n=0}^{\infty} l_{p,q,2n} z^n = 2\sum_{n=0}^{\infty} S_{2n}\left(e_1 + [-e_2]\right) z^n = \frac{2-2qz}{1-\left(p^2+2q\right)z + q^2z^2}.$$

And

$$\sum_{n=0}^{\infty} l_{p,q,2n+1} z^n = 2\sum_{n=0}^{\infty} S_{2n+1}\left(e_1 + [-e_2]\right) z^n = \frac{2p}{1-\left(p^2+2q\right)z + q^2z^2}.$$

Hence, we obtain the desired result. □

Corollary 4.2. *Taking $p = k$ and $q = 1$ in the Eqs.* (4.6) *and* (4.7)*, we get the new generating functions of even and odd k-Fibonacci-like numbers, respectively as follows:*

$$\sum_{n=0}^{\infty} l_{k,2n} z^n = \frac{2-2z}{1-\left(k^2+2\right)z + z^2}.$$

$$\sum_{n=0}^{\infty} l_{k,2n+1} z^n = \frac{2k}{1-\left(k^2+2\right)z + z^2}.$$

Remark 4.3. *Put $k = 1$ in the Corollary 4.2, we get the new generating functions of even and odd Fibonacci-like numbers.*

Case 2. *The substitution of* $\begin{cases} e_1 - e_2 = 2x \\ e_1e_2 = k \end{cases}$ *in* (4.1), (4.2) *and* (4.3)*, we obtain:*

$$\sum_{n=0}^{\infty} S_{2n-1}\left(e_1 + [-e_2]\right) z^n = \frac{2xz}{1-2\left(2x^2+k\right)z + k^2z^2}, \tag{4.8}$$

$$\sum_{n=0}^{\infty} S_{2n}\left(e_1 + [-e_2]\right) z^n = \frac{1-kz}{1-2\left(2x^2+k\right)z + k^2z^2}, \tag{4.9}$$

$$\sum_{n=0}^{\infty} S_{2n+1}\left(e_1 + [-e_2]\right) z^n = \frac{2x}{1-2\left(2x^2+k\right)z + k^2z^2}, \tag{4.10}$$

respectively, and we have the following proposition and theorem.

Proposition 4.4. *For $n \in \mathbb{N}$, the new generating functions of even and odd k-Pell polynomials are respectively given by:*

$$\sum_{n=0}^{\infty} P_{k,2n}(x) z^n = \frac{2xz}{1-2\left(2x^2+k\right)z + k^2z^2}, \tag{4.11}$$

with $P_{k,2n}(x) = S_{2n-1}(e_1 + [-e_2])$.

$$\sum_{n=0}^{\infty} P_{k,2n+1}(x) z^n = \frac{1-kz}{1-2\left(2x^2+k\right)z + k^2z^2}, \tag{4.12}$$

with $P_{k,2n+1}(x) = S_{2n}(e_1 + [-e_2])$.

Theorem 4.5. *For $n \in \mathbb{N}$, the new generating functions of even and odd k-Pell Lucas polynomials are respectively given by:*

$$\sum_{n=0}^{\infty} Q_{k,2n}(x) z^n = \frac{2-2\left(2x^2+k\right)z}{1-2(2x^2+k)z+k^2z^2}. \tag{4.13}$$

$$\sum_{n=0}^{\infty} Q_{k,2n+1}(x) z^n = \frac{2x+2kxz}{1-2(2x^2+k)z+k^2z^2}. \tag{4.14}$$

Proof. By [16] we have:

$$Q_{k,n}(x) = 2S_n(e_1+[-e_2]) - 2xS_{n-1}(e_1+[-e_2]).$$

Then, from the Eqs. (4.8), (4.9) and (4.10), we obtain:

$$\sum_{n=0}^{\infty} Q_{k,2n}(x) z^n = \sum_{n=0}^{\infty} \left(2S_{2n}(e_1+[-e_2]) - 2xS_{2n-1}(e_1+[-e_2])\right) z^n$$

$$= 2\sum_{n=0}^{\infty} S_{2n}(e_1+[-e_2]) z^n - 2x\sum_{n=0}^{\infty} S_{2n-1}(e_1+[-e_2]) z^n$$
$$= \frac{2(1-kz)}{1-2(2x^2+k)z+k^2z^2} - \frac{4x^2z}{1-2(2x^2+k)z+k^2z^2}$$
$$= \frac{2-2\left(2x^2+k\right)z}{1-2(2x^2+k)z+k^2z^2}.$$

And

$$\sum_{n=0}^{\infty} Q_{k,2n+1}(x) z^n = \sum_{n=0}^{\infty} \left(2S_{2n+1}(e_1+[-e_2]) - 2xS_{2n}(e_1+[-e_2])\right) z^n$$

$$= 2\sum_{n=0}^{\infty} S_{2n+1}(e_1+[-e_2]) z^n - 2x\sum_{n=0}^{\infty} S_{2n}(e_1+[-e_2]) z^n$$
$$= \frac{4x}{1-2(2x^2+k)z+k^2z^2} - \frac{2x(1-kz)}{1-2(2x^2+k)z+k^2z^2}$$
$$= \frac{2x+2kxz}{1-2(2x^2+k)z+k^2z^2}.$$

Which completes the proof. □

Remark 4.6. *Put $k = 1$ in the Eqs.* (4.11) − (4.14), *we get the new generating functions of even and odd Pell and Pell Lucas polynomials.*

Case 3. *The substitution of* $\begin{cases} e_1 - e_2 = 3 \\ e_1e_2 = -2 \end{cases}$ *in* (4.1), (4.2) *and* (4.3), *we obtain:*

$$\sum_{n=0}^{\infty} S_{2n-1}(e_1+[-e_2]) z^n = \frac{3z}{1-5z+4z^2}, \tag{4.15}$$

$$\sum_{n=0}^{\infty} S_{2n}(e_1+[-e_2]) z^n = \frac{1+2z}{1-5z+4z^2}, \tag{4.16}$$

$$\sum_{n=0}^{\infty} S_{2n+1}\left(e_1+[-e_2]\right) z^n = \frac{3}{1-5z+4z^2}, \tag{4.17}$$

respectively, and we have the following proposition and theorem.

Proposition 4.7. *For $n \in \mathbb{N}$, the new generating functions of even and odd Mersenne numbers are respectively given by:*

$$\sum_{n=0}^{\infty} M_{2n} z^n = \frac{3z}{1-5z+4z^2}, \tag{4.18}$$

with $M_{2n} = S_{2n-1}(e_1+[-e_2])$.

$$\sum_{n=0}^{\infty} M_{2n+1} z^n = \frac{1+2z}{1-5z+4z^2}, \tag{4.19}$$

with $M_{2n+1} = S_{2n}(e_1+[-e_2])$.

Theorem 4.8. *For $n \in \mathbb{N}$, the new generating functions of even and odd Mersenne Lucas numbers are respectively given by:*

$$\sum_{n=0}^{\infty} m_{2n} z^n = \frac{2-5z}{1-5z+4z^2}. \tag{4.20}$$

$$\sum_{n=0}^{\infty} m_{2n+1} z^n = \frac{3-6z}{1-5z+4z^2}. \tag{4.21}$$

Proof. We have:

$$m_n = 2S_n\left(e_1+[-e_2]\right) - 3S_{n-1}\left(e_1+[-e_2]\right), \text{ (see [12]).}$$

Then, from the Eqs. (4.15), (4.16) and (4.17), we obtain:

$$\begin{aligned}\sum_{n=0}^{\infty} m_{2n} z^n &= \sum_{n=0}^{\infty} \left(2S_{2n}\left(e_1+[-e_2]\right) - 3S_{2n-1}\left(e_1+[-e_2]\right)\right) z^n \\ &= 2\sum_{n=0}^{\infty} S_{2n}\left(e_1+[-e_2]\right) z^n - 3\sum_{n=0}^{\infty} S_{2n-1}\left(e_1+[-e_2]\right) z^n \\ &= \frac{2(1+2z)}{1-5z+4z^2} - \frac{9z}{1-5z+4z^2} \\ &= \frac{2-5z}{1-5z+4z^2}.\end{aligned}$$

And

$$\begin{aligned}\sum_{n=0}^{\infty} m_{2n+1} z^n &= \sum_{n=0}^{\infty} \left(2S_{2n+1}\left(e_1+[-e_2]\right) - 3S_{2n}\left(e_1+[-e_2]\right)\right) z^n \\ &= 2\sum_{n=0}^{\infty} S_{2n+1}\left(e_1+[-e_2]\right) z^n - 3\sum_{n=0}^{\infty} S_{2n}\left(e_1+[-e_2]\right) z^n \\ &= \frac{6}{1-5z+4z^2} - \frac{3(1+2z)}{1-5z+4z^2} = \frac{3-6z}{1-5z+4z^2}.\end{aligned}$$

Which completes the proof. □

5. Generating functions of the products of (p, q)-Fibonacci-like numbers with even and odd Mersenne numbers and k-Pell polynomials

We now consider the Lemma 3.2, Lemma 3.3 and Lemma 3.4 in order to derive a new generating functions of binary products of (p, q)-Fibonacci-like numbers with even and odd Mersenne and Mersenne Lucas numbers, k-Pell and k-Pell Lucas polynomials.

We consider the following sets:

$$A = \{a_1, -a_2\} \text{ and } E = \{e_1, -e_2\}.$$

By changing a_2 to $(-a_2)$ and e_2 to $(-e_2)$ in the Eqs. (3.2), (3.3) and (3.4), it becomes:

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{2n-1}(e_1 + [-e_2]) z^n = \frac{(a_1 - a_2)(e_1 - e_2) z + a_1 a_2 (e_1 - e_2)\left((e_1 - e_2)^2 + 2e_1 e_2\right) z^2}{(1 - a_1 e_1^2 z)(1 - a_1 e_2^2 z)(1 + a_2 e_1^2 z)(1 + a_2 e_2^2 z)}, \tag{5.1}$$

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{2n}(e_1 + [-e_2]) z^n = \frac{1 - e_1 e_2 (a_1 - a_2) z - e_1 e_2 a_1 a_2 \left((e_1 - e_2)^2 + e_1 e_2\right) z^2}{(1 - a_1 e_1^2 z)(1 - a_1 e_2^2 z)(1 + a_2 e_1^2 z)(1 + a_2 e_2^2 z)}, \tag{5.2}$$

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{2n+1}(e_1 + [-e_2]) z^n = \frac{e_1 - e_2 + a_1 a_2 (e_1 e_2)^2 (e_1 - e_2) z^2}{(1 - a_1 e_1^2 z)(1 - a_1 e_2^2 z)(1 + a_2 e_1^2 z)(1 + a_2 e_2^2 z)}, \tag{5.3}$$

respectively, and we have three cases.

Case 1. Let us now consider the following conditions for Eqs. (5.2) and (5.3):

$$\begin{cases} a_1 - a_2 = p \\ a_1 a_2 = q \end{cases} \text{ and } \begin{cases} e_1 - e_2 = p \\ e_1 e_2 = q \end{cases}.$$

Then it yields:

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{2n}(e_1 + [-e_2]) z^n = \frac{1 - pqz - q^2\left(p^2 + q\right) z^2}{1 - p(p^2 + 2q) z - q(p^4 + 3p^2 q + 2q^2) z^2 + pq^3 (p^2 + 2q) z^3 + q^6 z^4}. \tag{5.4}$$

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{2n+1}(e_1 + [-e_2]) z^n = \frac{p + pq^3 z^2}{1 - p(p^2 + 2q) z - q(p^4 + 3p^2 q + 2q^2) z^2 + pq^3 (p^2 + 2q) z^3 + q^6 z^4}. \tag{5.5}$$

Thus, we have the following proposition.

Proposition 5.1. *For $n \in \mathbb{N}$, the new generating functions of the products of (p, q)-Fibonacci-like numbers with even and odd (p, q)-Fibonacci-like numbers ($l_{p,q,n} l_{p,q,2n}$ and $l_{p,q,n} l_{p,q,2n+1}$) are respectively given by:*

$$\sum_{n=0}^{\infty} l_{p,q,n} l_{p,q,2n} z^n = \frac{4 - 4pqz - 4q^2\left(p^2 + q\right) z^2}{1 - p(p^2 + 2q) z - q(p^4 + 3p^2 q + 2q^2) z^2 + pq^3 (p^2 + 2q) z^3 + q^6 z^4}. \tag{5.6}$$

$$\sum_{n=0}^{\infty} l_{p,q,n} l_{p,q,2n+1} z^n = \frac{4p + 4pq^3 z^2}{1 - p(p^2 + 2q) z - q(p^4 + 3p^2 q + 2q^2) z^2 + pq^3 (p^2 + 2q) z^3 + q^6 z^4}. \tag{5.7}$$

Proof. We have:

$$l_{p,q,n} = 2S_n(a_1 + [-a_2]).$$

Then, using the Eqs. (5.4) and (5.5), we get:

$$\sum_{n=0}^{\infty} l_{p,q,n} l_{p,q,2n} z^n = 4 \sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{2n}(e_1 + [-e_2]) z^n$$

$$= \frac{4 - 4pqz - 4q^2\left(p^2+q\right)z^2}{1 - p\left(p^2+2q\right)z - q\left(p^4+3p^2q+2q^2\right)z^2 + pq^3\left(p^2+2q\right)z^3 + q^6z^4}.$$

And

$$\sum_{n=0}^{\infty} l_{p,q,n} l_{p,q,2n+1} z^n = 4 \sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{2n+1}(e_1 + [-e_2]) z^n$$

$$= \frac{4p + 4pq^3z^2}{1 - p\left(p^2+2q\right)z - q\left(p^4+3p^2q+2q^2\right)z^2 + pq^3\left(p^2+2q\right)z^3 + q^6z^4}.$$

As required. □

Corollary 5.2. *Putting $p = k$ and $q = 1$ in the Eqs.* (5.6) *and* (5.7) *gives the following new generating functions:*

$$\sum_{n=0}^{\infty} l_{k,n} l_{k,2n} z^n = \frac{4 - 4kz - 4\left(k^2+1\right)z^2}{1 - k\left(k^2+2\right)z - \left(k^4+3k^2+2\right)z^2 + k\left(k^2+2\right)z^3 + z^4}.$$

$$\sum_{n=0}^{\infty} l_{k,n} l_{k,2n+1} z^n = \frac{4k + 4kz^2}{1 - k\left(k^2+2\right)z - \left(k^4+3k^2+2\right)z^2 + k\left(k^2+2\right)z^3 + z^4}.$$

Remark 5.3. *Put $k = 1$ in the Corollary 5.2, yields the new generating functions of the products of Fibonacci-like numbers with even and odd Fibonacci-like numbers.*

Case 2. Let us now consider the following conditions for Eqs. (5.1) – (5.3):

$$\begin{cases} a_1 - a_2 = p \\ a_1 a_2 = q \end{cases} \text{ and } \begin{cases} e_1 - e_2 = 3 \\ e_1 e_2 = -2 \end{cases}.$$

Then it yields:

$$\sum_{n=0}^{\infty} S_n\left(a_1 + [-a_2]\right) S_{2n-1}\left(e_1 + [-e_2]\right) z^n = \frac{3pz + 15qz^2}{1 - 5pz - \left(17q - 4p^2\right)z^2 + 20pqz^3 + 16q^2z^4}. \quad (5.8)$$

$$\sum_{n=0}^{\infty} S_n\left(a_1 + [-a_2]\right) S_{2n}\left(e_1 + [-e_2]\right) z^n = \frac{1 + 2pz + 14qz^2}{1 - 5pz - \left(17q - 4p^2\right)z^2 + 20pqz^3 + 16q^2z^4}. \quad (5.9)$$

$$\sum_{n=0}^{\infty} S_n\left(a_1 + [-a_2]\right) S_{2n+1}\left(e_1 + [-e_2]\right) z^n = \frac{3 + 12qz^2}{1 - 5pz - \left(17q - 4p^2\right)z^2 + 20pqz^3 + 16q^2z^4}. \quad (5.10)$$

Thus, we have the following proposition and theorem.

Proposition 5.4. *For $n \in \mathbb{N}$, the new generating functions of the products of (p,q)-Fibonacci-like numbers with even and odd Mersenne numbers ($l_{p,q,n}M_{2n}$ and $l_{p,q,n}M_{2n+1}$) are respectively given by:*

$$\sum_{n=0}^{\infty} l_{p,q,n} M_{2n} z^n = \frac{6pz + 30qz^2}{1 - 5pz - \left(17q - 4p^2\right)z^2 + 20pqz^3 + 16q^2z^4}. \quad (5.11)$$

$$\sum_{n=0}^{\infty} l_{p,q,n} M_{2n+1} z^n = \frac{2 + 4pz + 28qz^2}{1 - 5pz - \left(17q - 4p^2\right)z^2 + 20pqz^3 + 16q^2z^4}. \quad (5.12)$$

Proof. We have:

$$l_{p,q,n} = 2S_n\,(a_1 + [-a_2]) \text{ and } M_n = S_{n-1}\,(e_1 + [-e_2])\,.$$

Then, using the Eqs. (5.8) and (5.9), we get:

$$\sum_{n=0}^{\infty} l_{p,q,n} M_{2n} z^n = 2\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{2n-1}(e_1 + [-e_2]) z^n$$

$$= \frac{6pz + 30qz^2}{1 - 5pz - (17q - 4p^2)\,z^2 + 20pqz^3 + 16q^2z^4}.$$

And

$$\sum_{n=0}^{\infty} l_{p,q,n} M_{2n+1} z^n = 2\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{2n}(e_1 + [-e_2]) z^n$$

$$= \frac{2 + 4pz + 28qz^2}{1 - 5pz - (17q - 4p^2)\,z^2 + 20pqz^3 + 16q^2z^4}.$$

Which completes the proof. □

Theorem 5.5. *For $n \in \mathbb{N}$, the new generating functions of the products of (p,q)-Fibonacci-like numbers with even and odd Mersenne Lucas numbers ($l_{p,q,n} m_{2n}$ and $l_{p,q,n} m_{2n+1}$) are respectively given by:*

$$\sum_{n=0}^{\infty} l_{p,q,n} m_{2n} z^n = \frac{4 - 10pz - 34qz^2}{1 - 5pz - (17q - 4p^2)\,z^2 + 20pqz^3 + 16q^2z^4}. \tag{5.13}$$

$$\sum_{n=0}^{\infty} l_{p,q,n} m_{2n+1} z^n = \frac{6 - 12pz - 36qz^2}{1 - 5pz - (17q - 4p^2)\,z^2 + 20pqz^3 + 16q^2z^4}. \tag{5.14}$$

Proof. We have:

$$l_{p,q,n} = 2S_n\,(a_1 + [-a_2]) \text{ and } m_n = 2S_n(e_1 + [-e_2]) - 3S_{n-1}(e_1 + [-e_2]).$$

Using the relationships (5.8), (5.9) and (5.10), we obtain:

$$\sum_{n=0}^{\infty} l_{p,q,n} m_{2n} z^n = \sum_{n=0}^{\infty} 2S_n(a_1 + [-a_2])\,(2S_{2n}(e_1 + [-e_2]) - 3S_{2n-1}(e_1 + [-e_2]))\,z^n$$

$$= 4\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{2n}(e_1 + [-e_2]) z^n - 6\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{2n-1}(e_1 + [-e_2]) z^n$$

$$= \frac{4\left(1 + 2pz + 14qz^2\right) - 6\left(3pz + 15qz^2\right)}{1 - 5pz - (17q - 4p^2)\,z^2 + 20pqz^3 + 16q^2z^4}$$

$$= \frac{4 - 10pz - 34qz^2}{1 - 5pz - (17q - 4p^2)\,z^2 + 20pqz^3 + 16q^2z^4}.$$

And

$$\sum_{n=0}^{\infty} l_{p,q,n} m_{2n+1} z^n = \sum_{n=0}^{\infty} 2S_n(a_1 + [-a_2])\,(2S_{2n+1}(e_1 + [-e_2]) - 3S_{2n}(e_1 + [-e_2]))\,z^n$$

$$= 4\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_{2n+1}(e_1 + [-e_2])z^n - 6\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_{2n}(e_1 + [-e_2])z^n$$

$$= \frac{4\left(3 + 12qz^2\right) - 6\left(1 + 2pz + 14qz^2\right)}{1 - 5pz - (17q - 4p^2)z^2 + 20pqz^3 + 16q^2z^4}$$

$$= \frac{6 - 12pz - 36qz^2}{1 - 5pz - (17q - 4p^2)z^2 + 20pqz^3 + 16q^2z^4}.$$

So, the proof is completed. □

Corollary 5.6. *Putting $p = k$ and $q = 1$ in the Eqs.* (5.11) − (5.14) *gives the following new generating functions:*

$$\sum_{n=0}^{\infty} l_{k,n}M_{2n}z^n = \frac{6kz + 30z^2}{1 - 5kz - (17 - 4k^2)z^2 + 20kz^3 + 16z^4}.$$

$$\sum_{n=0}^{\infty} l_{k,n}M_{2n+1}z^n = \frac{2 + 4kz + 28z^2}{1 - 5kz - (17 - 4k^2)z^2 + 20kz^3 + 16z^4}.$$

$$\sum_{n=0}^{\infty} l_{k,n}m_{2n}z^n = \frac{4 - 10kz - 34z^2}{1 - 5kz - (17 - 4k^2)z^2 + 20kz^3 + 16z^4}.$$

$$\sum_{n=0}^{\infty} l_{k,n}m_{2n+1}z^n = \frac{6 - 12kz - 36z^2}{1 - 5kz - (17 - 4k^2)z^2 + 20kz^3 + 16z^4}.$$

Remark 5.7. *Put $k = 1$ in the Corollary 5.6, yields the new generating functions of the products of Fibonacci-like numbers with even and odd Mersenne and Mersenne Lucas numbers.*

Case 3. Let us now consider the following conditions for Eqs. (5.1) − (5.3):

$$\begin{cases} a_1 - a_2 = p \\ a_1a_2 = q \end{cases} \text{ and } \begin{cases} e_1 - e_2 = 2x \\ e_1e_2 = k \end{cases}.$$

Then it yields:

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_{2n-1}(e_1 + [-e_2])z^n \tag{5.15}$$

$$= \frac{2pxz + 4qx\left(2x^2 + k\right)z^2}{1 - 2p(2x^2 + k)z - (16qx^4 + 16qkx^2 + k^2(2q - p^2))z^2 + 2pqk^2(2x^2 + k)z^3 + q^2k^4z^4}.$$

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_{2n}(e_1 + [-e_2])z^n \tag{5.16}$$

$$= \frac{1 - pkz - qk\left(4x^2 + k\right)z^2}{1 - 2p(2x^2 + k)z - (16qx^4 + 16qkx^2 + k^2(2q - p^2))z^2 + 2pqk^2(2x^2 + k)z^3 + q^2k^4z^4}.$$

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_{2n+1}(e_1 + [-e_2])z^n \tag{5.17}$$

$$= \frac{2x + 2qk^2xz^2}{1 - 2p(2x^2 + k)z - (16qx^4 + 16qkx^2 + k^2(2q - p^2))z^2 + 2pqk^2(2x^2 + k)z^3 + q^2k^4z^4}.$$

Thus, we have the following proposition and theorem.

Proposition 5.8. *For $n \in \mathbb{N}$, the new generating functions of the products of (p,q)-Fibonacci-like numbers with even and odd k-Pell polynomials ($l_{p,q,n}P_{k,2n}(x)$ and $l_{p,q,n}P_{k,2n+1}(x)$) are respectively given by:*

$$\sum_{n=0}^{\infty} l_{p,q,n}P_{k,2n}(x)z^n = \frac{4pxz + 8qx\left(2x^2+k\right)z^2}{1-2p\left(2x^2+k\right)z-\left(16qx^4+16qkx^2+k^2\left(2q-p^2\right)\right)z^2+2pqk^2\left(2x^2+k\right)z^3+q^2k^4z^4}. \quad (5.18)$$

$$\sum_{n=0}^{\infty} l_{p,q,n}P_{k,2n+1}(x)z^n = \frac{2-2pkz-2qk\left(4x^2+k\right)z^2}{1-2p\left(2x^2+k\right)z-\left(16qx^4+16qkx^2+k^2\left(2q-p^2\right)\right)z^2+2pqk^2\left(2x^2+k\right)z^3+q^2k^4z^4}. \quad (5.19)$$

Proof. We have:

$$l_{p,q,n} = 2S_n\left(a_1+[-a_2]\right) \text{ and } P_{k,n}(x) = S_{n-1}(e_1+[-e_2]).$$

Then, using the Eqs. (5.15) and (5.16), we get:

$$\sum_{n=0}^{\infty} l_{p,q,n}P_{k,2n}(x)z^n = 2\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_{2n-1}(e_1+[-e_2])z^n$$

$$= \frac{4pxz + 8qx\left(2x^2+k\right)z^2}{1-2p\left(2x^2+k\right)z-\left(16qx^4+16qkx^2+k^2\left(2q-p^2\right)\right)z^2+2pqk^2\left(2x^2+k\right)z^3+q^2k^4z^4}.$$

And

$$\sum_{n=0}^{\infty} l_{p,q,n}P_{k,2n+1}(x)z^n = 2\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_{2n}(e_1+[-e_2])z^n$$

$$= \frac{2-2pkz-2qk\left(4x^2+k\right)z^2}{1-2p\left(2x^2+k\right)z-\left(16qx^4+16qkx^2+k^2\left(2q-p^2\right)\right)z^2+2pqk^2\left(2x^2+k\right)z^3+q^2k^4z^4}.$$

Which completes the proof. □

Theorem 5.9. *For $n \in \mathbb{N}$, the new generating functions of the products of (p,q)-Fibonacci-like numbers with even and odd k-Pell Lucas polynomials ($l_{p,q,n}Q_{k,2n}(x)$ and $l_{p,q,n}Q_{k,2n+1}(x)$) are respectively given by:*

$$\sum_{n=0}^{\infty} l_{p,q,n}Q_{k,2n}(x)z^n = \frac{4-4p\left(2x^2+k\right)z-4q\left(8x^4+8kx^2+k^2\right)z^2}{1-2p\left(2x^2+k\right)z-\left(16qx^4+16qkx^2+k^2\left(2q-p^2\right)\right)z^2+2pqk^2\left(2x^2+k\right)z^3+q^2k^4z^4}. \quad (5.20)$$

$$\sum_{n=0}^{\infty} l_{p,q,n}Q_{k,2n+1}(x)z^n = \frac{4x+4pkxz+4qkx\left(4x^2+3k\right)z^2}{1-2p\left(2x^2+k\right)z-\left(16qx^4+16qkx^2+k^2\left(2q-p^2\right)\right)z^2+2pqk^2\left(2x^2+k\right)z^3+q^2k^4z^4}. \quad (5.21)$$

Proof. We have:

$$l_{p,q,n} = 2S_n\left(a_1+[-a_2]\right) \text{ and } Q_{k,n}(x) = 2S_n(e_1+[-e_2]) - 2xS_{n-1}(e_1+[-e_2]).$$

Using the relationships (5.15), (5.16) and (5.17), we obtain:

$$\sum_{n=0}^{\infty} l_{p,q,n}Q_{k,2n}(x)z^n = \sum_{n=0}^{\infty} 2S_n(a_1+[-a_2])\left(2S_{2n}(e_1+[-e_2]) - 2xS_{2n-1}(e_1+[-e_2])\right)z^n$$

$$= 4\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_{2n}(e_1+[-e_2])z^n - 4x\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_{2n-1}(e_1+[-e_2])z^n$$

$$= \frac{4\left(1-kpz-qk\left(4x^2+k\right)z^2\right)-4x\left(2pxz+4qx\left(2x^2+k\right)z^2\right)}{1-2p\left(2x^2+k\right)z-\left(16qx^4+16qkx^2+k^2\left(2q-p^2\right)\right)z^2+2pqk^2\left(2x^2+k\right)z^3+q^2k^4z^4}$$

$$= \frac{4-4p\left(2x^2+k\right)z-4q\left(8x^4+8kx^2+k^2\right)z^2}{1-2p\left(2x^2+k\right)z-\left(16qx^4+16qkx^2+k^2\left(2q-p^2\right)\right)z^2+2pqk^2\left(2x^2+k\right)z^3+q^2k^4z^4}.$$

And

$$\sum_{n=0}^{\infty} l_{p,q,n}Q_{k,2n+1}(x)z^n = \sum_{n=0}^{\infty} 2S_n(a_1+[-a_2])\left(2S_{2n+1}(e_1+[-e_2])-2xS_{2n}(e_1+[-e_2])\right)z^n$$

$$= 4\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_{2n+1}(e_1+[-e_2])z^n - 4x\sum_{n=0}^{\infty} S_n(a_1+[-a_2])S_{2n}(e_1+[-e_2])z^n$$

$$= \frac{4\left(2x+2qk^2xz^2\right)-4x\left(1-kpz-qk\left(4x^2+k\right)z^2\right)}{1-2p\left(2x^2+k\right)z-\left(16qx^4+16qkx^2+k^2\left(2q-p^2\right)\right)z^2+2pqk^2\left(2x^2+k\right)z^3+q^2k^4z^4}$$

$$= \frac{4x+4pkxz+4qkx\left(4x^2+3k\right)z^2}{1-2p\left(2x^2+k\right)z-\left(16qx^4+16qkx^2+k^2\left(2q-p^2\right)\right)z^2+2pqk^2\left(2x^2+k\right)z^3+q^2k^4z^4}.$$

So, the proof is completed. □

Corollary 5.10. *Taking $p = k$ and $q = 1$ in Eqs.* (5.18) − (5.21) *gives the following new generating functions:*

$$\sum_{n=0}^{\infty} l_{k,n}P_{k,2n}(x)z^n = \frac{4kxz+8x\left(2x^2+k\right)z^2}{1-2k\left(2x^2+k\right)z-\left(16x^4+16kx^2+k^2\left(2-k^2\right)\right)z^2+2k^3\left(2x^2+k\right)z^3+k^4z^4}.$$

$$\sum_{n=0}^{\infty} l_{k,n}P_{k,2n+1}(x)z^n = \frac{2-2k^2z-2k\left(4x^2+k\right)z^2}{1-2k\left(2x^2+k\right)z-\left(16x^4+16kx^2+k^2\left(2-k^2\right)\right)z^2+2k^3\left(2x^2+k\right)z^3+k^4z^4}.$$

$$\sum_{n=0}^{\infty} l_{k,n}Q_{k,2n}(x)z^n = \frac{4-4k\left(2x^2+k\right)z-4\left(8x^4+8kx^2+k^2\right)z^2}{1-2k\left(2x^2+k\right)z-\left(16x^4+16kx^2+k^2\left(2-k^2\right)\right)z^2+2k^3\left(2x^2+k\right)z^3+k^4z^4}.$$

$$\sum_{n=0}^{\infty} l_{k,n}Q_{k,2n+1}(x)z^n = \frac{4x+4k^2xz+4kx\left(4x^2+3k\right)z^2}{1-2k\left(2x^2+k\right)z-\left(16x^4+16kx^2+k^2\left(2-k^2\right)\right)z^2+2k^3\left(2x^2+k\right)z^3+k^4z^4}.$$

Remark 5.11. *Put $k = 1$ in the Eqs.* (5.18) − (5.21) *and Corollary 5.10, yields the new generating functions of the products of (p, q)-Fibonacci-like numbers and Fibonacci-like numbers with even and odd Pell and Pell Lucas polynomials.*

6. Conclusion

In this paper, by making use of Eq. (3.1), we have derived some new generating functions of odd and even (p, q)-Fibonacci-like numbers, Mersenne numbers, Mersenne Lucas numbers, k-Pell polynomials and k-Pell Lucas polynomials, and the products of (p, q)-Fibonacci-like numbers with odd and even certain numbers and polynomials. The derived theorems and propositions are based on symmetric functions and products of these numbers and polynomials.

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