Filomat 36:15 (2022), 5003–5028
https://doi.org/10.2298/FIL2215003N

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Characterizations of Double Hausdorff Matrices and Best Approximation of Conjugate of a Function in Generalized Hölder Space

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Abstract. In the present paper, we obtain characterization results of double Hausdorff matrices and apply these results to obtain error estimation of the function $\tilde{f}(x, y)$, conjugate to a function $f(x, y)$ (periodic with period 2π in both x and y) in generalized Hölder space by double Hausdorff means of its double conjugate Fourier series. Some important corollaries are also obtained from our main result.

1. Introduction

Error estimation of a function of one dimensional variable in Lipschitz, Besov, Hölder, generalized Hölder spaces has been studied by [1], [2], [3], [4], [15], [11], [13], [14], [16], [20], [21] etc. using different single and product summability means of Fourier series and conjugate Fourier series. The degree of approximation of a function of one dimensional variable in weighted Lipschitz class using characterizations of Hausdorff matrices of Fourier series, has been studied by [6].
Different double summability means of double Fourier series and double conjugate Fourier series have been investigated by the researchers [14], [17] and [24].
Error estimation of a function of two dimensional variable in Lipschitz spaces has been studied by [9] and [22] using double Nörlund means and double matrix means respectively of Fourier series.
Error estimation of a function of two dimensional variable in Hölder spaces has been studied by [12] and [23] using double Karamata means and double matrix means respectively of double Fourier series. The error estimation of a function of n-dimensional variable in Hölder space has also been studied by [23] using double matrix means of multiple Fourier series.
The double Hausdorff matrices of double sequences was firstly studied by [8]. Later, [19] and [10] have also studied double Hausdorff matrices of double sequences.
The concept of row monotonicity and row positivity of the matrices is very useful in different mathematical problems, e.g., in finding the solution of partial differential equations using finite difference method (See [18]).
The concept of row monotonicity and row positivity of summability matrices is also very useful in the error

2020 *Mathematics Subject Classification*. Primary 42A10; Secondary 40G05
Keywords. Generalized Hölder space ($H_r^{(\xi_1,\xi_2)}$; $r \geq 1$), double Hausdorff matrices, double conjugate Fourier series, error estimation of function, modulus of continuity.
Received: 22 August 2021; Revised: 17 March 2022; Accepted: 05 July 2022
Communicated by Miodrag Spalević
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estimation of a function in function spaces.
In this paper, we use double Hausdorff matrices and obtain their characterization results corresponding to row monotonicity and row positivity. Further, by using these characterizations of double Hausdorff matrices, we obtain best approximation of conjugate of a function $\tilde{f}(x,y)$ in generalized Hölder spaces $(H_r^{(\xi_1,\xi_2)}; r \geq 1)$ of double conjugate Fourier series.

The organization of the paper is as follows: In section 1, we give definitions and notations related to the present work of the paper. In section 2, we obtain characterizations of double Hausdorff matrices corresponding to row monotonicity and row positivity. In section 3, we propose our Theorem 3.1 in order to obtain the best approximation of conjugate of a function $\tilde{f}(x,y)$ of two dimensional variable in generalized Hölder spaces $(H_r^{(\xi_1,\xi_2)}; r \geq 1)$ using characterizations of double Hausdorff matrices of double conjugate Fourier series. In section 4, we prove five lemmas which are used in proving our Theorem 3.1. In section 5, we establish Theorem 3.1. In section 6, four corollaries are deduced from the Theorem 3.1 and in section 7, we verify Theorem 3.1 for different values of $d_n, d_m, d_{n,m}$ through the way of an example and observe that the Theorem 3.1 provides the best approximation of the conjugate function $\tilde{f}(x,y)$.

Let $f(x)$ be a 2π-periodic Lebesgue integral function of x over the interval $(-\pi,\pi)$. The Fourier series of function $f(x)$ is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{m=1}^{\infty}(a_m \cos mx + b_m \sin mx). \tag{1}$$

The conjugate series of (1) is given by

$$\sum_{m=1}^{\infty}(a_m \sin mx - b_m \cos mx) \tag{2}$$

and it is said to be conjugate Fourier series. It is well known that the corresponding conjugate function of (2) is defined as

$$\tilde{f}(x) = -\frac{1}{\pi}\int_0^{\pi} \frac{f(x+l) - f(x-l)}{2\tan(\frac{l}{2})} dl.$$

Let $f(x,y)$ be a function of (x,y), 2π-periodic in both the variables x and y, Lebesgue integrable and summable in the square $Q^2 := Q(-\pi,\pi;-\pi,\pi)$. The double Fourier series of a function $f(x,y)$ is given by

$$\begin{aligned} f(x,y) &\sim \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \lambda_{n,m}[\alpha_{n,m}\cos nx \cos my + \beta_{n,m}\sin nx \cos my + \gamma_{n,m}\cos nx \sin my + \delta_{n,m}\sin nx \sin my] \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \lambda_{n,m} A_{n,m}(x,y), \end{aligned} \tag{3}$$

where

$$\lambda_{n,m} = \begin{cases} \frac{1}{4} & n = m = 0, \\ \frac{1}{2} & n > 0, m = 0 \text{ and } n = 0, m > 0, \\ 1 & n > 0, m > 0, \end{cases}$$

and the coefficients $\alpha_{n,m}, \beta_{n,m}, \gamma_{n,m}$ and $\delta_{n,m}$ are given by

$$\alpha_{n,m} = \frac{1}{\pi^2}\iint_{Q^2} f(x,y)\cos nx \cos my dxdy,$$

$$\beta_{n,m} = \frac{1}{\pi^2} \iint\limits_{Q^2} f(x,y) \sin nx \cos my dxdy,$$

$$\gamma_{n,m} = \frac{1}{\pi^2} \iint\limits_{Q^2} f(x,y) \cos nx \sin my dxdy,$$

$$\delta_{n,m} = \frac{1}{\pi^2} \iint\limits_{Q^2} f(x,y) \sin nx \sin my dxdy$$

for $n = 0, 1, 2, \cdots$ and $m = 0, 1, 2, \cdots$.
One can associate three conjugate series to the double Fourier series (3) in the following ways:

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \lambda_{n,m}[-\beta_{n,m} \cos nx \cos my + \alpha_{n,m} \sin nx \cos my - \delta_{n,m} \cos nx \sin my + \gamma_{n,m} \sin nx \sin my]; \quad (4)$$

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \lambda_{n,m}[-\gamma_{n,m} \cos nx \cos my - \delta_{n,m} \sin nx \cos my + \alpha_{n,m} \cos nx \sin my + \beta_{n,m} \sin nx \sin my]; \quad (5)$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \lambda_{n,m}[\delta_{n,m} \cos nx \cos my - \gamma_{n,m} \sin nx \cos my - \beta_{n,m} \cos nx \sin my + \alpha_{n,m} \sin nx \sin my]; \quad (6)$$

where $\lambda_{n,m} = 1,\ \lambda_{n,0} = \lambda_{0,m} = \frac{1}{2}$ for $n, m \geq 1$ and $\lambda_{0,0} = \frac{1}{4}$.
The conjugate functions $\tilde{f}^{(1)}(x,y),\ \tilde{f}^{(2)}(x,y)$ and $\tilde{f}^{(3)}(x,y)$ corresponding to (4), (5) and (6) are given by

$$\tilde{f}^{(1)}(x,y) = -\frac{1}{\pi}\int_0^{\pi} \frac{f(x+s,y) - f(x-s,y)}{2\tan(\frac{s}{2})} ds, \quad (7)$$

$$\tilde{f}^{(2)}(x,y) = -\frac{1}{\pi}\int_0^{\pi} \frac{f(x,y+l) - f(x,y-l)}{2\tan(\frac{l}{2})} dl, \quad (8)$$

$$\tilde{f}^{(3)}(x,y) = \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} \frac{f(x+s,y+l) - f(x-s,y+l) - f(x+s,y-l) + f(x-s,y-l)}{2\tan(\frac{s}{2})2\tan(\frac{l}{2})} dsdl \quad (9)$$

respectively.
In this paper, we shall consider the symmetric square partial sums of series (6).
If H is a double Hausdorff matrix then

$$h_{n,p;m,q} = \begin{cases} \binom{n}{p}\binom{m}{q}\Delta_1^{n-p}\Delta_2^{m-q}\mu_{p,q}, & p = 0,1,\cdots n; q = 0,1,\cdots m, \\ 0, & p > n, q > m, \end{cases} \quad (10)$$

where $\{\mu_{p,q}\}$ is any real or complex sequences and for any sequence $\mu_{p,q}$, the operator Δ is defined by

$$\Delta\Delta\mu_{p,q} = \mu_{p,q} - \mu_{p+1,q} - \mu_{p,q+1} + \mu_{p+1,q+1}$$

and

$$\Delta^{n-p}\Delta^{m-q}\mu_{p,q} = \sum_{s=0}^{n-p}\sum_{l=0}^{m-q}(-1)^{p+q}\binom{n-p}{s}\binom{m-q}{l}\mu_{p+s,q+l}.$$

The necessary and sufficient condition for double Hausdorff matrix (H) to be conservative is the existence of a mass function $\chi(s,l) \in BV[0,1]\times[0,1]$ such that

$$\int_0^1\int_0^1 |d\chi(s,l)| < \infty$$

and

$$\mu_{n,m} = \int_0^1 \int_0^1 s^n l^m d\chi(s,l).$$

Without loss of generality, we may assume that $\chi(0,0) = 0$. If in addition, we have $\chi(1,1) = 1$, and the continuity conditions

$$\chi(s,+0) = \chi(s,0), \quad \chi(s,+0) = \lim_{l\to 0} \chi(s,l),$$

$$\chi(+0,l) = \chi(0,l), \quad \chi(+0,l) = \lim_{s\to 0} \chi(s,l)$$

are also satisfied, so that $\mu_{0,0} = 1$.
We say that $\mu_{n,m}$ is a regular moment constant ([8, 19]).
Now, we can re-write (10) as

$$h_{n,p;m,q} = \begin{cases} \binom{n}{p}\binom{m}{q} \int_0^1 \int_0^1 s^p(1-s)^{n-p} l^q (1-l)^{m-q} d\chi(s,l), & p = 0,1,\cdots n; q = 0,1,\cdots m; \\ 0, & p > n, q > m. \end{cases} \tag{11}$$

Let $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} b_{n,m}$ be a double conjugate Fourier series with $\tilde{s}_{n,m} = \sum_{j=0}^{n}\sum_{k=0}^{m} b_{j,k}$ as its $(n,m)^{th}$ partial sums. The double Hausdorff means $\tilde{t}^H_{n,m}$ is given by

$$\tilde{t}^H_{n,m} = \sum_{j=0}^{n}\sum_{k=0}^{m} h_{n,j;m,k}\tilde{s}_{j,k}.$$

If $\tilde{t}^H_{n,m} \to c$ as $n,m \to \infty$, then the double conjugate Fourier series $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} b_{n,m}$ with the sequence of $(n,m)^{th}$ partial sums $(s_{n,m})$ is said to be summable to some finit number c by the H method.

Remark 1.1. *A double Hausdorff matrix method reduces to*

(i) double Cesàro means (C,λ,σ) if mass function $\chi(s,l) = \lambda\sigma \int_0^s \int_0^l (1-s)^{\lambda-1}(1-l)^{\sigma-1} dsdl$

(ii) double Euler's means (E,ρ_n,ρ_m) if mass function $\chi(s,l) = \begin{cases} 0, & \text{if } s \in [0,a] \text{ and } l \in [0,b] \\ 1, & \text{if } s \in [a,1] \text{ and } l \in [b,1] \end{cases}$, where $a = \frac{1}{1+\rho_n}, \rho_n > 0$ and $b = \frac{1}{1+\rho_m}, \rho_m > 0$.

Note 1.

(i) Puting $\lambda = \sigma = 1$ in Remark 1.1 (i), (C,λ,σ) means reduces to $(C,1,1)$ means,

(ii) Puting $\rho_n = 1, \forall n$, and $\rho_m = 1, \forall m$, in Remark 1.1 (ii), (E,ρ_n,ρ_m) reduces to $(E,1,1)$ means.

The Hölder class for $f(x,y)$ continuous function periodic in both the variables with period 2π, is defined as

$$H_{(\alpha,\beta)} = \left\{ f : |f(x,y;s,l)| := |f(x+s,y+l) - f(x,y)| \le C_1(|s|^{\alpha} + |l|^{\beta}) \right\}$$

for some $0 < \alpha,\beta \le 1$ and for all x,y,s,l, where C_1 is a positive constant may depend on f, but not on x,y,s,l. This class of functions is also called Lipschitz class and denoted by $Lip(\alpha,\beta)$. It can be easily varified that $H_{(\alpha,\beta)}$ is a Banach space with the norm $\|\cdot\|_{(\alpha,\beta)}$ defined by

$$\|f\|_{\alpha,\beta} = \|f\|_C + \sup_{x\neq s, y\neq l} \Delta^{\alpha,\beta} f(x,y;s,l),$$

where

$$\Delta^{\alpha,\beta} f(x,y;s,l) = \frac{|f(x+s,y+l) - f(x,y)|}{|x-s|^{\alpha} + |y-l|^{\beta}} \quad (x \neq s, y \neq l).$$

By convention $\Delta^{0,0} f(x, y; s, l) = 0$ and

$$\|f\|_C = \sup_{(x,y)\in Q^2} |f(x, y)|.$$

The space of the functions $L^r[[0, 2\pi] \times [0, 2\pi]]$ is given by

$$L^r[[0, 2\pi] \times [0, 2\pi]] = \left\{ f : [[0, 2\pi] \times [0, 2\pi]] \to \mathbb{R} \times \mathbb{R} : \int_0^{2\pi} \int_0^{2\pi} |f(x, y)|^r dxdy < \infty, r \geq 1 \right\}.$$

The norm $\|\cdot\|_r$ is defined by

$$\|f\|_r := \begin{cases} \left\{ \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \left| f(x, y) \right|^r dxdy \right\}^{\frac{1}{r}}, & r \in [1, \infty); \\ \operatorname{ess\,sup}_{x,y\in[0,2\pi]} \left| f(x, y) \right|, & r = \infty. \end{cases}$$

Let $\xi_1, \xi_2 : [-\pi, \pi; -\pi, \pi] \to \mathbb{R} \times \mathbb{R}$ be an arbitrary function. The class of function $H_r^{(\xi_1,\xi_2)}$ is defined by

$$H_r^{(\xi_1,\xi_2)} = \left\{ f \in L^r[[0, 2\pi] \times [0, 2\pi]] : \sup_{s\neq 0, l\neq 0} \frac{\left\| f(x + s, y + l) - f(x, y) \right\|_r}{\xi_1(s) + \xi_2(l)} \right\},$$

where ξ_1 and ξ_2 are the moduli of continuity that is ξ_1 and ξ_2 are positive non-decreasing continuous function with the properties:

$$\lim_{s\to 0^+} \xi_1(s) = \xi_1(0) = 0; \ \lim_{l\to 0^+} \xi_2(l) = \xi_2(0) = 0$$

and

$$\xi_1(s_1 + s_2) \leq \xi_1(s_1) + \xi_1(s_2); \ \ \xi_2(l_1 + l_2) \leq \xi_2(l_1) + \xi_2(l_2).$$

We define

$$\|f\|_r^{(\xi_1,\xi_2)} = \|f\|_r + \sup_{s\neq 0, l\neq 0} \frac{\left\| f(x + s, y + l) - f(x, y) \right\|_r}{\xi_1(s) + \xi_2(l)}.$$

Clearly, $\|\cdot\|_r^{(\xi_1,\xi_2)}$ is a norm on $H_r^{(\xi_1,\xi_2)}$.
It can be varified that the completeness of the space $L^r[[0, 2\pi] \times [0, 2\pi]]$ implies the completeness of the space $H_r^{(\xi_1,\xi_2)}$.
We also define

$$\|f\|_r^{(\eta_1,\eta_2)} = \|f\|_r + \sup_{s\neq 0, l\neq 0} \frac{\left\| f(x + s, y + l) - f(x, y) \right\|_r}{\eta_1(s) + \eta_2(l)}.$$

Let $\left(\frac{\xi_1(s)}{\eta_1(s)}\right)$ and $\left(\frac{\xi_2(l)}{\eta_2(l)}\right)$ both be positive, non-decreasing. Then

$$\|f\|_r^{(\eta_1,\eta_2)} \leq \max\left(1, \frac{\xi_1(2\pi)}{\eta_1(2\pi)}, \frac{\xi_2(2\pi)}{\eta_2(2\pi)}\right) \|f\|_r^{(\xi_1,\xi_2)} < \infty.$$

Thus,

$$H_r^{(\xi_1,\xi_2)} \subseteq H_r^{(\eta_1,\eta_2)} \subseteq L^r[[0, 2\pi] \times [0, 2\pi]].$$

Note 2. If $\xi_1(s) = |s|^\alpha, \xi_2(l) = |l|^\beta$ and $r \to \infty$, then $H_r^{(\xi_1,\xi_2)}$ class reduces to $H_{\alpha,\beta}$ calss.

The degree of approximation of a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by a trigonometric polynomial $t_{n,m}$ of degree $(n, m)^{th}$ under norm $\|\cdot\|$ is defined by

$$\|t_{n,m} - f\|_\infty = \sup\{|t_{n,m} - f(x, y)|\} : x, y \in \mathbb{R}$$

and the degree of approximation $E_{n,m}(f)$ of a function $f \in L_r[[0, 2\pi] \times [0, 2\pi]]$ is given by

$$E_{n,m}(f) = \min \|t_{n,m} - f\|_r.$$

If $E_{n,m}(f) \to 0$ as $n, m \to \infty$, then $E_{n,m}(f)$ is said to be the best approximation of f.
We write

$$\Psi(s,l) = \psi(x,s;y,l) = \frac{1}{4}\left[f(x+s,y+l) - f(x+s,y-l) - f(x-s,y+l) + f(x-s,y-l)\right];$$

$$\Psi(x,y) = \int_0^x \int_0^y |\psi(u,v)| du dv;$$

$$\tilde{K}_{n,m}(s,l) = \frac{1}{4\pi^2} \sum_{p=0}^{n} h_{n,p} \frac{\cos(p+\frac{1}{2})s}{\sin(\frac{s}{2})} \sum_{q=0}^{m} h_{m,q} \frac{\cos(q+\frac{1}{2})l}{\sin(\frac{l}{2})}. \tag{12}$$

2. Characterizations of double Hausdorff matrices

By simple computation, (11) gives $h_{0,0;0,0} = 1$ for $n = 0, m = 0$ and for $n > 0,\ m > 0$, we have

$$h_{n,p;m,q} = \begin{cases} \frac{1}{B(1,n).B(1,m)} \int_{s=0}^{1} \int_{l=0}^{1} (1-l)^{m-1}(1-s)^{n-1}\chi(l)\chi(s) dl ds, & p=0, q=0; \\ \left(\frac{1}{p \cdot B(p,n-p+1)} \left\{\int_{s=0}^{1} s^{p-1}(1-s)^{n-p-1}(ns-p)\chi(s)ds\right\}\right) & \\ \cdot\left(\frac{1}{q \cdot B(q,m-q+1)} \left\{\int_{l=0}^{1} l^{q-1}(1-l)^{m-q-1}(ml-q)\chi(l)dl\right\}\right), & 0 < p < n \text{ and } 0 < q < m; \\ \left(1 - \frac{1}{B(n,1)} \int_{s=0}^{1} s^{n-1}\chi(s)ds\right)\left(1 - \frac{1}{B(m,1)} \int_{l=0}^{1} l^{m-1}\chi(l)dl\right), & p = n, q = m; \\ 0, & p > n, q > m; \end{cases} \tag{13}$$

where $B(n, m)$ denotes the beta function.
Now, we establish the following characterization results of Hausdorff matrix $H = h_{n,p;m,q}$.

Theorem 2.1. *If $H = (h_{n,p;m,q})$ be a double Hausdorff matrix (11) and $\chi(s,l)$ is strictly increasing function, then the matrix $h_{n,p;m,q} > 0$ for $0 \le p \le n;\ n = 0, 1, 2, \cdots$ and $0 \le q \le m;\ m = 0, 1, 2, \cdots$.*

Proof. We can see that $h_{0,0;0,0} = 1$ for $n, m = 0$. So that, we deal for $n > 1$. Since $\chi(s,l)$ is strictly increasing and $\chi(0,0) = 0,\ \chi(1,1) = 1$, we find that $0 < \chi(s,l) < 1$ for $s,l \in (0,1)$. Thus, for $p, q = 0$,

$$h_{n,0;m,0} = \frac{1}{B(1,m).B(1,n)} \int_{s=0}^{1} \int_{l=0}^{1} (1-l)^{m-1}(1-s)^{n-1}\chi(l)\chi(s) dl ds > 0$$

as

$$(1-l)^{m-1}\chi(l) > 0 \ \text{ for } \ l \in (0,1) \text{ and } (1-s)^{n-1}\chi(s) > 0 \ \text{ for } \ s \in (0,1).$$

For $p = n$ and $q = m$, we have

$$\begin{aligned} h_{n,n;m,m} &= \left(1 - \frac{1}{B(m,1)} \int_{l=0}^{1} l^{m-1}\chi(l)dl\right) \cdot \left(1 - \frac{1}{B(n,1)} \int_{s=0}^{1} s^{n-1}\chi(s)ds\right) \\ &> \left(1 - \frac{1}{B(m,1)} \int_{l=0}^{1} l^{m-1}dl\right) \cdot \left(1 - \frac{1}{B(n,1)} \int_{s=0}^{1} s^{n-1}ds\right), \end{aligned}$$

where $l^{m-1}\chi(l) > 0$ *for* $l \in (0,1)$ *and* $s^{n-1}\chi(s) > 0$ *for* $s \in (0,1)$.

$$h_{n,n;m,m} = \left(1 - \frac{1}{\frac{(m-1)!.0!}{m!}}\frac{1}{m}\right)\cdot\left(1 - \frac{1}{\frac{(n-1)!.0!}{n!}}\frac{1}{n}\right)$$

Thus,

$$h_{n,p;m,q} > 0.$$

For $0 < p < n$ *and* $0 < q < m$, *we have*

$$h_{n,p;m,q} = \frac{1}{p\cdot qB(p,n-p+1)\cdot B(q,m-q+1)}$$
$$\cdot\left\{\int_{s=0}^{1}\int_{l=0}^{1} s^{p-1}l^{q-1}(1-s)^{n-p-1}(1-l)^{m-q-1}(ns-p)(ml-q)\chi(s)\chi(l)dsdl\right\}. \quad (14)$$

We denote

$$M_{m,q}(l) = \frac{l^{q-1}(1-l)^{m-q-1}(ml-q)}{q\cdot B(q,m-q+1)} \quad (15)$$

and

$$N_{n,p}(s) = \frac{s^{p-1}(1-s)^{n-p-1}(ns-p)}{p\cdot B(p,n-p+1)} \quad (16)$$

We check that (15) and (16) become zero only at $l = \frac{q}{m}$ *and* $s = \frac{p}{n}$ *respectively within* $(0,1)$. *Also, we note that* $M_{m,q}(l) < 0$ *for* $l \in (0, \frac{q}{m})$; $N_{n,p}(s) < 0$ *for* $s \in (0, \frac{p}{n})$ *and* $M_{m,q}(l) > 0$ *for* $l \in (\frac{q}{m}, 1)$; $N_{n,p}(s) > 0$ *for* $s \in (\frac{p}{n}, 1)$. *From (14)-(16), we have*

$$h_{n,p;m,q} = \int_{s=0}^{1}\int_{l=0}^{1} M_{m,q}(l)\cdot N_{n,p}(s)\chi(l)\chi(s)dlds$$

Now,

$$\begin{aligned}
h_{n,p;m,q} &= \int_{s=0}^{1} N_{n,p}(s)\left[\int_{l=0}^{1} M_{m,q}(l)\chi(l)dl\right]\chi(s)ds \\
&= \int_{s=0}^{1} N_{n,p}(s)\left[\left\{\int_{l=0}^{\frac{q}{m}} + \int_{\frac{q}{m}}^{1}\right\} M_{m,q}(l)\chi(l)dl\right]\chi(s)ds \\
&= \int_{s=0}^{1} N_{n,p}(s)\left[\int_{l=0}^{\frac{q}{m}} M_{m,q}(l)\chi(l)dl + \int_{\frac{q}{m}}^{1} M_{m,q}(l)\chi(l)dl\right]\chi(s)ds \\
&> \int_{s=0}^{1} N_{n,p}(s)\left[\int_{l=0}^{\frac{q}{m}} M_{m,q}(l)\chi(l)dl - \chi\left(\frac{q}{m}\right)\int_{1}^{\frac{q}{m}} M_{m,q}(l)dl\right]\chi(s)ds \\
&= \int_{s=0}^{1} N_{n,p}(s)\left[\int_{l=0}^{\frac{q}{m}} M_{m,q}(l)\chi(l)dl - \chi\left(\frac{q}{m}\right)\left\{\int_{l=1}^{0} M_{m,q}(l)dl + \int_{l=0}^{\frac{q}{m}} M_{m,q}(l)dl\right\}\right]\chi(s)ds \\
&= \int_{s=0}^{1} N_{n,p}(s)\left[\int_{l=0}^{\frac{q}{m}} M_{m,q}(l)\chi(l)dl + \chi\left(\frac{q}{m}\right)\int_{l=0}^{1} M_{m,q}(l)dl - \chi\left(\frac{q}{m}\right)\int_{l=0}^{\frac{q}{m}} M_{m,q}(l)dl\right]\chi(s)ds \\
&= \int_{s=0}^{1} N_{n,p}(s)\left[\int_{l=0}^{\frac{q}{m}} M_{m,q}(l)\chi(l)dl - \chi\left(\frac{q}{m}\right)\int_{l=0}^{\frac{q}{m}} M_{m,q}(l)dl\right]\chi(s)ds \quad \because \int_{l=0}^{1} M_{m,q}(l)dl = 0
\end{aligned}$$

$$= \left[\int_0^{\frac{q}{m}} \left\{\chi(l) - \chi\left(\frac{q}{m}\right)\right\} M_{m,q}(l)dl\right] \cdot \left[\int_{s=0}^{1} N_{n,p}(s)\chi(s)ds\right]$$
$$= \left[\int_0^{\frac{q}{m}} \left\{\chi(l) - \chi\left(\frac{q}{m}\right)\right\} M_{m,q}(l)dl\right] \cdot \left[\int_{s=0}^{\frac{p}{n}} \left\{\chi(s) - \chi\left(\frac{p}{n}\right)\right\} N_{n,p}(s)ds\right] > 0$$

Thus,

$$h_{n,p;m,q} > 0.$$

□

Theorem 2.2. *Let $H = (h_{n,p;m,q})$ be a double Hausdorff matrix (11). Then*

(i) if $\chi(l)$ and $\chi(s)$ are convex, then for all m and n,

$$h_{n,0;m,0} \le h_{n,1;m,1} \le \cdots \le h_{n,p;m,q} \le h_{n,p+1;m,q+1} \le \cdots \le h_{n,n-1;m,m-1} \le h_{n,n;m,m}.$$

(ii) if $\chi(l)$ and $\chi(s)$ are concave, then for all m and n,

$$h_{n,0;m,0} \ge h_{n,1;m,1} \ge \cdots \ge h_{n,p;m,q} \ge h_{n,p+1;m,q+1} \ge \cdots \ge h_{n,n-1;m,m-1} \ge h_{n,n;m,m}.$$

Proof. (i) For $n, m = 1$, using the convexity of $\chi(l)$ and $\chi(s)$, we have $\chi(l) \le l$ for $l \in (0, 1)$ and $\chi(s) \le s$ for $s \in (0, 1)$ respectively. Now, we proceed with the following:

$$\begin{aligned}
h_{1,1;1,1} - h_{1,0;1,0} &= \left\{1 - \frac{1}{B(n,1)} \int_0^1 s^{n-1}\chi(s)ds\right\} \cdot \left\{1 - \frac{1}{B(m,1)} \int_0^1 l^{m-1}\chi(l)dl\right\} \\
&- \frac{1}{B(1,n) \cdot B(1,m)} \int_0^1 \int_0^1 (1-l)^{m-1}(1-s)^{n-1}\chi(l)\chi(s)dlds \\
&= \left\{1 - \frac{1}{B(1,1)} \int_0^1 s^{1-1}\chi(s)ds\right\} \cdot \left\{1 - \frac{1}{B(1,1)} \int_0^1 l^{1-1}\chi(l)dl\right\} \\
&- \frac{1}{B(1,1) \cdot B(1,1)} \int_0^1 \int_0^1 (1-l)^{1-1}(1-s)^{1-1}\chi(l)\chi(s)dlds \\
&= \left\{1 - \int_0^1 \chi(s)ds\right\} \cdot \left\{1 - \int_0^1 \chi(l)dl\right\} - \int_0^1 \int_0^1 \chi(l)\chi(s)dlds \\
&= 1 - \int_0^1 \chi(s)ds - \int_0^1 \chi(l)dl + \int_0^1 \int_0^1 \chi(l)\chi(s)dlds - \int_0^1 \int_0^1 \chi(l)\chi(s)dlds \\
&= 1 - \int_0^1 \chi(s)ds - \int_0^1 \chi(l)dl \\
&\ge 1 - \int_0^1 sds - \int_0^1 ldl \\
&= 1 - \frac{1}{2} - \frac{1}{2} \\
&= 0.
\end{aligned}$$

Now, for any $n = 2, 3, \cdots$, we need to show that $h_{n,0;m,0} \le h_{n,1;m,1}$ and $h_{n,n-1;m,m-1} \le h_{n,n;m,m}$. So, we proceed with the *following:*

$$\begin{aligned} h_{n,1;m,1} - h_{n,0;m,0} &= \frac{1}{B(1,n)\cdot B(1,m)}\left\{\int_{l=0}^{1}(1-l)^{m-2}(ml-1)\chi(l)dl\right\}\cdot\left\{\int_{s=0}^{1}(1-s)^{n-2}(ns-1)\chi(s)ds\right\} \\ &- \frac{1}{B(1,m)\cdot B(1,n)}\int_{s=0}^{1}\int_{l=0}^{1}(1-l)^{m-1}(1-s)^{n-1}\chi(l)\chi(s)dlds \\ &= mn\left[\int_0^1\int_0^1(1-l)^{m-2}(1-s)^{n-2}\{(ml-1)(ns-1)-(1-l)(1-s)\}\chi(l)\chi(s)dlds\right] \\ &= mn\left[\int_0^1\int_0^1(1-l)^{m-2}(1-s)^{n-2}\{(mn-1)sl+(1-m)l+(1-n)s\}\chi(l)\chi(s)dlds\right] \end{aligned}$$

It can be observe that $[(mn-1)sl+(1-m)l+(1-n)s]$ *becomes zero only at* $s=\frac{2}{n+1}$ *and* $l=\frac{2}{m+1}$. *It can also be observed that* $[(mn-1)sl+(1-m)l+(1-n)s]<0$ *for* $s\in\left(0,\frac{2}{n+1}\right)$ *and* $l\in\left(0,\frac{2}{m+1}\right)$. *Also,* $[(mn-1)sl+(1-m)l+(1-n)s]>0$ *for* $s\in\left(\frac{2}{m+1},1\right)$ *and* $l\in\left(\frac{2}{m+1},1\right)$.
Thus,

$$\begin{aligned} &h_{n,1;m,1} - h_{n,0;m,0} \\ &= mn\left\{\int_{s=0}^{\frac{2}{n+1}}\int_{l=0}^{\frac{2}{m+1}} + \int_{s=\frac{2}{n+1}}^{1}\int_{l=\frac{2}{m+1}}^{1}\right\}(1-l)^{m-2}(1-s)^{n-2}\Big[(mn-1)sl+(1-m)l+(1-n)s\Big]\chi(l)\chi(s)dlds \\ &= \int_{s=0}^{\frac{2}{n+1}} n(1-s)^{n-2}\left[\int_{l=0}^{l=\frac{2}{m+1}} m(1-l)^{m-2}\Big\{(mn-1)sl+(1-m)l+(1-n)s\Big\}.\chi(l)dl\right]\chi(s)ds \\ &+ \int_{s=\frac{2}{n+1}}^{1} n(1-s)^{n-2}\left[\int_{l=\frac{2}{m+1}}^{1} m(1-l)^{m-2}\Big\{(mn-1)sl+(1-m)l+(1-n)s\Big\}\chi(l)dl\right]\chi(s)ds \end{aligned} \tag{17}$$

By the convexity of $\chi(s)$ *and* $\chi(l)$, *we have*

$$\chi(s)\le\frac{(n+1)s}{2}\chi\left(\frac{2}{n+1}\right);\ s\in\left(0,\frac{2}{n+1}\right),\quad \chi(s)\ge\frac{(n+1)s}{2}\chi\left(\frac{2}{n+1}\right);\ s\in\left(\frac{2}{n+1},1\right)$$

and

$$\chi(l)\le\frac{(m+1)l}{2}\chi\left(\frac{2}{m+1}\right);\ l\in\left(0,\frac{2}{m+1}\right),\quad \chi(l)\ge\frac{(m+1)l}{2}\chi\left(\frac{2}{m+1}\right);\ l\in\left(\frac{2}{m+1},1\right).$$

Using above, (17) becomes

$$\begin{aligned} &h_{n,1;m,1} - h_{n,0;m,0} \\ &\ge \int_{s=\frac{2}{n+1}}^{1} n(1-s)^{n-2}\left[\int_{l=\frac{2}{m+1}}^{1} m(1-l)^{m-2}\Big\{((mn-1)s+(1-m))l+(1-n)s\Big\}\right. \\ &\left.\cdot\frac{(m+1)l}{2}\chi\left(\frac{2}{m+1}\right)dl\right]\frac{(n+1)s}{2}\chi\left(\frac{2}{n+1}\right)ds \\ &= 0 \end{aligned}$$

Thus,

$$h_{n,1;m,1}\ge h_{n,0;m,0}.$$

Further,

$$\begin{aligned}
h_{n,n;m,m} - h_{n,n-1;m,m-1} &= \left(1 - \frac{1}{B(m,1)}\int_{l=0}^{1} l^{m-1}\chi(l)dl\right)\cdot\left(1 - \frac{1}{B(n,1)}\int_{s=0}^{1} s^{n-1}\chi(s)ds\right)\\
&- \frac{1}{(n-1)B(n-1,2)}\left[\int_{s=0}^{1} s^{n-2}(1-s)^0(ns-n+1)\chi(s)ds\right]\\
&\cdot \frac{1}{(m-1)B(m-1,2)}\left[\int_{l=0}^{1} l^{m-2}(1-l)^0(ml-m+1)\chi(l)dl\right]\\
&= \left(1 - m\int_{l=0}^{1} l^{m-1}\chi(l)dl\right).\left(1 - n\int_{s=0}^{1} s^{n-1}\chi(s)ds\right)\\
&- nm\left[\left\{\int_{s=0}^{1} s^{n-2}(ns-n+1)\chi(s)ds\right\}\cdot\left\{\int_{l=0}^{1} l^{m-2}(ml-m+1)\chi(l)dl\right\}\right]\\
&= \left(1 - m\int_{l=0}^{1} l^{m-1}\chi(l)dl\right).\left(1 - n\int_{s=0}^{1} s^{n-1}\chi(s)ds\right)\\
&- nm\left[\int_{s=0}^{1}\int_{l=0}^{1}\left\{s^{n-2}l^{m-2}(ns-n+1)(ml-m+1)\right\}\chi(l)dl\chi(s)ds\right]
\end{aligned}$$

It can be observed that $\left\{s^{n-2}l^{m-2}(ns-n+1)(ml-m+1)\right\}$ *becomes zero only at* $l = \frac{m-1}{m}$ *and* $s = \frac{n-1}{n}$ *in* $(0,1)$. *It can also be observed that* $\left\{s^{n-2}l^{m-2}(ns-n+1)(ml-m+1)\right\} < 0$ *for* $s \in \left(0, \frac{n-1}{n}\right)$ *and* $l \in \left(0, \frac{m-1}{m}\right)$. *Also,* $\left\{s^{n-2}l^{m-2}(ns-n+1)(ml-m+1)\right\} > 0$ *for* $s \in \left(\frac{n-1}{n}, 1\right)$ *and* $l \in \left(\frac{m-1}{m}, 1\right)$.
Thus,

$$\begin{aligned}
&h_{n,n;m,m} - h_{n,n-1;m,m-1}\\
&= \left(1 - m\left\{\int_{l=0}^{\frac{m-1}{m}} + \int_{\frac{m-1}{m}}^{1}\right\} l^{m-1}\chi(l)dl\right).\left(1 - n\left\{\int_{s=0}^{\frac{n-1}{n}} + \int_{\frac{n-1}{n}}^{1}\right\} s^{n-1}\chi(s)ds\right)\\
&- nm\left[\left\{\int_{s=0}^{\frac{n-1}{n}}\int_{l=0}^{\frac{m-1}{m}} + \int_{s=\frac{n-1}{n}}^{1}\int_{l=\frac{m-1}{m}}^{1}\right\} s^{n-2}.l^{m-2}\cdot(ns-n+1)(ml-m+1)\chi(l)\chi(s)dlds\right]
\end{aligned} \tag{18}$$

By the convexity of $\chi(s)$ *and* $\chi(l)$, *we have*

$$\chi(s) \geq 1 - n\left(1 - \chi\left(\frac{n-1}{n}\right)\right)\frac{1-s}{2}, \quad s \in \left(0, \frac{n-1}{n}\right);$$

$$\chi(s) \leq 1 - n\left(1 - \chi\left(\frac{n-1}{n}\right)\right)\frac{1-s}{2}, \quad s \in \left(\frac{n-1}{n}, 1\right).$$

and

$$\chi(l) \geq 1 - m\left(1 - \chi\left(\frac{m-1}{m}\right)\right)\frac{1-l}{2}, \quad l \in \left(0, \frac{m-1}{m}\right);$$

$$\chi(l) \leq 1 - m\left(1 - \chi\left(\frac{m-1}{m}\right)\right)\frac{1-l}{2}, \quad l \in \left(\frac{m-1}{m}, 1\right).$$

Using above, (18) becomes

$$\begin{aligned}
h_{n,n;m,m} - h_{n,n-1;m,m-1} \geq & \left(1 - m\int_{l=0}^{1} l^{m-1}\left\{1 - m\left(1 - \chi\left(\frac{m-1}{m}\right)\right)\frac{1-l}{2}\right\}dl\right) \\
& \cdot\left(1 - n\int_{s=0}^{1} s^{n-1}\left\{1 - n\left(1 - \chi\left(\frac{n-1}{n}\right)\right)\frac{1-s}{2}\right\}ds\right) \\
& - nm\left[\int_{s=0}^{1} s^{n-2}(ns-n+1)\left\{\int_{l=0}^{1} l^{m-2}(ml-m+1)\right.\right. \\
& \left.\left.\cdot\left\{1 - m\left(1 - \chi\left(\frac{m-1}{m}\right)\right)\frac{1-l}{2}\right\}dl\right\}.\left\{1 - n\left(1 - \chi\left(\frac{n-1}{n}\right)\right)\frac{1-s}{2}\right\}ds\right] \\
& = 0
\end{aligned}$$

Thus,

$$h_{n,n;m,m} \geq h_{n,n-1;m,m-1}.$$

Lastly, we need to prove that $h_{n,p+1;m,q+1} \geq h_{n,p;m,q}$ *for* $n = 2,3,\cdots; m = 2,3,\cdots$ *and* $0 < p < n-1;\ 0 < q < m-1$. *Now, we proceed with the following:*

$$\begin{aligned}
& h_{n,p+1;m,q+1} - h_{n,p;m,q} \\
& = \frac{1}{(p+1).B(p+1,n-p)}\left[\int_{s=0}^{1} s^{p}(1-s)^{n-p-2}(ns-p-1)\chi(s)ds\right] \\
& .\frac{1}{(q+1).B(q+1,m-q)}\left[\int_{l=0}^{1} l^{q}(1-l)^{m-q-2}(ml-q-1)\chi(l)dl\right] \\
& - \frac{1}{p.B(p,n-p+1)}\left[\int_{s=0}^{1} s^{p-1}(1-s)^{n-p-1}(ns-p)\chi(s)ds\right] \\
& .\frac{1}{q.B(q,m-q+1)}\left[\int_{l=0}^{1} l^{q-1}(1-l)^{m-q-1}(ml-q)\chi(l)dl\right] \\
& = \int_{s=0}^{1}\int_{l=0}^{1} s^{p-1}l^{q-1}(1-s)^{n-p-2}(1-l)^{m-q-2}\left\{\frac{sl(ns-p-1)(ml-q-1)}{(p+1)(q+1)B(p+1,n-p).B(q+1,m-q)}\right. \\
& \left. - \frac{(1-s)(1-l)(ns-p)(ml-q)}{p.qB(p,n-p+1).B(q,m-q+1)}\right\}\chi(s)\chi(l)dsdl \\
& = \int_{s=0}^{1}\int_{l=0}^{1}\left[\left\{\frac{s^{p-1}l^{q-1}(1-s)^{n-p-2}(1-l)^{m-q-2}}{(p+1)(q+1)(n-p)(m-q)B(p+1,n-p).B(q+1,m-q)}\right\}\right. \\
& \left.\cdot\left\{sl(ns-p-1)(ml-q-1)(n-p)(m-q) - (1-s)(1-l)(ns-p)(ml-q)(p+1)(q+1)\right\}\right]\cdot\chi(l)\chi(s)dlds \\
& = \int_{0}^{1}\int_{0}^{1} P_{m,q}(l)\cdot R_{n,p}(s)\chi(l)\chi(s)dlds \qquad (19)
\end{aligned}$$

for $p = 1,2,\cdots,n-2;\ q = 1,2,\cdots,m-2$ *and* $n,m = 3,4,\cdots$
where

$$P_{m,q}(l) = \frac{l^{q-1}(1-l)^{m-q-2}}{(q+1)(m-q).B(q+1,m-q)}\left\{m(m+1)l^2 - 2m(q+1)l + q(q+1)\right\} \qquad (20)$$

and

$$R_{n,p}(s) = \frac{s^{p-1}(1-s)^{n-p-2}}{(p+1)(n-p).B(p+1,n-p)}\left\{n(n+1)s^2 - 2n(p+1)s + p(p+1)\right\}. \qquad (21)$$

$P_{m,q}(l)$ has following two zeros in $(0,1)$:

$$\lambda_1 = \frac{q+1}{m+1} - \frac{\sqrt{m(m-q)(q+1)}}{m(m+1)} \text{ and } \lambda_2 = \frac{q+1}{m+1} + \frac{\sqrt{m(m-q)(q+1)}}{m(m+1)}.$$

$R_{n,p}(s)$ has following two zeros in (0,1):

$$\omega_1 = \frac{p+1}{n+1} - \frac{\sqrt{n(n-p)(p+1)}}{n(n+1)} \text{ and } \omega_2 = \frac{p+1}{n+1} + \frac{\sqrt{n(n-p)(p+1)}}{n(n+1)}.$$

It is observed that $0 < \lambda_1 < \lambda_2 < 1$. Also, $P_{m,q}(l) < 0$ for $l \in (\lambda_1, \lambda_2)$ and $P_{m,q}(l) > 0$ in $\in ((0, \lambda_1) \cup (\lambda_2, 1))$.
Similarly, it is observed that $0 < \omega_1 < \omega_2 < 1$. Also, $R_{n,p}(s) < 0$ for (ω_1, ω_2) and $R_{n,p}(s) > 0$ in $(0, \omega_1) \cup (\omega_2, 1)$.
Now, we define the conditions on $\chi(l)$ and $\chi(s)$ which make the root of the corresponding matrix monotonic up to diagonal elements.
By convexity of $\chi(l)$ and $\chi(s)$, we have

$$\chi(l) \geq \chi(\lambda_1) - \frac{\chi(\lambda_2) - \chi(\lambda_1)}{\lambda_2 - \lambda_1}(l - \lambda_1), \qquad l \in (0, \lambda_1) \cup (\lambda_2, 1)$$

and

$$\chi(l) \leq \chi(\lambda_1) - \frac{\chi(\lambda_2) - \chi(\lambda_1)}{\lambda_2 - \lambda_1}(l - \lambda_1), \qquad l \in (\lambda_1, \lambda_2).$$

Also,

$$\chi(s) \geq \chi(\omega_1) - \frac{\chi(\omega_2) - \chi(\omega_1)}{\omega_2 - \omega_1}(l - \omega_1), \qquad s \in (0, \omega_1) \cup (\omega_2, 1)$$

and

$$\chi(s) \leq \chi(\omega_1) - \frac{\chi(\omega_2) - \chi(\omega_1)}{\omega_2 - \omega_1}(s - \omega_1), \qquad s \in (\omega_1, \omega_2).$$

Using positivity of $P_{m,q}(l)$ and $R_{n,p}(s)$ in $(0, \lambda_1) \cup (\lambda_2, 1)$ and $(0, \omega_1) \cup (\omega_2, 1)$ respectively and negativity of $P_{m,q}(l)$ and $R_{n,p}(s)$ in (λ_1, λ_2) and (ω_1, ω_2) respectively, (19) becomes

$$\begin{aligned}
&h_{n,p+1;m,q+1} - h_{n,p;m,q} \\
&= \left\{ \int_{l=0}^{\lambda_1} \int_{s=0}^{\omega_1} + \int_{l=\lambda_1}^{\lambda_2} \int_{s=\omega_1}^{\omega_2} + \int_{l=\lambda_2}^{1} \int_{s=\omega_2}^{1} \right\} P_{m,q}(l) \cdot R_{n,p}(s) \chi(l) \chi(s) dl ds. \\
&= \left(\int_{l=0}^{\lambda_1} P_{m,q}(l) \chi(l) dl \right) \cdot \left(\int_{s=0}^{\omega_1} R_{n,p}(s) \chi(s) ds \right) + \left(\int_{l=\lambda_1}^{\lambda_2} P_{m,q}(l) \chi(l) dl \right) \\
&\cdot \left(\int_{s=\omega_1}^{\omega_2} R_{n,p}(s) \chi(s) ds \right) + \left(\int_{l=\lambda_2}^{1} P_{m,q}(l) \chi(l) dl \right) \cdot \left(\int_{s=\omega_2}^{1} R_{n,p}(s) \chi(l) \chi(s) ds \right) \\
&\geq \int_0^1 \int_0^1 \left[P_{m,q}(l) \cdot R_{n,p}(s) \cdot \left\{ \chi(\lambda_1) - \frac{\chi(\lambda_2) - \chi(\lambda_1)}{\lambda_2 - \lambda_1}(l - \lambda_1) \right\} \right. \\
&\quad \left. \cdot \left\{ \chi(\omega_1) - \frac{\chi(\omega_2) - \chi(\omega_1)}{\omega_2 - \omega_1}(l - \omega_1) \right\} \right] dl ds \\
&= 0
\end{aligned}$$

Thus,

$$h_{n,p+1;m,q+1} \geq h_{n,p;m,q}.$$

In this way, first part of the theorem is proved.
Second part of the theorem, i.e., for concave $\chi(s)$ and $\chi(l)$, can be proved just by inverting the inequalities used for convex $\chi(s)$ and $\chi(l)$ in the first part of the proof. □

3. Best approximation of a conjugate function using characterizations of double Hausdorff matrices

Theorem 3.1. *Let $\tilde{f}(x,y)$ be a function, conjugate to a function $f(x,y)$ periodic with period 2π in both x and y, Lebesgue integrable on $Q(-\pi,\pi;-\pi,\pi)$. Then, the error estimation of $\tilde{f}(x,y)$ in the space $(H_r^{(\xi)}; r \geq 1)$ using double Hausdorff matrix (H) wherein mass function $\chi(s,l)$ is strictly increasing and convex or concave, of its double conjugate Fourier series, is given by*

$$\begin{aligned}
&\|\tilde{t}_{n,m}^{H}(x,y) - \tilde{f}(x,y)\|_r^{(\eta)} \\
&= O\Bigg[\left(\frac{1}{(n+1)(m+1)}\right)\Bigg(2\log(n+1)\cdot\log(m+1) + (m+1)d_m\log(n+1) + (n+1)d_n\log(m+1) \\
&\qquad + 2(n+1)(m+1)d_{n,m}\Bigg)\cdot\left(\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\left(\frac{1}{s^2l^2}\right)dlds\right)\Bigg], \qquad (22)
\end{aligned}$$

where $d_{n,m} = h_{n,n;m,m}$ if $\chi(s,l)$ is convex and $d_{n,m} = h_{n,0;m,0}$ if $\chi(s,l)$ is concave; $d_m = h_{m,m}$ if $\chi(s)$ is convex and $d_m = h_{m,0}$ if $\chi(s)$ is concave; and $d_n = h_{n,n}$ if $\xi(l)$ is converx and $d_n = h_{n,0}$ if $\xi(l)$ is concave.

4. Lemmas

Lemma 4.1. *$\tilde{K}_{n,m}(s,l) = O\left(\frac{1}{sl}\right)$ for $0 < s \leq \frac{1}{n+1}$ and $0 < l \leq \frac{1}{m+1}$.*

Proof. For $0 < s \leq \frac{1}{n+1}$, $0 < l \leq \frac{1}{m+1}$, $\sin(\frac{s}{2}) \geq \frac{s}{\pi}$, $\sin(\frac{l}{2}) \geq \frac{l}{\pi}$ and $|\cos ns| \leq 1$.

$$\begin{aligned}
\left|\tilde{K}_{n,m}(s,l)\right| &= \frac{1}{4\pi^2}\left|\sum_{p=0}^{n} h_{n,p}\frac{\cos(p+\frac{1}{2})s}{\sin(\frac{s}{2})}\sum_{q=0}^{m} h_{m,q}\frac{\cos(q+\frac{1}{2})l}{\sin(\frac{l}{2})}\right| \\
&= \frac{1}{4\pi^2}\left\{\sum_{p=0}^{n} h_{n,p}\frac{|\cos(p+\frac{1}{2})s|}{|\sin(\frac{s}{2})|}\right\}\cdot\left\{\sum_{q=0}^{m} h_{m,q}\frac{|\cos(q+\frac{1}{2})l|}{|\sin(\frac{l}{2})|}\right\} \\
&\leq \frac{\pi^2}{4\pi^2 sl}\sum_{p=0}^{n} h_{n,p}\cdot\sum_{q=0}^{m} h_{m,q}.
\end{aligned}$$

We have $\sum_{p=0}^{n} h_{n,p} = 1$ and $\sum_{q=0}^{m} h_{m,q} = 1$ [5, p.397], then

$$\left|\tilde{K}_{n,m}(s,l)\right| = O\left(\frac{1}{sl}\right).$$

□

Lemma 4.2. *If H is a double Hausdorff matrix with strictly increasing and convex or concave $\chi(s,l)$ then*

$$\tilde{K}_{n,m}(s,l) = O\left(\frac{d_m}{sl^2}\right) \text{ for } 0 < s \leq \frac{1}{n+1} \text{ and } \frac{1}{m+1} < l \leq \pi,$$

where $d_m = h_{m,m}$ if $\chi(l)$ is convex and $d_m = h_{m,0}$ if $\chi(l)$ is concave.

Proof. For $0 < s \leq \frac{1}{n+1}$, $\sin(\frac{s}{2}) \geq \frac{s}{\pi}$, $|\cos(ns)| \leq 1$ and for $\frac{1}{m+1} < l \leq \pi$, $\sin(\frac{l}{2}) \geq \frac{l}{\pi}$, $|\sin(ml)| \leq 1$, $|\sin(m+1)l| \leq 1$.

$$
\begin{aligned}
\left|\tilde{K}_{n,m}(s,l)\right| &= \frac{1}{4\pi^2}\left|\sum_{p=0}^{n} h_{n,p}\frac{\cos(p+\frac{1}{2})s}{\sin(\frac{s}{2})}\cdot\sum_{q=0}^{m} h_{m,q}\frac{\cos(q+\frac{1}{2})l}{\sin(\frac{l}{2})}\right| \\
&= \frac{1}{4\pi^2}\left|\sum_{p=0}^{n} h_{n,p}\frac{\cos(p+\frac{1}{2})s}{\sin(\frac{s}{2})}\cdot\frac{1}{2\sin^2(\frac{l}{2})}\sum_{q=0}^{m} h_{m,q}\cdot 2\cos\left(q+\frac{1}{2}\right)l\cdot\sin\left(\frac{l}{2}\right)\right| \\
&\leq \frac{1}{4\pi^2}\left\{\sum_{p=0}^{n} h_{n,p}\frac{|\cos(p+\frac{1}{2})s|}{|\sin(\frac{s}{2})|}\right\}\cdot\frac{1}{|2\sin^2(\frac{l}{2})|}\left|\sum_{q=0}^{m} h_{m,q}\cdot 2\cos\left(q+\frac{1}{2}\right)l\cdot\sin\left(\frac{l}{2}\right)\right| \\
&\leq \frac{1}{4\pi^2}\cdot\frac{\pi}{s}\sum_{p=0}^{n} h_{n,p}\cdot\frac{\pi^2}{2l^2}\left|\sum_{q=0}^{m} h_{m,q}\cdot\{\sin(q+1)l-\sin(ql)\}\right| \\
&\leq \frac{\pi}{8sl^2}\left|h_{m,m}\sin(m+1)l-\sum_{q=1}^{m}\left(h_{m,q}-h_{m,q-1}\right)\sin(ql)\right| \\
&\leq \frac{\pi}{8sl^2}\left(h_{m,m}-\sum_{q=1}^{m}\left(h_{m,q}-h_{m,q-1}\right)\right) \\
&\leq \frac{\pi}{8sl^2}(h_{m,m}-h_{m,1}+h_{m,0}-h_{m,2}+h_{m,1}-h_{m,3}+h_{m,2}-\cdots-h_{m,m}+h_{m,m-1}) \\
&\leq \frac{\pi}{8sl^2}h_{m,0} \\
&\leq \frac{\pi}{8sl^2}d_m \\
\left|\tilde{K}_{n,m}(s,l)\right| &= O\left(\frac{d_m}{sl^2}\right).
\end{aligned}
$$

□

Lemma 4.3. *If H is a double Hausdorff matrix with strictly increasing and convex or concave $\chi(s,l)$ then*

$$\tilde{K}_{n,m}(s,l) = O\left(\frac{d_n}{s^2 l}\right) \ for \ \frac{1}{n+1} < s \leq \pi \ and \ 0 < l \leq \frac{1}{m+1},$$

where $d_n = h_{n,n}$ if $\chi(s)$ is convex and $d_n = h_{n,0}$ if $\chi(s)$ is concave.

Proof. For $\frac{1}{n+1} < s \leq \pi$, $\sin(\frac{s}{2}) \geq \frac{s}{\pi}$, $|\sin(n+1)s| \leq 1$, $|\sin(ps)| \leq 1$ and for $0 < l \leq \frac{1}{m+1}$, $\sin(\frac{l}{2}) \geq \frac{l}{\pi}$, $|\cos(ml)| \leq 1$.

$$
\begin{aligned}
\left|\tilde{K}_{n,m}(s,l)\right| &= \frac{1}{4\pi^2}\left|\sum_{p=0}^{n} h_{n,p}\frac{\cos(p+\frac{1}{2})s}{\sin(\frac{s}{2})}\cdot\sum_{q=0}^{m} h_{m,q}\frac{\cos(q+\frac{1}{2})l}{\sin(\frac{l}{2})}\right| \\
&= \frac{1}{4\pi^2}\left|\frac{1}{2\sin^2(\frac{s}{2})}\sum_{p=0}^{n} h_{n,p}\cdot 2\cos\left(p+\frac{1}{2}\right)s\cdot\sin\left(\frac{s}{2}\right)\cdot\sum_{q=0}^{m} h_{m,q}\frac{\cos(q+\frac{1}{2})l}{\sin(\frac{l}{2})}\right| \\
&\leq \frac{1}{4\pi^2}\frac{1}{|2\sin^2(\frac{s}{2})|}\left|\sum_{p=0}^{n} h_{n,p}\cdot 2\cos\left(p+\frac{1}{2}\right)s\cdot\sin\left(\frac{s}{2}\right)\right|\cdot\left\{\sum_{q=0}^{m} h_{m,q}\frac{|\cos(q+\frac{1}{2})l|}{|\sin(\frac{l}{2})|}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{4\pi^2}\cdot\frac{\pi^2}{2s^2}\left|\sum_{p=0}^{n} h_{n,p}\cdot\{\sin(p+1)s-\sin(ps)\}\right|\cdot\frac{\pi}{l}\sum_{q=0}^{m} h_{m,q}\\
&\leq \frac{\pi}{8s^2 l}\left|h_{n,n}\sin(n+1)s-\sum_{p=1}^{n}\left(h_{n,p}-h_{n,p-1}\right)\sin(ps)\right|\\
&\leq \frac{\pi}{8s^2 l}\left(h_{n,n}-\sum_{p=1}^{n}\left(h_{n,p}-h_{n,p-1}\right)\right)\\
&\leq \frac{\pi}{8s^2 l}(h_{n,n}-h_{n,1}+h_{n,0}-h_{n,2}+h_{n,1}-h_{n,3}+h_{n,2}-\cdots-h_{n,n}+h_{n,n-1})\\
&\leq \frac{\pi}{8s^2 l}h_{n,0}\\
&\leq \frac{\pi}{8s^2 l}d_n
\end{aligned}
$$

$$
\left|\tilde{K}_{n,m}(s,l)\right| = O\left(\frac{d_n}{s^2 l}\right).
$$

□

Lemma 4.4. *If H is a double Hausdorff matrix with strictly increasing and convex or concave $\chi(s,l)$ then*

$$
\tilde{K}_{n,m}(s,l) = O\left(\frac{d_{n,m}}{s^2 l^2}\right) \ \textit{for}\ \frac{1}{n+1} < s \leq \pi \ \textit{and}\ \frac{1}{m+1} < l \leq \pi,
$$

where $d_{n,m} = h_{n,n;m,m}$ if $\chi(s)$ is convex and $d_{n,m} = h_{n,0;m,0}$ if $\chi(l)$ is concave.

Proof. For $\frac{1}{n+1} < s \leq \pi$, $\sin(\frac{s}{2}) \geq \frac{s}{\pi}$, $|\cos(ns)| \leq 1$, $|\sin(n+1)s| \leq 1$, $|\sin(ps)| \leq 1$ and for $\frac{1}{m+1} < l \leq \pi$, $\sin(\frac{l}{2}) \geq \frac{l}{\pi}$, $|\sin(ql)| \leq 1$, $|\sin(m+1)l| \leq 1$.

$$
\begin{aligned}
\left|\tilde{K}_{n,m}(s,l)\right| &= \frac{1}{4\pi^2}\left|\sum_{p=0}^{n} h_{n,p}\frac{\cos(p+\frac{1}{2})s}{\sin(\frac{s}{2})}\cdot\sum_{q=0}^{m} h_{m,q}\frac{\cos(q+\frac{1}{2})l}{\sin(\frac{l}{2})}\right|\\
&= \frac{1}{4\pi^2}\left[\frac{1}{2\sin^2(\frac{s}{2})}\sum_{p=0}^{n} h_{n,p}2\sin\left(\frac{2p+1}{2}\right)s\cdot\sin\left(\frac{s}{2}\right)\cdot\frac{1}{2\sin^2(\frac{l}{2})}\sum_{q=0}^{m} h_{m,q}2\sin\left(\frac{2q+1}{2}\right)l\cdot\sin\left(\frac{l}{2}\right)\right]\\
&\leq \frac{1}{4\pi^2}\cdot\frac{\pi^2}{2s^2}\left|\sum_{p=0}^{n} h_{n,p}\{\sin(p+1)s-\sin(p)s\}\right|\cdot\frac{\pi^2}{2l^2}\left|\sum_{p=0}^{m} h_{m,q}\{\sin(q+1)l-\sin(q)l\}\right|\\
&\leq \frac{\pi^4}{16\pi^2 s^2 l^2}\left|h_{n,n}\sin(n+1)s-\sum_{p=1}^{n}(h_{n,p}-h_{n,p-1})\sin(ps)\right|\\
&\qquad .\left|h_{m,m}\sin(m+1)l-\sum_{q=1}^{m}(h_{m,q}-h_{m,q-1})\sin(ql)\right|\\
&\leq \frac{\pi^2}{16s^2 l^2}\left(h_{n,n}-\sum_{p=1}^{n}(h_{n,p}-h_{n,p-1})\right).\left(h_{m,m}-\sum_{q=1}^{m}(h_{m,q}-h_{m,q-1})\right)\\
&\leq \frac{\pi^2}{16s^2 l^2}(h_{n,n}-h_{n,1}+h_{n,0}-h_{n,2}+h_{n,1}-h_{n,3}+h_{n,2}-\cdots-h_{n,n}+h_{n,n-1})\\
&\qquad\cdot(h_{m,m}-h_{m,1}+h_{m,0}-h_{m,2}+h_{m,1}-h_{m,3}+h_{m,2}-\cdots-h_{m,m}+h_{m,m-1})
\end{aligned}
$$

$$\leq \frac{\pi^2}{16s^2l^2}(h_{n,0}).(h_{m,0})$$
$$\leq \frac{\pi^2}{16s^2l^2}h_{n,0;m,0}$$
$$\leq \frac{\pi^2}{16s^2l^2}d_{n,m}$$
$$\left|\tilde{K}_{n,m}(s,l)\right| = O\left(\frac{d_{n,m}}{s^2l^2}\right).$$

□

Lemma 4.5. *Let $\tilde{f}(x,y) \in H_r^{(\xi_1,\xi_2)} (r \geq 1)$. Then for $0 < s \leq \pi$ and $0 < l \leq \pi$,*

(i) $\|\psi(\cdot,s;\cdot,l)\|_r = O\{\xi_1(s) + \xi_2(l)\}$;

(ii) $\|\psi(\cdot+u,s;\cdot+v,l) - \psi(\cdot,s;\cdot,l)\|_r = \begin{cases} O(\xi_1(s) + \xi_2(l)), \\ O(\xi_1(|u|) + \xi_2(|v|)); \end{cases}$

(iii) For positive, non-decreasing $\eta_1, \eta_2, s \leq |u|, l \leq |v|$, we obtain

$$\|\psi(\cdot+u,s;\cdot+v,l) - \psi(\cdot,s;\cdot,l)\|_r = \begin{cases} O\left((\eta_1(s) + \eta_2(l))\left(\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\right)\right), \\ O\left((\eta_1(|u|) + \eta_2(|v|))\left(\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\right)\right). \end{cases}$$

Proof. *(i)*

$$|\psi(x,s;y,l)| = \frac{1}{4}|\left[f(x+s,y+l) - f(x+s,y-l) - f(x-s,y+l) + f(x-s,y-l)\right]|$$
$$\leq \frac{1}{4}|f(x+s,y+l) - f(x+s,y-l) - f(x-s,y+l) + f(x+s,y+l) + 2f(x,y) - 2f(x,y)|$$
$$\leq \frac{1}{4}[|f(x+s,y+l) - f(x,y)| + |f(x+s,y-l) - f(x,y)|$$
$$+ |f(x-s,y+l) - f(x,y)| + |f(x-s,y-l) - f(x,y)|]$$
$$\left\|\psi(\cdot,s;\cdot,l)\right\|_r \leq \frac{1}{4}[\|f(x+s,y+l) - f(x,y)\|_r + \|f(x+s,y-l) - f(x,y)\|_r$$
$$+ \|f(x-s,y+l) - f(x,y)\|_r + \|f(x-s,y-l) - f(x,y)\|_r]$$
$$= O\{\xi_1(s) + \xi_2(l)\} + O\{\xi_1(s) + \xi_2(l)\} + O\{\xi_1(s) + \xi_2(l)\} + O\{\xi_1(s) + \xi_2(l)\}$$
$$\left\|\psi(\cdot,s;\cdot,l)\right\|_r = O\left\{\xi_1(s) + \xi_2(l)\right\}.$$

(ii) $|\psi(x+u,s;y+v,l) - \psi(x,s;y,l)|$

$$= \frac{1}{4}[|\{f(x+u+s,y+v+l) - f(x+u+s,y+v-l) - f(x+u-s,y+v+l)$$
$$+ f(x+u-s,y+v-l)\} - \{f(x+s,y+l) - f(x+s,y-l) - f(x-s,y+l) + f(x-s,y-l)\}|]$$
$$\leq \frac{1}{4}[|f(x+u+s,y+v+l) - f(x+s,y+l)| + |f(x+u+s,y+v-l) - f(x+s,y-l)|$$
$$+ |f(x+u-s,y+v+l) - f(x-s,y+l)| + |f(x+u-s,y+v-l) - f(x-s,y-l)|]$$

Applying generalized Minkowski's inequality, we have

$$\begin{aligned}
&\left\|\psi(\cdot+u,s;\cdot+v,l)-\psi(\cdot,s;\cdot,l)\right\|_r\\
&\leq \frac{1}{4}[\|f(\cdot+u+s,\cdot+v+l)-f(\cdot+s,\cdot+l)\|_r+\|f(\cdot+u+s,\cdot+v-l)-f(\cdot+s,\cdot-l)\|_r\\
&+\|f(\cdot+u-s,\cdot+v+l)-f(\cdot-s;\cdot+l)\|_r+\|f(\cdot+u-s,\cdot+v-l)-f(\cdot-s,\cdot-l)\|_r]\\
&=[O\{\xi_1(s)+\xi_2(l)\}+O\{\xi_1(s)+\xi_2(l)\}+O\{\xi_1(s)+\xi_2(l)\}+O\{\xi_1(s)+\xi_2(l)\}]\\
&=O\{\xi_1(s)+\xi_2(l)\}.
\end{aligned}$$

Also,

$$\begin{aligned}
&\left\|\psi(\cdot+u,s;\cdot+v,l)-\psi(\cdot,s;\cdot,l)\right\|_r\\
&\leq \frac{1}{4}[\|f(\cdot+s+u,\cdot+l+v)-f(\cdot+u,\cdot+v)\|_r+\|f(\cdot+s+u,\cdot+l-v)-f(\cdot+u,\cdot-v)\|_r\\
&+\|f(\cdot+s-u,\cdot+l+v)-f(\cdot-u,\cdot+v)\|_r+\|f(\cdot+s-u,\cdot+l-v)-f(\cdot-u,\cdot-v)\|_r]\\
&=[O\{\xi_1(|u|)+\xi_2(|v|)\}+O\{\xi_1(|u|)+\xi_2(|v|)\}+O\{\xi_1(|u|)+\xi_2(|v|)\}+O\{\xi_1(|u|)+\xi_2(|v|)\}]\\
&=O\{\xi_1(|u|)+\xi_2(|v|)\}.
\end{aligned}$$

(iii) Since η_1 and η_2 are positive and non-decreasing, $s\leq|u|, l\leq|v|$, then using Lemma 4.5(ii), we get

$$\begin{aligned}
\left\|\psi(\cdot+u,s;\cdot+v,l)-\psi(\cdot,s;\cdot,l)\right\|_r &= O[\xi_1(s)+\xi_2(l)]\\
&= O\left[(\eta_1(s)+\eta_2(l))\cdot\left(\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\right)\right].
\end{aligned}$$

Since $\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}$ is positive and non-decreasing, $s\geq|u|, l\geq|v|$, then $\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\geq\frac{\xi_1(u)+\xi_2(v)}{\eta_1(u)+\eta_2(v)}$. Thus, using Lemma 4.5(ii), we get

$$\begin{aligned}
\left\|\psi(\cdot+u,s;\cdot+v,l)-\psi(\cdot,s;\cdot,l)\right\|_r &= O[\xi_1(|u|)+\xi_2(|v|)]\\
&= O\left[(\eta_1(|u|)+\eta_2(|v|))\cdot\left(\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\right)\right].
\end{aligned}$$

□

5. Proof of the Theorem 3.1

Proof. The $(p,q)^{th}$ partial sums $\tilde{s}_{p,q}(x,y)$ of the series (6) is given by

$$\begin{aligned}
\tilde{S}_{p,q}(x,y)-\tilde{f}(x,y) &= \frac{1}{\pi^2}\int_0^\pi\int_0^\pi\psi(x,s;y,l)\frac{\cos(p+\frac{1}{2})s}{2\sin(\frac{s}{2})}\frac{\cos(q+\frac{1}{2})l}{2\sin(\frac{l}{2})}dsdl\\
&= \frac{1}{4\pi^2}\int_0^\pi\int_0^\pi\psi(x,s;y,l)\frac{\cos(p+\frac{1}{2})s}{\sin(\frac{s}{2})}\frac{\cos(q+\frac{1}{2})l}{\sin(\frac{l}{2})}dsdl. \qquad (23)
\end{aligned}$$

The $\tilde{t}^H_{n,m}(x,y)$ is double Hausdorff matrix mean of $\tilde{S}_{n,m}(x,y)$ and taking in view (23), we write

$$\begin{aligned}\tilde{t}^H_{n,m}(x,y)-\tilde{f}(x,y) &= \sum_{p=0}^{m}\sum_{q=0}^{n} h_{n,m;p,q}\left\{\tilde{S}_{p,q}(x,y)-\tilde{f}(x,y)\right\}\\ &= \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\psi(x,s;y,l)\sum_{p=0}^{n}\sum_{q=0}^{m} h_{n,m;p,q}\frac{\cos(p+\frac{1}{2})s}{2\sin(\frac{s}{2})}\cdot\frac{\cos(q+\frac{1}{2})l}{2\sin(\frac{l}{2})}dsdl\\ &= \int_0^{\pi}\int_0^{\pi}\psi(x,s;y,l)\tilde{K}_{n,m}(s,l)dsdl.\end{aligned}$$

Let

$$\begin{aligned}\tilde{l}_{n,m}(x,y) &= \tilde{t}^H_{n,m}(x,y)-\tilde{f}(x,y)\\ &= \int_0^{\pi}\int_0^{\pi}\psi(x,s;y,l)\tilde{K}_{n,m}(s,l)dsdl.\end{aligned}$$

Then,

$$\tilde{l}_{n,m}(x+u,y+v)-\tilde{l}_{n,m}(x,y)=\int_0^{\pi}\int_0^{\pi}\left(\psi(x+u,s;y+v,l)-\psi(x,s;y,l)\right)\tilde{K}_{n,m}(s,l)dsdl.$$

Using generalized Minkowski's inequality [7], we have

$$\begin{aligned}&\left\|\tilde{l}_{n,m}(\cdot+u,\cdot+v)-\tilde{l}_{n,m}(\cdot,\cdot)\right\|_r\\ &\leq \int_0^{\pi}\int_0^{\pi}\|\psi(\cdot+u,s;\cdot+v,l)-\psi(\cdot,s;\cdot,l)\|_r\tilde{K}_{n,m}(s,l)dsdl\\ &=\left(\int_0^{\frac{1}{n+1}}\int_0^{\frac{1}{m+1}}+\int_0^{\frac{1}{n+1}}\int_{\frac{1}{m+1}}^{\pi}+\int_{\frac{1}{n+1}}^{\pi}\int_0^{\frac{1}{m+1}}+\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\right)\cdot\|\psi(\cdot+u,s;\cdot+v,l)-\psi(\cdot,s;\cdot,l)\|_r\\ &\quad\cdot\tilde{K}_{n,m}(s,l)dsdl\\ &=A+B+C+D.\end{aligned}\tag{24}$$

Using Lemmas 4.1 and 4.5(iii), we get

$$\begin{aligned}A &= \int_0^{\frac{1}{n+1}}\int_0^{\frac{1}{m+1}}\|\psi(\cdot+u,s;\cdot+v,l)-\psi(\cdot,s;\cdot,l)\|_r\,\tilde{K}_{n,m}(s,l)dsdl\\ &= O\left(\int_0^{\frac{1}{n+1}}\int_0^{\frac{1}{m+1}}(\eta_1(|u|)+\eta_2(|v|))\left(\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\right)\cdot\left(\frac{1}{sl}\right)dsdl\right)\\ &= O\left(\eta_1(|u|)+\eta_2(|v|)\left\{\int_0^{\frac{1}{m+1}}\left(\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2(l)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2(l)}\right)\cdot\frac{1}{l}dl\right\}\int_0^{\frac{1}{n+1}}\frac{1}{s}ds\right)\\ &= O\left(\log(n+1)\cdot(\eta_1(|u|)+\eta_2(|v|))\left(\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1(\frac{1}{n+1})+\eta_2\left(\frac{1}{m+1}\right)}\right)\int_0^{\frac{1}{m+1}}\frac{1}{l}dl\right)\\ &= O\left(\log(n+1)\cdot\log(m+1)\cdot(\eta_1(|u|)+\eta_2(|v|))\cdot\left(\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1(\frac{1}{n+1})+\eta_2(\frac{1}{m+1})}\right)\right).\end{aligned}\tag{25}$$

Using Lemmas 4.2 and 4.5(iii), we get

$$
\begin{aligned}
B &= \int_0^{\frac{1}{n+1}} \int_{\frac{1}{m+1}}^{\pi} \|\psi(\cdot+u,s;,\cdot+v,l)-\psi(\cdot,s;\cdot,l)\|_r \, \tilde{K}_{n,m}(s,l)dsdl \\
&= O\left(\int_0^{\frac{1}{n+1}} \int_{\frac{1}{m+1}}^{\pi} (\eta_1(|u|)+\eta_2(|v|))\left(\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\right)\cdot \frac{d_m}{sl^2}dsdl\right) \\
&= O\left(d_m\cdot(\eta_1(|u|)+\eta_2(|v|))\left\{\int_{\frac{1}{m+1}}^{\pi}\left(\frac{\xi_1(\frac{1}{n+1})+\xi_2(l)}{\eta_1(\frac{1}{n+1})+\eta_2(l)}\right)\cdot\frac{1}{l^2}dl\right\}\int_0^{\frac{1}{n+1}}\frac{1}{s}ds\right) \\
&= O\left(d_m\log(n+1)\cdot(\eta_1(|u|)+\eta_2(|v|))\int_{\frac{1}{m+1}}^{\pi}\left(\frac{\xi_1(\frac{1}{n+1})+\xi_2(l)}{\eta_1(\frac{1}{n+1})+\eta_2(l)}\right)\cdot\frac{1}{l^2}dl\right).
\end{aligned} \tag{26}
$$

Using Lemmas 4.3 and 4.5(iii), we get

$$
\begin{aligned}
C &= \int_{\frac{1}{n+1}}^{\pi} \int_0^{\frac{1}{m+1}} \|\psi(\cdot+u,s;\cdot+v,l)-\psi(\cdot,s;\cdot,l)\|_r \, \tilde{K}_{n,m}(s,l)dsdl \\
&= O\left(\int_{\frac{1}{n+1}}^{\pi} \int_0^{\frac{1}{m+1}} (\eta_1(|u|)+\eta_2(|v|))\left(\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\right)\cdot \frac{d_n}{s^2 l}dsdl\right) \\
&= O\left(d_n\,(\eta_1(|u|)+\eta_2(|v|))\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi_1(s)+\xi_2(\frac{1}{m+1})}{\eta_1(s)+\eta_2(\frac{1}{m+1})}\right)\cdot\frac{1}{s^2}ds\int_0^{\frac{1}{m+1}}\frac{1}{l}dl\right) \\
&= O\left(d_n\log(m+1)\cdot(\eta_1(|u|)+\eta_2(|v|))\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi_1(s)+\xi_2(\frac{1}{m+1})}{\eta_1(s)+\eta_2(\frac{1}{m+1})}\right)\cdot\frac{1}{s^2}ds\right).
\end{aligned} \tag{27}
$$

Using Lemmas 4.4 and 4.5(iii), we get

$$
\begin{aligned}
D &= \int_{\frac{1}{n+1}}^{\pi} \int_{\frac{1}{m+1}}^{\pi} \|\psi(\cdot+u,s;\cdot+v,l)-\psi(\cdot,s;\cdot,l)\|_r \, \tilde{K}_{n,m}(s,l)dsdl \\
&= O\left(\int_{\frac{1}{n+1}}^{\pi} \int_{\frac{1}{m+1}}^{\pi} (\eta_1(|u|)+\eta_2(|v|))\left(\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\right)\cdot \frac{d_{n,m}}{s^2 l^2}dsdl\right) \\
&= O\left(d_{n,m}\,(\eta_1(|u|)+\eta_2(|v|))\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\left(\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\right)\cdot\frac{1}{s^2l^2}dsdl\right).
\end{aligned} \tag{28}
$$

Combining (24)-(28), we have

$$
\begin{aligned}
&\|\tilde{I}_{n,m}(x+u,y+v)-\tilde{I}_{n,m}(x,y)\|_r \\
&= O\left(\log(n+1)\cdot\log(m+1)\cdot(\eta_1(|u|)+\eta_2(|v|))\cdot\left(\frac{\xi_1(\frac{1}{n+1})+\xi_2(\frac{1}{m+1})}{\eta_1(\frac{1}{n+1})+\eta_2(\frac{1}{m+1})}\right)\right) \\
&+ O\left(d_m\log(n+1)\cdot(\eta_1(|u|)+\eta_2(|v|))\int_{\frac{1}{m+1}}^{\pi}\left(\frac{\xi_1(\frac{1}{n+1})+\xi_2(l)}{\eta_1(\frac{1}{n+1})+\eta_2(l)}\right)\cdot\frac{1}{l^2}dl\right) \\
&+ O\left(d_n\log(m+1)\cdot(\eta_1(|u|)+\eta_2(|v|))\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi_1(s)+\xi_2(\frac{1}{m+1})}{\eta_1(s)+\eta_2(\frac{1}{m+1})}\right)\cdot\frac{1}{s^2}ds\right) \\
&+ O\left(d_{n,m}\,(\eta_1(|u|)+\eta_2(|v|))\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\left(\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\right)\cdot\frac{1}{s^2l^2}dsdl\right).
\end{aligned} \tag{29}
$$

Thus,

$$\sup_{u\neq0,v\neq0}\frac{\left\|\tilde{l}_{n,m}(x+u,y+v)-\tilde{l}_{n,m}(x,y)\right\|_r}{\eta_1(|u|)+\eta_2(|v|)}$$

$$=O\left(\log(n+1)\cdot\log(m+1)\cdot\left(\frac{\xi_1(\frac{1}{n+1})+\xi_2(\frac{1}{m+1})}{\eta_1(\frac{1}{n+1})+\eta_2(\frac{1}{m+1})}\right)\right)$$
$$+O\left(d_m\log(n+1)\int_{\frac{1}{m+1}}^{\pi}\left(\frac{\xi_1(\frac{1}{n+1})+\xi_2(l)}{\eta_1(\frac{1}{n+1})+\eta_2(l)}\right)\cdot\frac{1}{l^2}dl\right)$$
$$+O\left(d_n\log(m+1)\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi_1(s)+\xi_2(\frac{1}{m+1})}{\eta_1(s)+\eta_2(\frac{1}{m+1})}\right)\cdot\frac{1}{s^2}ds\right)$$
$$+O\left(d_{n,m}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\left(\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\right)\cdot\frac{1}{s^2l^2}dsdl\right).\tag{30}$$

Now, we have

$$\left\|\tilde{l}_{n,m}(x,y)\right\|_r=\left\|\left(\tilde{t}^H_{n,m}(x,y)-\tilde{f}(x,y)\right)\right\|_r$$
$$=\left(\int_0^{\pi}\int_0^{\pi}\left\|\psi(x,s;y,l)\right\|_r\tilde{K}_{n,m}(s,l)dsdl\right)$$
$$\leq\left(\int_0^{\frac{1}{n+1}}\int_0^{\frac{1}{m+1}}+\int_0^{\frac{1}{n+1}}\int_{\frac{1}{m+1}}^{\pi}+\int_{\frac{1}{n+1}}^{\pi}\int_0^{\frac{1}{m+1}}+\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\right)\cdot\left\|\psi(x,s;y,l)\right\|_r\cdot\left|\tilde{K}_{n,m}(s,l)\right|dsdl$$

Using Lemmas 4.1 to 4.4 and 4.5(i), we have

$$\left\|\tilde{l}_{n,m}(x,y)\right\|_r$$
$$=O\left(\int_0^{\frac{1}{n+1}}\int_0^{\frac{1}{m+1}}(\xi_1(s)+\xi_2(l))\ \frac{1}{sl}dsdl\right)+O\left(\int_0^{\frac{1}{n+1}}\int_{\frac{1}{m+1}}^{\pi}(\xi_1(s)+\xi_2(l))\frac{d_m}{sl^2}dsdl\right)$$
$$+O\left(\int_{\frac{1}{n+1}}^{\pi}\int_0^{\frac{1}{m+1}}(\xi_1(s)+\xi_2(l))\frac{d_n}{s^2l}dsdl\right)+O\left(\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}(\xi_1(s)+\xi_2(l))\frac{d_{n,m}}{s^2l^2}dsdl\right)$$

$$=O\left(\left\{\int_0^{\frac{1}{m+1}}\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2(l)}{l}dl\right\}\int_0^{\frac{1}{n+1}}\frac{1}{s}ds\right)+O\left(d_m\left\{\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2(l)}{l^2}dl\right\}\int_0^{\frac{1}{n+1}}\frac{1}{s}ds\right)$$
$$+O\left(d_n\left\{\int_{\frac{1}{n+1}}^{\pi}\frac{\xi_1(s)+\xi_2\left(\frac{1}{m+1}\right)}{s^2}ds\right\}\int_0^{\frac{1}{m+1}}\frac{1}{l}dl\right)+O\left(d_{n,m}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{(\xi_1(s)+\xi_2(l))}{s^2l^2}dsdl\right)$$
$$=O\left(\log(n+1)\cdot\log(m+1)\left\{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)\right\}\right)+O\left(d_m\log(n+1)\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2(l)}{l^2}dl\right)$$
$$+O\left(d_n\log(m+1)\int_{\frac{1}{n+1}}^{\pi}\frac{\xi_1(s)+\xi_2\left(\frac{1}{m+1}\right)}{s^2}ds\right)+O\left(d_{n,m}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{s^2l^2}dsdl\right),\tag{31}$$

Now, we have

$$\left\|\tilde{l}_{n,m}(\cdot,\cdot)\right\|_r^{\eta}=\left\|\tilde{l}_{n,m}(x,y)\right\|_r+\sup_{u\neq0,v\neq0}\frac{\left\|\tilde{l}_{n,m}(x+u,y+v)-\tilde{l}_{n,m}(x,y)\right\|_r}{\eta_1(|u|)+\eta_2(|v|)}\tag{32}$$

Using (30) and (31) in (32), we get

$$\begin{aligned}
&\left\|\tilde{l}_{n,m}(\cdot,\cdot)\right\|_r^{(\eta)}\\
&= O\left(\log(n+1)\log(m+1)\left\{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)\right\}\right)+O\left(d_m\log(n+1)\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2(l)}{l^2}dl\right)\\
&+O\left(d_n\log(m+1)\int_{\frac{1}{n+1}}^{\pi}\frac{\xi_1(s)+\xi_2\left(\frac{1}{m+1}\right)}{s^2}ds\right)+O\left(d_{n,m}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{s^2l^2}dsdl\right)\\
&+O\left(\log(n+1)\log(m+1)\frac{\xi_1(\frac{1}{n+1})+\xi_2(\frac{1}{m+1})}{\eta_1(\frac{1}{n+1})+\eta_2(\frac{1}{m+1})}\right)+O\left(d_m\log(n+1)\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(\frac{1}{n+1})+\xi_2(l)}{\eta_1(\frac{1}{n+1})+\eta_2(l)}\left(\frac{1}{l^2}\right)dl\right)\\
&+O\left(d_n\log(m+1)\int_{\frac{1}{n+1}}^{\pi}\frac{\xi_1(s)+\xi_2(\frac{1}{m+1})}{\eta_1(s)+\eta_2(\frac{1}{m+1})}\left(\frac{1}{s^2}\right)ds\right)+O\left(d_{n,m}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\left(\frac{1}{s^2l^2}\right)dsdl\right). \qquad (33)
\end{aligned}$$

Since $\xi_1(s)+\xi_2(l)=\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}$ and $\eta_1(s)+\eta_2(l)\le(\eta_1(\pi)+\eta_2(\pi))\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}$, $0<s\le\pi$, $0<l\le\pi$, we get

$$\begin{aligned}
&\left\|\tilde{l}_{n,m}(\cdot,\cdot)\right\|_r^{(\eta)}\\
&= O\left(\log(n+1)\log(m+1)\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\right)+O\left(d_m\log(n+1)\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2(l)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2(l)}\left(\frac{1}{l^2}\right)dl\right)\\
&+O\left(d_n\log(m+1)\int_{\frac{1}{n+1}}^{\pi}\frac{\xi_1(s)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1(s)+\eta_2\left(\frac{1}{m+1}\right)}\left(\frac{1}{s^2}\right)ds\right)+O\left(d_{n,m}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\left(\frac{1}{s^2l^2}\right)dsdl\right). \qquad (34)
\end{aligned}$$

Since $\xi_1(s)+\xi_2(l)$ and $\eta_1(s)+\eta_2(l)$ are moduli of continuity, $\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}$ is positive and non-decreasing and

$$\begin{aligned}
\left(d_m\log(n+1)\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2(l)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2(l)}\left(\frac{1}{l^2}\right)dl\right) &\ge d_m\log(n+1)\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\int_{\frac{1}{m+1}}^{\pi}\frac{1}{l^2}dl\\
&\ge d_m\log(n+1)\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{2\left(\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)\right)}\cdot(m+1)
\end{aligned}$$

Then,

$$O\left(d_m\log(n+1)\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2(l)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2(l)}\left(\frac{1}{l^2}\right)dl\right)=O\left(d_m\log(n+1)\frac{\left(\frac{m+1}{2}\right)\left\{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)\right\}}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\right). \qquad (35)$$

Since $\xi_1(s)+\xi_2(l)$ and $\eta_1(s)+\eta_2(l)$ are moduli of continuity, $\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}$ is positive and non-decreasing and

$$\begin{aligned}
\left(d_n\log(m+1)\int_{\frac{1}{n+1}}^{\pi}\frac{\xi_1(s)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1(s)+\eta_2\left(\frac{1}{m+1}\right)}\left(\frac{1}{s^2}\right)ds\right) &\ge \left(d_n\log(m+1)\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\int_{\frac{1}{n+1}}^{\pi}\frac{1}{s^2}ds\right)\\
&\ge \left(d_n\log(m+1)\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{2\left(\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)\right)}\right)\cdot(n+1).
\end{aligned}$$

Then,

$$O\left(d_n\log(m+1)\int_{\frac{1}{n+1}}^{\pi}\frac{\xi_1(s)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1(s)+\eta_2\left(\frac{1}{m+1}\right)}\left(\frac{1}{s^2}\right)ds\right)=O\left(d_n\log(m+1)\frac{\left(\frac{n+1}{2}\right)\left\{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)\right\}}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\right). \qquad (36)$$

Since $\xi_1(s)+\xi_2(l)$ and $\eta_1(s)+\eta_2(l)$ are moduli of continuity, $\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}$ is positive and non-decreasing and

$$\left(d_{n,m}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\left(\frac{1}{s^2l^2}\right)dsdl\right)\geq\left(\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}d_{n,m}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{1}{s^2l^2}dsdl\right)$$
$$\geq\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{4\left(\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)\right)}d_{n,m}(n+1)(m+1).$$

Then,

$$O\left(\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\right)=O\left(\frac{1}{(n+1)(m+1)}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\left(\frac{1}{s^2l^2}\right)dsdl\right). \tag{37}$$

Combining (34)-(37), we get

$$\begin{aligned}
&\left\|\tilde{l}_{n,m}(\cdot,\cdot)\right\|_r^{(\eta)}\\
&=O\left(\log(n+1)\log(m+1)\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\right)+O\left(d_m\log(n+1)\frac{\left(\frac{m+1}{2}\right)\left\{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)\right\}}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\right)\\
&+O\left(d_n\log(m+1)\frac{\left(\frac{n+1}{2}\right)\left\{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)\right\}}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\right)+O\left(d_{n,m}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\left(\frac{1}{s^2l^2}\right)dsdl\right)\\
&=O\Bigg[\left(\log(n+1)\cdot\log(m+1)+\frac{(m+1)}{2}d_m\log(n+1)+\frac{(n+1)}{2}d_n\log(m+1)\right)\\
&\quad\cdot\left(\frac{\xi_1\left(\frac{1}{n+1}\right)+\xi_2\left(\frac{1}{m+1}\right)}{\eta_1\left(\frac{1}{n+1}\right)+\eta_2\left(\frac{1}{m+1}\right)}\right)\Bigg]+O\left(d_{n,m}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\left(\frac{1}{s^2l^2}\right)dsdl\right)\\
&=O\Bigg[\left(\log(n+1).\log(m+1)+\frac{(m+1)}{2}d_m\log(n+1)+\frac{(n+1)}{2}d_n\log(m+1)\right)\\
&\quad\cdot\left(\left(\frac{1}{(n+1)(m+1)}\right)\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\left(\frac{1}{s^2l^2}\right)dlds\right)\Bigg]\\
&\quad+O\left(d_{n,m}\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\left(\frac{1}{s^2l^2}\right)dlds\right)\\
&=O\Bigg[\left(\frac{1}{(n+1)(m+1)}\right)\left(\log(n+1)\cdot\log(m+1)+\frac{(m+1)}{2}d_m\log(n+1)+\frac{(n+1)}{2}d_n\log(m+1)\right)\\
&\qquad+d_{n,m}\Bigg]\cdot\left(\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\left(\frac{1}{s^2l^2}\right)dsdl\right)\\
&=O\Bigg[\left(\frac{1}{2(n+1)(m+1)}\right)\Big(2\log(n+1)\cdot\log(m+1)+(m+1)d_m\log(n+1)+(n+1)d_n\log(m+1)\\
&\qquad+2(n+1)(m+1)d_{n,m}\Big)\cdot\left(\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\left(\frac{1}{s^2l^2}\right)dlds\right)\Bigg]
\end{aligned}$$

$$= O\left[\left(\frac{1}{(n+1)(m+1)}\right)\left(2\log(n+1)\cdot\log(m+1)+(m+1)d_m\log(n+1)+(n+1)d_n\log(m+1)\right.\right.$$
$$\left.\left.+2(n+1)(m+1)d_{n,m}\right)\cdot\left(\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\left(\frac{1}{s^2l^2}\right)dlds\right)\right]$$

This proves the theorem. □

6. Corollaries

Corollary 6.1. *Following Note 1 (i), we obtain*

$$\left\|\tilde{t}_{n,m}^{(C,1,1)}(x,y)-\tilde{f}(x,y)\right\|_r^{(\eta)} = O\left[\left(\frac{1}{(n+1)(m+1)}\right)\left(2\log(n+1)\cdot\log(m+1)+(m+1)d_m\log(n+1)\right.\right.$$
$$\left.\left.+(n+1)d_n\log(m+1)+2(n+1)(m+1)d_{n,m}\right)\cdot\left(\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\left(\frac{1}{s^2l^2}\right)dlds\right)\right],$$

where $\xi_1(s)+\xi_2(l)$ and $\eta_1(s)+\eta_2(l)$ denote the moduli of continuity such that $\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}$ is positive and non-decreasing.

Corollary 6.2. *Following Note 1 (ii), we obtain*

$$\left\|\tilde{t}_{n,m}^{(E,r,r)}(x,y)-\tilde{f}(x,y)\right\|_r^{(\eta)} = O\left[\left(\frac{1}{(n+1)(m+1)}\right)\left(2\log(n+1)\cdot\log(m+1)+(m+1)d_m\log(n+1)\right.\right.$$
$$\left.\left.+(n+1)d_n\log(m+1)+2(n+1)(m+1)d_{n,m}\right)\cdot\left(\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}\left(\frac{1}{s^2l^2}\right)dlds\right)\right],$$

where $\xi_1(s)+\xi_2(l)$ and $\eta_1(s)+\eta_2(l)$ denote the moduli of continuity such that $\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)}$ is positive and non-decreasing.

Corollary 6.3. *Let $\tilde{f}\in H_{(\alpha,\beta),r}; r\geq 1$ and suppose $\xi_1(s)+\xi_2(l)=(sl)^{\alpha},\ \eta_1(s)+\eta_2(l)=(sl)^{\beta}, 0\leq\beta<\alpha\leq 1$, then*

$$\left\|\tilde{t}_{n,m}^{H}(x,y)-\tilde{f}(x,y)\right\|_r^{(\eta)}$$
$$=\begin{cases} O\left[\left(2\log(n+1)\cdot\log(m+1)+(m+1)d_m\log(n+1)+(n+1)d_n\log(m+1)\right.\right. \\ \quad\left.\left.+2(n+1)(m+1)d_{n,m}\right)(n+1)^{\beta-\alpha}(m+1)^{\beta-\alpha}\right], & if\ \ 0\leq\beta<\alpha<1, \\ O\left[\left(\frac{1}{(n+1)(m+1)}\right)\left(2\log(n+1)\cdot\log(m+1)+(m+1)d_m\log(n+1)+(n+1)d_n\log(m+1)\right.\right. \\ \quad\left.\left.+2(n+1)(m+1)d_{n,m}\right)\log(n+1)\pi\cdot\log(m+1)\pi\right], & if\ \ \beta=0,\alpha=1. \end{cases}$$

Proof. Putting $\xi_1(s)+\xi_2(l)=(sl)^{\alpha},\ \eta_1(s)+\eta_2(l)=(sl)^{\beta}, 0\leq\beta<\alpha\leq 1$, in Theorem 3.1.

$$\left\|\tilde{t}_{n,m}^{H}(x,y)-\tilde{f}(x,y)\right\|_r^{(\eta)} = O\left[\left(\frac{1}{(n+1)(m+1)}\right)\left(2\log(n+1)\cdot\log(m+1)+(m+1)d_m\right.\right.$$
$$\left.\left.\cdot\log(n+1)+(n+1)d_n\log(m+1)+2(n+1)(m+1)d_{n,m}\right)\left(\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}s^{\alpha-\beta-2}l^{\alpha-\beta-2}dlds\right)\right],$$

$$\implies \left\|\tilde{t}^H_{n,m}(x,y)-\tilde{f}(x,y)\right\|^{(\eta)}_r$$

$$=\begin{cases} O\Bigg[\bigg(2\log(n+1)\cdot\log(m+1)+(m+1)d_m\log(n+1)+(n+1)d_n\log(m+1) \\ \qquad +2(n+1)(m+1)d_{n,m}\bigg)\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi} s^{\alpha-\beta-2}\cdot l^{\alpha-\beta-2}dlds\Bigg], & if\ 0\le\beta<\alpha<1, \\ O\Bigg[\left(\frac{1}{(n+1)(m+1)}\right)\bigg(2\log(n+1)\cdot\log(m+1)+(m+1)d_m\log(n+1)+(n+1)d_n \\ \qquad \log(m+1)+2(n+1)(m+1)d_{n,m}\bigg)\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{1}{sl}dlds\Bigg], & if\ \beta=0,\alpha=1, \end{cases}$$

$$\therefore\ \left\|\tilde{t}^H_{n,m}(x,y)-\tilde{f}(x,y)\right\|^{(\eta)}_r$$

$$=\begin{cases} O\Bigg[\bigg(2\log(n+1)\cdot\log(m+1)+(m+1)d_m\log(n+1)+(n+1)d_n\log(m+1) \\ \qquad +2(n+1)(m+1)d_{n,m}\bigg)(n+1)^{\beta-\alpha}(m+1)^{\beta-\alpha}\Bigg], & if\ 0\le\beta<\alpha<1, \\ O\Bigg[\left(\frac{1}{(n+1)(m+1)}\right)\bigg(2\log(n+1)\log(m+1)+(m+1)d_m\log(n+1)+(n+1)d_n \\ \log(m+1)+2(n+1)(m+1)d_{n,m}\bigg)\{\log(n+1)\pi\}\cdot\{\log(m+1)\pi\}\Bigg], & if\ \beta=0,\alpha=1, \end{cases}$$

□

Corollary 6.4. *Let $\tilde{f}\in H_{(\alpha,\beta),r}; r\ge 1,\ a,b\in\mathbb{R}$ and suppose $\xi_1(s)+\xi_2(l)=\frac{(sl)^\alpha}{(\log\frac{1}{s})^a\cdot(\log\frac{1}{l})^a}$, $\eta_1(s)+\eta_2(l)=\frac{(sl)^\beta}{(\log\frac{1}{s})^b\cdot(\log\frac{1}{l})^b}$ and $0\le\beta<\alpha\le 1,\ 0<s,l\le\pi$, then*

$$\left\|\tilde{t}^H_{n,m}(x,y)-\tilde{f}(x,y)\right\|^{(\eta)}_r$$

$$=\begin{cases} O\Bigg[\bigg(2\log(n+1)\cdot\log(m+1)+(m+1)d_m\log(n+1)+(n+1)d_n\log(m+1) \\ \qquad +2(n+1)(m+1)d_{n,m}\bigg)\{\log(n+1)\cdot\log(m+1)\}^{b-a}\Bigg], & if\ \ \alpha=\beta,\ a-b>-1, \\ 0\Bigg[\bigg(2\log(n+1)\cdot\log(m+1)+(m+1)d_m\log(n+1)+(n+1)d_n\log(m+1) \\ \qquad +2(n+1)(m+1)d_{n,m}\bigg)\{\log(n+1)\cdot\log(m+1)\}\Bigg], & if\ \ \alpha=\beta,\ a-b=-1. \end{cases}$$

Proof. Putting $\xi_1(s)+\xi_2(l)=\frac{(sl)^\alpha}{(\log\frac{1}{s})^a\cdot(\log\frac{1}{l})^a}$, $\eta_1(s)+\eta_2(l)=\frac{(sl)^\beta}{(\log\frac{1}{s})^b\cdot(\log\frac{1}{l})^b}$ and $0\le\beta<\alpha\le 1,\ 0<s,l\le\pi$, in Theorem 3.1.

$$\left\|\tilde{t}^H_{n,m}(x,y)-\tilde{f}(x,y)\right\|^{(\eta)}_r = O\Bigg[\left(\frac{1}{(n+1)(m+1)}\right)\bigg(2\log(n+1)\log(m+1)+(m+1)d_m\log(n+1)$$
$$+(n+1)d_n\log(m+1)+2(n+1)(m+1)d_{n,m}\bigg)\left(\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}(sl)^{\alpha-\beta-2}\left(\log\frac{1}{s}\right)^{b-a}\left(\log\frac{1}{l}\right)^{b-a}dlds\right)\Bigg],$$

$$\implies \left\|\tilde{t}^{H}_{n,m}(x,y)-\tilde{f}(x,y)\right\|_r^{(\eta)}$$

$$= \begin{cases} O\Bigg[\Big(2\log(n+1)\cdot\log(m+1)+(m+1)d_m\log(n+1)+(n+1)d_n\log(m+1)+2(n+1) \\ (m+1)d_{n,m}\Big)\Big(\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}(sl)^{\alpha-\beta-2}\Big(\log\frac{1}{s}\Big)^{b-a}\cdot\Big(\log\frac{1}{l}\Big)^{b-a}dlds\Big)\Bigg], & if\ \alpha=\beta, a-b>-1, \\ O\Bigg[\Big(\frac{1}{(n+1)(m+1)}\Big)\Big(2\log(n+1)\cdot\log(m+1)+(m+1)d_m\log(n+1)+(n+1)d_n\log(m+1) \\ +2(n+1)(m+1)d_{n,m}\Big)\Big(\int_{\frac{1}{n+1}}^{\pi}\int_{\frac{1}{m+1}}^{\pi}\frac{1}{(sl)^2}(\log\frac{1}{s})(\log\frac{1}{l})dlds\Big)\Bigg], & if\ \alpha=\beta, a-b=-1. \end{cases}$$

$$\therefore \left\|\tilde{t}^{H}_{n,m}(x,y)-\tilde{f}(x,y)\right\|_r^{(\eta)}$$

$$= \begin{cases} O\Bigg[\Big(2\log(n+1)\cdot\log(m+1)+(m+1)d_m\log(n+1)+(n+1)d_n\log(m+1) \\ +2(n+1)(m+1)d_{n,m}\Big)\{\log(n+1)\cdot\log(m+1)\}^{b-a}\Bigg], & if\ \alpha=\beta,\ a-b>-1, \\ O\Bigg[\Big(2\log(n+1)\cdot\log(m+1)+(m+1)d_m\log(n+1)+(n+1)d_n\log(m+1) \\ +2(n+1)(m+1)d_{n,m}\Big)\{\log(n+1)\cdot\log(m+1)\}\Bigg], & if\ \alpha=\beta,\ a-b=-1. \end{cases}$$

□

7. Verification

Let us consider $\frac{\xi_1(s)+\xi_2(l)}{\eta_1(s)+\eta_2(l)} = s^2l^2$.

7.1. For $n=m=3, d_3=h_{3,3}=0.4, d_{3,3}=h_{3,3;3,3}=0.16$, (22) gives $E_{1,1}\sim 4.0612673289$.
7.2. For $n=m=4, d_4=h_{4,4}=0.3333333333, d_{3,3}=h_{3,3;3,3}=0.1111111111$, (22) gives $E_{2,2}\sim 3.0675023513$.
7.3. For $n=m=10, d_{10}=h_{10,10}=0.16666667, d_{10,10}=h_{10,10;10,10}=0.0.0277777778$, (22) gives $E_{3,3}\sim 0.9775587569$.
7.4. For $n=m=100, d_{100}=h_{100,100}=0.0196078431, d_{100,100}=h_{100,100;100,100}$ $=0.0003844675$, (22) gives $E_{4,4}\sim 0.022894417$.
7.5. For $n=m=1000, d_{1000}=h_{1000,1000}=0.001996008, d_{1000,1000}=h_{1000,1000;1000,1000}=0.000003984$, (22) gives $E_{5,5}\sim 0.0003738504$.
From the above verification, we observed that error estimate ($E_{n,m}$) approaches to zero as n,m approach infinity. Thus, Theorem 3.1 provide the best approximation of the function $\tilde{f}(x,y)$.

Acknowledgements

The first author expresses his gratitude towards his mother for her blessings. The first author also expresses his gratitude towards his father in heaven whose soul is always guiding and encouraging him. The second author also expresses his gratitude towards his parents for their blessings and encouragement.

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