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The Sherman-Morrison-Woodbury Formula for the Generalized Inverses

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Abstract. In this paper, we investigate the Sherman-Morrison-Woodbury formula for the {1}-inverses and the {2}-inverses of bounded linear operators on a Hilbert space. Some conditions are established to guarantee that $(A+YGZ^*)^{\odot} = A^{\odot} - A^{\odot}Y(G^{\odot} + Z^*A^{\odot}Y)^{\odot}Z^*A^{\odot}$ holds, where $A^{\odot}$ stands for any kind of standard inverse, {1}-inverse, {2}-inverse, Moore-Penrose inverse, Drazin inverse, group inverse, core inverse and dual core inverse of A.

1. Introduction

Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces over the same field. We use $\mathscr{B}(\mathscr{H},\mathscr{K})$ to denote the set of all bounded linear operators from $\mathscr{H}$ to $\mathscr{K}$, and set $\mathscr{B}(\mathscr{H}) = \mathscr{B}(\mathscr{H},\mathscr{H})$. For $A \in \mathscr{B}(\mathscr{H},\mathscr{K})$, let A^*, $\mathscr{R}(A)$ and $\mathscr{N}(A)$ be the adjoint, the range and the null space of A, respectively. Let $A \in \mathscr{B}(\mathscr{H})$ and $G \in \mathscr{B}(\mathscr{K})$ both be invertible, and $Y, Z \in \mathscr{B}(\mathscr{H},\mathscr{K})$. Then $A + YGZ^*$ is invertible if and only if $G^{-1} + Z^*A^{-1}Y$ is invertible. In this case,

$$(A + YGZ^*)^{-1} = A^{-1} - A^{-1}Y(G^{-1} + Z^*A^{-1}Y)^{-1}Z^*A^{-1}. \quad (1)$$

The formula (1) is called Sherman-Morrison-Woodbury formula (for short SMW formula). The SMW formula was discovered by Sherman and Morrison [1], Woodbury [2], Bartlett [3] and Bodewig [4]. The original SMW formula was considered for matrices and is valid if the matrix A is invertible. The SMW formula has been used in various fields, see for example, [5]-[8]. In particular, Hager [5] applied it to statistics, networks, structural analysis, asymptotic analysis, optimization and partial differential equations.

Let $A \in \mathscr{B}(\mathscr{H},\mathscr{K})$, if there exists $X \in \mathscr{B}(\mathscr{K},\mathscr{H})$ satisfying the following four operator equations (see for example, [9–11]):

$$(1)\ AXA = A,\ (2)\ XAX = X,\ (3)\ (AX)^* = AX,\ (4)\ (XA)^* = XA,$$

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then X is called the Moore-Penrose inverse of A. The Moore-Penrose inverse of A is unique if it exists and is denoted by $A^{\dagger}$. In addition, X satisfying equation (i) is called a $\{i\}$-inverse of A and is denoted by $X \in A\{i\}$, where $i \in \{1,2,3,4\}$. We use A^- and A^+ to denote a $\{1\}$-inverse and a $\{2\}$-inverse of A, respectively. An operator $A \in \mathscr{B}(\mathscr{H},\mathscr{K})$ is $\{1\}$-invertible (or Moore-Penrose invertible) if and only if $\mathscr{R}(A)$ is closed in $\mathscr{K}$. For $A \in \mathscr{B}(\mathscr{H},\mathscr{K})$, there exists $B \in \mathscr{B}(\mathscr{K},\mathscr{H})$ such that $B \neq 0$ and $B \in A\{2\}$ if and only if $A \neq 0$.

The Drazin inverse of $A \in \mathscr{B}(\mathscr{H})$ is the element $X \in \mathscr{B}(\mathscr{H})$ such that

$$AX = XA,\ XAX = X,\ A - A^2X \text{ is nilpotent.}$$

Such an X is unique if it exists and is denoted by A^D. If $A - A^2X = 0$, then X is called the group inverse of A.

Let $A \in \mathscr{B}(\mathscr{H})$. Baksalary and Trenkler [12] introduced the core inverse for a complex matrix. Rakić et al. [13] generalized this concept to bounded linear operators on a Hilbert space. $A^{\#} \in \mathscr{B}(\mathscr{H})$ is called the core inverse of A if it satisfies

$$AA^{\#}A = A,\ A^{\#}AA^{\#} = A^{\#},\ (AA^{\#})^* = AA^{\#},\ A(A^{\#})^2 = A^{\#},\ A^{\#}A^2 = A.$$

And $A_{\#} \in \mathscr{B}(\mathscr{H})$ is called the dual core inverse of A if it satisfies

$$AA_{\#}A = A,\ A_{\#}AA_{\#} = A_{\#},\ (A_{\#}A)^* = A_{\#}A,\ (A_{\#})^2A = A_{\#},\ A^2A_{\#} = A.$$

Several authors generalized the original SMW formula to singular or rectangular matrices by the concept of Moore-Penrose inverses (see for example, [14–17]). Even the extension of SMW formula is available for bounded linear operators on Hilbert space (see for example, [18–20]). In this paper, we generalized the SMW formula to the $\{1\}$-inverse case and the $\{2\}$-inverse case. Moreover, we obtain the SMW formula for the Moore-Penrose inverse, Drazin inverse, group inverse, core inverse and dual core inverse. Therefore, some results in [18] and [19] are completed.

2. Main results

Let us first present some auxiliary lemmas and results for the further reference.

Lemma 2.1. *([21, Lemma 10] and [9]) If $A \in \mathscr{B}(\mathscr{H})$ and $P = P^2 \in \mathscr{B}(\mathscr{H})$, then*
(i) $PA = A \Leftrightarrow \mathscr{R}(A) \subset \mathscr{R}(P)$;
(ii)$AP = A \Leftrightarrow \mathscr{N}(P) \subset \mathscr{N}(A)$.

Lemma 2.2. *Let $A \in \mathscr{B}(\mathscr{H},\mathscr{K})$. Then we have the following results:*
(i) $\mathscr{R}(AA^-) = \mathscr{R}(A)$ and $\mathscr{N}(A^-A) = \mathscr{N}(A)$, where $A^- \in A\{1\}$;
(ii)[19, Lemma 2] $\mathscr{R}(A^+A) = \mathscr{R}(A^+)$ and $\mathscr{N}(AA^+) = \mathscr{N}(A^+)$, where $A^+ \in A\{2\}$;
(iii) if A is core invertible, then $\mathscr{R}(AA^{\#}) = \mathscr{R}(A) = \mathscr{R}(A^{\#}) = \mathscr{R}(A^{\#}A)$;
(iv) if A is core invertible, then $\mathscr{N}(AA_{\#}) = \mathscr{N}(A) = \mathscr{N}(A_{\#}) = \mathscr{N}(A_{\#}A)$.

Proof. (i). From $\mathscr{R}(A) = \mathscr{R}(AA^-A) \subset \mathscr{R}(AA^-) \subset \mathscr{R}(A)$ and $\mathscr{N}(A) \subset \mathscr{N}(A^-A) \subset \mathscr{N}(AA^-A) = \mathscr{N}(A)$, we get that $\mathscr{R}(AA^-) = \mathscr{R}(A)$ and $\mathscr{N}(A^-A) = \mathscr{N}(A)$.

(iii). Since $A^{\#} \in A\{1,2\}$, we have $\mathscr{R}(AA^{\#}) = \mathscr{R}(A)$ and $\mathscr{R}(A^{\#}) = \mathscr{R}(A^{\#}A)$ according to (i) and (ii). Moreover,

$$\mathscr{R}(A) = \mathscr{R}(A^{\#}A^2) \subset \mathscr{R}(A^{\#}) = \mathscr{R}(A(A^{\#})^2) \subset \mathscr{R}(A),$$

thus $\mathscr{R}(A) = \mathscr{R}(A^{\#})$.

(iv). Since $A_{\#} \in A\{1,2\}$, we have $\mathscr{N}(AA_{\#}) = \mathscr{N}(A)_{\#}$ and $\mathscr{N}(A) = \mathscr{N}(A_{\#}A)$ according to (i) and (ii). Furthermore,

$$\mathscr{N}(A) \subset \mathscr{N}((A_{\#})^2A) = \mathscr{N}(A_{\#}) \subset \mathscr{N}(A^2A_{\#}) = \mathscr{N}(A),$$

hence $\mathscr{N}(A) = \mathscr{N}(A^{\#})$. □

Lemma 2.3. *[19, Theorem 3] Let $A, G \in \mathscr{B}(\mathscr{H})$ such that $A \neq 0$ and $G \neq 0$. Also let $Y, Z \in \mathscr{B}(\mathscr{K}, \mathscr{H})$, $B = A + YGZ^*$, $T = G^+ + Z^*A^+Y$ such that $B \neq 0$ and $T \neq 0$. If*

$$\mathscr{R}(A^+) \subset \mathscr{R}(B^+), \ \mathscr{N}(A^+) \subset \mathscr{N}(B^+),$$
$$\mathscr{N}(G^+) \subset \mathscr{N}(Y), \ \mathscr{N}(T^+) \subset \mathscr{N}(G),$$

then $B^+ = A^+ - A^+YT^+Z^*A^+$.

Duan [19] proved that Lemma 2.3 is valid for standard inverse, Moore-Penrose inverse, Drazin inverse and group inverse. It is worth mentioning that Lemma 2.3 is also valid for core inverse and dual core inverse. Now we give the following result which which in way a mimics dual of Lemma 2.3.

Theorem 2.4. *Let $A, G \in \mathscr{B}(\mathscr{H})$ such that $A \neq 0$ and $G \neq 0$. Also let $Y, Z \in \mathscr{B}(\mathscr{K}, \mathscr{H})$, $B = A + YGZ^*$, $T = G^+ + Z^*A^+Y$ such that $B \neq 0$ and $T \neq 0$. If*

$$\mathscr{N}(B^+) \subset \mathscr{N}(A^+), \ \mathscr{R}(B^+) \subset \mathscr{R}(A^+),$$
$$\mathscr{R}(G) \subset \mathscr{R}(T^+), \ \mathscr{R}(Z^*) \subset \mathscr{R}(G^+),$$

then $B^+ = A^+ - A^+YT^+Z^*A^+$.

Proof. By Lemma 2.1 and 2.2, these four conditions $\mathscr{N}(BB^+) = \mathscr{N}(B^+) \subset \mathscr{N}(A^+), \mathscr{R}(B^+) \subset \mathscr{R}(A^+) = \mathscr{R}(A^+A)$, $\mathscr{R}(G) \subset \mathscr{R}(T^+) = \mathscr{R}(T^+T)$ and $\mathscr{R}(Z^*) \subset \mathscr{R}(G^+) = \mathscr{R}(G^+G)$ are equivalent to $A^+BB^+ = A^+$, $A^+AB^+ = B^+$, $T^+TG = G$ and $G^+GZ^* = Z^*$, respectively.

Since

$$\begin{aligned} TGZ^*B^+ &= (G^+ + Z^*A^+Y)GZ^*B^+ \\ &= G^+GZ^*B^+ + Z^*A^+YGZ^*B^+ \\ &= Z^*B^+ + Z^*A^+(B - A)B^+ \\ &= Z^*B^+ + Z^*A^+BB^+ - Z^*A^+AB^+ \\ &= Z^*B^+ + Z^*A^+ - Z^*B^+ \\ &= Z^*A^+, \end{aligned}$$

we obtain

$$GZ^*B^+ = T^+TGZ^*B^+ = T^+Z^*A^+. \tag{2}$$

Therefore,

$$\begin{aligned} A^+ &= A^+BB^+ = A^+(A + YGZ^*)B^+ \\ &= A^+AB^+ + A^+YGZ^*B^+ \\ &\overset{(2)}{=} B^+ + A^+YT^+Z^*A^+, \end{aligned}$$

which shows that $B^+ = A^+ - A^+YT^+Z^*A^+$. □

Let $A^{\odot}$ stand for any kind of the following standard inverse A^{-1}, Moore-Penrose inverse $A^{\dagger}$, Drazin inverse A^D, group inverse $A^{\#}$, core inverse $A^{\circledast}$ and dual core inverse $A_{\circledast}$. Since $A^{\odot} \in A\{2\}$, we can obtain the following corollary.

Corollary 2.5. *Let $A, G \in \mathscr{B}(\mathscr{H})$ such that $A^{\odot}$ and $G^{\odot}$ exist. Also let $Y, Z \in \mathscr{B}(\mathscr{K}, \mathscr{H})$, $B = A + YGZ^*$, $T = G^{\odot} + Z^*A^{\odot}Y$ such that $B^{\odot}$ and $T^{\odot}$ exist. If*

$$\mathscr{N}(B^{\odot}) \subset \mathscr{N}(A^{\odot}), \ \mathscr{R}(B^{\odot}) \subset \mathscr{R}(A^{\odot}),$$
$$\mathscr{R}(G) \subset \mathscr{R}(T^{\odot}), \ \mathscr{R}(Z^*) \subset \mathscr{R}(G^{\odot}),$$

then $B^{\odot} = A^{\odot} - A^{\odot}YT^{\odot}Z^*A^{\odot}$.

If G and T are invertible in Corollary 2.5, then we can get the following result.

Corollary 2.6. *Let $A, G \in \mathscr{B}(\mathscr{H})$ such that $A^{\odot}$ exists and G is invertible. Also let $Y, Z \in \mathscr{B}(\mathscr{K}, \mathscr{H})$, $B = A + YGZ^*$, $T = G^{-1} + Z^*A^{\odot}Y$ such that $B^{\odot}$ exists and T is invertible. If*

$$\mathscr{N}(B^{\odot}) \subset \mathscr{N}(A^{\odot}), \;\; \mathscr{R}(B^{\odot}) \subset \mathscr{R}(A^{\odot}),$$

then $B^{\odot} = A^{\odot} - A^{\odot}YT^{-1}Z^*A^{\odot}$.

Now we establish new conditions to guarantee the validity of SWM formula for {2}-inverse and {1}-inverse. Let $A^{\triangledown}$ denote a {1}-inverse or a {2}-inverse of A. Suppose that $A, G \in \mathscr{B}(\mathscr{H})$ with $A^{\triangledown}$ and $G^{\triangledown}$ exist and $Y, Z \in \mathscr{B}(\mathscr{K}, \mathscr{H})$, let $B = A + YGZ^*$ and $T = G^{\triangledown} + Z^*A^{\triangledown}Y$ with $T^{\triangledown}$ exists. We use the notation a_i $(i = 1, \ldots, 8)$ to stand for the following conditions:

a_1 : $\mathscr{R}(YT^{\triangledown}Z^*A^{\triangledown}) \subset \mathscr{R}(AA^{\triangledown})$

a_2 : $\mathscr{N}(TT^{\triangledown}) \subset \mathscr{N}(YG)$ or $\mathscr{R}(Z^*A^{\triangledown}) \subset \mathscr{R}(TT^{\triangledown})$

a_3 : $\mathscr{N}(GG^{\triangledown}) \subset \mathscr{N}(Y)$ or $\mathscr{R}(T^{\triangledown}Z^*A^{\triangledown}) \subset \mathscr{R}(GG^{\triangledown})$

a_4 : $\mathscr{R}(YGZ^*) \subset \mathscr{R}(AA^{\triangledown})$

a_5 : $\mathscr{N}(A^{\triangledown}A) \subset \mathscr{N}(A^{\triangledown}YT^{\triangledown}Z^*)$

a_6 : $\mathscr{N}(T^{\triangledown}T) \subset \mathscr{N}(A^{\triangledown}Y)$ or $\mathscr{R}(GZ^*) \subset \mathscr{R}(T^{\triangledown}T)$

a_7 : $\mathscr{N}(G^{\triangledown}G) \subset \mathscr{N}(A^{\triangledown}YT^{\triangledown})$ or $\mathscr{R}(Z^*) \subset \mathscr{R}(G^{\triangledown}G)$

a_8 : $\mathscr{N}(A^{\triangledown}A) \subset \mathscr{N}(YGZ^*)$

Let $X = A^{\triangledown} - A^{\triangledown}YT^{\triangledown}Z^*A^{\triangledown}$.

Case I. If a_1, a_2 and a_3 hold, then

$$AA^{\triangledown}YT^{\triangledown}Z^*A^{\triangledown} \overset{a_1}{=} YT^{\triangledown}Z^*A^{\triangledown} \tag{3}$$

and

$$\begin{aligned} YGZ^*A^{\triangledown}YT^{\triangledown}Z^*A^{\triangledown} &= YG(Z^*A^{\triangledown}Y)T^{\triangledown}Z^*A^{\triangledown} \\ &= YG(T - G^{\triangledown})T^{\triangledown}Z^*A^{\triangledown} \\ &= YGTT^{\triangledown}Z^*A^{\triangledown} - YGG^{\triangledown}T^{\triangledown}Z^*A^{\triangledown} \\ &\overset{a_2, a_3}{=} YGZ^*A^{\triangledown} - YT^{\triangledown}Z^*A^{\triangledown}, \end{aligned} \tag{4}$$

we obtain

$$\begin{aligned} BX &= (A + YGZ^*)(A^{\triangledown} - A^{\triangledown}YT^{\triangledown}Z^*A^{\triangledown}) \\ &= AA^{\triangledown} - AA^{\triangledown}YT^{\triangledown}Z^*A^{\triangledown} + YGZ^*A^{\triangledown} - YGZ^*A^{\triangledown}YT^{\triangledown}Z^*A^{\triangledown} \\ &\overset{(3)(4)}{=} AA^{\triangledown} - YT^{\triangledown}Z^*A^{\triangledown} + YGZ^*A^{\triangledown} - YGZ^*A^{\triangledown} + YT^{\triangledown}Z^*A^{\triangledown} \\ &= AA^{\triangledown}. \end{aligned} \tag{5}$$

Case II. If a_5, a_6 and a_7 hold, then

$$A^{\triangledown}YT^{\triangledown}Z^*A^{\triangledown}A \overset{a_5}{=} A^{\triangledown}YT^{\triangledown}Z^* \tag{6}$$

and

$$\begin{aligned} A^{\triangledown}YT^{\triangledown}Z^*A^{\triangledown}YGZ^* &= A^{\triangledown}YT^{\triangledown}(Z^*A^{\triangledown}Y)GZ^* \\ &= A^{\triangledown}YT^{\triangledown}(T - G^{\triangledown})GZ^* \\ &= A^{\triangledown}YT^{\triangledown}TGZ^* - A^{\triangledown}YT^{\triangledown}G^{\triangledown}GZ^* \\ &\overset{a_6, a_7}{=} A^{\triangledown}YGZ^* - A^{\triangledown}YT^{\triangledown}Z^*, \end{aligned} \tag{7}$$

we obtain

$$\begin{aligned} XB &= (A^{\nabla} - A^{\nabla}YT^{\nabla}Z^{*}A^{\nabla})(A + YGZ^{*}) \\ &= A^{\nabla}A + A^{\nabla}YGZ^{*} - A^{\nabla}YT^{\nabla}Z^{*}A^{\nabla}A - A^{\nabla}YT^{\nabla}Z^{*}A^{\nabla}YGZ^{*} \\ &\overset{(6)(7)}{=} A^{\nabla}A + A^{\nabla}YGZ^{*} - A^{\nabla}YT^{\nabla}Z^{*} - A^{\nabla}YGZ^{*} + A^{\nabla}YT^{\nabla}Z^{*} \\ &= A^{\nabla}A. \end{aligned} \tag{8}$$

If $A^{\nabla} = A^{+}$ is a $\{2\}$-inverse in the above notations, then we have the following result.

Theorem 2.7. *Let $A, G \in \mathscr{B}(\mathscr{H})$ such that $A \neq 0$ and $G \neq 0$. Let $Y, Z \in \mathscr{B}(\mathscr{K}, \mathscr{H})$, $B = A+YGZ^{*}$, $T = G^{+}+Z^{*}A^{+}Y$. If $T \neq 0$ and any of the following items holds:*
(i) a_1, a_2, a_3;
(ii) a_5, a_6, a_7,
then $A^{+} - A^{+}YT^{+}Z^{*}A^{+} \in B\{2\}$.

Proof. Let $X = A^{+} - A^{+}YT^{+}Z^{*}A^{+}$.
(i). According to Case I, we have

$$XBX \overset{(5)}{=} (A^{+} - A^{+}YT^{+}Z^{*}A^{+})AA^{+} = A^{+} - A^{+}YT^{+}Z^{*}A^{+} = X,$$

thus $X \in B\{2\}$.

(ii). According to Case II, we get

$$XBX \overset{(8)}{=} A^{+}A(A^{+} - A^{+}YT^{+}Z^{*}A^{+}) = A^{+} - A^{+}YT^{+}Z^{*}A^{+} = X,$$

that is to say, $X \in B\{2\}$. □

Corollary 2.8. *[19, Theorem 5] Let $A, G \in \mathscr{B}(\mathscr{H})$ such that $A \neq 0$ and $G \neq 0$. Let $Y, Z \in \mathscr{B}(\mathscr{K}, \mathscr{H})$, $B = A+YGZ^{*}$, $T = G^{+} + Z^{*}A^{+}Y$. If $T \neq 0$ and any of the following items holds:*
(i) $\mathscr{R}(Y) \subset \mathscr{R}(AA^{+})$, $\mathscr{R}(Z^{*}) \subset \mathscr{R}(TT^{+})$, $\mathscr{N}(G^{+}) \subset \mathscr{N}(Y)$;
(ii) $\mathscr{N}(A^{+}A) \subset \mathscr{N}(Z^{*})$, $\mathscr{N}(T^{+}T) \subset \mathscr{N}(Y)$, $\mathscr{R}(Z^{*}) \subset \mathscr{R}(G^{+})$,
then $A^{+} - A^{+}YT^{+}Z^{*}A^{+} \in B\{2\}$.

Proof. (i). The conditions $\mathscr{R}(Y) \subset \mathscr{R}(AA^{+})$ and $\mathscr{R}(Z^{*}) \subset \mathscr{R}(TT^{+})$ imply the conditions a_1 and a_2, respectively. $\mathscr{N}(GG^{+}) = \mathscr{N}(G^{+}) \subset \mathscr{N}(Y)$ satisfies the condition a_3, thus the result is valid by Theorem 2.7.
(ii). Similarly, $\mathscr{N}(A^{+}A) \subset \mathscr{N}(Z^{*})$, $\mathscr{N}(T^{+}T) \subset \mathscr{N}(Y)$ and $\mathscr{R}(Z^{*}) \subset \mathscr{R}(G^{+}) = \mathscr{R}(G^{+}G)$ give the conditions a_5, a_6 and a_7, respectively. Therefore, we obtain the conclusion by Theorem 2.7. □

We present new conditions under which that generalized SMW formula is satisfied for $\{1\}$-inverse. If $A^{\nabla} = A^{-}$ is a $\{1\}$-inverse of A in the above notations, then we obtain the following result.

Theorem 2.9. *Suppose that $A, G \in \mathscr{B}(\mathscr{H})$ with $\mathscr{R}(A)$ and $\mathscr{R}(G)$ are both closed. Let $Y, Z \in \mathscr{B}(\mathscr{K}, \mathscr{H})$, $B = A + YGZ^{*}$, $T = G^{-} + Z^{*}A^{-}Y$. If $\mathscr{R}(T)$ is closed and any of the following items holds:*
(i) a_1, a_2, a_3, a_4;
(ii) a_5, a_6, a_7, a_8,
then $\mathscr{R}(B)$ is closed with $A^{-} - A^{-}YT^{-}Z^{*}A^{-} \in B\{1\}$.

Proof. Let $X = A^{-} - A^{-}YT^{-}Z^{*}A^{-}$.
(i). According to Case I, we obtain

$$BXB \overset{(5)}{=} AA^{-}(A + YGZ^{*}) = AA^{-}A + AA^{-}YGZ^{*} \overset{a_4}{=} A + YGZ^{*} = B,$$

which shows that $X \in B\{1\}$.

(ii). According to Case II, we obtain

$$BXB \overset{(8)}{=} (A + YGZ^*)A^-A = AA^-A + YGZ^*A^-A \overset{a_8}{=} A + YGZ^* = B,$$

hence $X \in B\{1\}$. □

Let $A^{\ominus}$ stand for any kind of standard inverse A^{-1}, Moore-Penrose inverse $A^{\dagger}$, Drazin inverse A^D, group inverse $A^{\#}$ and core inverse $A^{\text{#}}$. Replace all the superscripts $\triangledown$ with $\ominus$ in items a_1-a_7, then we can obtain the following corollary.

Corollary 2.10. *Let $A, G \in \mathscr{B}(\mathscr{H})$ with $A^{\ominus}, G^{\ominus}$ exist, $Y, Z \in \mathscr{B}(\mathscr{K}, \mathscr{H})$, $B = A + YGZ^*$, $T = G^{\ominus} + Z^*A^{\ominus}Y$. If $T^{\ominus}$ exists and the following items holds:*

$$a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,\ a_7,$$

then $B^{\ominus}$ exists with $B^{\ominus} = A^{\ominus} - A^{\ominus}YT^{\ominus}Z^*A^{\ominus}$.

Proof. Let $X = A^{\ominus} - A^{\ominus}YT^{\ominus}Z^*A^{\ominus}$. According to Case I and Case II, conditions a_1, a_2, a_3, a_5, a_6, a_7 follow that X satisfies $BX = AA^{\ominus}$ and $XB = A^{\ominus}A$.

If $\ominus$ denotes the standard inverse, then the result is valid obviously.

If $\ominus$ denotes the Moore-Penrose inverse, then $BX = AA^{\dagger}$ and $XB = A^{\dagger}A$ show that BX and XB are both Hermitian. Moreover, Theorem 2.7 and 2.9 show that $XBX = X$ and $BXB = B$, respectively. Thus $X = B^{\dagger}$.

If $\ominus$ denotes the Drazin inverse (resp., group inverse), then $BX = AA^D = A^DA = XB$. Moreover,

$$XBX = A^DA(A^D - A^DYT^DZ^*A^D) = X$$

and

$$B - B^2X = (I - BX)B = (I - AA^D)(A + YGZ^*) \overset{a_4}{=} (I - AA^D)A$$

is nilpotent (resp., $B - B^2X = 0$ for the group inverse). Thus $X = B^D$ (resp., $X = B^{\#}$).

If $\ominus$ denotes the core inverse, then $XB = A^{\text{#}}A$, and $BX = AA^{\text{#}}$ shows that BX is Hermitian. Furthermore, since $\mathscr{R}(AA^{\text{#}}) = \mathscr{R}(A) = \mathscr{R}(A^{\text{#}}) = \mathscr{R}(A^{\text{#}}A)$ by Lemma 2.2,

$$BXB = AA^{\text{#}}(A + YGZ^*) = A + AA^{\text{#}}YGZ^* \overset{a_4}{=} A + YGZ^* = B,$$

$$XBX = A^{\text{#}}A(A^{\text{#}} - A^{\text{#}}YT^{\text{#}}Z^*A^{\text{#}}) = A^{\text{#}} - A^{\text{#}}YT^{\text{#}}Z^*A^{\text{#}} = X,$$

$$BX^2 = AA^{\text{#}}(A^{\text{#}} - A^{\text{#}}YT^{\text{#}}Z^*A^{\text{#}}) = A^{\text{#}} - A^{\text{#}}YT^{\text{#}}Z^*A^{\text{#}} = X,$$

$$XB^2 = A^{\text{#}}A(A + YGZ^*) = A + A^{\text{#}}AYGZ^* \overset{a_4}{=} A + YGZ^* = B.$$

Hence $X = B^{\text{#}}$. □

Let $A^{\oslash}$ stand for any kind of standard inverse A^{-1}, Moore-Penrose inverse $A^{\dagger}$, Drazin inverse A^D, group inverse $A^{\#}$ and dual core inverse $A_{\text{#}}$. Replace all the superscripts $\triangledown$ with $\oslash$ in items a_1-a_3, a_5-a_8, then we can obtain the following corollary.

Corollary 2.11. *Let $A, G \in \mathscr{B}(\mathscr{H})$ with $A^{\oslash}, G^{\oslash}$ exist, $Y, Z \in \mathscr{B}(\mathscr{K}, \mathscr{H})$, $B = A + YGZ^*$, $T = G^{\oslash} + Z^*A^{\oslash}Y$. If $T^{\oslash}$ exists and the following items holds:*

$$a_1,\ a_2,\ a_3,\ a_5,\ a_6,\ a_7,\ a_8,$$

then $B^{\oslash}$ exists with $B^{\oslash} = A^{\oslash} - A^{\oslash}YT^{\oslash}Z^*A^{\oslash}$.

Proof. Let $X = A^{\oslash} - A^{\oslash}YT^{\oslash}Z^*A^{\oslash}$. According to Case I and Case II, conditions a_1, a_2, a_3, a_5, a_6, a_7 deduce that X satisfies $BX = AA^{\oslash}$ and $XB = A^{\oslash}A$.

If $\oslash$ denotes the standard inverse, then the result is valid obviously.

If $\oslash$ denotes the Moore-Penrose inverse, then $BX = AA^{\dagger}$ and $XB = A^{\dagger}A$ show that BX and XB are both Hermitian. Moreover, Theorem 2.7 and 2.9 show that $XBX = X$ and $BXB = B$, respectively. Thus $X = B^{\dagger}$.

If ⊘ denotes the Drazin inverse (resp., group inverse), then $BX = AA^D = A^DA = XB$. Moreover,

$$XBX = A^DA(A^D - A^DYT^DZ^*A^D) = X$$

and

$$B - B^2X = B(I - BX) = (A + YGZ^*)(I - AA^D) \stackrel{a_8}{=} A(I - AA^D)$$

is nilpotent (resp., $B - B^2X = 0$ for the group inverse). Thus $X = B^D$ (resp., $X = B^\#$).

If ⊘ denotes the dual core inverse, then $BX = AA_{\#}$, and $XB = A_{\#}A$ shows that XB is Hermitian. Furthermore, since $\mathscr{N}(AA_{\#}) = \mathscr{N}(A_{\#}) = \mathscr{N}(A) = \mathscr{N}(A_{\#}A)$ by Lemma 2.2,

$$BXB = (A + YGZ^*)A_{\#}A = A + YGZ^*A_{\#}A \stackrel{a_8}{=} A + YGZ^* = B,$$

$$XBX = (A_{\#} - A_{\#}YT_{\#}Z^*A_{\#})AA_{\#} = A_{\#} - A_{\#}YT_{\#}Z^*A_{\#} = X,$$

$$X^2B = (A_{\#} - A_{\#}YT_{\#}Z^*A_{\#})A_{\#}A = A_{\#} - A_{\#}YT_{\#}Z^*A_{\#} = X,$$

$$B^2X = (A + YGZ^*)AA_{\#} = A + YGZ^*AA_{\#} \stackrel{a_8}{=} A + YGZ^* = B.$$

So $X = B_{\#}$. □

Corollary 2.12. *[19, Corollary 6] Let $A, G \in \mathscr{B}(\mathscr{H})$ with $A^{\odot}, G^{\odot}$ exist, $Y, Z \in \mathscr{B}(\mathscr{K}, \mathscr{H})$, $B = A + YGZ^*$, $T = G^{\odot} + Z^*A^{\odot}Y$. If $T^{\odot}$ exists and the following conditions holds:*

$$\mathscr{R}(Y) \subset \mathscr{R}(A),\ \mathscr{R}(Z^*) \subset \mathscr{R}(T),\ \mathscr{N}(G^{\odot}) \subset \mathscr{N}(Y),$$

$$\mathscr{N}(A) \subset \mathscr{N}(Z^*),\ \mathscr{N}(T) \subset \mathscr{N}(Y),\ \mathscr{R}(Z^*) \subset \mathscr{R}(G^{\odot}).$$

then $B^{\odot}$ exists with $B^{\odot} = A^{\odot} - A^{\odot}YT^{\odot}Z^*A^{\odot}$.

Proof. The hypothesis $\mathscr{R}(Y) \subset \mathscr{R}(A) = \mathscr{R}(AA^{\odot})$ implies conditions a_1 and a_4, $\mathscr{N}(A^{\odot}A) = \mathscr{N}(A) \subset \mathscr{N}(Z^*)$ implies conditions a_5 and a_8. In addition, $\mathscr{R}(Z^*) \subset \mathscr{R}(T) = \mathscr{R}(TT^{\odot})$, $\mathscr{N}(GG^{\odot}) = \mathscr{N}(G^{\odot}) \subset \mathscr{N}(Y)$, $\mathscr{N}(T^{\odot}T) = \mathscr{N}(T) \subset \mathscr{N}(Y)$, $\mathscr{R}(Z^*) \subset \mathscr{R}(G^{\odot} = \mathscr{R}(G^{\odot}G)$ can yield conditions a_2, a_3, a_6 and a_7, respectively. Therefore, the conclusion is true by applying Corollary 2.10 and 2.11. □

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