Filomat 36:15 (2022), 5323–5335
https://doi.org/10.2298/FIL2215323G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A-Statistical Convergence with a Rate and Applications to Approximation

Mustafa Gülfırat[a]

[a]*Ankara University, Faculty of Science, Department of Mathematics, 06100, Ankara, Turkey.*

Abstract. $A = (a_{nk})$ be a regular summability matrix. In the present paper we deal with subspaces of the space of $A-$statistically convergent sequences obtained by the rate at which the $A-$statistical limit tends to zero. We prove that a sequence is the $A-$strongly convergent if and only if it is the $A-$statistically convergent and the $A-$uniformly integrable with the rate of $o(a_n)$ where $a = (a_n)$ is a positive nonincreasing sequence. We also make a link between the $A-$strong convergence and the $A-$distributional convergence with the rate of $o(a_n)$. Finally, as an application we present an approximation theorem of Korovkin type.

1. Introduction

Strong, statistical and distributional convergences are of some interest in the convergence theories. Some studies on the statistical convergence may be found in [4–8, 10, 12, 14–16, 24]. Recently Duman, Khan and Orhan [8], introduced the concept of A-statistical convergence with a rate at which the $A-$statistical limit tends to zero where $A = (a_{nk})$ is a nonnegative regular matrix (see also [7]). In the present paper we mainly deal with subspaces of the space of $A-$statistically convergent sequences obtained by the rate at which the $A-$statistical limit tends to zero. We prove that a sequence is the $A-$strongly convergent if and only if it is the $A-$statistically convergent and the $A-$uniformly integrable with the rate of $o(a_n)$ where $a = (a_n)$ is a positive nonincreasing sequence. We also make a link between the $A-$strong convergence and the $A-$distributional convergence with the rate of $o(a_n)$. Some criteria for the $A-$statistical convergence with the rate of $o(a_n)$ is also given. Finally, as an application, an approximation theorem of Korovkin type is considered.

We pause to collect some notation. If the natural density of the set $E := \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ is zero then we say that the sequence (x_k) is statistically convergent to L (see, e.g.[9], [10]). Replacing the Cesaro matrix $(C, 1)$ by a nonnegative regular matrix $A = (a_{nk})$ Freedman and Sember [10] extended the notion of natural density to the $A-$density for a subset E of positive integers. Recall that an infinite matrix $A = (a_{nk})$ is said to be regular if the sequence $Ax := ((Ax)_n) = (\sum_{k=1}^{\infty} a_{nk}x_k)$, exists (i.e., the series on the right hand side is convergent for each n) and $\lim (Ax)_n = \lim x_n$ for each convergent sequence $x = (x_n)$. A characterization of regularity of the matrix $A = (a_{nk})$ may be found in [2]. Using this idea Connor [3], Kolk [16], Miller [19] examined the A-statistical convergence. In [21] a criterion for the statistical convergence was given. Later on it was weakened by Salat [20] when x satisfies a certain condition (see, also, [4]).

2020 *Mathematics Subject Classification*. 40A35, 40G15, 40F05, 60B10, 41A36
Keywords. Density, Statistical convergence, Uniform integrability, Strong convergence, Distributional convergence, Korovkin type approximation
Received: 12 September 2021; Revised: 09 November 2021; Accepted: 10 November 2021
Communicated by Eberhard Malkowsky
Email address: mgulfirat@ankara.edu.tr (Mustafa Gülfırat)

Let $A = (a_{nk})$ be a nonnegative regular summability matrix and let $a = (a_n)$ be a positive nonincreasing sequence. Following [8] we say that the sequence $x = (x_k)$ is A-statistically convergent to the number L with the rate of $o(a_n)$ if for each $\varepsilon > 0$,

$$\lim_n \frac{1}{a_n} \sum_{k:|x_k - L| \geq \varepsilon} a_{nk} = 0.$$

In this case we write $st_{A,a} - \lim x = L$ or $x_k - L = st_A - o(a_k)$, as $k \to \infty$. We also consider the following two subspaces of A-statistically convergent sequences:

$$st_{A,a} := \{x = (x_k) : st_{A,a} - \lim x = L \text{ for some } L\},$$

$$st^0_{A,a} := \{x = (x_k) : st_{A,a} - \lim x = 0\}.$$

Also Demirci, Khan and Orhan [7] proved under certain conditions that $st^0_{A,a}$ and $st_{A,a}$ cannot be endowed with a locally convex $FK-$topology.

In Section 2 we study the A-density with the rate of $o(a_n)$ and present some basic properties of this concept. Section 3 is reserved for the $A-$strong convergence, the $A-$uniform integrability and the $A-$distributional convergence with the rate of $o(a_n)$. In Section 4 we give some criteria for the $A-$statistical convergence with the rate of $o(a_n)$. In the last section as an application we prove an approximation theorem of Korovkin type.

2. $A-$density with the rate of $o(a_n)$

This section collects some results concerning the $A-$density with the rate of $o(a_n)$.

Definition 2.1. *Let $A = (a_{nk})$ be a nonnegative regular matrix and let $a = (a_n)$ be a positive nonincreasing sequence. Let E be a subset of positive integers. The upper $\overline{\delta}_{A,a}(E)$ and lower $\underline{\delta}_{A,a}(E)$ densities of E are respectively defined by*

$$\overline{\delta}_{A,a}(E) = \limsup_n \tfrac{1}{a_n} \sum_{k \in E} a_{nk}, \quad \text{and} \quad \underline{\delta}_{A,a}(E) = \liminf_n \tfrac{1}{a_n} \sum_{k \in E} a_{nk}.$$

If $\overline{\delta}_{A,a}(E) = \underline{\delta}_{A,a}(E)$ then we say that E has $A-$density with the rate of $o(a_n)$.

Throughout the paper we assume that $\delta_{A,a}(\mathbb{N}) = \alpha$ is finite. Note that α cannot be zero since $A = (a_{nk})$ is a nonnegative regular matrix.

Proposition 2.2. *For subsets E, G of positive integers we have*
i) $E \subseteq G \Rightarrow \delta_{A,a}(E) \leq \delta_{A,a}(G)$,
ii) $\delta_{A,a}(\varnothing) = 0$,
iii) if either $\delta_{A,a}(E)$ or $\delta_{A,a}(\mathbb{N} \backslash E)$ exists then $\delta_{A,a}(\mathbb{N} \backslash E) = \alpha - \delta_{A,a}(E)$.

Hence the sequence $x = (x_k)$ is the $A-$statistically convergent to L with the rate of $o(a_n)$ provided that for each $\varepsilon > 0$ the set

$$E(\varepsilon) := \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\},$$

has the $A-$density zero with the rate of $o(a_n)$, i.e., $\delta_{A,a}(E(\varepsilon)) = 0$.

Fridy and Khan [12] proved that the A-statistical convergence is a regular method if and only if the columns of A go to zero. It is important to note that the $A-$statistical convergence with the rate of $o(a_n)$ is a regular method if and only if $a_{nk} = o(a_n)$, as $n \to \infty$, for every $k \in \mathbb{N}$. In the sequel the method will be assumed to be regular.

The next result is an improvement of a result of Demirci [5].

Theorem 2.3. *Let A and B be nonnegative regular matrices and $a = (a_n)$ be a positive nonincreasing sequence. Assume that*

$$\limsup_n \frac{1}{a_n} \sum_{k=1}^{\infty} |a_{nk} - b_{nk}| = 0.$$

Then $\overline{\delta}_{A,a}(K) = 0$ if and only if $\overline{\delta}_{B,a}(K) = 0$ for every $K \subseteq \mathbb{N}$.

Proof. If $\overline{\delta}_{A,a}(K) = 0$, then $\limsup_n \frac{1}{a_n} \sum_{k \in K} a_{nk} = 0$. Since

$$\begin{aligned} \left| \tfrac{1}{a_n} \textstyle\sum_{k\in K} a_{nk} - \tfrac{1}{a_n} \textstyle\sum_{k\in K} b_{nk} \right| &\leq \tfrac{1}{a_n} \textstyle\sum_{k\in K} |a_{nk} - b_{nk}| \\ &\leq \tfrac{1}{a_n} \textstyle\sum_{k=1}^{\infty} |a_{nk} - b_{nk}|, \end{aligned}$$

we get from the hypothesis that

$$\limsup_n \left| \frac{1}{a_n} \sum_{k\in K} a_{nk} - \frac{1}{a_n} \sum_{k\in K} b_{nk} \right| = 0.$$

This implies that $\overline{\delta}_{A,a}(K) = 0$ if and only if $\overline{\delta}_{B,a}(K) = 0$. □

3. Strong, Distributional Convergences and Uniform Integrability

In this section we consider the $A-$strong convergence and the $A-$uniform integrability with a rate. We prove that a sequence is the $A-$strongly convergent if and only if it is the $A-$statistically convergent and the $A-$uniformly integrable with the rate of $o(a_n)$ where $a = (a_n)$ is a positive nonincreasing sequence. We also make a link between the $A-$strong convergence and the $A-$distributional convergence with the rate of $o(a_n)$. Recall that strong summability arises in the study of the summability of Fourier series [13].

Definition 3.1. *Let $A = (a_{nk})$ be a nonnegative regular matrix and let $a = (a_n)$ be a positive nonincreasing sequence. Let $W_a(A)$ be defined by*

$$W_a(A) := \{x : \lim_n \frac{1}{a_n} \sum_{k=1}^{\infty} a_{nk} |x_k - L| = 0 \text{ for some } L\}.$$

If $x \in W_a(A)$, then we say that x is $A-$strongly summable to L with the rate of $o(a_n)$.

Definition 3.2. *Let $A = (a_{nk})$ be a nonnegative regular matrix and let $a = (a_n)$ be a positive nonincreasing sequence. A sequence $x = (x_k)$ is said to be $A-$uniformly integrable with the rate of $o(a_n)$ if*

$$\lim_{t\to\infty} \sup_n \frac{1}{a_n} \sum_{k:|x_k|>t} |a_{nk}| |x_k| = 0.$$

By $U_{A,a}$ we denote the set of all $A-$uniformly integrable sequences with the rate of $o(a_n)$.

It is clear from the definition that any bounded sequence $x = (x_k)$ is the $A-$uniformly integrable with the rate of $o(a_n)$.

Definition 3.3. *A real sequence x is defined to be A−distributionally convergent to αF with the rate of* $o(a_n)$ *where F is a probality distribution on* $\mathbb{R}$*, if*

$$\lim_n \frac{1}{a_n} \sum_{k:x_k \leq t} a_{nk} = \alpha F(t),$$

for each t at which F is continuous.

The following theorem is motivated by the Summer seminar lectures given by M.K. Khan on "Probabilistic Methods in the Theory of Summability" at Ankara University during 21 August-1 September 2006 ([14]).

The class of summability matrices with nonnegative entries is denoted by M^+.

The next result characterizes the uniform integrability with the rate of $o(a_n)$.

Theorem 3.4. *Let* $x = (x_k)$ *be a real sequence and let* $A \in M^+$ *and let* $a = (a_n)$ *be a positive nonincreasing sequence. The following statements are equivalent:*
1) $x \in U_{A,a}$*,*
2) i) $\sup_n \frac{1}{a_n} \sum_{k=1}^{\infty} a_{nk} |x_k| < \infty$*,*
ii) For any $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that for any subset E of nonnegative integers for which*

$$\sup_n \frac{1}{a_n} \sum_{k \in E} a_{nk} < \delta,$$

we have

$$\sup_n \frac{1}{a_n} \sum_{k \in E} a_{nk} |x_k| < \varepsilon.$$

Proof. Let $x \in U_{A,a}$. Then for an arbitrarily given $\varepsilon > 0$ we may choose a $t_0 \in \mathbb{R}$ with

$$\sup_n \frac{1}{a_n} \sum_{k:|x_k|>t} a_{nk} |x_k| < \frac{\varepsilon}{2} \text{ for each } t \geq t_0.$$

From this we have

$$\begin{aligned} \sup_n \tfrac{1}{a_n} \sum_{k=1}^{\infty} a_{nk} |x_k| &\leq \sup_n \tfrac{1}{a_n} \sum_{k:|x_k|\leq t_0} a_{nk} |x_k| + \sup_n \tfrac{1}{a_n} \sum_{k:|x_k|>t_0} a_{nk} |x_k| \\ &\leq t_0 \sup_n \tfrac{1}{a_n} \sum_{k=1}^{\infty} a_{nk} + \tfrac{\varepsilon}{2} \\ &< \infty, \end{aligned}$$

which yields (*i*).

To show Part (*ii*), we take $\delta = \varepsilon/2t_0$, and for any set E of nonnegative integers, we let

$$\sup_n \frac{1}{a_n} \sum_{k \in E} a_{nk} < \delta.$$

Hence, we obtain

$$\begin{aligned}\sup_n \tfrac{1}{a_n}\sum_{k\in E} a_{nk}\left|x_k\right| &\leq \sup_n \tfrac{1}{a_n}\sum_{\substack{k:|x_k|>t_0\\ k\in E}} a_{nk}\left|x_k\right| + \sup_n \tfrac{1}{a_n}\sum_{\substack{k:|x_k|\leq t_0\\ k\in E}} a_{nk}\left|x_k\right| \\ &\leq \sup_n \tfrac{1}{a_n}\sum_{k:|x_k|>t_0} a_{nk}\left|x_k\right| + t_0 \sup_n \tfrac{1}{a_n}\sum_{k\in E} a_{nk} \\ &\leq \tfrac{\varepsilon}{2} + t_0\delta \\ &= \varepsilon,\end{aligned}$$

which yields (ii).

Now, we show that Part (2) implies Part (1). In Part (2) (i), we let

$$M := \sup_n \frac{1}{a_n}\sum_{k=1}^{\infty} a_{nk}\left|x_k\right| < \infty.$$

Moreover by Part (ii), the statement, for any $\varepsilon > 0$, there is $\delta > 0$, such that $\sup_n \frac{1}{a_n}\sum_{k\in E} a_{nk} < \delta$, implies the condition

$$\sup_n \frac{1}{a_n}\sum_{k=1}^{\infty} a_{nk}\left|x_k\right| < \infty.$$

Hence for this $\varepsilon > 0$, take $t_0 = \frac{M}{\delta}$.

Next, consider the set $E(t) := \{k : |x_k| \geq t\}$. So we have for any fixed $t \geq t_0$ that

$$\begin{aligned}\sup_n \tfrac{1}{a_n}\sum_{k\in E(t)} a_{nk} &\leq \tfrac{1}{t}\sup_n \tfrac{1}{a_n}\sum_{k=1}^{\infty} a_{nk}\left|x_k\right| \\ &\leq \tfrac{M}{t} \\ &\leq \tfrac{M}{t_0} \\ &= \delta.\end{aligned}$$

This means that Part (ii) can be applied, with $E = E(t)$, and we conclude

$$\sup_n \frac{1}{a_n}\sum_{k\in E(t)} a_{nk}\left|x_k\right| < \varepsilon,$$

for $t \geq t_0$. This implies that $x \in U_{A,a}$. □

The following result characterizes the $A-$strong convergence with the rate of $o(a_n)$.

Theorem 3.5. *Let $A = (a_{nk})$ be a nonnegative regular matrix, let $a = (a_n)$ be a positive nonincreasing sequence and let $x = (x_k)$ be a real number sequence. Then the following statements are equivalent:*
i) $\lim_n \frac{1}{a_n}\sum_{k=1}^{\infty} a_{nk}\left|x_k\right| = 0$,
ii) $st_{A,a} - \lim x = 0$ *and* $x \in U_{A,a}$,
iii) The sequnece x is the $A-$distributionally convergent to αF with the rate of $o(a_n)$ and $x \in U_{A,a}$, where $F = \chi_{[0,\infty)}$.

Proof. $(ii) \Rightarrow (i)$: Since $st_{A,a} - \lim x = 0$ and $x \in U_{A,a}$ for any $\varepsilon > 0$ and any $t > 0$ we have

$$\begin{aligned}
\limsup_n \tfrac{1}{a_n} \sum_{k:|x_k|\le t} a_{nk} |x_k| &\le \limsup_n \tfrac{1}{a_n} \sum_{k:\varepsilon<|x_k|\le t} a_{nk} |x_k| + \limsup_n \tfrac{1}{a_n} \sum_{k:|x_k|\le \min(t,\varepsilon)} a_{nk} |x_k| \\
&\le t \limsup_n \tfrac{1}{a_n} \sum_{k:|x_k|>\varepsilon} a_{nk} + \varepsilon \limsup_n \tfrac{1}{a_n} \sum_{k=1}^{\infty} a_{nk} \\
&\le \varepsilon\alpha.
\end{aligned}$$

From this we also get

$$\begin{aligned}
\limsup_n \tfrac{1}{a_n} \sum_k a_{nk} |x_k| &\le \limsup_n \tfrac{1}{a_n} \sum_{k:|x_k|\le t} a_{nk} |x_k| + \limsup_n \tfrac{1}{a_n} \sum_{k:|x_k|>t} a_{nk} |x_k| \\
&\le \varepsilon\alpha + \limsup_n \tfrac{1}{a_n} \sum_{k:|x_k|>t} a_{nk} |x_k| . \qquad (3.1)
\end{aligned}$$

Since $x \in U_{A,a}$ by (3.1) we obtain (by letting $t \to \infty$) that

$$\limsup_n \sum_k \frac{1}{a_n} a_{nk} |x_k| \le \varepsilon\alpha.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\lim_n \frac{1}{a_n} \sum_k a_{nk} |x_k| = 0.$$

$(i) \Rightarrow (ii)$: For any $\varepsilon > 0$, it is clear that

$$\tfrac{1}{a_n} \sum_{k:|x_k|>\varepsilon} a_{nk} \quad \le \quad \tfrac{1}{\varepsilon a_n} \sum_k a_{nk} |x_k| ,$$

and by Part (i), this implies

$$st_{A,a} - \lim x = 0.$$

To complete the proof, it remains to show that $x \in U_{A,a}$.
By Part (i), for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\frac{1}{a_n} \sum_{k=1}^{\infty} a_{nk} |x_k| < \varepsilon \text{ for all } n \ge N.$$

Since $\sup_n \frac{1}{a_n} \sum_k a_{nk} |x_k| < \infty$, for each $n = 1, 2, ..., N-1$ we may choose a positive integer K large enough for which

$$\frac{1}{a_n} \sum_{k>K} a_{nk} |x_k| < \varepsilon,$$

for all $n < N$.

When $t > \max\{|x_1|, |x_2|, ..., |x_K|\}$, we observe that

$$\sup_n \frac{1}{a_n} \sum_{k:|x_k|>t} a_{nk} |x_k| < \varepsilon,$$

which means that $x \in U_{A,a}$.

$(ii) \Rightarrow (iii)$: By Part (i), we have

$$\lim_n \frac{1}{a_n} \sum_k a_{nk} |x_k| = 0. \tag{3.2}$$

Case I: Let $t < 0$. If $x_k \leq t$, then $-\frac{|x_k|}{t} \geq 1$.
Thus we get

$$\begin{aligned} \frac{1}{a_n} \sum_{k:x_k \leq t} a_{nk} &\leq -\frac{1}{t}\frac{1}{a_n} \sum_{k:x_k \leq t} a_{nk} |x_k| \\ &\leq -\frac{1}{t}\frac{1}{a_n} \sum_{k=1}^{\infty} a_{nk} |x_k| . \end{aligned}$$

Combining this with (3.2), we have

$$\lim_n \frac{1}{a_n} \sum_{k:x_k \leq t} a_{nk} = 0 = \alpha F(t),$$

thus $F(t) = 0$ for all $t < 0$.

Case II: Let $t > 0$. One can get

$$\begin{aligned} \frac{1}{a_n} \sum_k a_{nk} &= \frac{1}{a_n} \sum_{k:x_k \leq t} a_{nk} + \frac{1}{a_n} \sum_{k:x_k > t} a_{nk} \\ &\leq \frac{1}{a_n} \sum_{k:x_k \leq t} a_{nk} + \frac{1}{t}\frac{1}{a_n} \sum_{k:x_k > t} a_{nk} x_k \\ &\leq \frac{1}{a_n} \sum_{k:x_k \leq t} a_{nk} + \frac{1}{t}\frac{1}{a_n} \sum_k a_{nk} |x_k| . \end{aligned}$$

Letting $n \to \infty$, we obtain

$$\begin{aligned} \alpha &\leq \lim_{n\to\infty} \frac{1}{a_n} \sum_{k:x_k \leq t} a_{nk} \\ &\leq \lim_{n\to\infty} \frac{1}{a_n} \sum_{k=1}^{\infty} a_{nk} = \alpha, \end{aligned}$$

which implies $F(t) = 1$ for all $t > 0$.

$(iii) \Rightarrow (ii)$: For all $\varepsilon > 0$ we get

$$\begin{aligned} \frac{1}{a_n} \sum_{k:|x_k|>\varepsilon} a_{nk} &= \frac{1}{a_n} \sum_{k:x_k<-\varepsilon} a_{nk} + \frac{1}{a_n} \sum_{k:x_k>\varepsilon} a_{nk} \\ &\leq \frac{1}{a_n} \sum_{k:x_k \leq -\varepsilon} a_{nk} + \frac{1}{a_n} \sum_{k=1}^{\infty} a_{nk} - \frac{1}{a_n} \sum_{k:x_k \leq \varepsilon} a_{nk}. \end{aligned}$$

By letting $n \to \infty$ we obtain

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{k:|x_k|>\varepsilon} a_{nk} \leq 0 + \alpha - \alpha = 0,$$

which means that $st_{A,a} - \lim x = 0$. □

4. Criteria

In this section, motivated by those of Demirci [4], Schoenberg [21], Şahin Bayram [23] we give a criterion for the $A-$statistical convergence with the rate of $o(a_n)$. Later on we will also improve this result.

Definition 4.1. *Let $A = (a_{nk})$ be a nonnegative regular summability matrix. $A_a x$ is the sequence whose nth term is given by $(A_a x)_n = \frac{1}{a_n} \sum_{k=1}^{\infty} a_{nk} x_k$, where we assume that the series $\sum_{k=1}^{\infty} a_{nk} x_k$ is convergent for each $n \in \mathbb{N}$. If*

$$\lim_n \frac{1}{a_n} \sum_{k=1}^{\infty} a_{nk} x_k = L,$$

then we say that x is $A-$summable to L with the rate of $o(a_n)$. In this case we write $A_a - \lim x = L$.

Let ℓ_∞ denote the space of all bounded sequences.

Theorem 4.2. *Let $A = (a_{nk})$ be a nonnegative regular matrix and let $a = (a_n)$ be a positive nonincreasing sequence. If $st_{A,a} - \lim x = L$ then $A_a - \lim x = \alpha L$ for every $x \in \ell_\infty$.*

Proof. Let $st_{A,a} - \lim x = L$ and for any $\varepsilon > 0$, we let $K = \{k : |x_k - L| \geq \varepsilon\}$. Then

$$\lim_n \frac{1}{a_n} \sum_{k\in K} a_{nk} = 0.$$

For every $x \in \ell_\infty$ we have

$$\begin{aligned}
\left|(A_a x)_n - \alpha L\right| &\leq \tfrac{1}{a_n} \sum_{k=1}^{\infty} a_{nk} |x_k - L| + |L| \left| \tfrac{1}{a_n} \sum_{k=1}^{\infty} a_{nk} - \alpha \right| \\
&= \tfrac{1}{a_n} \sum_{k\in K} a_{nk} |x_k - L| + \tfrac{1}{a_n} \sum_{k\notin K} a_{nk} |x_k - L| + |L| \left| \tfrac{1}{a_n} \sum_{k=1}^{\infty} a_{nk} - \alpha \right| \\
&\leq \sup_k |x_k - L| \tfrac{1}{a_n} \sum_{k\in K} a_{nk} + \varepsilon \tfrac{1}{a_n} \sum_{k=1}^{\infty} a_{nk} + |L| \left| \tfrac{1}{a_n} \sum_{k=1}^{\infty} a_{nk} - \alpha \right|.
\end{aligned}$$

Letting $n \to \infty$ we get that $\left|(A_a x)_n - \alpha L\right| \leq \varepsilon\alpha$. Since $\varepsilon > 0$ is arbitrary we conclude that $A_a - \lim x = \alpha L$. □

Lemma 4.3. *Let $a = (a_n)$ be a positive nonincreasing sequence. If the sequence $x = (x_k)$ is the $A-$statistically convergent to the number L with the rate of $o(a_n)$ and the function g defined on $\mathbb{R}$, is continuous at $y = L$, then $st_{A,a} - \lim g(x) = g(L)$.*

Since the proof uses same technique as in [21], we omit the details (see, also, [4]).
Now we are ready to give an analog of Schoenberg's criterion.

Theorem 4.4. *Let $A = (a_{nk})$ be a nonnegative regular matrix and let $a = (a_n)$ be a positive nonincreasing sequence. The sequence $x = (x_k)$ is the $A-$statistically convergent to the number L with the rate of $o(a_n)$ if and only if we get*

$$\lim \frac{1}{a_n} \sum_{k=1}^{\infty} a_{nk} e^{itx_k} = \alpha e^{itL}, \tag{4.1}$$

for every real t.

Proof. Let $st_{A,a} - \lim x = L$ and for a fixed $t \in \mathbb{R}$, $g(x) = e^{itx}$. Note that the function g is a continuous function of x. Then we have by Lemma 4.3 that

$$st_{A,a} - \lim e^{itx_k} = e^{itL}.$$

Since $\left(e^{itx_k}\right) \in l_\infty$, we conclude that

$$A_a - \lim e^{itx_k} = \alpha e^{itL},$$

by Theorem 4.2.

Conversely suppose that (4.1) holds. As in [21], we define a continuous function M by

$$M(y) = \begin{cases} 0 & , \quad y \le -1 \\ 1+y & , \quad -1 < y < 0 \\ 1-y & , \quad 0 \le y < 1 \\ 0 & , \quad 1 \le y. \end{cases}$$

Since the M is a Lebesgue integrable function, its Fourier transformation is given by

$$\begin{aligned} f(t) &= \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} M(y) e^{-ity} dy, \quad t \in \mathbb{R} \\ &= \tfrac{1}{\sqrt{2\pi}} \left(\tfrac{\sin(t/2)}{t/2}\right)^2. \end{aligned}$$

Moreover inverse Fourier Transformation of the function f is

$$\begin{aligned} M(y) &= \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ity} dt \\ &= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} \left(\tfrac{\sin(t/2)}{t/2}\right)^2 e^{ity} dt. \end{aligned} \tag{4.2}$$

To complete the proof, we need to show that $st_{A,a} - \lim x = 0$. Let $\varepsilon > 0$ and $K := K(\varepsilon) := \{k \in \mathbb{N} : |x_k| \ge \varepsilon\}$. Substituting $\frac{t}{\varepsilon} = u$, we obtain

$$M\left(\frac{y}{\varepsilon}\right) = \frac{\varepsilon}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin(\varepsilon t/2)}{\varepsilon t/2}\right)^2 e^{ity} dt.$$

Hence

$$\frac{1}{a_n} \sum_{k=1}^{\infty} a_{nk} M\left(\frac{x_k}{\varepsilon}\right) = \frac{\varepsilon}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin(\varepsilon t/2)}{\varepsilon t/2}\right)^2 \left(\frac{1}{a_n} \sum_{k=1}^{\infty} a_{nk} e^{itx_k}\right) dt.$$

We remark that (4.2) is an absolutely convergent integral. By the Lebesgue dominated convergence theorem we see that

$$\begin{aligned} \lim_n \frac{1}{a_n} \sum_{k=1}^{\infty} a_{nk} M\left(\frac{x_k}{\varepsilon}\right) &= \frac{\varepsilon}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin(\varepsilon t/2)}{\varepsilon t/2}\right)^2 \left(\lim \frac{1}{a_n} \sum_{k=1}^{\infty} a_{nk} e^{itx_k}\right) dt \\ &= \frac{\varepsilon}{2\pi} \alpha \int_{-\infty}^{\infty} \left(\frac{\sin(\varepsilon t/2)}{\varepsilon t/2}\right)^2 dt \\ &= \alpha M(0) \\ &= \alpha. \end{aligned}$$

Considering the definition of the function M, we get

$$\begin{aligned} \frac{1}{a_n} \sum_{k=1}^{\infty} a_{nk} M\left(\frac{x_k}{\varepsilon}\right) &= \frac{1}{a_n} \sum_{k:-1<\frac{x_k}{\varepsilon}<0} a_{nk} M\left(\frac{x_k}{\varepsilon}\right) + \frac{1}{a_n} \sum_{k:0\le\frac{x_k}{\varepsilon}<1} a_{nk} M\left(\frac{x_k}{\varepsilon}\right) \\ &\le \frac{1}{a_n} \sum_{k\in\mathbb{N}} a_{nk} - \frac{1}{a_n} \sum_{k\in K} a_{nk}. \end{aligned} \tag{4.3}$$

Taking limit as $n \to \infty$ on the both sides of (4.3) and using the fact that $\delta_{A,a}(\mathbb{N}) = \alpha$, we now see that

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{k\in K} a_{nk} = 0.$$

This concludes the proof. □

The next theorem is an analogue of Salat's result [20].
Let

$$S^*_{A,a} := \left\{ x : \left(\frac{1}{a_n} \sum_{k=1}^{\infty} a_{nk} |x_k| \right) \in \ell_\infty \right\}.$$

We show that condition (4.1) in Theorem 4.4 can be weakened provided that x is in $S^*_{A,a}$.

Theorem 4.5. *Let $A = (a_{nk})$ be a nonnegative regular matrix and let $a = (a_n)$ be a positive nonincreasing sequence. If $x \in S^*_{A,a}$, then the sequence x is the $A-$statistically convergent to the number L with the rate of $o(a_n)$ if and only if for each rational number t we get*

$$\lim_n \frac{1}{a_n} \sum_{k=1}^{\infty} a_{nk} e^{itx_k} = \alpha e^{itL}. \tag{4.4}$$

Proof. The necessity follows from Theorem 4.4. Sufficiency. For each rational number t, let (4.4) hold and t_0 be an arbitrary real number. We need to show that

$$\lim_n \frac{1}{a_n} \sum_{k=1}^{\infty} e^{it_0 x_k} = \alpha e^{it_0 L}. \tag{4.5}$$

Now let

$$C_n(t_0, t) := \frac{1}{a_n} \sum_{k=1}^{\infty} e^{it_0 x_k} - \frac{1}{a_n} \sum_{k=1}^{\infty} a_{nk} e^{itx_k}.$$

Observe that

$$|C_n(t_0,t)| \leq \frac{1}{a_n}\sum_{k=1}^{\infty} a_{nk}\sqrt{(\cos t_0x_k - \cos tx_k)^2 + (\sin t_0x_k - \sin tx_k)^2}.$$

By the Mean Value Theorem we have

$$|C_n(t_0,t)| \leq |t-t_0|\frac{1}{a_n}\sum_{k=1}^{\infty} a_{nk}|x_k|.$$

Since $x \in S^*_{A,a}$, there exists $M > 0$ such that

$$|C_n(t_0,t)| \leq |t-t_0|M. \tag{4.6}$$

We observe that

$$\left|\frac{1}{a_n}\sum_{k=1}^{\infty} a_{nk}e^{it_0x_k} - \alpha e^{it_0L}\right| \leq \left|\frac{1}{a_n}\sum_{k=1}^{\infty} a_{nk}e^{itx_k} - \alpha e^{itL}\right| + \alpha\left|e^{itL} - e^{it_0L}\right| + |C_n(t_0,t)|.$$

Let $\varepsilon > 0$. By the continuity of $g(x) = \alpha e^{ixL}$, one can get that there exists a rational number t such that

$$\left|e^{itL} - e^{it_0L}\right| < \frac{\varepsilon}{3\alpha}, \tag{4.7}$$

and by (4.6) we have

$$|C_n(t_0,t)| < \frac{\varepsilon}{3}. \tag{4.8}$$

Finally, combining (4.4) and (4.7) and (4.8) we conclude that (4.5) holds. Since $t_0 \in \mathbb{R}$ is arbitrary, hence $st_{A,a} - \lim x = L$. □

5. An Application to Approximation Theory

The main purpose of this section is to present an application of the rates of the $A-$statistical convergence to Korovkin type approximation theory. Note that Korovkin type approximation theorems provide conditions under which a given sequence of positive linear operators, acting on some function space, converges strongly to the identity operator [17]. Firstly we recall, for the reader's convenience, some definitions and notation stated in [1] and [18]. Let X be a compact metric space. The collection of all continuous real valued functions on X will be denoted by $C(X)$ equipped with norm $\|f\| = \sup_{x\in X}|f(x)|$. A linear operator $L : C(X) \to C(X)$ is called positive if $L(f) \geq 0$ provided that $f \geq 0$.
The diagonal $\Delta(f)$ of $f \in C(X)$ in X is defined by

$$\Delta(f) = \{(x,t) \in X \times X : f(x) = f(t)\}.$$

Let $\alpha \in C(X)$ and $Z(\alpha)$ be the set of zeros of α i.e.,

$$Z(\alpha) = \{x \in X : \alpha(x) = 0\}.$$

If γ is a positive function in $C(X \times X)$ such that $Z(\gamma) \subset \Delta(f)$, then γ is called a bounding function for $f \in C(X)$. In addition for each $t \in X$ we write $\gamma_t(x) := \gamma(x,t)$.

Lemma 5.1. *Let $A = (a_{jn})$ be a nonnegative regular matrix and let $a = (a_j)$ be a positive nonincreasing sequence and let γ be a bounding function for $f \in C(X)$. Suppose that $\{L_n\}$ be a sequence of positive operators from $C(X)$ into $C(X)$. If (i) $st_{A,a} - \lim\|L_n(1) - 1\| = 0$,*
(ii) $st_{A,a} - \lim\|L_n(\gamma_t)\| = 0$
then $st_{A,a} - \lim\|L_nf - f\| = 0$.

Proof. Following [18], we immediately get
$\left|L_n(f)(t) - f(t)\right| \leq \varepsilon + \left(\varepsilon + \left|f(t)\right|\right)\left|L_n(1)(t) - 1\right| + ML_n(\gamma_t)(t)$.

This gives the inequality

$$\left\|L_n(f) - f\right\| \leq \varepsilon + B\left(\left\|L_n(1) - 1\right\| + \left\|L_n(\gamma_t)\right\|\right), \tag{5.1}$$

where $B := \max\left\{\varepsilon + \left\|f\right\|, M\right\}$.

Let $r > 0$. Hence there exist some $\varepsilon > 0$ such that $\varepsilon < r$. Define the sets
$D := \left\{n : \left\|L_n(1) - 1\right\| + \left\|L_n(\gamma_t)\right\| \geq r - \varepsilon\right\}$,
$D_1 := \left\{n : \left\|L_n(1) - 1\right\| \geq \frac{r-\varepsilon}{2B}\right\}$,
$D_2 := \left\{n : \left\|L_n(\gamma_t)\right\| \geq \frac{r-\varepsilon}{2B}\right\}$.

Then we have $D \subset D_1 \cup D_2$. Now (5.1) yields that

$$\frac{1}{a_j}\sum_{n:\left\|L_n(f)-f\right\|\geq r} a_{jn} \leq \frac{1}{a_j}\sum_{n\in D} a_{jn} \leq \frac{1}{a_j}\sum_{n\in D_1} a_{jn} + \frac{1}{a_j}\sum_{n\in D_2} a_{jn}.$$

Letting $j \to \infty$ on the both sides and using (i) and (ii), we obtain that $st_{A,a} - \lim\left\|L_n f - f\right\| = 0$. □

Letting $X = [a, b]$ and taking $\gamma_t(x) := (x - t)^2$ as a bounding function of an arbitrary $f \in C[a, b]$ then Lemma 5.1 allows us to conclude the following:

Theorem 5.2. *Let $A = \left(a_{jn}\right)$ be a nonnegative regular matrix and let $a = \left(a_j\right)$ be a positive nonincreasing sequence and let $L_n : C(X) \to C(X)$ be a sequence of positive linear operators. If*

$$st_{A,a} - \lim\left\|L_n f_i - f_i\right\| = 0, \ (i = 0, 1, 2)$$

then, we get

$$st_{A,a} - \lim\left\|L_n f - f\right\| = 0,$$

for any function $f \in C(X)$, where $f_i(y) = y^i$.

Acknowledgements

We would like to thank the referee for his/her suggestions that improved the presentation of the paper.

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