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# Existence of a Renormalized Solutions to a Nonlinear System in Orlicz Spaces

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**Abstract.** In this paper, we will be concerned with the existence of renormalized solutions to the following parabolic-elliptic system

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au = \sigma(u)|\nabla\varphi|^2 & \text{in } Q_T = \Omega\times(0,T), \\ -\text{div}(\sigma(u)\nabla\varphi) = \text{div}F(u) & \text{in } Q_T, \\ u = 0 & \text{on } \partial\Omega\times(0,T), \\ \varphi = 0 & \text{on } \partial\Omega\times(0,T), \\ u(\cdot,0) = u_0 & \text{in } \Omega, \end{cases}$$

where $Au = -\text{div}\, a(x,t,u,\nabla u)$ is a Leray-Lions operator defined on the inhomogeneous Orlicz-Sobolev space $W_0^{1,x}L_M(Q_T)$ into its dual, $M$ is a N-function related to the growth of $a$. $M$ does not satisfy the $\Delta_2$-condition, and $\sigma$ and $F$ are two Carathéodory functions defined in $Q_T\times\mathbb{R}$.

## 1. Introduction

We consider the following parabolic-elliptic nonlinear system

$$\begin{cases} \dfrac{\partial u}{\partial t} - \text{div}\, a(x,t,u,\nabla u) = \sigma(u)|\nabla\varphi|^2 & \text{in } Q_T = \Omega\times(0,T), \\ -\text{div}(\sigma(u)\nabla\varphi) = \text{div}(F(u)) & \text{in } Q_T, \\ u = 0 & \text{on } \partial\Omega\times(0,T), \\ \varphi = 0 & \text{on } \partial\Omega\times(0,T), \\ u(\cdot,0) = u_0 & \text{in } \Omega. \end{cases} \tag{1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$, $T>0$. $a$, $\sigma$, and $F$ are Carathéodory functions defined in $Q_T\times\mathbb{R}$, and the function $u_0$ is given.

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Let us start with the case of $F = 0$ in which the system corresponds to the study of the electrical heating of a conductor where $u$ is the temperature of the conductor and $\varphi$ is the electric potential. Since $\sigma(u)$ can reach zero, $u$ will not be bounded in $Q_T$, and no a priori estimate of $\nabla\varphi$ will be available and, consequently, $\varphi$ may not belong to Sobolev spaces. That's why X. Xu introduces the capacity solution of (1) in $W_0^1 L^p(Q_T)$ for $p \geq 2$; and later, other authors applied this concept to more general contexts by considering either weaker assumptions or mixed boundary conditions (See [22, 28–30]). The existence of a capacity solution of (1), in the Orlicz spaces, has been proved by H. Moussa et al. in [24].

Now, our goal, in this work, is to study the system more generally by taking $F \neq 0$. This problem may be a generalization of the thermistor problem arising in electromagnetism.

The interest by several researchers has increased in recent years in the problems of thermistors [5, 14], with various assumptions, but both the assumptions $a$ and $\sigma$ are always assumed to be bounded in all these referred works. Under these assumptions, the search for a weak solution is completely inappropriate, so we will show the existence of a renormalized solution in the inhomogeneous Orlicz-Sobolev spaces.

R.J. DiPerna, and P.L. Lions, [17] introduced the notion of renormalized solutions in the study of the Boltzmann equation. It was then adapted to the study of certain nonlinear elliptical or parabolic problems and evolutionary problems in fluid mechanics. Later this concept was applied more generally to the nonlinear elliptic equations [13, 25, 26], as well as to the existence and uniqueness of a renormalized solution to nonlinear parabolic equations. We refer to [1–3, 9–11, 21, 32, 33] for more details. Note that the main reason for choosing such solutions is to find an appropriate solution, which answers the existence and uniqueness questions and at the same time ensures the physical solution of (1). Recall that there are more solutions in this sense, including the entropic solutions introduced by Ph. Bénilan et al. in [8] and the SOLA (Solution Obtained as Limit of Approximation) solutions developed by A. Dall'Aglio in [16].

Problem (1) does not admit weak solutions under the assumptions (5)-(14), due to the unboundedness of $a$ and $\sigma$, and to the fact that $\sigma(u)|\nabla\varphi|^2 \in L^1(Q_T)$. It is our purpose, in this paper, to prove the existence of renormalized solutions, for problem (1) in the setting of the Inhomogeneous Orlicz-Sobolev spaces, without assuming the $\Delta_2$-condition on the N-function $M$. Indeed, in general, these spaces fail to be separable or reflexive if $M$ and its conjugate $\overline{M}$ satisfy a $\Delta_2$-condition. In addition to this difficulty, we aim to analyze problem (1) with the assumption that the diffusion of $a$ and $\sigma$ is unbounded, and no asymptotic conduct on $a$, $\sigma$, and $F$ is expected. Furthermore, we may encounter the fact that the parabolic equation needs special treatment due to the nonlinear right-hand side belonging to $L^1(Q_T)$.
The main tool, used here, is the truncation techniques, a generalized Minty method in the functional setting of non-reflexive spaces and approximate solutions.
The result of this paper can be applied to models like the following example

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_M u = r u^{\zeta} e^{\frac{-s}{k_B u}} |\nabla u|^2 & \text{in} \quad Q_T, \\ \operatorname{div}(\kappa(u)\nabla\varphi) = 0 & \text{in} \quad \Omega, \end{cases} \tag{2}$$

where $\Delta_M u = -\operatorname{div}\left((1+|u|)^2 Du \frac{\log(e+Du)}{|Du|}\right)$, $M(t) = t\log(e+t)$ is an $N$ -function, $F = 0$, $\varphi$ represent the electric motive force, $u$ the temperature inside the electrical conductor, and $\kappa(u) = r u^{\zeta} e^{\frac{-s}{k_B u}}$, the electrical conductivity where it means the ability of electrical material to pass charges, $u > 0$, $r, s \in \mathbb{R}^+$, $\zeta \in [-1, 1)$ and $k_B$ is the Boltzmann constant. Other applications of the stationary case of the thermostat problem can be found in [7, 31].

The plan of the paper is as follows: In Section 2, we recall some well-known preliminary properties and results of Orlicz-Sobolev spaces; in Section 3, we precisely make all the basic assumptions on $a$, $\sigma$, $F$; in Section 4, we define a renormalized solution of (1) and in Section 5, we will provide and prove the main result of this article (Theorem 5.1).

## 2. Preliminaries

Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, that is, $M$ is continuous, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$, and $\frac{M(t)}{t} \to +\infty$ as $t \to +\infty$. Equivalently, $M$ admits the representation $M(t) = \int_0^t a(s)ds$, where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, right continuous, with $a(0) = 0, a(t) > 0$ for $t > 0$, and $a(t) \to +\infty$ as $t \to +\infty$. The N-function $\overline{M}$ conjugate to $M$ is defined by $\overline{M}(t) = \int_0^t \overline{a}(s)ds$, where $\overline{a} : \mathbb{R}^+ \to \mathbb{R}^+$, is given by $\overline{a}(t) = \sup_{s\geq 0}\{s : a(s) \leq t\}$.
As $M$ is convex, we have,

$$M(\beta t) \leq \beta M(t) \text{ for all } 0 \leq \beta \leq 1,$$
$$M(\beta t) \geq \beta M(t) \text{ for all } \beta \geq 1.$$

We will extend these N-functions into even functions on all $\mathbb{R}$.
Let $P$ and $Q$ be two N-functions. $P \ll Q$ means that $P$ grows essentially less rapidly than $Q$, that is, for each $\epsilon > 0$, $\frac{P(t)}{Q(\epsilon t)} \to 0$ as $t \to +\infty$. This is the case if and only if $\lim_{t\to+\infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0$.
The Orlicz class $K_M(\Omega)$ $\big($resp. the Orlicz space $L_M(\Omega)\big)$, is defined as the set of (equivalence classes of) real valued measurable functions $u$ on $\Omega$ such that

$$\int_\Omega M(|u(x)|)dx < +\infty \left(\text{resp. } \int_\Omega M\left(\frac{|u(x)|}{\lambda}\right)dx < +\infty \quad \text{for some} \quad \lambda > 0\right).$$

The set $L_M(\Omega)$ is Banach space under the norm

$$\|u\|_M = \inf\left\{\lambda > 0 : \int_\Omega M\left(\frac{|u(x)|}{\lambda}\right)dx \leq 1\right\},$$

and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. The dual $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega uvdx$ and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|u\|_{\overline{M},\Omega}$. We now turn to the Orlicz-Sobolev space, $W^1L_M(\Omega)$ [resp. $W^1E_M(\Omega)$ ] is the space of all functions $u$ such that $u$ and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ [resp. $E_M(\Omega)$ ]. It is a Banach space under the norm $\|u\|_{1,M} = \sum_{|\alpha|\leq 1} \|D^\alpha u\|_M$.

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by $\Pi L_M$, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.

Let $W^{-1}L_{\overline{M}}(\Omega)$ [resp. $W^{-1}E_{\overline{M}}(\Omega)$ ] denote the space of distributions on $\Omega$ which can be written as sums of derivatives of order $\leq 1$ of functions in $L_{\overline{M}}(\Omega)$ [resp. $E_{\overline{M}}(\Omega)$]. It is a Banach space under the usual quotient norm (for more details see [4]).

**The inhomogeneous Orlicz-Sobolev spaces.**

Let $M$ be an N-function, for each $\alpha \in \mathbb{N}^N$, denote by $\nabla_x^\alpha$ the distributional derivative on $Q_T$ of order $\alpha$ with respect to the variable $x \in \mathbb{R}^N$. The inhomogeneous Orlicz-Sobolev spaces are defined as follows

$$W^{1,x}L_M(Q_T) = \left\{u \in L_M(Q_T) : \nabla_x^\alpha u \in L_M(Q_T), \forall \alpha \in \mathbb{N}^N, |\alpha| \leq 1\right\},$$

$$W^{1,x}E_M(Q_T) = \left\{u \in E_M(Q_T) : \nabla_x^\alpha u \in E_M(Q_T), \forall \alpha \in \mathbb{N}^N, |\alpha| \leq 1\right\}.$$

The last space is a subspace of the first one, and both are Banach spaces under the norm $\|u\| = \sum_{|\alpha|\leq 1} \left\|\nabla_x^\alpha u\right\|_{M,Q_T}$.
The space $W_0^{1,x}E_M(Q_T)$ is defined as the (norm) closure $W^{1,x}E_M(Q_T)$ of $\mathcal{D}(Q_T)$. We can easily show that when

$\Omega$ has the segment property, then each element $u$ of the closure of $\mathcal{D}(Q_T)$ with respect of the weak* topology $\sigma(\Pi L_M, \Pi E_{\bar{M}})$ is a limit in $W_0^{1,x}E_M(Q_T)$, of some subsequence in $\mathcal{D}(Q_T)$ for the modular convergence. This implies that $\overline{\mathcal{D}(Q_T)}^{\sigma(\Pi L_M,\Pi E_{\bar{M}})} = \overline{\mathcal{D}(Q_T)}^{\sigma(\Pi L_M,\Pi L_{\bar{M}})}$. This space will be denoted by $W_0^{1,x}L_M(Q_T)$. Furthermore, $W_0^{1,x}E_M(Q_T) = W_0^{1,x}L_M(Q_T) \cap \Pi E_M$, and the dual space of $W_0^{1,x}E_M(Q_T)$ will be denoted by

$$W^{-1,x}L_{\bar{M}}(Q_T) = \left\{ f = \sum_{|\alpha|\leq 1} \nabla_x^\alpha f_\alpha : f_\alpha \in L_{\bar{M}}(Q_T) \right\}$$

This space will be equipped with the usual quotient norm $\|f\| = \inf \sum_{|\alpha|\leq 1} \left\|f_\alpha\right\|_{\bar{M},Q_T}$.

In the sequel, we have to use the following results which concern mollification with respect to time and space variable and some trace results. Thus, we define for all $\mu > 0$ and all $(x,t) \in Q_T$:

$$u_\mu(x,t) = \mu \int_{-\infty}^{t} \tilde{u}(x,s)\exp(\mu(s-t))ds \text{ where } \tilde{u}(x,s) = u(x,s)\chi_{(0,T)}, \tag{3}$$

**Lemma 2.1.** *(see [19])*

1. *If $u \in L_M(Q_T)$, then $u_\mu \to u$ as $\mu \to +\infty$ in $L_M(Q_T)$ for the modular convergence.*
2. *If $u \in W^{1,x}L_M(Q_T)$, then $u_\mu \to u$ as $\mu \to +\infty$ in $W^{1,x}L_M(Q_T)$ for the modular convergence.*
3. *If $u \in W^{1,x}L_M(Q_T)$, then $\frac{\partial u_\mu}{\partial t} = \mu\left(u - u_\mu\right) \in W^{1,x}L_M(Q_T)$.*

We will use the following technical lemmas.

**Lemma 2.2.** *(see [19]) Let $M$ be an N-function. Let $(u_n)$ be a sequence of $W^{1,x}L_M(Q_T)$ such that*

$$u_n \to u \text{ weakly in } W^{1,x}L_M(Q_T) \text{ for } \sigma\left(\Pi L_M, \Pi E_{\overline{M}}\right) \text{ and } \frac{\partial u_n}{\partial t} = h_n + k_n \text{ in } \mathcal{D}'(Q_T),$$

*with $(h_n)$ bounded in $W^{-1,x}L_M(Q_T)$ and $(k_n)$ bounded in the space* $\mathrm{M}(Q_T)$ *of measurable on $Q_T$. Then*

$$u_n \longrightarrow u \text{ strongly in } L^1_{loc}(Q_T).$$

*If further $u_n \in W_0^{1,x}L_M(Q_T)$ then $u_n \longrightarrow u$ strongly in $L^1(Q_T)$.*

**Lemma 2.3.** *(see [20]) Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ with the segment property. Then*

$$\left\{ u \in W_0^{1,x}L_{\bar{M}}(Q_T) \mid \frac{\partial u}{\partial t} \in W^{-1,x}L_{\bar{M}}(Q_T) + L^1(Q_T) \right\} \subset C\left([0,T], L^1(\Omega)\right)$$

**Lemma 2.4.** *(see [18]) For all $u \in W_0^1L_M(Q_T)$ with meas$(Q_T) < +\infty$, one has*

$$\int_{Q_T} M\left(\frac{|u|}{\lambda}\right)dxdt \leq \int_{Q_T} M(|\nabla u|)dxdt. \tag{4}$$

*where $\lambda = diam(Q_T)$, is the diameter of $Q_T$.*

## 3. Assumptions

Throughout the paper, we assume that the following assumptions hold true:
Let $M$ and $P$ be two $N$-functions such that $P \ll M$.
The operator $A : D(A) \subset W_0^{1,x}L_M(Q_T) \to W^{-1,x}L_{\overline{M}}(Q_T)$ is defined by $Au = -\text{div}\, a(x,t,u,\nabla u)$ where $a$ :

$Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that for almost every $(x,t) \in Q_T$ and for every $s, s_1, s_2 \in \mathbb{R}$, $\xi, \xi^* \in \mathbb{R}^N$,

$$|a(x,t,s,\xi)| \leq \nu[a_0(x,t) + \overline{M}^{-1}P(k_1|s|) + \overline{M}^{-1}M(k_2|\xi|)], \tag{5}$$

$$|a(x,t,s_1,\xi) - a(x,t,s_2,\xi)| \leq \nu[a_1(x,t) + |s_1| + |s_2| + \overline{P}^{-1}(k_3M(|\xi|))], \tag{6}$$

$$(a(x,t,s,\xi) - a(x,t,s,\xi^*))(\xi - \xi^*) \geq \alpha M(|\xi - \xi^*|), \tag{7}$$

$$a(x,t,s,0) = 0, \tag{8}$$

where $a_0(.,.) \in E_{\overline{M}}(Q_T)$, $a_1(.,.) \in E_P(Q_T)$ and $\alpha, \nu, \sigma_0, k_i > 0$ (i=1, 2, 3), are given positif real numbers; $\sigma : Q_T \times \mathbb{R} \to \mathbb{R}$ and $F : Q_T \times \mathbb{R} \to \mathbb{R}^N$ are two Carathéodory functions and there exists a nondecreasing function $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\max(\sigma(x,t,s), |F(x,t,s)|) \leq \theta(|s|), \text{ for all } s \in \mathbb{R}, \text{ a.e. in } Q_T, \tag{9}$$

$$\sigma(x,t,s) \geq \sigma_0, \text{ for all } s \in \mathbb{R}, \text{ a.e. in } Q_T; \tag{10}$$

There exists a function $\beta \in L^1(Q_T)$ such that

$$|F(x,t,s)|^2 \leq \beta(x,t)\sigma(x,t,s), \text{ for all } s \in \mathbb{R}, \text{ a.e. in } Q_T, \tag{11}$$

and

$$\max_{\{k \leq |s| \leq 2k\}} \text{ess sup}_{\{(x,t,\xi) \in Q_T \times \mathbb{R}^N : a(x,t,s,\xi).\xi \neq 0\}} \frac{1}{k} \frac{\sigma(x,t,s)|\xi|^2}{a(x,t,s,\xi).\xi} = \omega(k), \tag{12}$$

where $\omega(k) \to 0$ as $k \to +\infty$.
We assume that there exist two positive constants $\gamma_0$ and $\gamma_1$ such that

$$\begin{aligned} &|u|^2 \leq \gamma_0 M(u), \quad \text{for all} \quad u \geq 0, \\ &|u|^2 \leq \gamma_1 P(u), \quad \text{for all} \quad u \geq 0. \end{aligned} \tag{13}$$

$$u_0 \in L^1(\Omega). \tag{14}$$

**Remark 3.1.** *A condition* (13) *give the following continuous inclusions:*

$$L_M(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L_{\overline{M}}(\Omega), \text{ and } L_P(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L_{\overline{P}}(\Omega).$$

*Moreover, we deduce that*

$$W_0^1 L_M(\Omega) \hookrightarrow H_0^1(\Omega), \text{and } H^{-1}(\Omega) \hookrightarrow W^{-1}L_{\overline{M}}(\Omega).$$

**Example 3.2.** *The N-function $M(t) = t\log(e + t)$ verifies the previous results.*

**Lemma 3.3.** *(see [20]) With the assumptions* (5)-(8)*, let $(z_n)$ be a sequence in $W_0^{1,x}L_M(Q_T)$ such that*

$$z_n \to z \text{ in } W_0^{1,x}L_M(Q_T) \text{ for } \sigma(\Pi L_M, \Pi E_{\bar{M}}),$$

$$(a(x,t,z_n,\nabla z_n))_n \text{ is bounded in } (L_{\bar{M}}(Q_T))^N,$$

$$\int_{Q_T} [a(x,t,z_n,\nabla z_n) - a(x,t,z_n,\nabla z\chi_s)][\nabla z_n - \nabla z\chi^s]\,dxdt \to 0,$$

*where $\chi^s$ is the characteristic function of $Q_s = \{(x,t) \in Q_T : |\nabla z| \leq s\}$.*
*Then*

$$\nabla z_n \to \nabla z \text{ a.e. in } Q_T,$$

$$\lim_{n\to\infty} \int_{Q_T} a(x,t,z_n,\nabla z_n)\nabla z_n dxdt = \int_{Q_T} a(x,t,z,\nabla z)\nabla z dxdt,$$

$$M(|\nabla z_n|) \to M(|\nabla z|) \text{ in } L^1(Q_T).$$

## 4. Definition of a Renormalized solution

As already mentioned in the introduction, problem (1.1) does not admit a weak solution under assumptions (5)– (12), then in the following paragraph, we will present the definition of a renormalized solution of (1).

**Definition 4.1.** *A couple of functions* $(u, \varphi)$ *is called a renormalized solution to problem* (1) *if it satisfies the following conditions:*

$(R_1)$ $u \in L^\infty(0, T; L^1(\Omega)), \varphi \in L^2\left(0, T; H_0^1(\Omega)\right)$, *and* $\sigma(u)|\nabla\varphi|^2 \in L^1(Q_T)$,

$(R_2)$ $T_k(u) \in W_0^{1,x}L_M(Q_T)$ *for all* $k > 0$,

$(R_3)$ $\lim\limits_{m\to+\infty} \int_{\{m\leq|u(x,t)|<m+1\}} a(x, t, u, \nabla u)\nabla u dxdt = 0$,

$(R_4)$ *For every* $S \in C^\infty(\mathbb{R})$ *with* $suppS'$ *is compact,*

$$\begin{aligned}\frac{\partial S(u)}{\partial t} - div\ (S'(u)a(x, t, u, \nabla u)) + S''(u)a(x, t, u, \nabla u)\nabla u = \sigma(u)|\nabla\varphi|^2 S'(u) \quad in\ \mathcal{D}'(Q_T),\\ S(u(\cdot, 0)) = S(u_0)\ in\ \Omega,\end{aligned} \tag{15}$$

$(R_5)$ *For all* $\psi \in L^2\left(0, T; H_0^1(\Omega)\right)$, *such that* $\sigma(u)|\nabla\psi|^2 \in L^1(Q_T)$, *we have*

$$\int_{Q_T} \sigma(u)\nabla\varphi\nabla\psi dxdt = -\int_{Q_T} F(u)\nabla\psi dxdt.$$

**Remark 4.2.** *Equation* (15) *is formally obtained through pointwise multiplication of a parabolic equation in* (1) *by* $S'(u)$. *Note that due to* $(R_1)$ *and* $(R_2)$ *each term in* (15) *has a meaning in* $L^1(Q_T) + W^{-1,x}L_{\bar{M}}(Q_T)$. *Indeed, if k is such that* supp $S' \subset [-k, k]$, *the following identifications are made in* (15)

- *We have* $S'(u)a(x, t, u, \nabla u) = S'(u)a\left(x, t, T_k(u), \nabla T_k(u)\right)$ *a.e in* $Q_T$. *Since* $S'(u) \in L^\infty(Q_T)$ *and with* (5) *and* $(R_1)$, *we obtain*
$$S'(u)a\left(x, t, T_k(u), \nabla T_k(u)\right) \in \left(L_{\bar{M}}\left(Q_T\right)\right)^N.$$
- *Also* $S''(u)a(x, t, u, \nabla u)\nabla u = S''(u)a\left(x, t, T_k(u), \nabla T_k(u)\right)\nabla T_k(u)$ *and by the same arguments as above, we get*
$$S''(u)a\left(x, t, T_k(u), \nabla T_k(u)\right)\nabla T_k(u) \in L^1\left(Q_T\right).$$
- *And by* $\sigma(u)|\nabla\varphi|^2 \in L^1(Q_T)$, *one has* $S'(u)\sigma(u)|\nabla\varphi|^2 \in L^1\left(Q_T\right)$.

## 5. Main result

**Theorem 5.1.** *Under assumptions* (5)-(14), *the system* (1) *admits at least a renormalized solution* $(u, \varphi)$, *in the sens of a Definition 4.1.*

**Proof**

**Step 1: Truncated problem.**

For each $n > 0$, we define the following approximations:

$$a_n(x, t, s, \xi) = a(x, t, T_n(s), \sigma_n(x, t, s) = \sigma\left(x, t, T_n(s)\right) \text{ and } F_n(x, t, s) = F\left(x, t, T_n(s)\right),$$

a.e. $(x, t) \in Q_T, \forall s \in \mathbb{R}, \forall\xi \in \mathbb{R}^N$.

$$u_{0n} \in C_0^\infty(\Omega) : \|u_{0n}\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)} \text{ and } u_{0n} \to u_0 \text{ in } L^1(\Omega) \text{ as } n \text{ tends to } +\infty. \tag{16}$$

The approximate problems are stated as follows:

$$\varphi_n \in L^\infty\left(0, T; H_0^1(\Omega)\right),\ u_n \in W_0^{1,x}L_M(Q_T),\ \frac{\partial u_n}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q_T), u_n(\cdot, 0) = u_{0n}, \tag{17}$$

$$\int_0^T \left\langle \frac{\partial u_n}{\partial t}, \phi \right\rangle dt + \int_{Q_T} a_n(x,t,u_n,\nabla u_n)\nabla\phi dxdt = \int_{Q_T} T_n\left(\sigma_n(u_n)\left|\nabla\varphi_n\right|^2\right)\phi dxdt, \quad \text{for all } \phi \in W_0^{1,x}L_M(Q_T), \tag{18}$$

$$\int_\Omega \sigma_n(u_n)\nabla\varphi_n\nabla\psi dx = -\int_\Omega F_n(u_n)\nabla\psi dx, \text{ for all } \psi \in H_0^1(\Omega), \text{ a.e. } t \in (0,T). \tag{19}$$

Under the conditions (5)-(14), and by applying the same arguments as in [24], we conclude that it exists $(u_n, \varphi_n)$ verifying (17)-(19).

**Step 2: A priori estimates.**

The estimates derived in this step rely on usual techniques for problems of type (18).

**Lemma 5.2.** *Assume that* (5)-(14) *are satisfied, and let* $(u_n, \varphi_n)$ *be a solution of* (17)-(19). *Then, for all* $n, k > 0$, *we have*

*i)*
$$\int_{Q_T} \sigma_n(u_n)\left|\nabla\varphi_n\right|^2 dxdt \le C_1, \tag{20}$$

*ii)*
$$\int_{Q_T} M(|\nabla T_k(u_n)|) \le kC_2, \tag{21}$$

*where* $C_1$ *and* $C_2$ *are two positives constants independent of k and n.*

*iii)*
$$\lim_{k\to+\infty} meas\{(x,t) \in Q_T \ : \ |u(x,t)| \ge k\} = 0. \tag{22}$$

*Proof.* *i*) Taking $\psi = \varphi_n$ as a test function in (19) and using Schwartz inequality, we obtain

$$\begin{aligned}\int_{Q_T} \sigma_n(u_n)\left|\nabla\varphi_n\right|^2 dxdt &= -\int_{Q_T} F_n(u_n)\nabla\varphi_n dxdt \\ &\le \left(\int_{Q_T} \sigma_n(u_n)^{-1}|F_n(u_n)|^2 dxdt\right)^{\frac{1}{2}} \times \left(\int_{Q_T} \sigma_n(u_n)\left|\nabla\varphi_n\right|^2 dxdt\right)^{\frac{1}{2}},\end{aligned}$$

using (11), we obtain

$$\int_{Q_T} \sigma_n(u_n)\left|\nabla\varphi_n\right|^2 dxdt \le \int_{Q_T} \sigma_n(u_n)^{-1}|F_n(u_n)|^2 dxdt \le \int_{Q_T} \beta(x,t)dxdt = C_1.$$

As a consequence, $(\sigma_n(u_n)\left|\nabla\varphi_n\right|^2)$ is bounded in $L^1(Q_T)$.

*ii*) Choosing $\phi = T_k(u_n)\chi_{(0,\tau)}$ in (18), we obtain

$$\begin{aligned}&\left\langle \frac{\partial u_n}{\partial t}, T_k(u_n)_{\chi(0,\tau)} \right\rangle + \int_{Q_\tau} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dxdt \\ &= \int_{Q_\tau} T_n\left(\sigma_n(u_n)|\nabla\varphi_n|^2\right)T_k(u_n)dxdt,\end{aligned} \tag{23}$$

which implies that

$$\begin{aligned}&\int_\Omega S_k(u_n(\tau))dx + \int_{Q_\tau} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dxdt \\ &= \int_{Q_\tau} T_n\left(\sigma_n(u_n)|\nabla\varphi_n|^2\right)T_k(u_n)dxdt + \int_\Omega S_k(u_{0n})dx,\end{aligned} \tag{24}$$

where $S_k(t) = \displaystyle\int_0^t T_k(s)\,ds$.
Since $0 \le S_k(t) \le k|t|$ and by (20), we have

$$\int_{Q_T} M(|\nabla T_k(u_n)|) \le k\left[C + \|u_0\|_{L^1(\Omega)}\right].$$

*iii*) Due to Lemma 2.4 and (21), one has

$$\begin{aligned} M\left(\frac{k}{\gamma}\right) \text{meas}\,\{(x,t) \in Q_T : |u_n| > k\} &= \int_{\{|u_n|>k\}} M\left(\frac{|T_k(u_n)|}{\gamma}\right) dxdt \\ &\le \int_{Q_T} M(|\nabla T_k(u_n)|)\,dxdt \le kC_2, \end{aligned}$$

this implies that meas $\{(x,t) \in Q_T : |u_n| > k\} \le \frac{kC_2}{M\left(\frac{k}{\gamma}\right)}$, and we get (22). □

**Remark 5.3.** *From* (20) *and* (10), *we have*

$$\varphi_n \to \varphi \text{ weakly in } L^2(0,T;H_0^1(\Omega)). \tag{25}$$

**Proposition 5.4.** *Let* $(u_n, \varphi_n)$ *be a solution of the approximate problem* (17)-(19), *then we have the following properties:*

$$u_n \to u \text{ a.e in } Q_T, \tag{26}$$

$$(a(x,t,T_k(u_n), \nabla T_k(u_n)))_n \text{ is bounded in } (L_{\bar{M}}(Q_T))^N. \tag{27}$$

*Proof.* - To prove the almost every convergence of $u_n$, we consider $\zeta_k$ is a $C^2(\mathbb{R})$ non-decrasing function such that $\zeta_k(s) = s$, if $|s| \le \frac{k}{2}$ and $\zeta_k(s) = k$, if $|s| \ge k$. Multiplying the parabolic equation in (18) by $\zeta_k'(u_n)$, we get

$$\begin{aligned} \frac{\partial(\zeta_k(u_n))}{\partial t} &= \text{div}\left(a_n(x,t,u_n,\nabla u_n)\,\zeta_k'(u_n)\right) - a_n(x,t,u_n,\nabla u_n)\,\zeta_k''(u_n)\,\nabla u_n \\ &\quad + f_n \zeta_k'(u_n) \text{ in } \mathcal{D}'(Q_T). \end{aligned} \tag{28}$$

where $f_n = T_n(\sigma_n(u_n)|\nabla\varphi_n|^2)$.
Thanks to (21) and the fact that $\zeta_k'$ has compact support, implies that $\zeta_k(u_n)$ is bounded in $W_0^{1,x}L_M(Q_T)$ while its time derivative $\frac{\partial(\zeta_k(u_n))}{\partial t}$ is bounded in $L^1(Q_T) + W^{-1,x}L_{\bar{M}}(Q_T)$, hence Lemma 2.2 allows us to conclude that $\zeta_k(u_n)$ is compact in $L^1(Q_T)$. Due to the choice of $\zeta_k$, we conclude that for each $k$, the sequence $T_k(u_n)$ converges almost everywhere in $Q_T$, which implies that $u_n$ converges almost everywhere to some measurable function $u$ in $Q_T$. Therefore, we can see that there exists a measurable function $u$ such that for every $k > 0$ and a subsequence, still denoted $u_n$, we have

$$u_n \longrightarrow u \text{ a. e. in } Q_T,$$

and

$$T_k(u_n) \to T_k(u) \text{ weakly in } W_0^{1,x}L_M(Q_T) \text{ for } \sigma(\Pi L_M, \Pi E_{\bar{M}}), \text{ and strongly in } L^1(Q_T). \tag{29}$$

- We prove that $(a(x,t,T_k(u_n), \nabla T_k(u_n)))_n$ is a bounded in $L_{\bar{M}}(Q_T)^N$.

Let $\psi \in W_0^{1,x}E_M(Q_T)^N$ be arbitrary with $\|\nabla\psi\|_{(M)} = \dfrac{1}{k_2+1}$. Given the monotonicity of $a$, one easily has

$$\begin{aligned}
&\int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla\psi dxdt\\
&\leq \int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dxdt\\
&-\int_{Q_T} a(x,t,T_k(u_n),\nabla\psi)(\nabla T_k(u_n)-\nabla\psi)dxdt\\
&\leq C+\int_{Q_T}|a(x,t,T_k(u_n),\nabla\psi)||\nabla T_k(u_n)|dxdt+\int_{\Omega}|a(x,t,T_k(u_n),\nabla\psi)||\nabla\psi|dxdt.
\end{aligned} \tag{30}$$

For the first integral in the right side, we use the Young's inequality to have

$$\int_{Q_T}|a(x,t,T_k(u_n),\nabla\psi)||\nabla T_k(u_n)|dxdt \leq 3\nu\int_{Q_T}[\overline{M}\left(\frac{a(x,t,T_k(u_n),\nabla\psi)}{3\nu}\right)+M(|\nabla T_k(u_n)|)]dxdt,$$

using (5), we have

$$3\nu\overline{M}\left(\frac{a(x,t,T_k(u_n),\nabla\psi)}{3\nu}\right)\leq \nu(\overline{M}(a_0(x,t))+P(k_1T_k(u_n))+M(k_2\nabla\psi)),$$

since $(T_k(u_n))$ is bounded in $W^1_{0,x}L_M(Q_T)$, and owing to Poincare's inequality, there exist $\lambda>0$ such that $\int_{Q_T} M\left(\frac{T_k(u_n)}{\lambda}\right)dxdt \leq 1$ for all $n\in\mathbb{N}^*$. Also, since $P \ll M$, there exists $s_0>0$ such that $P(k_1 s)\leq P(k_1 s_0)+M\left(\frac{s}{\lambda}\right)$ for all $s\in\mathbb{R}$.
Consequently,

$$3\nu\int_{Q_T}\overline{M}\left(\frac{a(x,t,T_k(u_n),\nabla\psi)}{3\nu}\right)dxdt \leq \nu\int_{Q_T}(\overline{M}(a_0(x,t))+P(k_1T_k(u_n))+M(k_2\nabla\psi))dxdt\leq C,$$

and thus $\int_{Q_T}|a(x,T_k(u_n),\nabla\psi)||\nabla T_k(u_n)|dxdt \leq C$, for all $n\in\mathbb{N}^*$ and $\psi\in W_0^{1,x}E_M(Q_T)^N$ such that $\|\nabla\psi\|_{(M)} = \dfrac{1}{k_2+1}$. On the other hand, the second integral in (30), namely $\int_{Q_T}|a(x,t,T_k(u_n),\nabla\psi)||\nabla\psi|dxdt\leq C$ can be dealt with the same way, so that it is easy to check that it is also bounded. Gathering all these estimates, and using the dual norm, one easily deduces that

$$(a(x,t,T_k(u_n),\nabla T_k(u_n)))_n \quad \text{is bounded in} \quad L_{\overline{M}}(Q_T)^N. \tag{31}$$

Thus, up to a subsequence, still denoted $u_n$ in the same way, there exists $\phi_k\in(L_{\bar{M}}(Q_T))^N$ such that

$$a(x,t,T_k(u_n),\nabla T_k(u_n))\to\phi_k \text{ in } (L_{\bar{M}}(Q_T))^N \text{ for } \sigma(\Pi L_{\bar{M}},\Pi E_M). \tag{32}$$

□

**Step 3: Almost everywhere convergence of the gradients.**
This step is devoted to introducing, for $k\geq 0$ fixed, a time regularization $w^i_{\mu,j}$ of the function $T_k(u)$ and to establish the following proposition,

**Proposition 5.5.** *Let $(u_n,\varphi_n)$ be a solution of (18). Then for any $k>0$,*

$$\nabla u_n\to\nabla u \text{ a.e. in } Q_T, \tag{33}$$

$$a(x,t,T_k(u_n),\nabla T_k(u_n))\to a(x,t,T_k(u),\nabla T_k(u)) \text{ weakly in } (L_{\bar{M}}(Q_T))^N, \tag{34}$$

$$M(|\nabla T_k(u_n)|)\to M(|\nabla T_k(u)|) \text{ strongly in } L^1(Q_T). \tag{35}$$

*Proof.* This proof is devoted to introduce, for $k > 0$ fixed, a time regularization of the function $T_k(u_n)$ in order to perform the monotonicity method. This kind of regularization has been first introduced by R. Landes (See [23]). More recently, it has been exploited in [12] and [15] to solve some nonlinear evolution problems with $L^1$ or measure data.
Let $v_j \in D(Q_T)$ be a sequence such that $v_j \to u$ in $W_0^{1,x}L_M(Q_T)$ for the modular convergence and let $\psi_i \in D(\Omega)$ be a sequence that converges strongly to $u_0$ in $L^1(\Omega)$.
Let $w^i_{\mu,j} = T_k\left(v_j\right)_\mu + e^{-\mu t}T_k(\psi_i)$ where $T_k\left(v_j\right)_\mu$ is the mollification with respect to the time of $T_k\left(v_j\right)$. Note that $w^i_{\mu,j}$ is a smooth function having the following properties,

$$\frac{\partial w^i_{\mu,j}}{\partial t} = \mu\left(T_k\left(v_j\right) - w^i_{\mu,j}\right), \quad w^i_{\mu,j}(0) = T_k(\psi_i), \quad \left|w^i_{\mu,j}\right| \leq k \tag{36}$$

$$w^i_{\mu,j} \to T_k(u)_\mu + e^{-\mu t}T_k(\psi_i) \text{ in } W_0^{1,x}L_M(Q_T), \tag{37}$$

for the modular convergence as $j \to \infty$,

$$T_k(u)_\mu + e^{-\mu t}T_k(\psi_i) \to T_k(u) \text{ in } W_0^{1,x}L_M(Q_T). \tag{38}$$

for the modular convergence as $\mu \to \infty$.
Define now the function $l_m$ on $\mathbb{R}$, by

$$l_m(s) = \begin{cases} 1 & \text{if } |s| \leq m, \\ m+1-|s| & \text{if } m \leq |s| \leq m+1, \\ 0 & \text{if } |s| \geq m+1, \end{cases}$$

for any $m \geq k$. Using the admissible test function $\Phi^{\mu,i}_{n,j,m} = \left(T_k(u_n) - w^\mu_{i,j}\right)l_m(u_n)$ as a test function in the parabolic equation in (18) leads to

$$\begin{aligned} &\left\langle \frac{\partial u_n}{\partial t}, \varphi^{\mu,i}_{n,j,m}\right\rangle + \int_{Q_T} a(x,t,u_n,\nabla u_n)\left(\nabla T_k(u_n) - \nabla w^\mu_{i,j}\right)l_m(u_n)\,dxdt \\ &\quad + \int_{Q_T} a(x,t,u_n,\nabla u_n)\left(T_k(u_n) - w^\mu_{i,j}\right)\nabla u_n l'_m(u_n)\,dxdt = \int_{Q_T} f_n\Phi^{\mu,i}_{n,j,m}dxdt. \end{aligned} \tag{39}$$

Let $\varepsilon(n,j,\mu,i) > 0$ be a positive sequence such that

$$\lim_{i\to\infty}\lim_{\mu\to\infty}\lim_{j\to\infty}\lim_{n\to\infty} \varepsilon(n,j,\mu,i) = 0.$$

The definition of the sequence $w^\mu_{i,j}$ makes it possible to establish the following lemma.

**Lemma 5.6.** *Let $\Phi^{\mu,i}_{n,j,m} = \left(T_k(u_n) - w^\mu_{i,j}\right)l_m(u_n)$. For any $k \geq 0$, we have*

$$\left\langle \frac{\partial u_n}{\partial t}, \Phi^{\mu,i}_{n,j,m}\right\rangle \geq \varepsilon(n,j,\mu,i). \tag{40}$$

*Proof.* See [6]. □

Now we turn to complete the proof of Proposition 5.5. First, it is easy to see that

$$\int_{Q_T} f_n\Phi^{\mu,i}_{n,j,m}dxdt = \varepsilon(n,j,\mu). \tag{41}$$

Indeed, by the almost everywhere convergence of $u_n$, we have that $\left(T_k(u_n) - w_{i,j}^{\mu}\right) l_m(u_n)$ converges to $\left(T_k(u) - w_{i,j}^{\mu}\right) l_m(u)$ in $L^\infty(Q_T)$ weak-*, and then

$$\int_{Q_T} f_n \left(T_k(u_n) - w_{i,j}^{\mu}\right) l_m(u_n)\, dxdt \to \int_{Q_T} f_n \left(T_k(u) - w_{i,j}^{\mu}\right) l_m(u) dxdt,$$

So that

$$\left(T_k(u) - w_{i,j}^{\mu}\right) l_m(u) \to \left(T_k(u) - T_k(u)_\mu - e^{-\mu t} T_k(\psi_i)\right) \text{ in } L^\infty(Q_T) \text{ weak-* as } j \to \infty.$$

Then, we deduce that

$$\int_{Q_T} f_n \left(T_k(u_n) - w_{i,j}^{\mu}\right) l_m(u_n)\, dxdt = \varepsilon(n, j, \mu), \tag{42}$$

Concerning the third term of the right-hand side of (39), we obtain

$$\int_{\{m \le |u_n| \le m+1\}} a(x,t,u_n,\nabla u_n) \nabla u_n l'_m(u_n) \left(T_k(u_n) - w_{i,j}^{\mu}\right) dxdt \le 2k \int_{\{m \le |u_n| \le m+1\}} a(x,t,u_n,\nabla u_n) \nabla u_n dxdt. \tag{43}$$

Then, since $(a(x,t,T_{m+1}(u_n), \nabla T_{m+1}(u_n)))$ is bounded in $L_{\overline{M}}(Q_T)^N$, we deduce

$$\int_{\{m \le |u_n| \le m+1\}} a(x,t,u_n,\nabla u_n) \nabla u_n l'_m(u_n) \left(T_k(u_n) - w_{i,j}^{\mu}\right) dxdt \le \varepsilon(n, \mu, m). \tag{44}$$

Finally, by means of (40)–(44), we obtain

$$\int_{Q_T} a(x,t,u_n,\nabla u_n) \nabla u_n l'_m(u_n) \left(T_k(u_n) - w_{i,j}^{\mu}\right) l_m(u_n)\, dxdt \le \varepsilon(n, j, \mu, m), \tag{45}$$

we able to write

$$\begin{aligned}
&\int_{Q_T} a(x,t,u_n,\nabla u_n) \left(\nabla T_k(u_n) - \nabla w_{i,j}^{\mu}\right) l_m(u_n)\, dxdt \\
&\quad = \int_{Q_T} a(x,t,T_k(u_n), \nabla T_k(u_n)) \left(\nabla T_k(u_n) - \nabla w_{i,j}^{\mu}\right) l_m(u_n)\, dxdt \\
&\quad - \int_{\{|u_n| > k\}} a(x,t,u_n,\nabla u_n) \nabla w_{i,j}^{\mu} l_m(u_n)\, dxdt,
\end{aligned}$$

since $l_m(u_n) = 0$ if $|u_n| \ge m+1$, one has

$$\begin{aligned}
&\int_{Q_T} a(x,t,u_n,\nabla u_n) \left(\nabla T_k(u_n) - \nabla w_{i,j}^{\mu}\right) l_m(u_n)\, dxdt \\
&\quad = \int_{Q_T} a(x,t,T_k(u_n), \nabla T_k(u_n)) \left(\nabla T_k(u_n) - \nabla w_{i,j}^{\mu}\right) l_m(u_n)\, dxdt \\
&\quad - \int_{\{|u_n| > k\}} a(x,t,T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla w_{i,j}^{\mu} l_m(u_n)\, dxdt = I_1 + I_2.
\end{aligned} \tag{46}$$

We pass to the limit in (46). Since $a(x,t,T_{m+1}(u_n), \nabla T_{m+1}(u_n))_n$ is bounded in $(L_{\bar{M}}(Q_T))^N$, we have $a(x,t,T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \to \phi_m$ weakly in $L_{\bar{M}}(Q_T)$ in $\sigma(\Pi L_{\bar{M}}, \Pi E_M)$ as $n$ tends to infinity.
Since $\nabla w_{i,j}^{\mu} l_m(u_n)_{\chi_{\{|u_n|>k\}}}$ converges to $\nabla w_{i,j}^{\mu} l_m(u)_{\chi_{\{|u|>k\}}}$ strongly in $E_M(\Omega)$ as $n$ tends to infinity, it follows that

$$I_2 = \int_{Q_T} \phi_m \nabla w_{i,j}^{\mu} l_m(u)_{\chi_{\{|u|>k\}}} dxdt + \varepsilon(n).$$

By letting $j \to \infty$, we get

$$I_2 = \int_{Q_T} \phi_m \left(\nabla T_k(u)_\mu - e^{-\mu t} \nabla T_k(\psi_i)\right) l_m(u) \chi_{\{|u|>k\}} dxdt + \varepsilon(n, j),$$

which, by letting $\mu \to \infty$, implies that

$$I_2 = \int_{Q_T} \phi_m \nabla T_k(u) l_m(u) \chi_{\{|u|>k\}} dxdt + \varepsilon(n, j, \mu).$$

One can easily show that

$$\begin{aligned} I_1 &= \int_Q \left[a(x,t,T_k(u_n), \nabla T_k(u_n)) - a\left(x,t,T_k(u_n), \nabla T_k(v_j)\chi_j^s\right)\right] . \left[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\right] l_m(u_n)\, dxdt \\ &+ \int_{Q_T} a\left(x,t,T_k(u_n), \nabla T_k(v_j)\chi_j^s\right)\left[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\right] l_m(u_n)\, dxdt \\ &+ \int_{Q_T} a(x,t,T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j)\chi_j^s l_m(u_n)\, dxdt \\ &- \int_{Q_T} a(x,t,T_k(u_n), \nabla T_k(u_n)) \nabla w_{i,j}^\mu l_m(u_n)\, dxdt = J_1 + J_2 + J_3 + J_4, \end{aligned} \tag{47}$$

where $\chi_j^s$ denotes the characteristic function of the subset $\Omega_s^j = \left\{(x,t) \in Q_T : |\nabla T_k(v_j)| \leq s\right\}$.
Starting with $J_2$, observe that

$$\begin{aligned} J_2 &= \int_{Q_T} a\left(x,t,T_k(u_n), \nabla T_k(v_j)\chi_j^s\right) \nabla T_k(u_n) l_m(u_n)\, dxdt \\ &\quad - \int_{Q_T} a\left(x,t,T_k(u_n), \nabla T_k(v_j)\chi_j^s\right) \nabla T_k(v_j)\chi_j^s l_m(u_n)\, dxdt. \end{aligned}$$

Since $a\left(x,t,T_k(u_n), \nabla T_k(v_j)\chi_j^s\right) l_m(u_n) \to a\left(x,t,T_k(u), \nabla T_k(v_j)\chi_j^s\right) l_m(u)$ strongly in $(E_{\bar{M}})^N$ and $\nabla T_k(u_n) \to \nabla T_k(u)$ weakly in $(L_M(Q_T))^N$ for $\sigma(\Pi L_M, \Pi E_{\bar{M}})$. Moreover, it is easy to show that

$$\int_{Q_T} a\left(x,t,T_k(u_n), \nabla T_k(v_j)\chi_j^s\right) \nabla T_k(v_j)\chi_j^s l_m(u_n)\, dxdt \to \int_{Q_T} a\left(x,t,T_k(u), \nabla T_k(v_j)\chi_j^s\right) \nabla T_k(v_j)\chi_j^s l_m(u) dxdt,$$

as $n$ tends to $+\infty$. We get

$$J_2 = \int_{Q_T} a\left(x,t,T_k(u), \nabla T_k(v_j)\chi_j^s\right)\left[\nabla T_k(u) - \nabla T_k(v_j)\chi_j^s\right] l_m(u) dxdt + \varepsilon(n),$$

since $\nabla T_k(v_j)\chi_j^s l_m(u) \to \nabla T_k(u)\chi^s l_m(u)$ strongly in $(E_M(Q_T))^N$ as $j \to \infty$ and
$a\left(x,t,T_k(u), \nabla T_k(v_j)\chi_j^s\right) \to a(x,t,T_k(u), \nabla T_k(u)\chi^s)$ strongly in $(L_{\bar{M}}(Q_T))^N$ as $j$ goes to $\infty$, we have

$$J_2 = \varepsilon(n, j). \tag{48}$$

By letting $n \to \infty$ and since $a(x,t,T_k(u_n), \nabla T_k(u_n)) \to \phi_k$ weakly in $(L_{\bar{M}}(Q_T))^N$ and $l_m(u_n) = 1$ in $\{(x,t) \in Q_T : |u_n| \leq k\}$, we have

$$J_3 = \int_{Q_T} \phi_k \nabla T_k(v_j)\chi_j^s dxdt + \varepsilon(n),$$

which gives

$$J_3 = \int_{Q_T} \phi_k \nabla T_k(u) \chi_s dxdt + \varepsilon(n, j), \tag{49}$$

by letting $j \to \infty$.
Concerning $J_4$, we can write

$$J_4 = -\int_{Q_T} \phi_k \nabla w^{\mu}_{i,j} \mu l_m(u) dxdt + \varepsilon(n), \tag{50}$$

which implies that, by letting $j \to \infty$,

$$J_4 = \int_{Q_T} \phi_k \left[\nabla T_k(u) - e^{-\mu t} \nabla T_k(u)\right] dxdt + \varepsilon(n, j), \tag{51}$$

by letting $\mu \to \infty$, we obtain

$$J_4 = -\int_{Q_T} \phi_k \nabla T_k(u) dxdt + \varepsilon(n, j, \mu, s). \tag{52}$$

We conclude that

$$\begin{aligned}
&\int_{Q_T} a(x, t, T_k(u_n), \nabla T_k(u_n)) \left[\nabla T_k(u_n) - \nabla w^{\mu}_{i,j}\right] l_m(u_n) dxdt \\
&\quad = \int_{Q_T} \left[a(x, t, T_k(u_n), \nabla T_k(u_n)) - a\left(x, t, T_k(u_n), \nabla T_k\left(v_j\right) \chi^s_j\right)\right] \\
&\qquad \times \left[\nabla T_k(u_n) - \nabla T_k\left(v_j\right) \chi^s_j\right] l_m(u_n) dxdt + \varepsilon(n, j, \mu, s).
\end{aligned} \tag{53}$$

Now, observe that

$$\begin{aligned}
&\int_{Q_T} [a(x, t, T_k(u_n), \nabla T_k(u_n)) - a(x, t, T_k(u_n), \nabla T_k(u)\chi_s)] . [\nabla T_k(u_n) - \nabla T_k(u)\chi^s] l_m(u_n) dxdt \\
&= \int_{Q_T} \left[a(x, t, T_k(u_n), \nabla T_k(u_n)) - a\left(x, t, T_k(u_n), \nabla T_k\left(v_j\right) \chi^s_j\right)\right] . [\nabla T_k(u_n) - \nabla T_k\left(v_j\right) \chi^s_j] l_m(u_n) dxdt \\
&+ \int_{Q_T} a\left(x, t, T_k(u_n), \nabla T_k\left(v_j\right) \chi^s_j\right) \left[\nabla T_k(u_n) - \nabla T_k\left(v_j\right) \chi^s_j\right] l_m(u_n) dxdt \\
&- \int_{Q_T} a(x, t, T_k(u_n), \nabla T_k(u)\chi^s) [\nabla T_k(u_n) - \nabla T_k(u)\chi^s] l_m(u_n) dxdt \\
&+ \int_{Q_T} a(x, t, T_k(u_n), \nabla T_k(u_n)) \left[\nabla T_k\left(v_j\right) \chi^s_j - \nabla T_k(u)\chi^s\right] l_m(u_n) dxdt.
\end{aligned}$$

Passing to the limit in $n$ and $j$, in the last three terms on the right-hand side of the last equality, we get

$$\begin{aligned}
&\int_{Q_T} a\left(x, t, T_k(u_n), \nabla T_k\left(v_j\right) \chi^s_j\right) \left[\nabla T_k(u_n) - \nabla T_k\left(v_j\right) \chi^s_j\right] l_m(u_n) dxdt \\
&\quad - \int_{Q_T} a(x, t, T_k(u_n), \nabla T_k(u)\chi_s) [\nabla T_k(u_n) - \nabla T_k(u)\chi^s] l_m(u_n) dxdt = \varepsilon(n, j),
\end{aligned}$$

and

$$\int_{Q_T} a(x, t, T_k(u_n), \nabla T_k(u_n)) \left[\nabla T_k\left(v_j\right) \chi^s_j - \nabla T_k(u)\chi^s\right] l_m(u_n) dxdt = \varepsilon(n, j). \tag{54}$$

This implies that

$$\begin{aligned}
&\int_{Q_T} [a(x, t, T_k(u_n), \nabla T_k(u_n)) - a(x, t, T_k(u_n), \nabla T_k(u)\chi^s)] . [\nabla T_k(u_n) - \nabla T_k(u)\chi^s] l_m(u_n) dxdt \\
&= \int_{Q_T} \left[a(x, t, T_k(u_n), \nabla T_k(u_n)) - a\left(x, t, T_k(u_n), \nabla T_k\left(v_j\right) \chi^s_j\right)\right] \cdot \left[\nabla T_k(u_n) - \nabla T_k\left(v_j\right) \chi^s_j\right] l_m(u_n) dxdt + \varepsilon(n, j).
\end{aligned}$$

(55)

On the other hand, we have

$$
\begin{aligned}
&\int_{Q_T} [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)] . [\nabla T_k(u_n) - \nabla T_k(u)\chi^s]\,dxdt \\
&= \int_{Q_T} [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)] . [\nabla T_k(u_n) - \nabla T_k(u)\chi^s]\, l_m(u_n)\,dxdt \\
&+ \int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n)) [\nabla T_k(u_n) - \nabla T_k(u)\chi^s] (1 - l_m(u_n))\,dxdt \\
&- \int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u)\chi^s) [\nabla T_k(u_n) - \nabla T_k(u)\chi_s] (1 - l_m(u_n))\,dxdt,
\end{aligned}
\tag{56}
$$

since $l_m(u_n) = 1$ in $\{|u_n| \leq m\}$ and $\{|u_n| \leq k\} \subset \{|u_n| \leq m\}$ for $m$ large enough, we deduce from (56) that

$$
\begin{aligned}
&\int_{Q_T} [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)] . [\nabla T_k(u_n) - \nabla T_k(u)\chi^s]\,dxdt \\
&= \int_{Q_T} [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)] . [\nabla T_k(u_n) - \nabla T_k(u)\chi^s]\,dxdt \\
&+ \int_{\{|u_n|>k\}} a(x,t,T_k(u_n),\nabla T_k(u)\chi^s) \cdot \nabla T_k(u)\chi^s (1 - l_m(u_n))\,dxdt.
\end{aligned}
$$

It is easy to see that the last term of the last equality tends to zero as $n \to +\infty$, which implies that

$$
\begin{aligned}
&\int_{Q_T} [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)] [\nabla T_k(u_n) - \nabla T_k(u)\chi^s] \\
&= \int_{Q_T} [a(x,t,T_k(u),\nabla T_k(u)) - a(x,t,T_k(u),\nabla T_k(u)\chi_s)] [\nabla T_k(u) - \nabla T_k(u)\chi^s]\, l_m(u_n)\,dxdt + \varepsilon(n,j).
\end{aligned}
$$

Combining (38), (47), (48), (49), (52) and (56), we obtain

$$
\int_{Q_T} [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)] . [\nabla T_k(u_n) - \nabla T_k(u)\chi^s]\,dxdt \leq \varepsilon(n,j,\mu,m,s). \tag{57}
$$

Pass to the limit in (57) as $n$, $j$, $m$, $s$ tend to infinity, we get

$$
\lim_{s\to\infty}\lim_{n\to\infty} \int_{Q_T} [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi^s)] . [\nabla T_k(u_n) - \nabla T_k(u)\chi^s]\,dxdt = 0. \tag{58}
$$

This implies by Lemma 3.3, the desired statement and therfore the proof of Proposition 5.5 is achieved.
**Step 4: Strong convergence of $\sigma_n(u_n)^{1/2}\nabla\varphi_n$ in $L^2(Q_T)^N$.**
In this step, we turn our attention to $(\varphi_n)$ and $\varphi$.

**Proposition 5.7.** *Let $(u_n, \varphi_n)$ be a solution of* (18)*, then we have*

$$
\sigma_n(u_n)^{1/2}\nabla\varphi_n \to \sigma(u)^{1/2}\nabla\varphi \text{ strongly in } L^2(Q_T)^N. \tag{59}
$$

*Proof.* First of all, we show that

$$
\sigma_n(u_n)^{1/2}\nabla\varphi_n \to \sigma(u)^{1/2}\nabla\varphi \text{ weakly in } L^2(Q_T)^N. \tag{60}
$$

Indeed, from (20), there exists a subsequence, noted still $(u_n, \varphi_n)$ and $\Phi \in L^2(Q_T)^N$, such that

$$
\sigma_n(u_n)^{1/2}\nabla\varphi_n \to \Phi \text{ weakly in } L^2(Q_T)^N, \tag{61}
$$

Using (26) and (10), it yields

$$\sigma_n(u_n)^{-1/2} \to \sigma(u)^{-1/2} \text{ weakly-* in } L^\infty(Q_T) \text{ and a.e. in } Q_T. \tag{62}$$

Putting

$$\nabla\varphi_n = \sigma_n(u_n)^{-1/2} \times \sigma_n(u_n)^{1/2} \times \nabla\varphi_n, \tag{63}$$

and passing to the limit, by gathering (61)-(63), we get $\Phi = \sigma(u)^{1/2}\nabla\varphi$, and this shows the statement (60). Notice that, in particular, $\sigma(u)|\nabla\varphi|^2 \in L^1(Q_T)$.
From (60), it is enough to show that

$$\int_{Q_T} \sigma_n(u_n) \left|\nabla\varphi_n\right|^2 dxdt \to \int_{Q_T} \sigma(u)|\nabla\varphi|^2 dxdt. \tag{64}$$

To do this, we first introduce the function $S_k \in W^{1,\infty}(\mathbb{R})$, $k > 0$, defined as

$$S_k(s) = \begin{cases} 1 & \text{if } |s| \leq k, \\ \frac{2k-|s|}{k} & \text{if } k < |s| \leq 2k, \\ 0 & \text{if } |s| > 2k. \end{cases} \tag{65}$$

Note that $\operatorname{supp}(S_k) = [-2k, 2k]$ and $S_k' = \frac{1}{k}\left(\chi_{(-2k,-k)} - \chi_{(k,2k)}\right)$. Next, we take $\psi = S_k(u_n)\, T_M(\varphi) \in L^\infty\left(0, T; H_0^1(\Omega)\right)$ as a test function in (19). Upon integrating over $(0, T)$, we obtain

$$\begin{aligned} &\int_{Q_T} \sigma_n(u_n)\nabla\varphi_n\nabla T_M(\varphi)S_k(u_n)\,dxdt + \int_{Q_T} \sigma_n(u_n)\nabla\varphi_n\nabla u_n S_k'(u_n)\,T_M(\varphi)dxdt \\ &= -\int_{Q_T} F_n(u_n)\nabla T_M(\varphi)S_k(u_n)\,dxdt - \int_{Q_T} F_n(u_n)\nabla u_n S_k'(u_n)\,T_M(\varphi)dxdt. \end{aligned}$$

The terms of this equality are denoted by $L_1 - L_4$ respectively, and are examined independently.
For $L_1$: As $\sigma_n(u_n)\,S_k(u_n) = \sigma_n(T_{2k}(u_n))\,S_k(u_n) \in L^\infty(Q_T)$ and is bounded in this space, using (26) it yields

$$\sigma_n(u_n)\,S_k(u_n) \to \sigma(u)S_k(u) \text{ weakly-* in } L^\infty(Q_T) \text{ and a.e. in } Q_T$$

Making $n \to \infty$ from (25), we obtain

$$\int_{Q_T} \sigma_n(u_n)\nabla\varphi_n\nabla T_M(\varphi)S_k(u_n)\,dxdt \to \int_{Q_T} \sigma(u)\nabla\varphi T_M(\varphi)S_k(u)dxdt.$$

Owing to Lebesgue's theorem, we deduce

$$\lim_{M\to\infty}\lim_{k\to\infty}\lim_{n\to\infty}\int_{Q_T} \sigma_n(u_n)\nabla\varphi_n\nabla T_M(\varphi)S_k(u_n)\,dxdt = \int_{Q_T} \sigma(u)|\nabla\varphi|^2 dxdt.$$

For $L_2$: We start by obtaining a new estimate for $(u_n)$. Let $H_k \in W^{1,\infty}(\mathbb{R})$ be the function

$$H_k(s) = \begin{cases} 0 & \text{if } |s| \leq k, \\ \frac{|s|-k}{k} & \text{if } k < |s| \leq 2k, \\ \frac{|s|}{s} & \text{if } |s| > 2k, \end{cases}$$

by taking $\psi = H_k(u_n)$ as a test function in (18), we obtain

$$\int_\Omega \tilde{H}_k(u_n(T))\,dx + \frac{1}{k}\int_{E_n^k} a_n(x, t, u_n, \nabla u_n)\nabla u_n dxdt = \int_{Q_T} f_n H_k(u_n)\,dxdt + \int_\Omega \tilde{H}_k(T_n(u_0))\,dx,$$

where $\tilde{H}_k(s) = \int_0^s H_k(\tau)\mathrm{d}\tau$ and $E_n^k = \{k < |u_n| < 2k\}$.
we can easily prove that there exists a constant $C$ such that

$$\frac{1}{k}\int_{Q_T} a_n\left(x,t,u_n,\nabla u_n\right)\nabla u_n \chi_{E_n^k} dxdt \leq C.$$

that is,

$$\left(\left(\frac{1}{k}a_n\left(x,t,u_n,\nabla u_n\right)\nabla u_n \chi_{E_n^k}\right)^{1/2}\right) \text{ is bounded in } L^2(Q_T)^N. \tag{66}$$

Going back to $L_2$,

$$L_2 = \int_{Q_T} \sigma_n(u_n)^{1/2}\nabla\varphi_n \sigma_n(u_n)^{1/2}\left(a_n(x,t,u_n,\nabla u_n)\nabla u_n\right)^{-1/2} \times \left(a_n(x,t,u_n,\nabla u_n)\nabla u_n\right)^{1/2}\nabla u_n S_k'\left(u_j\right)T_M(\varphi)dxdt.$$

Thus

$$\begin{aligned}|L_2| &\leq M\int_{Q_T} |\sigma_n(u_n)^{1/2}\nabla\varphi_n\left(\frac{1}{k}a_n\left(x,t,u_n,\nabla u_n\right)\nabla u_n\chi_{E_n^k}\right)^{1/2} \\ &\quad\times\left(\frac{\sigma_n(u_n)}{ka_n\left(x,t,u_n,\nabla u_n\right)\nabla u_n\chi_{E_n^k}}\right)^{1/2}\nabla u_n\chi_{E_n^k}| \\ &\leq M\left\|\sigma_n(u_n)^{1/2}\nabla\varphi_n\right\|_{L^2(Q_T)}\cdot\left\|\left(\frac{1}{k}a_n\left(x,t,u_n,\nabla u_n\right)\nabla u_n\chi_{E_n^k}\right)^{1/2}\right\|_{L^2(Q_T)} \\ &\quad\times\left\|\left(\frac{\sigma_n(u_n)}{ka_n\left(x,t,u_n,\nabla u_n\right)\nabla u_n\chi_{E_n^k}}\right)^{1/2}\nabla u_n\chi_{E_n^k}\right\|_{L^\infty(Q_T)}.\end{aligned}$$

Using the first estimate in Lemma 5.2, (12), and (66), we deduce

$$|L_2| \leq C\omega(k),$$

which implies

$$\lim_{k\to\infty}\limsup_{n\to\infty}\int_{Q_T}\sigma_n\left(u_n\right)\nabla\varphi_n\nabla u_n S_k'\left(u_n\right)T_M(\varphi)dxdt = 0.$$

For $L_3$: Lebesgue's theorem easily shows that

$$\lim_{n\to\infty}\int_{Q_T} F_n\left(u_n\right)\nabla T_M(\varphi)S_k\left(u_n\right)dxdt = \int_{Q_T} F(u)\nabla T_M(\varphi)S_k(u)dxdt.$$

We now express this last integral as

$$\int_{Q_T} F(u)\sigma(u)^{-1/2}\sigma(u)^{1/2}\nabla T_M(\varphi)S_k(u)dxdt,$$

owing to (11), (61), and reapplying Lebesgue's theorem, first in $k$, then in $M$, to deduct that

$$\lim_{M\to\infty}\lim_{k\to\infty}\lim_{n\to\infty}\int_{Q_T} F_n\left(u_n\right)\nabla T_M(\varphi)S_k\left(u_n\right)dxdt = \int_{Q_T} F(u)\nabla\varphi dxdt. \tag{67}$$

For $L_4$: Applying the same techniques as in $L_2$ and $L_3$, it is obvious that

$$\lim_{k\to\infty}\limsup_{n\to\infty}\int_{Q_T} F_n\left(u_n\right)\nabla u_n S_k'\left(u_n\right)T_M(\varphi)dxdt = 0.$$

Gathering (63)–(67),

$$\int_{Q_T} \sigma(u)|\nabla\varphi|^2 dxdt = -\int_{Q_T} F(u)\nabla\varphi dxdt. \tag{68}$$

On the other hand, taking $\psi = \varphi_n$ in (19) and integrating over $(0, T)$, we obtain

$$\int_{Q_T} \sigma_n(u_n)\left|\nabla\varphi_n\right|^2 dxdt = -\int_{Q_T} F_n(u_n)\nabla\varphi_n dxdt,$$

since $F_n(u_n)\nabla\varphi_n = F_n(u_n)\sigma_n(u_n)^{-1/2}\sigma(u_n)^{1/2}\nabla\varphi_n$, and bearing in mind (10), (26) and (61), we conclude that

$$\int_{Q_T} F_n(u_n)\nabla\varphi_n dxdt \to \int_{Q_T} F(u)\nabla\varphi dxdt, \tag{69}$$

Combining (68)-(69) results in (60), which is $,\sigma_n(u_n)^{1/2}\nabla\varphi_n \to \sigma(u)^{1/2}\nabla\varphi$ strongly in $L^2(Q_T)^N$. This also implies that

$$f_n = T_n\left(\sigma_n(u_n)\left|\nabla\varphi_n\right|^2\right) \to \sigma(u)|\nabla\varphi|^2 \text{ strongly in } L^1(Q_T). \tag{70}$$

□

**Step 5: Proving that $u$ satisfies $(R_3)$.**

**Lemma 5.8.** *The limit $u$ of $u_n$ satisfies*

$$\lim_{m\to+\infty}\int_{\{m\le|u|\le m+1\}} a(x,t,u,\nabla u)\nabla u dxdt = 0. \tag{71}$$

*Proof.* Taking $T_1(u_n - T_m(u_n))$ as a test function in (18), we obtain

$$\left\langle \frac{\partial u_n}{\partial t}, T_1(u_n - T_m(u_n))\right\rangle + \int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt = \int_Q f_n T_1(u_n - T_m(u_n))\,dxdt, \tag{72}$$

$$\begin{aligned}\int_\Omega U_m(u_n(T))\,dx &+ \int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt\\ &\le \int_Q \left|f_n T_1(u_n - T_m(u_n))\right| dxdt + \int_\Omega U_m(u_{0n})\,dx,\end{aligned} \tag{73}$$

where $U_m^n(r) = \displaystyle\int_0^r \frac{\partial u_n}{\partial t} T_1(s - T_m(s))\,ds$.

In order to pass to the limit, as $n$ tends to $+\infty$, in (73), we use $U_m^n(u_n(T)) \ge 0$, (16), (26) and (70), we obtain that

$$\begin{aligned}\lim_{n\to+\infty}&\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt\\ &\le \int_{\{|u|>m\}} \sigma(u)|\nabla\varphi|^2 dxdt + \int_{\{|u_0|>m\}} |u_0|\,dx.\end{aligned} \tag{74}$$

Finally, by (14) and letting $m$ to $+\infty$, we get

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n dxdt = 0. \tag{75}$$

To this end, observe that for any fixed $m \geq 0$, we get

$$\begin{aligned}\int_{\{m\leq|u_n|\leq m+1\}} a\left(x,t,u_n,\nabla u_n\right)\nabla u_n dxdt &= \int_{Q_T} a\left(x,t,u_n,\nabla u_n\right)\left[\nabla T_{m+1}\left(u_n\right)-\nabla T_m\left(u_n\right)\right]dxdt\\ &= \int_{Q_T} a\left(x,t,T_{m+1}\left(u_n\right),\nabla T_{m+1}\left(u_n\right)\right)\nabla T_{m+1}\left(u_n\right)dxdt\\ &- \int_{Q_T} a\left(x,t,T_m\left(u_n\right),\nabla T_m\left(u_n\right)\right)\nabla T_m\left(u_n\right)dxdt.\end{aligned}$$

Under Proposition 5.5, one can pass to the limit as $n$ tends to $+\infty$ for fixed $m \geq 0$, to obtain

$$\begin{aligned}\lim_{n\to+\infty}\int_{\{m\leq|u_n|\leq m+1\}} a\left(x,t,u_n,\nabla u_n\right)\nabla u_n dxdt &= \int_{Q_T} a\left(x,t,T_{m+1}(u),\nabla T_{m+1}(u)\right)\nabla T_{m+1}(u)dxdt\\ &- \int_{Q_T} a\left(x,t,T_m(u),\nabla T_m(u)\right)\nabla T_m(u)dxdt\\ &= \int_{\{m\leq|u|\leq m+1\}} a(x,t,u,\nabla u)\nabla u dxdt.\end{aligned}\tag{76}$$

Taking the limit, as $m$ tends to $+\infty$, in (76) and using the estimate (75), it possible to conclude that (71) holds true, and the proof of Lemma 5.8 is complete. □

**Step 6: Showing that $u$ satisfies $(R_4)$.**
In this step, $u_n$ is shown to satisfy (77). Let $S$ be a function in $C^\infty(\mathbb{R})$ such that $S'$ has a compact support. Let $k$ be a positive real number such that supp($S'$)$\subset [-k,k]$. Pointwise multiplication of the parabolic equation in (18), by $S'\left(u_n\right)$, leads to

$$\tfrac{\partial S(u_n)}{\partial t} - \operatorname{div}\left(S'\left(u_n\right)a_n\left(x,t,u_n,\nabla u_n\right)\right) + S''\left(u_n\right)a_n\left(x,t,u_n,\nabla u_n\right)\nabla u_n = f_n S'\left(u_n\right).\tag{77}$$

In what follows, we pass to the limit as $n$ tends to $+\infty$ in each term of (77),

- since $S'$ is bounded, and $S\left(u_n\right)$ converges to $S(u)$ a.e. in $Q_T$ and in $L^\infty\left(Q_T\right)$ weak-∗. Then $\frac{\partial S(u_n)}{\partial t}$ converges to $\frac{\partial S(u)}{\partial t}$ in $\mathcal{D}'\left(Q_T\right)$ as $n$ tends to $+\infty$.
- Since supp($S$)$\subset [-k,k]$, we have $S'\left(u_n\right)a_n\left(x,t,u_n,\nabla u_n\right) = S'\left(u_n\right)a_n\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)$ a.e. in $Q_T$.
  The pointwise convergence of $u_n$ to $u$ as $n$ tends to $+\infty$, the bounded character of $S''$, Proposition 5.2 and (34) of Proposition 5.5 imply that, as $n$ tends to $+\infty$,

  $$S'\left(u_n\right)a_n\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right) \to S'(u)a\left(x,t,T_k(u),\nabla T_k(u)\right),$$

  weakly in $\left(L_{\bar{M}}\left(Q_T\right)\right)^N$, for $\sigma\left(\Pi L_{\bar{M}},\Pi E_M\right)$ because $S(u)=0$ for $|u|\geq k$ a. e. in $Q_T$.
  And the term $S'(u)a\left(x,t,T_k(u),\nabla T_k(u)\right) = S'(u)a(x,t,u,\nabla u)$ a. e. in $Q_T$.
- Since supp($S'$)$\subset [-k,k]$, we have

  $$S''\left(u_n\right)a_n\left(x,t,u_n,\nabla u_n\right)\nabla u_n = S''\left(u_n\right)a_n\left(x,t,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\nabla T_k\left(u_n\right) \text{ a. e. in } Q_T.$$

  The pointwise convergence of $S''\left(u_n\right)$ to $S''(u)$ as $n$ tends to $+\infty$, the bounded character of $S''$, Proposition 5.2 and (34) imply that, as $n$ tends to $+\infty$

  $$S'\left(u_n\right)a_n\left(x,t,u_n,\nabla u_n\right)\nabla u_n \to S'(u)a\left(x,t,T_k(u),\nabla T_k(u)\right)\nabla T_k(u) \text{ weakly in } L^1\left(Q_T\right),$$

  and

  $$S''(u)a\left(x,t,T_k(u),\nabla T_k(u)\right)\nabla T_k(u) = S''(u)a(x,t,u,\nabla)\nabla u \text{ a. e. in } Q_T.$$

- Due to (70) and (26), $f_n S'(u_n)$ converges to $fS'(u)$ strongly in $L^1(Q_T)$, as $n$ tends to $+\infty$.

The above convergence result allows us to pass to the limit, as $n$ tends to $+\infty$, in equation (77) and to conclude that $u$ satisfies $(R_4)$. Remark that, $S'$ has a compact support, implies that $S(u_n)$ is bounded in $L^\infty(Q_T)$. By (77) and the above considerations on the behavior of the terms of this equation show that $\frac{\partial S(u_n)}{\partial t}$ is bounded in $L^1(Q_T) + W^{-1,x}L_M(Q_T)$. A consequence, an Aubin's type Lemma (See e.g., [27], Corollary 4) (See also [19]) implies that $S(u_n)(t=0)$ lies in a compact set of $C^0\left([0,T];L^1(\Omega)\right)$. As a result, $S(u_n)(t=0)$ converges to $S(u)(t=0)$ strongly in $L^1(\Omega)$. Due to (16), we conclude that $S(u_n)(t=0) = S(u_n(x,0))$ converges to $S(u)(t=0)$ strongly in $L^1(\Omega)$. Then, we state that $S(u)(t=0) = S(u_0)$ in $\Omega$.
Finally, in order to show $(R_5)$, we just take $\psi = S_k(u_n)T_M(\phi)$ in (19), where $S_k$ is defined in (65) and $\phi \in L^2\left(0,T;H_0^1(\Omega)\right)$ is such that $\sigma(u)|\nabla\phi|^2 \in L^1(Q_T)$. Therefore, we can proceed as in $L_1 - L_4$ above: taking the iterate limits, in $n$, $k$ and in $M$ respectively, and the last expression, allows us to have $(R_5)$. □

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