Filomat 36:16 (2022), 5573–5589
https://doi.org/10.2298/FIL2216573S

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Well-Posedness and Ulam's Stability of Functional Equations in $\mathcal{F}$-Metric Space with an Application

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Abstract. In this paper, we consider a fixed point problem related to some contraction mappings and introduce new classes of Picard operators for such mappings in the framework of $\mathcal{F}$-metric space, yielding some interesting and novel results. As application of the obtained results, we investigate the Hyers-Ulam stability of a fixed point problem, a Cauchy functional equation, and an integral equation. Also, we present the well-posedness of the fixed point problem and integral equation. Some illustrative examples are also provided to support the new findings.

1. Introduction

In the theory of Ulam's stability, one can find the efficient tools to evaluate the errors, that is, to study the existence of an exact solution of the perturbed functional equation which is not far from the given function (see [10, 13, 24] and references cited therein). In the subject of analysis, the study of solutions and stability results of functional equations is a popular topic. In nonlinear analysis, notably in fixed point theory, the stability results of functional equations are used. The study of the stability of functional equations has many applications in economics and optimization theory (see [6]). Recently, many research publications addressed the stability of several forms of functional equations (see [16, 20–23, 28–30]).

One of the most important branches of nonlinear analysis is fixed point theory. The Banach contraction principle, or fixed point theorem, was initially stated in Banach's thesis [3], where it was used to prove the existence of an integral equation solution. Due to its simplicity and applicability, it has now become a highly common method for solving existence issues in many disciplines of mathematical analysis. Because of its relevance and simplicity, Banach's contraction principle has generated several interesting extensions and generalizations (see Boyd and Wong [4], Geraghty [12], Amini-Harandi and Petrusel [2], Jleli and Samet [14], Wardowski and Dung [31], Wardowski [32] and others).

In the last few years, there are many interesting modifications (or generalizations) of the metric space concept appeared in the literature, such as Czerwik [7] introduced the notion of b-metric with a coefficient 2, and this notion was further generalized by the author in [8] with a coefficient $K \geq 1$ in 1994. Matthews [19] gave the concept of partial metric space, Branciari [5] introduced a notion of a v-generalized metric space, Khamsi et al. [17] reintroduced the notion of b-metric as metric type, Fagin et al. [9] gave the notion of $s-relaxed_p$ metric (see, also [18]) and thereafter many researchers gave different and wonderful

2020 *Mathematics Subject Classification*. 39B52, 39B82, 47H10
Keywords. Fixed point, Picard operator, W_b-contraction, φ_b-contraction, $\mathcal{F}$-metric space, Hyers-Ulam stability
Received: 13 October 2021; Revised: 25 April 2022; Accepted: 10 June 2022
Communicated by Vasile Berinde
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concepts. Recently, Jleli et al. [15] presented a fascinating generalization called as $\mathcal{F}$-metric space. They demonstrated that every metric space is an $\mathcal{F}$-metric space, but not the other way around, proving that $\mathcal{F}$-metric space is more general than metric space. They found a similar result for s-relaxed$_p$ metric space with the use of realistic examples. They looked at the comparison between b-metric and $\mathcal{F}$-metric spaces, constructed a natural topology on $\mathcal{F}$-metric spaces, and showed that the closed ball is closed with regard to the proposed topology after imposing a sufficient condition. Finally, they proved a Banach contraction fixed point theorem in $\mathcal{F}$-metric spaces.

In this paper, we present some results for the existence of the Picard operator for the class of contraction mappings and investigate the Hyers-Ulam stability of fixed point problem, Cauchy functional equation, and integral equation as applications. We also obtain the well-posedness of a fixed point problem and integral equation. Some examples are provided for the usability of the results.

2. Preliminaries

Let $\mathcal{F}$ be the set of function $f : (0, +\infty) \to \mathbb{R}$ satisfying the following conditions:

(Θ_1) f is non-decreasing, that is $0 < \lambda < \mu$ implies $f(\lambda) \leq f(\mu)$,

(Θ_2) for every sequence $\{\mu_n\} \subset (0, +\infty)$, we have

$$\lim_{n\to\infty} \mu_n = 0 \Leftrightarrow \lim_{n\to\infty} f(\mu_n) = -\infty.$$

Definition 2.1. *[15] Let $\mathscr{E} \neq \emptyset$, and $\mathscr{D} : \mathscr{E} \times \mathscr{E} \to [0, \infty)$ be a map. Assume that there exists $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ such that*

($\mathscr{D}_1$) *$(\varpi, \varkappa) \in \mathscr{E} \times \mathscr{E}, \mathscr{D}(\varpi, \varkappa) = 0 \Leftrightarrow \varpi = \varkappa$,*

($\mathscr{D}_2$) *$\mathscr{D}(\varpi, \varkappa) = \mathscr{D}(\varkappa, \varpi)$, for all $(\varpi, \varkappa) \in \mathscr{E} \times \mathscr{E}$,*

($\mathscr{D}_3$) *for every $(\varpi, \varkappa) \in \mathscr{E} \times \mathscr{E}$, for every $r \in \mathbb{N}, r > 1$, and for every $(\varpi_i)_{i=1}^r \subset \mathscr{E}$ with $(\varpi_i, \varpi_{i+1}) = (\varpi, \varkappa)$, we have*

$$\mathscr{D}(\varpi, \varkappa) > 0 \Rightarrow f(\mathscr{D}(\varpi, \varkappa)) \leq f\left(\sum_{i=1}^{r-1} \mathscr{D}(\varpi_i, \varpi_{i+1})\right) + \alpha.$$

Then $\mathscr{D}$ is said to be an $\mathcal{F}$-metric ($\mathcal{F}M$) on $\mathscr{E}$, and the pair $(\mathscr{E}, \mathscr{D})$ is said to be an $\mathcal{F}$-metric space ($\mathcal{F}MS$).

Definition 2.2. *[15] Let $(\mathscr{E}, \mathscr{D})$ be an $\mathcal{F}MS$ and $\mathscr{A}$ be a nonempty subset of $\mathscr{E}$. $\mathscr{A}$ is said to be $\mathcal{F}$-open if for every $\varpi \in \mathscr{A}$, there is some $\delta > 0$ such that $\mathscr{S}(\varpi, \delta) \subset \mathscr{A}$, where*

$$\mathscr{S}(z, \delta) = \{p \in \mathscr{E} : \mathscr{D}(\varpi, p) < \delta\}.$$

We say that a subset $\mathscr{A}$ of $\mathscr{E}$ is $\mathcal{F}$-closed if $\mathscr{E} \setminus \mathscr{A}$ is $\mathcal{F}$-open. The family of all $\mathcal{F}$-open subset of $\mathscr{E}$ is denoted by $\mathcal{T}_{\mathcal{F}}$.

Definition 2.3. *[15] Let $(\mathscr{E}, \mathscr{D})$ be an $\mathcal{F}MS$. A sequence $\{\varpi_r\}$ is $\mathcal{F}$-convergent to $\varpi \in \mathscr{E}$ if $\{\varpi_r\}$ is convergent to $\mathscr{E}$ with respect to topology $\mathcal{T}_{\mathcal{F}}$.*

Definition 2.4. *[15] Let $(\mathscr{E}, \mathscr{D})$ be an $\mathcal{F}MS$ and $\{\varpi_{r_1}\}$ be a sequence in $\mathscr{E}$. A sequence $\{\varpi_{r_1}\}$ is said to be $\mathcal{F}$-Cauchy if*

$$\lim_{r_2, r_1\to\infty} \mathscr{D}(\varpi_{r_1}, \varpi_{r_2}) = 0.$$

Definition 2.5. *(see [11, 25, 26]) Let $(\mathscr{E}, \mathscr{D})$ be an $\mathcal{F}MS$ and $\mathscr{T} : \mathscr{E} \to \mathscr{E}$ be a self mapping. A sequence $\{\varpi_r\}$ defined by $\varpi_{r+1} = \mathscr{T}(\varpi_r) = \mathscr{T}^r\varpi_0$ is called a Picard sequence (PS) based at point $\varpi_0 \in \mathscr{E}$. A self mapping $\mathscr{T}$ is said to be a Picard operator (PO) if it has a unique fixed point $\varkappa \in \mathscr{E}$ and $\varkappa = \lim_{r\to\infty} \mathscr{T}^r\varpi$ for all $\varpi \in \mathscr{E}$.*

Definition 2.6. *[15] Let $(\mathscr{E}, \mathscr{D})$ be an $\mathcal{F}MS$ and $\{\varpi_r\}$ be a sequence in $\mathscr{E}$. A sequence $\{\varpi_r\}$ is said to be $\mathcal{F}$-complete if every $\mathcal{F}$-Cauchy sequence in $\mathscr{E}$ is $\mathcal{F}$-convergent to a certain point in $\mathscr{E}$.*

Definition 2.7. *[15] Let $(\mathscr{E}, \mathscr{D})$ be an $\mathcal{F}MS$ and $\mathscr{A}$ be a non empty subset of $\mathscr{E}$. $\mathscr{A}$ is said to be $\mathcal{F}$-compact if $\mathscr{A}$ is compact with respect to the topology $\mathcal{T}_{\mathcal{F}}$ on $\mathscr{E}$.*

3. W_b- Picard Operator

To start with, we have the following notations:

Definition 3.1. *Suppose that Φ denote the family of functions $\varphi : [0,\infty) \to [0,\infty)$ satisfying following conditions:*

(Wi) φ *is non-decreasing, that is* $0 < \lambda < \mu$ *implies* $\varphi(\lambda) \leq \varphi(\mu)$.
(Wii) $\varphi^r(t) \to 0$ *as* $r \to \infty$, *for* $t \in [0,\infty)$.

Definition 3.2. *Let $(\mathscr{E},\mathscr{D})$ be an $\mathcal{F}MS$. If $\mathscr{D}$ satisfying the following condition:*

$(\mathscr{D}_4)$ $\quad \mathscr{D}(\lambda\varkappa, \lambda\varpi) \leq| \lambda | \mathscr{D}(\varkappa,\varpi),\ \lambda \in \mathbb{R}.$

Then $\mathscr{D}$ is said to be an $\mathcal{F}$-linear metric ($\mathcal{F}LM$) on $\mathscr{E}$, and the pair $(\mathscr{E},\mathscr{D})$ is said to be an $\mathcal{F}$-linear metric space ($\mathcal{F}LMS$).

Now, we give W_b-contraction which will be used in our results.

Definition 3.3. *Let $(\mathscr{E},\mathscr{D})$ be an $\mathcal{F}MS$ and $\mathscr{T} : \mathscr{E} \to \mathscr{E}$ be a self mapping. A mapping $\mathscr{T}$ is called W_b-contraction if there exists a $\varphi \in \Phi$ such that for all $\varpi,\varkappa \in \mathscr{E}$, we have*

$$\mathscr{D}(\mathscr{T}\varpi, \mathscr{T}\varkappa) \leq \varphi(\mathscr{D}(\varpi,\varkappa)). \tag{1}$$

Remark 3.4. *Here, we note that W_b-contraction mapping has at most one fixed point (FP). Take $\mathscr{E} = [0,\infty)$. Suppose that $\mathscr{T}$ has two distinct FPs say $e_1, e_2 \in \mathscr{E}$, such that $\mathscr{D}(e_1,e_2) > 0$. Using (1) we have*

$$\begin{aligned}\mathscr{D}(e_1,e_2) &= \mathscr{D}(\mathscr{T}e_1, \mathscr{T}e_2) \leq \varphi(\mathscr{D}(e_1,e_2)) = \varphi(\mathscr{D}(\mathscr{T}e_1, \mathscr{T}e_2)) \leq \varphi^2(\mathscr{D}(e_1,e_2)) \leq ...\\ &\leq \varphi^r(\mathscr{D}(e_1,e_2)).\end{aligned}$$

Taking $r \to \infty$, and using Definition 3.1, we have, $\mathscr{D}(e_1,e_2) \leq 0$. Hence $\mathscr{D}(e_1,e_2) = 0$, that is, $e_1 = e_2$.

Lemma 3.5. *Suppose that $\mathscr{T}$ is a W_b-contraction in $\mathcal{F}MS$ $(\mathscr{E},\mathscr{D})$. Then for every PS $\{\varpi_r\} \subset \mathscr{E}$ defined in Definition 2.5, we have $\mathscr{D}(\varpi_r,\varpi_{r+1}) \to 0$ as $r \to \infty$, where $\varpi_r \neq \varpi_{r+1}$.*

Proof. Let ϖ_0 be an arbitrary element. Define the PS $\{\varpi_r\} \subset \mathscr{E}$ defined by $\varpi_{r+1} = \mathscr{T}\varpi_r = \mathscr{T}^r\varpi_0$ for all $r \in \mathbb{N} \cup \{0\}$. We may assume that $\mathscr{D}(\varpi_0,\varpi_1) > 0$. Since $\mathscr{T}$ is a W_b contraction, we have

$$\begin{aligned}\mathscr{D}(\varpi_r,\varpi_{r+1}) &= \mathscr{D}(\mathscr{T}\varpi_{r-1}, \mathscr{T}\varpi_r)\\ &\leq \varphi(\mathscr{D}(\varpi_{r-1},\varpi_r)) = \varphi(\mathscr{D}(\mathscr{T}\varpi_{r-2}, \mathscr{T}\varpi_{r-1}))\\ &\leq \varphi^2(\mathscr{D}(\varpi_{r-2},\varpi_{r-1}))\\ &\ \vdots\\ &\leq \varphi^{r-1}(\mathscr{D}(\mathscr{T}\varpi_0, \mathscr{T}\varpi_1))\\ &\leq \varphi^r(\mathscr{D}(\varpi_0,\varpi_1)).\end{aligned}$$

Therefore, we have $\mathscr{D}(\varpi_r,\varpi_{r+1}) \leq \varphi^r(\mathscr{D}(\varpi_0,\varpi_1))$, for all $r \in \mathbb{N}$. Taking limit $r \to \infty$ and using Definition 3.1, we have

$$\lim_{r\to\infty} \mathscr{D}(\varpi_r,\varpi_{r+1}) \leq \lim_{r\to\infty} \varphi^r(D(\varpi_0,\varpi_1)) \to 0.$$

Hence $\lim_{r\to\infty} \mathscr{D}(\varpi_r,\varpi_{r+1}) = 0$. □

Lemma 3.6. *If all the hypotheses of Lemma 3.5 hold. Then the PS is an $\mathcal{F}$-Cauchy sequence.*

Proof. Let $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ be such that $(\mathscr{D}_3)$ is satisfied. Suppose that $\varepsilon > 0$ is given. By (Θ_1), there exists a $\eta > 0$ such that for $0 < t < \eta$, we have

$$f(t) < f(\varepsilon) - \alpha. \tag{2}$$

Let ϖ_0 be an arbitrary element. Define the PS $\{\varpi_r\} \subset \mathscr{E}$ defined by $\varpi_{r+1} = \mathscr{T}\varpi_r = \mathscr{T}^r\varpi_0$ for all $r \in \mathbb{N} \cup \{0\}$. We may assume that $\mathscr{D}(\varpi_0, \varpi_1) > 0$. Using Lemma 3.5, we have $\lim_{r\to\infty} \mathscr{D}(\varpi_r, \varpi_{r+1}) = 0$. Further, we have

$$\sum_{i=r}^{s-1} \mathscr{D}(\varpi_i, \varpi_{i+1}) = \mathscr{D}(\varpi_r, \varpi_{r+1}) + \mathscr{D}(\varpi_{r+1}, \varpi_{r+2}) + \ldots + \mathscr{D}(\varpi_{s-1}, \varpi_s). \tag{3}$$

It implies that

$$\sum_{i=r}^{s-1} \mathscr{D}(\varpi_i, \varpi_{i+1}) \leq \varphi^r(\mathscr{D}(\varpi_0, \varpi_1)) + \varphi^{r+1}(\mathscr{D}(\varpi_0, \varpi_1)) + \ldots + \varphi^{s-1}(\mathscr{D}(\varpi_0, \varpi_1)). \tag{4}$$

Hence, we have

$$\sum_{i=r}^{s-1} \mathscr{D}(\varpi_i, \varpi_{i+1}) \leq \frac{\varphi^r(\mathscr{D}(\varpi_0, \varpi_1))}{1 - \varphi(\mathscr{D}(\varpi_0, \varpi_1))}.$$

Since $\lim_{r\to\infty} \frac{\varphi^r(\mathscr{D}(\varpi_0, \varpi_1))}{1 - \varphi(\mathscr{D}(\varpi_0, \varpi_1))} = 0$, for a given $\eta > 0$ there exists $N \in \mathbb{N}$ such that $0 < \frac{\varphi^r(\mathscr{D}(\varpi_0, \varpi_1))}{1 - \varphi(\mathscr{D}(\varpi_0, \varpi_1))} < \eta$, for $r \geq N$. Hence by (2) and (Θ_1), we obtain

$$f\left(\sum_{i=n}^{m-1} \mathscr{D}(\varpi_i, \varpi_{i+1})\right) \leq f\left(\frac{\varphi^r(\mathscr{D}(\varpi_0, \varpi_1))}{1 - \varphi(\mathscr{D}(\varpi_0, \varpi_1))}\right) < f(\varepsilon) - \alpha, \quad s > r \geq N. \tag{5}$$

Using $(\mathscr{D}_3)$ and (5), we obtain

$$f(\mathscr{D}(\varpi_r, \varpi_s)) \leq f\left(\sum_{i=r}^{s-1} \mathscr{D}(\varpi_i, \varpi_{i+1})\right) + \alpha < f(\varepsilon).$$

Using Θ_1, we have

$$\mathscr{D}(\varpi_r, \varpi_s) < \varepsilon,$$

for $s, r \geq N$. Hence $\{\varpi_r\}$ is $\mathcal{F}$- Cauchy. □

Theorem 3.7. *Every W_b-contraction in an $\mathcal{F}$-complete $\mathcal{F}MS$ $(\mathscr{E}, \mathscr{D})$ is a PO.*

Proof. Let ϖ_0 be an arbitrary element. Using Lemma 3.6, PS $\{\varpi_r\}$ is an $\mathcal{F}$−Cauchy sequence. Since $(\mathscr{E}, \mathscr{D})$ is $\mathcal{F}$-complete, there exists $\varpi^* \in \mathscr{E}$ such that $\{\varpi_r\}$ is $\mathcal{F}$- convergent to ϖ^*, that is

$$\lim_{r\to\infty} \mathscr{D}(\varpi_r, \varpi^*) = 0. \tag{6}$$

Now we have to prove that ϖ^* is a FP of $\mathscr{T}$. We argue by contradiction, suppose that $\mathscr{D}(\mathscr{T}\varpi^*, \varpi^*) > 0$. By $(\mathscr{D}_3)$, we have

$$\begin{aligned} f(\mathscr{D}(\mathscr{T}\varpi^*, \varpi^*)) &\leq f\left[\mathscr{D}(\mathscr{T}\varpi^*, \mathscr{T}\varpi_r) + \mathscr{D}(\mathscr{T}\varpi_r, \varpi^*)\right] + \alpha \\ &\leq f\left[\varphi(\mathscr{D}(\varpi^*, \varpi_r)) + \mathscr{D}(\varpi_{r+1}, \varpi^*)\right] + \alpha. \end{aligned}$$

Taking $r \to \infty$ and using (Θ_2) and (6), we have

$$\lim_{r\to\infty} f\left[\varphi(\mathscr{D}(\varpi^*, \varpi_r)) + \mathscr{D}(\varpi_{r+1}, \varpi^*)\right] + \alpha = -\infty.$$

Therefore, $f(\mathscr{D}(\mathscr{T}\varpi^*, \varpi^*)) \leq -\infty$ or $\mathscr{D}(\mathscr{T}\varpi^*, \varpi^*) \leq -\infty$, which is contradiction. Therefore, we have $\mathscr{D}(\mathscr{T}\varpi^*, \varpi^*) = 0$, that is $T\varpi^* = \varpi^*$. Hence the result. □

Example 3.8. *Let* $\mathscr{E} = [0,3]$ *and* $\mathscr{D} : \mathscr{E} \times \mathscr{E} \to \mathscr{E}$ *be the mapping defined by*

$$\mathscr{D}(x,y) = (x-y)^2,$$

for all $(x,y) \in \mathscr{E} \times \mathscr{E}$ *is* $\mathcal{F}$*-complete* $\mathcal{F}MS$ *with* $f(t) = ln(t)$ *and* $\alpha = ln(3)$.
$\mathscr{T} : \mathscr{E} \to \mathscr{E}$ *be a mapping defined by*

$$\mathscr{T}x = \frac{x}{2} + 1,$$

for all $x \in \mathscr{E}$. *Define* $\varphi : [0,3] \to [0,3]$ *as* $\varphi(t) = \frac{t}{4} + a$, *where* $a \leq \frac{1}{5}$, *so it satisfies the following conditions:*

(i) φ *is non-decreasing,*
(ii) $\varphi^r(t) \to 0$ *as* $r \to \infty$, *for* $t \in [0,3]$.

Then $\mathscr{T}$ *has a* W_b*–contraction and has a unique FP.*

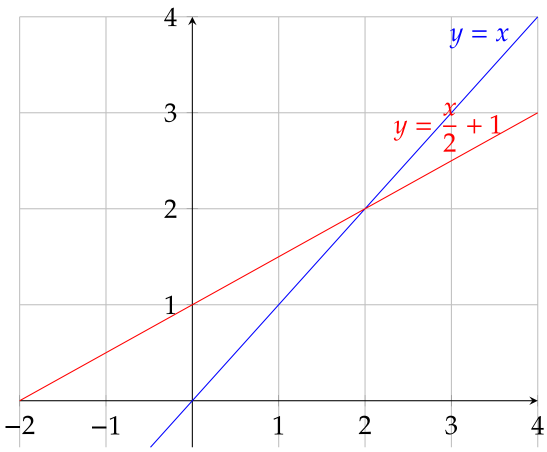

Figure 1: Graph of $x = \frac{x}{2} + 1$, showing the intersecting point (fixed point) of curve is 2.

Proof. Let $\mathscr{T}x = \frac{x}{2} + 1$, for each $x \in \mathscr{E}$. Consider

$$\begin{aligned}\mathscr{D}(\mathscr{T}x, \mathscr{T}y) &= \left(\frac{x}{2} + 1 - \frac{y}{2} - 1\right)^2 \\ &= \left(\frac{x}{2} - \frac{y}{2}\right)^2 \\ &= \frac{(x-y)^2}{4} \\ &= \frac{\mathscr{D}(x,y)}{4} \\ &\leq \frac{\mathscr{D}(x,y)}{4} + a.\end{aligned}$$

Therefore, $\mathscr{D}(\mathscr{T}x, \mathscr{T}y) \leq \varphi(\mathscr{D}(x,y))$, $\varphi(t) = \frac{t}{4} + a$ is a non decreasing function. Therefore $\mathscr{T}$ has a W_b-contraction. Since $(\mathscr{E}, \mathscr{D})$ is $\mathcal{F}$-complete therefore $\mathscr{T}$ has a unique FP and FP of $x = \frac{x}{2} + 1$ is 2 (see Fig. 1). □

3.1. Hyers-Ulam (HU) stability

Many authors have investigated the HU stability of various functional equations in various abstract spaces. In this section, we analyze the HU stability result by generalizing the results of [1, 27] in the setting of $\mathcal{FMS}$ using FP techniques.

Definition 3.9. *Let $\mathscr{T} : \mathscr{E} \to \mathscr{E}$ be an operator on an $\mathcal{FMS}$ $(\mathscr{E}, \mathscr{D})$. The FP equation*

$$\varpi = \mathscr{T}(\varpi), \quad \varpi \in \mathscr{E} \tag{7}$$

is HU stable if there exists a strictly increasing and surjective function $\beta : [0, \infty) \to [0, \infty)$ with $\beta(t) = t - \varphi(t)$, $t \in [0, \infty)$, where φ is a non-decreasing function $\varphi : [0, \infty) \to [0, \infty)$ and $\lim_{r\to\infty} \varphi^r(t) = 0$ and such that for each $\varepsilon > 0$ and each solution $\varkappa^*$ of the inequality $\mathscr{D}(\varkappa, \mathscr{T}(\varkappa)) < \varepsilon$, for each $\varkappa \in \mathscr{E}$, there exists a solution ϖ^* of equation (7) such that

$$\mathscr{D}(\varkappa^*, \varpi^*) < \beta^{-1}(\varepsilon).$$

Definition 3.10. *If the FP problem (7) for $\mathscr{T}$ meets the following criteria, then it is well-posed (WP)*

(p1) *$\mathscr{T}$ has a unique FP $\varpi^* \in \mathscr{E}$,*

(p2) *if for any sequence $\{\varpi_r\}$ in $\mathscr{E}$ such that*

$$\lim_{r\to\infty} \mathscr{D}(\mathscr{T}\varpi_r, \varpi_r) = 0,$$

then

$$\lim_{r\to\infty} \mathscr{D}(\varpi_r, \varpi^*) = 0.$$

Theorem 3.11. *Assume that all of Theorem 3.7's hypotheses are true. Then the following conditions are hold:*

(A1) *The FP problem (7) is HU stable, that is, if for each $\varepsilon > 0$ and each solution $\varkappa^*$ of the inequality $\mathscr{D}(\varkappa, \mathscr{T}(\varkappa)) < \varepsilon$, for each $\varkappa \in \mathscr{E}$, there exists a solution ϖ^* of equation (7) such that*

$$\mathscr{D}(\varkappa^*, \varpi^*) < \beta^{-1}(\varepsilon).$$

(A2) *If $\{\varpi_r\}$ is a sequence in $\mathscr{E}$ such that $\lim_{r\to\infty} \mathscr{D}(\mathscr{T}\varpi_r, \varpi_r) = 0$ and ϖ^* is a FP of $\mathscr{T}$, then the FP problem (7) is WP.*

Proof. $(A1)$ Using Theorem 3.7, there is a unique $\varpi^* \in \mathscr{E}$ such that $\varpi^* = \mathscr{T}\varpi^*$ that is $\varpi^* \in \mathscr{E}$ is solution of the FP equation ($\varpi = \mathscr{T}\varpi$). Assume that $\varepsilon > 0$ and $\varkappa^* \in \mathscr{E}$. Using $(\mathscr{D}_3)$, we have

$$\begin{aligned} f(\mathscr{D}(\varkappa^*, \varpi^*)) &\le f[\mathscr{D}(\varkappa^*, \mathscr{T}\varkappa^*) + \mathscr{D}(\mathscr{T}\varkappa^*, \varpi^*)] + \alpha \\ &\le f[\varepsilon + \mathscr{D}(\mathscr{T}\varkappa^*, \mathscr{T}\varpi^*)] + \alpha \\ &\le f[\varepsilon + \varphi(\mathscr{D}(\varkappa^*, \varpi^*))] + \alpha. \end{aligned}$$

Hence using property of (Θ_1), we have $\mathscr{D}(\varkappa^*, \varpi^*) \le \varepsilon + \varphi(\mathscr{D}(\varkappa^*, \varpi^*))$, or $\mathscr{D}(\varkappa^*, \varpi^*) - \varphi\mathscr{D}(\varkappa^*, \varpi^*) \le \varepsilon$. Further, we have $\beta(\mathscr{D}(\varkappa^*, \varpi^*)) \le \varepsilon$. Hence

$$\mathscr{D}(\varkappa^*, \varpi^*) \le \beta^{-1}(\varepsilon),$$

which completes the proof.

$(A2)$ If $\{\xi_r\}$ is a sequence in $\mathscr{E}$ such that $\lim_{r\to\infty} \mathscr{D}(\mathscr{T}\xi_r, \xi_r) = 0$ and ϖ^* is a unique FP of $\mathscr{T}$ (using Theorem 3.7). From the triangle inequality and contractive condition, we have

$$\begin{aligned} f(\mathscr{D}(\xi_r, \varpi^*)) &\le f[\mathscr{D}(\xi_r, \mathscr{T}\xi_r) + \mathscr{D}(\mathscr{T}\xi_r, \varpi^*)] + \alpha \\ &\le f[\mathscr{D}(\xi_r, \mathscr{T}\xi_r) + \mathscr{D}(\mathscr{T}\xi_r, \mathscr{T}\varpi^*)] + \alpha \\ &\le f[\mathscr{D}(\xi_r, \mathscr{T}\xi_r) + \varphi(\mathscr{D}(\xi_r, \varpi^*))] + \alpha. \end{aligned}$$

On the same lines of above cases, we have
$\beta(\mathscr{D}(\xi_r, \varpi^*)) \leq \mathscr{D}(\xi_r, \mathscr{T}\xi_r)$. Taking limit $r \to \infty$, we get

$$\lim_{r\to\infty} \beta(\mathscr{D}(\xi_r, \varpi^*)) \leq \lim_{r\to\infty} \mathscr{D}(\xi_r, \mathscr{T}\xi_r).$$

Therefore, $\lim_{r\to\infty} \beta(\mathscr{D}(\xi_r, \varpi^*)) = 0$. Hence $\mathscr{D}(\xi_r, x^*) = 0$. This shows that the FP problem (7) is WP.
□

Theorem 3.12. *Assume that all of Theorem 3.7's hypotheses are true. If $\mathscr{R} : \mathscr{E} \to \mathscr{E}$ is a map such that there exists $\Lambda > 0$ with*

$$\mathscr{D}(\mathscr{T}\xi, \mathscr{R}\xi) < \Lambda,$$

for all $\xi \in \mathscr{E}$, then for any FP $\varkappa^*$ of $\mathscr{R}$, we have

$$\mathscr{D}(\varpi^*, \varkappa^*) \leq \beta^{-1}(\Lambda).$$

Proof. Suppose that $\mathscr{R} : \mathscr{E} \to \mathscr{E}$ is a map such that there exists $\Lambda > 0$, with $\mathscr{D}(\mathscr{T}\xi, \mathscr{R}\xi) < \Lambda$, for all $\xi \in \mathscr{E}$. Choose $\varkappa^*$ be the FP of $\mathscr{R}$ then by the property of $(\mathscr{D}_3)$, we have

$$\begin{aligned} f(\mathscr{D}(\varpi^*, \varkappa^*)) &\leq f(\mathscr{D}(\varpi^*, \varkappa^*)) + \alpha \\ &\leq f(\mathscr{D}(\mathscr{T}\varpi^*, \mathscr{R}\varkappa^*)) + \alpha \\ &\leq f[\mathscr{D}(\mathscr{T}\varpi^*, \mathscr{T}\varkappa^*) + \mathscr{D}(\mathscr{T}\varkappa^*, \mathscr{R}\varkappa^*)] + \alpha \\ &\leq f[\varphi(\mathscr{D}(\varpi^*, \varkappa^*)) + \mathscr{D}(\mathscr{T}\varkappa^*, \mathscr{R}\varkappa^*)] + \alpha \\ &\leq f[\varphi(\mathscr{D}(\varpi^*, \varkappa^*)) + \Lambda] + \alpha. \end{aligned}$$

Using the property of Θ_1, we have

$$\mathscr{D}(\varpi^*, \varkappa^*) \leq \varphi(\mathscr{D}(\varpi^*, \varkappa^*)) + \Lambda.$$

It implies $\mathscr{D}(\varpi^*, \varkappa^*) - \varphi(\mathscr{D}(\varpi^*, \varkappa^*)) \leq \Lambda$. Therefore, we get $\beta(\mathscr{D}(\varpi^*, \varkappa^*)) \leq \Lambda$ or $\mathscr{D}(\varpi^*, \varkappa^*) \leq \beta^{-1}(\Lambda)$. Hence the result. □

3.2. *Stability of Cauchy functional equation*

Theorem 3.13. *Let $(\mathscr{E}, \mathscr{D})$ be an $\mathcal{F}$LMS and $\mathscr{T} : \mathscr{E} \to \mathscr{E}$ be a W_b−contraction.*
Assume that $R : \mathscr{E} \to \mathscr{E}$ is a mapping such that for each $\varepsilon_1 > 0$

$$\mathscr{D}(R(\varkappa + \varpi), R(\varkappa) + R(\varpi)) < \varepsilon_1, \tag{8}$$

for all $\varpi, \varkappa \in \mathscr{E}$.
Then there exists a unique function $q : \mathscr{E} \to \mathscr{E}$ satisfies

$$\mathscr{D}(R(\varkappa), q(\varkappa)) < \beta^{-1}(\varepsilon), \tag{9}$$

where $\varepsilon = \dfrac{\varepsilon_1}{2}$.

Proof. Put $\varkappa = \varpi$ in (8), we have

$$\mathscr{D}(R(2\varkappa), 2R(\varkappa)) < \varepsilon_1. \tag{10}$$

Since, by the Definition 3.2

$$\mathscr{D}(R(2\varkappa), 2R(\varkappa)) \leq |2| \mathscr{D}\left(\frac{1}{2} R(2\varkappa), R(\varkappa)\right) < \varepsilon_1.$$

Therefore, we have

$$\mathscr{D}\left(\frac{1}{2}R(2\varkappa), R(\varkappa)\right) < \frac{\varepsilon_1}{2}. \tag{11}$$

Now we define an operator $\mathscr{T} : \mathscr{E} \to \mathscr{E}$ by

$$\mathscr{T}R(\varkappa) = \frac{1}{2}R(\varkappa). \tag{12}$$

Then (11) becomes

$$\mathscr{D}\left(\mathscr{T}R(\varkappa), R(\varkappa)\right) < \varepsilon, \tag{13}$$

where $\frac{\varepsilon_1}{2} = \varepsilon$. Now we have to prove that there exists a unique function $q : \mathscr{E} \to \mathscr{E}$ satisfies

$$\mathscr{D}(R(\varkappa), q(\varkappa)) < \beta^{-1}(\varepsilon).$$

To prove this, using Theorem 3.7, there is a unique $q(\varkappa)$ such that $q(\varkappa) = \mathscr{T}q(\varkappa)$.

Assume that $\varepsilon > 0$ and $\varkappa \in \mathscr{E}$. Using $(\mathscr{D}_3)$, we have

$$\begin{aligned} f(\mathscr{D}(q(\varkappa), R(\varkappa))) &\leq f[\mathscr{D}(q(\varkappa), \mathscr{T}R(\varkappa)) + \mathscr{D}(\mathscr{T}R(\varkappa), R(\varkappa))] + \alpha \\ &\leq f[\mathscr{D}(\mathscr{T}q(\varkappa), \mathscr{T}R(\varkappa)) + \varepsilon] + \alpha \\ &\leq f[\varphi(\mathscr{D}(q(\varkappa), R(\varkappa))) + \varepsilon] + \alpha. \end{aligned}$$

Hence using property of (Θ_1), we have $\mathscr{D}(q(\varkappa), R(\varkappa)) \leq \varepsilon + \varphi(\mathscr{D}(q(\varkappa), R(\varkappa)))$, or $\mathscr{D}(q(\varkappa), R(\varkappa)) - \varphi(\mathscr{D}(q(\varkappa), R(\varkappa))) \leq \varepsilon$. Further, we have $\beta(\mathscr{D}(q(\varkappa), R(\varkappa))) \leq \varepsilon$. Hence

$$\mathscr{D}(q(\varkappa), R(\varkappa)) \leq \beta^{-1}(\varepsilon).$$

Finally we have to prove the uniqueness part. To prove this, suppose that there exist a function $q_1 : \mathscr{E} \to \mathscr{E}$ $(q \neq q_1)$ such that

$$\mathscr{D}(R(\varkappa), q_1(\varkappa)) < \beta^{-1}(\varepsilon).$$

Now Using $(\mathscr{D}_3)$, we have

$$\begin{aligned} f(\mathscr{D}(q(\varkappa), q_1(\varkappa))) &\leq f[\mathscr{D}(q(\varkappa), R(\varkappa)) + \mathscr{D}(R(\varkappa), q_1(\varkappa))] \\ &\leq f[\beta^{-1}(\varepsilon) + \beta^{-1}(\varepsilon)], \end{aligned}$$

using property of (Θ_1), we have

$$\mathscr{D}(q(\varkappa), q_1(\varkappa) \leq \beta^{-1}(\varepsilon) + \beta^{-1}(\varepsilon). \tag{14}$$

Since $\beta : [0, \infty) \to [0, \infty)$ is a strictly increasing and surjective function, so for any given $\varepsilon \in [0, \infty)$ there exists a $\delta > 0$ such that $\beta(\frac{\delta}{2}) = \varepsilon$ or $\beta^{-1}(\varepsilon) = \frac{\delta}{2}$. Therefore equation (14) implies that

$$\mathscr{D}(q(\varkappa), q_1(\varkappa)) \leq \delta,$$

which completes the proof. □

4. φ_b- Picard Operator

To start with, we have the following notations:

Definition 4.1. *Suppose that Φ denote the family of functions $\varphi : [0,\infty) \to [0,\infty)$ satisfying following conditions:*

(φi) φ is non-decreasing, that is $0 < \lambda < \mu$ implies $\varphi(\lambda) \leq \varphi(\mu)$,
(φii) $\varphi(t) = 0$ implies $t = 0$.

Now, we give φ_b-contraction which will be used in our results.

Definition 4.2. *Let $(\mathscr{E}, \mathscr{D})$ be an $\mathcal{F}MS$ and $\mathscr{T} : \mathscr{E} \to \mathscr{E}$ be a self mapping. A mapping $\mathscr{T}$ is called φ_b-contraction if there exists a $\varphi \in \Phi$ such that for all $\varpi, \varkappa \in \mathscr{E}$ and $k \in [0,1)$, we have*

$$\varphi(\mathscr{D}(\mathscr{T}\varpi, \mathscr{T}\varkappa)) \leq k\varphi(\mathscr{D}(\varpi, \varkappa)). \tag{15}$$

Remark 4.3. *Here, we note that φ_b-contraction mapping has at most one FP. Take $\mathscr{E} = [0,\infty)$. Assume that $\mathscr{T}$ has two distinct FPs say $e_1, e_2 \in \mathscr{E}$, such that $\mathscr{D}(e_1, e_2) > 0$. Using (15) we have*

$$\begin{aligned}\varphi(\mathscr{D}(e_1, e_2)) &= \varphi(\mathscr{D}(\mathscr{T}e_1, \mathscr{T}e_2)) \leq k\varphi(\mathscr{D}(e_1, e_2)) = k(\varphi(\mathscr{D}(\mathscr{T}e_1, \mathscr{T}e_2)))\\ &\leq k^2(\varphi(\mathscr{D}(e_1, e_2))) \leq \ldots \leq k^r(\varphi(\mathscr{D}(e_1, e_2))).\end{aligned}$$

Taking $r \to \infty$, and using Definition 4.1, we have, $\varphi(\mathscr{D}(e_1, e_2)) \leq 0$. Hence $\varphi(\mathscr{D}(e_1, e_2)) = 0$, that is, $e_1 = e_2$.

Lemma 4.4. *Suppose that $\mathscr{T}$ is a φ_b-contraction in an $\mathcal{F}MS$ $(\mathscr{E}, \mathscr{D})$. Then for every PS $\{\varpi_r\} \subset \mathscr{E}$ defined in Definition 2.5, we have $\mathscr{D}(\varpi_r, \varpi_{r+1}) \to 0$ as $r \to \infty$, where $\varpi_r \neq \varpi_{r+1}$.*

Proof. Let ϖ_0 be an arbitrary element. Define the PS $\{\varpi_r\} \subset \mathscr{E}$ defined by $\varpi_{r+1} = \mathscr{T}\varpi_r = \mathscr{T}^r\varpi_0$ for all $r \in \mathbb{N} \cup \{0\}$. We may suppose that $\mathscr{D}(\varpi_0, \varpi_1) > 0$. Since $\mathscr{T}$ is a φ_b contraction, we have

$$\begin{aligned}\varphi(\mathscr{D}(\varpi_r, \varpi_{r+1})) &= \varphi(\mathscr{D}(\mathscr{T}\varpi_{r-1}, \mathscr{T}\varpi_r))\\ &\leq k(\varphi(\mathscr{D}(\varpi_{r-1}, \varpi_r))) = k(\varphi(\mathscr{D}(\mathscr{T}\varpi_{r-2}, \mathscr{T}\varpi_{r-1})))\\ &\leq k^2(\varphi(\mathscr{D}(\varpi_{r-2}, \varpi_{r-1})))\\ &\ \vdots\\ &\leq k^{(r-1)}(\varphi(\mathscr{D}(\mathscr{T}\varpi_0, \mathscr{T}\varpi_1)))\\ &\leq k^r(\varphi(\mathscr{D}(\varpi_0, \varpi_1))).\end{aligned}$$

Therefore, we have $\varphi(\mathscr{D}(\varpi_r, \varpi_{r+1})) \leq k^r(\varphi(\mathscr{D}(\varpi_0, \varpi_1)))$, for all $r \in \mathbb{N}$. Taking limit $r \to \infty$ and using Definition 4.1, we have

$$\lim_{r\to\infty} \varphi(\mathscr{D}(\varpi_r, \varpi_{r+1})) \leq \lim_{r\to\infty} k^r(\varphi(\mathscr{D}(\varpi_0, \varpi_1))) \to 0.$$

Hence $\lim_{r\to\infty} \mathscr{D}(\varpi_r, \varpi_{r+1}) = 0$. □

Lemma 4.5. *If all the hypotheses of Lemma 4.4 hold. Then the PS is an $\mathcal{F}$-Cauchy sequence.*

Proof. Suppose that $(f, \alpha) \in \mathcal{F} \times [0,\infty)$ is such that $(\mathscr{D}_3)$ is satisfied. Suppose that $\varepsilon > 0$ is given. By (Θ_1), there exists a $\eta > 0$ such that for $0 < t < \eta$, we have

$$f(t) < f(\varepsilon) - \alpha. \tag{16}$$

Let ϖ_0 be an arbitrary element. Define the PS $\{\varpi_r\} \subset \mathscr{E}$ defined by $\varpi_{r+1} = \mathscr{T}\varpi_r = \mathscr{T}^r\varpi_0$ for all $r \in \mathbb{N} \cup \{0\}$. We may suppose that $\mathscr{D}(\varpi_0, \varpi_1) > 0$. Using Lemma 4.4, we have $\lim_{r\to\infty} \mathscr{D}(\varpi_r, \varpi_{r+1}) = 0$. Further, we have

$$\sum_{i=r}^{s-1} \varphi(\mathscr{D}(\varpi_i, \varpi_{i+1})) = \varphi(\mathscr{D}(\varpi_r, \varpi_{r+1})) + \varphi(\mathscr{D}(\varpi_{r+1}, \varpi_{r+2})) + \dots + \varphi(\mathscr{D}(\varpi_{s-1}, \varpi_s)). \tag{17}$$

It implies that

$$\sum_{i=r}^{s-1} \varphi(\mathscr{D}(\varpi_i, \varpi_{i+1})) \leq k^r(\varphi(\mathscr{D}(\varpi_0, \varpi_1))) + k^{r+1}(\varphi(\mathscr{D}(\varpi_0, \varpi_1))) + \dots + k^{s-1}(\varphi(\mathscr{D}(\varpi_0, \varpi_1))). \tag{18}$$

Hence, we have

$$\sum_{i=r}^{s-1} \varphi(\mathscr{D}(\varpi_i, \varpi_{i+1})) \leq \frac{k^r}{1-k}(\varphi(\mathscr{D}(\varpi_0, \varpi_1))).$$

Since $\lim_{r\to\infty} \frac{k^r}{1-k}(\varphi(\mathscr{D}(\varpi_0, \varpi_1))) = 0$, for a given $\eta > 0$ there exists $N \in \mathbb{N}$ such that $0 < \frac{k^r}{1-k}(\varphi(\mathscr{D}(\varpi_0, \varpi_1))) < \eta$, for $r \geq N$. Hence by (16) and (Θ_1), we obtain

$$f\left(\sum_{i=r}^{s-1} \varphi(\mathscr{D}(\varpi_i, \varpi_{i+1}))\right) \leq f\left(\frac{k^r}{1-k}(\varphi(\mathscr{D}(\varpi_0, \varpi_1)))\right) < f(\varepsilon) - \alpha, \quad s > r \geq N. \tag{19}$$

Using $(\mathscr{D}_3)$ and (19), we obtain

$$f(\varphi(\mathscr{D}(\varpi_r, \varpi_s))) \leq f\left(\sum_{i=r}^{s-1} \varphi(\mathscr{D}(\varpi_i, \varpi_{i+1}))\right) + \alpha < f(\varepsilon).$$

Using Θ_1, we have

$$\varphi(\mathscr{D}(\varpi_r, \varpi_s)) < \varepsilon.$$

Hence

$$\mathscr{D}(\varpi_r, \varpi_s) < \varphi^{-1}(\varepsilon),$$

for $s, r \geq N$. Hence $\{\varpi_r\}$ is $\mathcal{F}$- Cauchy.
□

Theorem 4.6. *Every φ_b-contraction in an $\mathcal{F}$-complete $\mathcal{F}MS$ $(\mathscr{E}, \mathscr{D})$ is a PO.*

Proof. Let ϖ_0 be an arbitrary element. Using Lemma 4.5, PS $\{\varpi_r\}$ is an $\mathcal{F}-$ Cauchy sequence. Since $(\mathscr{E}, \mathscr{D})$ is $\mathcal{F}$-complete, there exists $\varpi^* \in \mathscr{X}$ such that $\{\varpi_r\}$ is $\mathcal{F}$- convergent to ϖ^*, that is

$$\lim_{r\to\infty} \mathscr{D}(\varpi_r, \varpi^*) = 0. \tag{20}$$

Now we have to show that ϖ^* is a fixed point of $\mathscr{T}$. We argue by contradiction, suppose that $\varphi(\mathscr{D}(\mathscr{T}\varpi^*, \varpi^*)) > 0$. By $(\mathscr{D}_3)$, we have

$$\begin{aligned} f(\varphi(\mathscr{D}(\mathscr{T}\varpi^*, \varpi^*))) &\leq f\left[\varphi(\mathscr{D}(\mathscr{T}\varpi^*, \mathscr{T}\varpi_r)) + \varphi(\mathscr{D}(\mathscr{T}\varpi_r, \varpi^*))\right] + \alpha \\ &\leq f\left[k(\varphi(\mathscr{D}(\varpi^*, \varpi_r))) + \varphi(\mathscr{D}(\varpi_{r+1}, \varpi^*))\right] + \alpha. \end{aligned}$$

Taking $r \to \infty$ and using (Θ_2) and (20), we have

$$\lim_{r\to\infty} f\left[k(\varphi(\mathscr{D}(\varpi^*,\varpi_r))) + \varphi(D(\varpi_{r+1},\varpi^*))\right] + \alpha = -\infty.$$

Therefore, $f(\varphi(\mathscr{D}(\mathscr{T}\varpi^*,\varpi^*))) \leq -\infty$ or $\varphi(\mathscr{D}(\mathscr{T}\varpi^*,\varpi^*)) \leq -\infty$, which is contradiction. Therefore, we have $\varphi(\mathscr{D}(\mathscr{T}\varpi^*,\varpi^*)) = 0$ implies $\mathscr{D}(\mathscr{T}\varpi^*,\varpi^*) = 0$, that is $\mathscr{T}\varpi^* = \varpi^*$. Hence the result. □

Example 4.7. *Let $\mathscr{E} = [0,3]$ and $\mathscr{D} : \mathscr{E}\times\mathscr{E} \to \mathscr{E}$ be the mapping defined by*

$$\mathscr{D}(x,y) = (x-y)^2 \quad if \quad (x,y) \in [0,3]\times[0,3],$$

for all $(x,y) \in \mathscr{E}\times\mathscr{E}$ is $\mathcal{F}$-complete $\mathcal{F}$MS with $f(t) = ln(t)$ and $\alpha = ln(3)$.
$\mathscr{T} : \mathscr{E} \to \mathscr{E}$ be a mapping defined by

$$\mathscr{T}x = \frac{x}{e^{x+\frac{1}{2}}},$$

for all $x \in \mathscr{E}$. Define $\varphi : [0,\infty) \to [0,\infty)$ as $\varphi(t) = t$, so it satisfies the following conditions:

(i) φ is non-decreasing,
(ii) $\varphi(t) = 0$ implies $t = 0$, for $t \in [0,\infty)$.

Then $\mathscr{T}$ has a φ_b- contraction and has a unique FP.

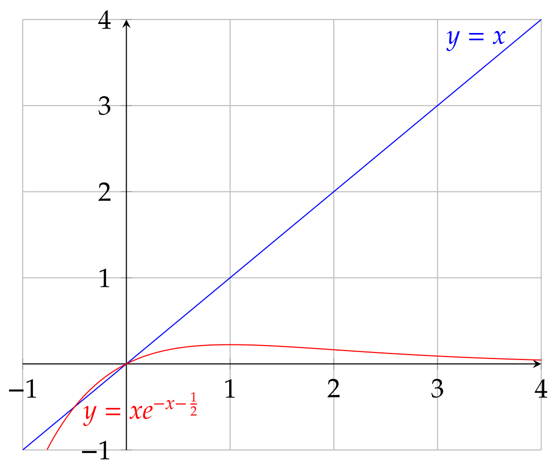

Figure 2: Graph of $x = \dfrac{x}{e^{x+\frac{1}{2}}}$, showing the intersecting point.

Proof. If $(x,y) \in [0,3]\times[0,3]$,

$$\mathscr{D}(\mathscr{T}x,\mathscr{T}y) = \left(\frac{x}{e^{x+\frac{1}{2}}} - \frac{y}{e^{y+\frac{1}{2}}}\right)^2.$$

By the mean value theorem, there exists a real number h between x and y, such that

$$\begin{aligned}\mathscr{D}(\mathscr{T}x,\mathscr{T}y) &= \left(-\frac{1}{e^{h+\frac{1}{2}}}\right)^2 \mid x-y\mid^2 \\ &\leq \frac{1}{e}\mid x-y\mid^2 \\ &= \frac{1}{e}\mathscr{D}(x,y).\end{aligned}$$

Therefore, we deduce that

$$\mathscr{D}(\mathscr{T}x, \mathscr{T}y) \leq \frac{1}{e}\mathscr{D}(x, y).$$

Therefore, $\varphi(\mathscr{D}(\mathscr{T}x, \mathscr{T}y)) \leq k(\varphi(\mathscr{D}(x, y)))$, $\varphi(t) = t$ is a non decreasing function and $k = \frac{1}{e}$. Therefore $\mathscr{T}$ has a φ_b-contraction. Since $(\mathscr{E}, \mathscr{D})$ is $\mathcal{F}$-complete, therefore $\mathscr{T}$ has a unique fixed point and fixed point of $x = \frac{x}{e^{x+\frac{1}{2}}}$ is 0 in $[0, 3]$ (see Fig. 2). □

4.1. HU stability

Definition 4.8. *Let $\mathscr{T} : \mathscr{E} \to \mathscr{E}$ be an operator on an $\mathcal{F}MS$ $(\mathscr{E}, \mathscr{D})$. The FP equation*

$$\varpi = \mathscr{T}(\varpi), \;\; \varpi \in \mathscr{E} \tag{21}$$

is HU stable if there exists a strictly increasing and surjective function $\beta : [0, \infty) \to [0, \infty)$ with $\beta(t) = t - \varphi^{-1}(k\varphi(t))$, $t \in [0, \infty)$, $k \in [0, 1)$, where φ is a non-decreasing function $\varphi : [0, \infty) \to [0, \infty)$ and $\varphi(t) = 0$ implies $t = 0$ and such that for each $\varepsilon > 0$ and each solution $\varkappa^*$ of the inequality $\mathscr{D}(\varkappa, \mathscr{T}(\varkappa)) < \varepsilon$, for each $\varkappa \in \mathscr{E}$, there exists a solution ϖ^* of equation (21) *such that*

$$\mathscr{D}(\varkappa^*, \varpi^*) < \beta^{-1}(\varepsilon).$$

Definition 4.9. *If the FP (21) for $\mathscr{T}$ meets the following criteria, it is WP*

(p1) *$\mathscr{T}$ has a unique FP $\varpi^* \in \mathscr{E}$,*

(p2) *if for any sequence $\{\varpi_r\}$ in $\mathscr{E}$ such that*

$$\lim_{r\to\infty} \mathscr{D}(\mathscr{T}\varpi_r, \varpi_r) = 0,$$

then

$$\lim_{r\to\infty} \mathscr{D}(\varpi_r, \varpi^*) = 0.$$

Theorem 4.10. *Assume that all of Theorem 4.6's hypotheses are true. Then the following conditions hold:*

$(A1)$ *The FP problem (21) is HU stable, that is, if for each $\varepsilon > 0$ and each solution $\varkappa^*$ of the inequality $\mathscr{D}(\varkappa, T(\varkappa)) < \varepsilon$, for each $\varkappa \in \mathscr{E}$, there exists a solution ϖ^* of equation (21) such that*

$$\mathscr{D}(\varkappa^*, \varpi^*) < \beta^{-1}(\varepsilon).$$

$(A2)$ *If $\{\varpi_n\}$ is a sequence in $\mathscr{E}$ such that $\lim_{n\to\infty} \mathscr{D}(\mathscr{T}\varpi_n, \varpi_n) = 0$ and ϖ^* is a FP of $\mathscr{T}$, then the FP problem (21) is WP.*

Proof. $(A1)$ Using Theorem 4.6, there is a unique $\varpi^* \in \mathscr{E}$ such that $\varpi^* = \mathscr{T}\varpi^*$ that is $\varpi^* \in \mathscr{E}$ is solution of the FP equation $(\varpi = \mathscr{T}\varpi)$. Assume that $\varepsilon > 0$ and $\varkappa^* \in \mathscr{E}$. Using $(\mathscr{D}_3)$, we have

$$\begin{aligned} f(\mathscr{D}(\varkappa^*, \varpi^*)) &\leq f[\mathscr{D}(\varkappa^*, \mathscr{T}\varkappa^*) + \mathscr{D}(\mathscr{T}\varkappa^*, \varpi^*)] + \alpha \\ &\leq f[\varepsilon + \mathscr{D}(\mathscr{T}\varkappa^*, \mathscr{T}\varpi^*)] + \alpha \\ &\leq f[\varepsilon + \varphi^{-1}(k(\varphi(\mathscr{D}(\varkappa^*, \varpi^*))))] + \alpha. \end{aligned}$$

Hence using property of (Θ_1), we have $\mathscr{D}(\varkappa^*, \varpi^*) \leq \varepsilon + \varphi^{-1}(k(\varphi(\mathscr{D}(\varkappa^*, \varpi^*))))$, or $\mathscr{D}(\varkappa^*, \varpi^*) - \varphi^{-1}(k(\varphi(\mathscr{D}(\varkappa^*, \varpi^*)))) \leq \varepsilon$. Further, we have $\beta(\mathscr{D}(\varkappa^*, \varpi^*)) \leq \varepsilon$. Hence

$$\mathscr{D}(\varkappa^*, \varpi^*) \leq \beta^{-1}(\varepsilon),$$

which completes the proof.

(*A*2) If $\{\xi_r\}$ is a sequence in $\mathscr{E}$ such that $\lim_{r\to\infty} \mathscr{D}(\mathscr{T}\xi_r, \xi_r) = 0$ and ϖ^* is a unique FP of $\mathscr{T}$ (using Theorem 4.6). From the triangle inequality and contractive condition, we have

$$\begin{aligned} f(\mathscr{D}(\xi_r, \varpi^*)) &\leq f[\mathscr{D}(\xi_r, \mathscr{T}\xi_r) + \mathscr{D}(\mathscr{T}\xi_r, \varpi^*)] + \alpha \\ &\leq f[\mathscr{D}(\xi_r, \mathscr{T}\xi_r) + \mathscr{D}(\mathscr{T}\xi_r, \mathscr{T}\varpi^*)] + \alpha \\ &\leq f[\mathscr{D}(\xi_r, \mathscr{T}\xi_r) + \varphi^{-1}(k(\varphi(\mathscr{D}(\xi_r, \varpi^*))))] + \alpha. \end{aligned}$$

On the same lines of above cases, we have
$\beta(\mathscr{D}(\xi_r, \varpi^*)) \leq \mathscr{D}(\xi_r, \mathscr{T}\xi_r)$. Taking limit $r \to \infty$, we get

$$\lim_{r\to\infty} \beta(\mathscr{D}(\xi_r, \varpi^*)) \leq \lim_{r\to\infty} \mathscr{D}(\xi_r, \mathscr{T}\xi_r).$$

Therefore, $\lim_{r\to\infty} \beta(\mathscr{D}(\xi_r, \varpi^*)) = 0$. Hence $\mathscr{D}(\xi_r, x^*) = 0$. This shows that the FP problem (21) is WP. □

Theorem 4.11. *Assume that all of Theorem 4.6's hypotheses are true. If $\mathscr{R} : \mathscr{E} \to \mathscr{E}$ is a map such that there exists $\Lambda > 0$ with*

$$\mathscr{D}(\mathscr{T}\xi, \mathscr{R}\xi) < \Lambda,$$

for all $\xi \in \mathscr{X}$, then for any fixed point $\varkappa^*$ of $\mathscr{R}$, we have

$$\mathscr{D}(\varpi^*, \varkappa^*) \leq \beta^{-1}(\Lambda).$$

Proof. Suppose that $\mathscr{R} : \mathscr{E} \to \mathscr{E}$ is a map such that there exists $\Lambda > 0$, with $\mathscr{D}(\mathscr{T}\xi, \mathscr{R}\xi) < \Lambda$, for all $\xi \in \mathscr{E}$. Choose $\varkappa^*$ be the FP of $\mathscr{R}$ then by the property $(\mathscr{D}_3)$, we have

$$\begin{aligned} f(\mathscr{D}(\varpi^*, \varkappa^*)) &\leq f(\mathscr{D}(\varpi^*, \varkappa^*)) + \alpha \\ &\leq f(\mathscr{D}(\mathscr{T}\varpi^*, \mathscr{R}\varkappa^*)) + \alpha \\ &\leq f[\mathscr{D}(\mathscr{T}\varpi^*, \mathscr{T}\varkappa^*) + \mathscr{D}(\mathscr{T}\varkappa^*, \mathscr{R}\varkappa^*)] + \alpha \\ &\leq f[\varphi^{-1}(k(\varphi(\mathscr{D}(\varpi^*, \varkappa^*)))) + D(\mathscr{T}\varkappa^*, \mathscr{R}\varkappa^*)] + \alpha \end{aligned}$$

Therefore, we have

$$f(\mathscr{D}(\varpi^*, \varkappa^*)) \leq f[\varphi^{-1}(k(\varphi(\mathscr{D}(\varpi^*, \varkappa^*)))) + \Lambda] + \alpha.$$

Using the property of Θ_1, we have

$$\mathscr{D}(\varpi^*, \varkappa^*) \leq \varphi^{-1}(k(\varphi(\mathscr{D}(\varpi^*, \varkappa^*)))) + \Lambda.$$

It implies $\mathscr{D}(\varpi^*, \varkappa^*) - \varphi^{-1}(k(\varphi(\mathscr{D}(\varpi^*, \varkappa^*)))) \leq \Lambda$. Therefore, we get $\beta(\mathscr{D}(\varpi^*, \varkappa^*)) \leq \Lambda$, or $\mathscr{D}(\varpi^*, \varkappa^*) \leq \beta^{-1}(\Lambda)$. Hence the result. □

4.2. Stability of Cauchy functional equation

Theorem 4.12. *Let $(\mathscr{E}, \mathscr{D})$ be an $\mathcal{F}LMS$ and $\mathscr{T} : \mathscr{E} \to \mathscr{E}$ be a φ_b−contraction.*

Assume that $R : \mathscr{E} \to \mathscr{E}$ is a mapping such that for each $\varepsilon_1 > 0$,

$$\mathscr{D}(R(\varkappa + \varpi), R(\varkappa) + R(\varpi)) < \varepsilon_1, \tag{22}$$

for all $\varpi, \varkappa \in \mathscr{E}$.

Then there exists a unique function $q : \mathscr{E} \to \mathscr{E}$ satisfies

$$\mathscr{D}(R(\varkappa), q(\varkappa)) < \beta^{-1}(\varepsilon), \tag{23}$$

where $\frac{\varepsilon_1}{2} = \varepsilon$.

Proof. Put $\varkappa = \varpi$ in (22), we have

$$\mathscr{D}\left(R(2\varkappa), 2R(\varkappa)\right) < \varepsilon_1. \tag{24}$$

Since, by the Definition 3.2

$$\mathscr{D}\left(R(2\varkappa), 2R(\varkappa)\right) \leq |2|\mathscr{D}\left(\frac{1}{2}R(2\varkappa), R(\varkappa)\right) < \varepsilon_1.$$

Therefore, we have

$$\mathscr{D}\left(\frac{1}{2}R(2\varkappa), R(\varkappa)\right) < \frac{\varepsilon_1}{2}. \tag{25}$$

Now we define an operator $\mathscr{T} : \mathscr{E} \to \mathscr{E}$ by

$$\mathscr{T}(R(\varkappa)) = \frac{1}{2}R(\varkappa). \tag{26}$$

Then (25) becomes

$$\mathscr{D}\left(\mathscr{T}(R(\varkappa)), R(\varkappa)\right) < \varepsilon, \tag{27}$$

where $\frac{\varepsilon_1}{2} = \varepsilon$. Now we have to prove that there exists a unique function $q : \mathscr{E} \to \mathscr{E}$ satisfies

$$\mathscr{D}(R(\varkappa), q(\varkappa)) < \beta^{-1}(\varepsilon).$$

To prove this, Using Theorem 4.6, there is a unique $q(\varkappa)$ such that $q(\varkappa) = \mathscr{T}q(\varkappa)$.
Assume that $\varepsilon > 0$ and $\varkappa \in \mathscr{E}$. Using $(\mathscr{D}_3)$, we have

$$\begin{aligned} f(\mathscr{D}(q(\varkappa), R(\varkappa))) &\leq f[\mathscr{D}(q(\varkappa), \mathscr{T}R(\varkappa)) + \mathscr{D}(\mathscr{T}R(\varkappa), R(\varkappa))] + \alpha \\ &\leq f[\mathscr{D}(\mathscr{T}q(\varkappa), \mathscr{T}R(\varkappa)) + \varepsilon] + \alpha \\ &\leq f[\varphi^{-1}(k(\varphi(\mathscr{D}(q(\varkappa), R(\varkappa))))) + \varepsilon] + \alpha. \end{aligned}$$

Hence using property of (Θ_1), we have $\mathscr{D}(q(\varkappa), R(\varkappa)) \leq \varepsilon + \varphi^{-1}(k(\varphi(\mathscr{D}(q(\varkappa), R(\varkappa)))))$, or $\mathscr{D}(q(\varkappa), R(\varkappa)) - \varphi^{-1}(k(\varphi\mathscr{D}(q(\varkappa), R(\varkappa)))) \leq \varepsilon$. Further, we have $\beta(\mathscr{D}(q(\varkappa), R(\varkappa))) \leq \varepsilon$. Hence

$$\mathscr{D}(q(\varkappa), R(\varkappa)) \leq \beta^{-1}(\varepsilon).$$

Finally we have to prove the uniqueness part. To prove this, suppose that there exists a function $q_1 : \mathscr{E} \to \mathscr{E}$ $(q \neq q_1)$ such that

$$\mathscr{D}(R(\varkappa), q_1(\varkappa)) < \beta^{-1}(\varepsilon).$$

Now Using $(\mathscr{D}_3)$, we have

$$\begin{aligned} f(\mathscr{D}(q(\varkappa), q_1(\varkappa))) &\leq f(\mathscr{D}(q(\varkappa), R(\varkappa)) + \mathscr{D}(R(\varkappa), q_1(\varkappa))) \\ &\leq f(\beta^{-1}(\varepsilon) + \beta^{-1}(\varepsilon)), \end{aligned}$$

using property of (Θ_1), we have

$$\mathscr{D}(q(\varkappa), q_1(\varkappa)) \leq \beta^{-1}(\varepsilon) + \beta^{-1}(\varepsilon), \tag{28}$$

Since $\beta : [0, \infty) \to [0, \infty)$ is a strictly increasing and surjective function, so for any given $\varepsilon \in [0, \infty)$ there exists a $\delta > 0$ such that $\beta(\frac{\delta}{2}) = \varepsilon$ or $\beta^{-1}(\varepsilon) = \frac{\delta}{2}$. So equation (28) implies that

$$\mathscr{D}(q(\varkappa), q_1(\varkappa)) \leq \delta,$$

which completes the proof. □

5. Stability of integral equation

Let $\mathscr{E} = C[s,t]$ be the set of all real continuous functions on $[s,t]$ equipped with $\mathcal{FM}$

$$\mathscr{D}(\varpi,\varkappa) = \|\varpi - \varkappa\|_\infty.$$

It is well known that $(\mathscr{E},\mathscr{D})$ is an $\mathcal{F}$-complete $\mathcal{F}MS$ with $f(t) = \ln(t)$ and $\alpha = 0$. We consider the integral equation

$$\varpi(b) = \int_s^t K(b,a,\varpi(a))d(a), \tag{29}$$

where $K : [s,t]\times[s,t]\times\mathbb{R} \to \mathbb{R}$. Let $\mathscr{T} : \mathscr{E} \to \mathscr{E}$ be a mapping defined by

$$\mathscr{T}\varpi(b) = \int_s^t K(b,a,\varpi(a))d(a), \tag{30}$$

for all $\varpi \in \mathscr{E}, a, b \in [s,t]$.

Theorem 5.1. *Suppose that $K : [s,t]\times[s,t]\times\mathbb{R} \to \mathbb{R}$ is continuous function satisfying the following condition*

$$| K(b,a,\varpi(a)) - K(b,a,\varkappa(a)) |^2 \leq F(b,a)\ln\left(\frac{| \varpi(a) - \varkappa(a) |^2}{4} + 1\right), \tag{31}$$

where $F : [s,t]\times[s,t] \to \mathbb{R}$ is a continuous function and for all $\varpi \in \mathscr{E}, a, b \in [s,t]$, we have

$$\int_s^t F(b,a)da \leq \frac{1}{t-s}.$$

Then, the integral equation (29) has a solution in $\mathscr{E}$.

Proof. Let $\varpi,\varkappa \in \mathscr{E}$. Using the equation (31), and using the Cauchy Schwarz inequality, we have

$$\begin{aligned}
| \mathscr{T}\varpi(b) - \mathscr{T}\varkappa(b) |^2 &= \left(\int_s^t | K(b,a,\varpi(a)) - K(b,a,\varkappa(a)) | \, da\right)^2 \\
&\leq \int_s^t 1^2 da \int_s^t | K(b,a,\varpi(a)) - K(b,a,\varkappa(a)) |^2 \, da \\
&\leq (t-s)\int_s^t F(b,a)\ln\left(\frac{| \varpi(a) - \varkappa(a) |^2}{4} + 1\right)da \\
&= (t-s)\int_s^t F(b,a)\ln\left(\frac{\mathscr{D}(\varpi,\varkappa)^2}{4} + 1\right)da \\
&= (t-s)\ln\left(\frac{\mathscr{D}(\varpi,\varkappa)^2}{4} + 1\right)\int_s^t F(b,a)da \\
&\leq \ln\left(\frac{\mathscr{D}(\varpi,\varkappa)^2}{4} + 1\right) \\
&\leq \frac{\mathscr{D}(\varpi,\varkappa)^2}{4}.
\end{aligned}$$

Therefore, we deduce that

$$\mathscr{D}(\mathscr{T}\varpi,\mathscr{T}\varkappa) \leq \frac{\mathscr{D}(\varpi,\varkappa)}{2}.$$

Therefore, $\varphi(\mathscr{D}(\mathscr{T}\varpi,\mathscr{T}\varkappa)) \leq k\varphi(\mathscr{D}(\varpi,\varkappa)), \varphi(t) = t$ is a non-decreasing function and $k = \frac{1}{2}$. Therefore $\mathscr{T}$ has a φ_b-contraction. Since $(\mathscr{E},\mathscr{D})$ is $\mathcal{F}$-complete, therefore $\mathscr{T}$ has a integral fixed point. □

Theorem 5.2. *Assume that all of Theorem 5.1's hypotheses are true. Then the following conditions hold:*

($A1$) *The integral equation (29) is HU stable, that is, if for each $\varepsilon > 0$ and each solution $\varkappa^*$ of the inequality $\mathscr{D}(\varkappa, T(\varkappa)) < \frac{\varepsilon}{2}$, for each $\varkappa \in \mathscr{E}$, there exists a solution ϖ^* of equation (29) such that*

$$\mathscr{D}(\varkappa^*, \varpi^*) < \varepsilon.$$

($A2$) *If $\{\varpi_n\}$ is a sequence in $\mathscr{E}$ such that $\lim_{n\to\infty} \mathscr{D}(\mathscr{T}\varpi_n, \varpi_n) = 0$ and ϖ^* is a of $\mathscr{T}$ then the integral equation (29) is WP.*

Proof.

($A1$) Using Theorem 5.1, there is a unique $\varpi^* \in \mathscr{E}$ such that $\varpi^* = \mathscr{T}\varpi^*$ that is $\varpi^* \in \mathscr{E}$ is solution of the integral equation ($\varpi = \mathscr{T}\varpi$). Assume that $\varepsilon > 0$ and $\varkappa^* \in \mathscr{E}$. Using ($\mathscr{D}_3$), we have

$$\begin{aligned} f(\mathscr{D}(\varkappa^*, \varpi^*)) &\le f[\mathscr{D}(\varkappa^*, \mathscr{T}\varkappa^*) + \mathscr{D}(\mathscr{T}\varkappa^*, \varpi^*)] + \alpha \\ &\le f[\frac{\varepsilon}{2} + \mathscr{D}(\mathscr{T}\varkappa^*, \mathscr{T}\varpi^*)] + \alpha. \end{aligned}$$

Hence using property of (Θ_1), we have

$$\begin{aligned} \mathscr{D}(\varkappa^*, \varpi^*) &\le \frac{\varepsilon}{2} + \mathscr{D}(\mathscr{T}\varkappa^*, \mathscr{T}\varpi^*), \\ &\le \frac{\varepsilon}{2} + \frac{\mathscr{D}(\varkappa^*, \varpi^*)}{2}. \end{aligned}$$

So we have

$$\mathscr{D}(\varkappa^*, \varpi^*) \le \varepsilon,$$

which completes the proof.

($A2$) If $\{\xi_r\}$ is a sequence in $\mathscr{E}$ such that $\lim_{r\to\infty} \mathscr{D}(\mathscr{T}\xi_r, \xi_r) = 0$ and ϖ^* is a unique fixed point of $\mathscr{T}$ (using Theorem 5.1). From the contractive condition and property ($\mathscr{D}_3$), we have

$$\begin{aligned} f(\mathscr{D}(\xi_r, \varpi^*)) &\le f[\mathscr{D}(\xi_r, \mathscr{T}\xi_r) + \mathscr{D}(\mathscr{T}\xi_r, \varpi^*)] + \alpha \\ &\le f[\mathscr{D}(\xi_r, \mathscr{T}\xi_r) + \mathscr{D}(\mathscr{T}\xi_r, \mathscr{T}\varpi^*)] + \alpha. \end{aligned}$$

On the same lines of above cases, we have
$\frac{1}{2}(\mathscr{D}(\xi_r, \varpi^*)) \le \mathscr{D}(\xi_r, \mathscr{T}\xi_r)$. Taking limit $r \to \infty$, we get

$$\lim_{r\to\infty} \frac{1}{2}(\mathscr{D}(\xi_r, \varpi^*)) \le \lim_{r\to\infty} \mathscr{D}(\xi_r, \mathscr{T}\xi_r).$$

Therefore, $\lim_{r\to\infty} \frac{1}{2}(\mathscr{D}(\xi_r, \varpi^*)) = 0$. Hence $\mathscr{D}(\xi_r, x^*) = 0$. This shows that the integral equation (29) is well-posed. □

Significance. We discuss the existence of PO in the setting of $\mathcal{F}MS$ and obtain some results on the HU stability and WP of the FP problem, Cauchy functional equation, and integral equations.

Acknowledgements The authors are grateful for the research funding 02011/11/2020/ NBHM (RP)/ R&D-II/7830 from the NBHM, DAE, India and DST-FIST (Govt. of India) for the grant SR/FST/MS-1/2017/13 to support this research work.

Competing Interests. The authors declare that they have no competing interests.

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