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Existence of ω-Periodic Solutions for Second Order Delay Differential Equation in Banach Spaces

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Abstract. The propose of the paper is devoted to study the existence of ω-periodic solutions for second-order delay differential equation in abstract Banach space. Firstly, we build a new maximum principle for the ω-periodic solutions of the corresponding linear equation. Secondly, with the help of this maximum principle, we study the existence of the minimal and maximal periodic solutions for our concerns problem by means of perturbation method and monotone iterative technique of the lower and upper solutions. In addition, an example is presented to show the application of our main results.

1. Introduction

The existence of periodic solutions is an important aspect of functional differential equation, many mathematicians have studied it via different tools and methods, for example the fixed point theory [1–3, 8, 26], the method of upper and lower solutions and the monotone iterative technique [9, 16, 26], the continuation method of topological degree [21, 22, 25]. Recently, Wang and Luo [28] discussed the existence of positive periodic solutions for the second order differential equation

$$-u''(t) + b(t)u(t) = g(t)f(t, u(t), u(t-\tau)), \quad t \in \mathbb{R},$$

where $g(t) \in C(\mathbb{R}, \mathbb{R})$, $f(t, x) \in C(\mathbb{R} \times [0, \infty), [0, \infty))$, $g(t + \omega) = g(t)$ and $f(t + \omega, u) = f(t, u)$. By using the fixed point index theory of cones, the existence results of singular positive periodic solutions and multiple positive periodic solutions are obtained.

Delay differential equation theory is an important branch of differential equation theory. It has a wide range of physical, biological, economic, engineering background and practical mathematical model. Therefore, in the past few decades, they have emerged as an important research field and studied many properties of their solutions, we refer to [12, 29]. The problem of periodic solutions of delay differential equations is an important research field, because they can consider the seasonal fluctuations in the model, and some researchers have studied it in recent years. In particular, when $E = \mathbb{R}$, several authors have

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been consider the existence of periodic solutions of the second-order differential equations with delays, we refer to the reader to see [29]-[30]. More authors have discussed the existence of periodic solutions for second-order differential equations without delay in the scalar space $\mathbb{R}$ or general abstract space E, see [7]-[32].

It is well known that the monotone iterative method for upper and lower solutions is an effective and flexible mechanism. The monotone sequence between the upper and lower approximate solutions converged to the upper and lower solutions of the minimum and maximum solution is obtained. In [14, 15], Jiang et.al considered the periodic problem of the second-order functional differential equation with time delay

$$-u''(t) = f(t, u(t), u(t-\tau)), \quad t \in \mathbb{R},$$

where $f \in C(\mathbb{R}^3, \mathbb{R}), \tau \in C(\mathbb{R}, [0, +\infty))$ and $\tau(t) = \tau(t+\omega)$. By exploring monotone iterative technique, they obtained the existence results of ω-periodic solutions.

In [4], authors investigated the existence of positive periodic solutions for a class of nonlinear second order ordinary differential equations of the form

$$-u''(t) + b(t)u(t) = g(t)f(t, u(t)), \quad t \in \mathbb{R},$$

where $b(t)$ and $g(t)$ are continuous ω-periodic positive functions, and $f \in C(\mathbb{R} \times [0, \infty), [0, \infty))$.

In [20], authors investigated the existence of positive periodic solutions for a class of second order delayed differential equations in Banach spaces

$$-u''(t) = f(t, u(t), u(t-\tau)), \quad t \in \mathbb{R},$$

By using monotone iterative technique, they obtained the existence of ω-periodic solutions.

Inspired and motivated by the jobs mentioned above, in this paper, we deal with the existence of ω-periodic solutions for second-order functional differential equation with delay in E

$$-u''(t) + b(t)u(t) = g(t)f(t, u(t), u(t-\tau)), \quad t \in \mathbb{R}, \tag{1.1}$$

where E is an ordered Banach space, $f : \mathbb{R} \times E \times E \to E$ is a continuous function which is ω-periodic in t and τ is a positive constant which denotes the time delay, $b(t)$ and $g(t)$ are continuous ω-periodic positive functions. Here, by an ω-periodic positive solution, we mean a function $u^*(t)$ which satisfies (1.1) and $u^*(t+\omega) = u^*(t), u^*(t) > 0$ for $t \in (0, \omega)$.

The highlights and advantages of this paper are presented in two aspects. on the one hand, we construct two monotone iterative sequences and prove that the sequences monotonically converge to the minimal and maximal periodic solutions for delayed differential equations. In this paper, the technology we use is different from those in [4, 20]. Our results are more general than those in [4, 20]. As well as we considered the periodic problem(1.1)in more general Banach space, therefore, it has more extensive application background. On the other hand, by means of the perturbation theorem of unit operator, we establish a new maximum principle for the ω-periodic solutions of the corresponding linear delay equation, it improves and improves the result of paper [4, 20].

The paper is organized as follows: In Section 2, we introduce some notations and build a maximum principle for the corresponding linear equation. In Section 3 we present existence result for our concern problem (1.1). In Section 4, we give an example to illustrate the feasibility of our results.

2. Preliminaries

Throughout this paper, let E be an ordered Banach space, whose positive cone $K = \{u \in E | u \geq \theta\}$ is normal with normal constant N. Denote $C^m_\omega(\mathbb{R}, E)$ the m-th order continuous differentiable ω-periodic E-value function space for $m \in \mathbb{N}$. Let $C_\omega(\mathbb{R}, E)$ denote the space $\{u \in C(\mathbb{R}, E) | u(t+\omega) = u(t), t \in \mathbb{R}\}$ endowed the maximum norm $\|u\|_C = \max_{t\in[0,\omega]} \|u(t)\|$. Evidently, $C_\omega(\mathbb{R}, E)$ is an order Banach space with the partial order " $\leq$ " deduced by the positive cone $K_C = \{u \in C_\omega(\mathbb{R}, E) | u(t) \geq \theta, t \in \mathbb{R}\}$ and P_C is also normal with the normal constant N. For $v, w \in C_\omega(\mathbb{R}, E)$ with $v \leq w$, we use $[v, w]$ to denote the order interval $\{u|v \leq u \leq w\}$ in $C_\omega(\mathbb{R}, E)$, $[v(t), w(t)]$ to denote the order interval $\{u(t)|v(t) \leq u(t) \leq w(t), t \in \mathbb{R}\}$ in E. We define the following constants:

$$l = \sqrt{\min_{0\leq t\leq\omega} b(t)}, \quad L = \sqrt{\max_{0\leq t\leq\omega} b(t)}.$$

Firstly, we consider the second-order differential equation with delay in E:

$$-u''(t) + b(t)u(t) + Mu(t-\tau) = h(t), \ \ t \in \mathbb{R}, \ h \in C_\omega(\mathbb{R}, E), \tag{2.1}$$

where $M \geq 0$.

By Lemma 2.1 in [19], for any $h \in C_\omega(\mathbb{R}, E)$, the second-order linear differential equation

$$-u''(t) + L^2u(t) = h(t), \ \ t \in \mathbb{R}$$

has a unique ω-periodic solution $u(t)$ which is given by

$$u(t) = \int_{t-\omega}^{t} \Psi(t-s)h(s)ds := T_0h(t), \quad t \in \mathbb{R}, \tag{2.2}$$

where

$$\Psi(t) = \frac{\cosh L(t-\frac{\omega}{2})}{2L\sinh\frac{L\omega}{2}} \tag{2.3}$$

is unique solution of linear second order boundary value problem

$$\begin{cases} -u''(t) + L^2u(t) = h(t), \\ u(0) = u(\omega), \ \ u'(0) = u'(\omega) - 1, \end{cases} \tag{2.4}$$

From (2.3), we compute that

$$0 \leq \frac{1}{2L\sinh\frac{L\omega}{2}} \leq \Psi(t) \leq \frac{\cosh\frac{L\omega}{2}}{2L\sinh\frac{L\omega}{2}}, \quad t \in [0, \omega]. \tag{2.5}$$

Clearly, the solution operator $T_0 : C_\omega(\mathbb{R}, E) \to C_\omega(\mathbb{R}, E)$ is a positive linear continuous operator.

Next, by simple computation, for every $h \in C_\omega(\mathbb{R}, E)$, we can get

$$\begin{aligned} \|T_0h(t)\| &= \left\| \int_{t-\omega}^{t} \Psi(t-s)h(s)ds \right\| \\ &\leq \int_{t-\omega}^{t} \Psi(t-s)ds\|h\|_C \\ &= \int_0^{\omega} \Psi(\omega-s)ds\|h\|_C \\ &= \frac{1}{L^2}\|h\|_C. \end{aligned}$$

Thus, $\|T_0\| \leq \frac{1}{L^2}$.

On the other hand, we can choose $e_0 \in E$ with $e_0 \neq \theta$, let $h_0(t) = e_0$, then $h_0 \in C_\omega(\mathbb{R}, E)$ and $\|h_0\|_C = \|e_0\|$, so

$$\|T_0h_0(t)\| = \| \int_{t-\omega}^{t} \Psi(t-s)h_0(s)ds\| = \int_0^{\omega} \Psi(s)ds \cdot \|e_0\| = \frac{1}{L^2}\|h_0\|_C,$$

Thus, we can get

$$\|T_0\| = \int_0^{\omega} \Psi(s)ds = \frac{1}{L^2}. \tag{2.6}$$

Moreover, for every $h \in C_\omega(\mathbb{R}, P)$, from (2.5) we can get

$$T_0h(t) = \int_{t-\omega}^{t} \Psi(t-s)h(s)ds \geq \frac{1}{2L\sinh\frac{L\omega}{2}} \int_0^\omega h(s)ds,$$

and on the other hand, we have

$$T_0h(t) = \int_{t-\omega}^{t} \Psi(t-s)h(s)ds \leq \frac{\cos\frac{L\omega}{2}}{2L\sinh\frac{L\omega}{2}} \int_0^\omega h(s)ds.$$

This implies that

$$T_0h(t) \geq \frac{1}{\cosh\frac{L\omega}{2}} T_0h(s), \ \ \forall\, t, s \in \mathbb{R}. \tag{2.7}$$

Lemma 2.1. *For any $h \in C_\omega(\mathbb{R}, E)$, if $\frac{L^2}{2} \leq M < l^2 \leq \sigma^2 L^2, \left(\sigma = \frac{1}{\cosh\frac{L\omega}{2}}\right)$, then the differential Eq.(2.1) exists a unique ω-periodic solution $u := Th \in C_\omega(\mathbb{R}, E)$. Moreover, the solution operator $T : C_\omega(\mathbb{R}, E) \to C_\omega(\mathbb{R}, E)$ is a positive bounded linear operator with the spectral radius $r(T) \leq \frac{1}{l^2-M}$.*

Proof. From (2.2), it is well that for $u \in C_\omega(\mathbb{R}, E)$ is an ω-periodic solution of Eq. (2.1) if and only if

$$u(t) = \int_{t-\omega}^{t} \Psi(t-s)(h(s) - Mu(s-\tau) - b(s)u(s) + L^2u(s))ds, \ \ t \in \mathbb{R}. \tag{2.8}$$

Define an operator $B_1 : C_\omega(\mathbb{R}, E) \to C_\omega(\mathbb{R}, E)$ as following

$$B_1u(t) = Mu(t-\tau). \tag{2.9}$$

Clearly, $B_1 : C_\omega(\mathbb{R}, E) \to C_\omega(\mathbb{R}, E)$ is a positive linear bounded operator with $\|B_1\| = M$.

Define the operator $P : C_\omega(\mathbb{R}, E) \to C_\omega(\mathbb{R}, E)$ by

$$Pu(t) = (L^2 - b(t))u(t), \ \ u \in C_\omega(\mathbb{R}, E).$$

Then, P is a positive linear bounded operator and $\|P\| \leq L^2 - l^2$.

Thus, from (2.3), (2.8) and (2.9), we have

$$(I + T_0B_1 - T_0P)u(t) = T_0h(t), \ \ t \in \mathbb{R}. \tag{2.10}$$

By (2.5), we have

$$\|T_0B_1 - T_0P\| \leq \|T_0\| \cdot \|B_1\| + \|T_0\| \cdot \|P\| \leq \frac{M}{L^2} + \frac{L^2 - l^2}{L^2} = \frac{M + L^2 - l^2}{L^2} < 1.$$

By the perturbation theorem of unit operator, $I + T_0B_1 - T_0P$ has a bounded inverse operator $(I + T_0B_1 - T_0P)^{-1}$ which is represented as

$$(I + T_0B_1 - T_0P)^{-1} = \sum_{i=0}^{\infty}(-1)^i(T_0B_1 - T_0P)^i = \sum_{i=0}^{\infty}(T_0B_1 - T_0P)^{2i}(I - T_0B_1 + T_0P),$$

and

$$\|(I + T_0B_1 - T_0P)^{-1}\| \leq \frac{1}{1 - \|T_0B_1 - T_0P\|} \leq \frac{L^2}{l^2 - M}. \tag{2.11}$$

Thus, operator Eq. (2.10) has unique solution given by

$$\begin{aligned} u(t) &= (I + T_0B_1 - T_0P)^{-1}T_0h(t) \\ &= (I + T_0B_1 - T_0P)^{-1}\int_{t-\omega}^{t} \Psi(t-s)h(s)ds \\ &:= Th(t),\ t \in \mathbb{R}, \end{aligned} \tag{2.12}$$

it is also ω-periodic solution of the linear Eq. (2.1). Clearly, $T : C_\omega(\mathbb{R}, E) \to C_\omega(\mathbb{R}, E)$ is a bounded linear operator. By (2.11) and (2.12), we have

$$\begin{aligned} \|(Th)(t)\| &\le \Big\|(I + T_0B_1 - T_0P)^{-1}\int_{t-\omega}^{t} \Psi(t-s)h(s)ds\Big\| \\ &\le \frac{L^2}{l^2 - M}\int_0^{\omega} \Psi(t-s)h(s)ds \\ &\le \frac{1}{l^2 - M}\|h\|_C, \end{aligned} \tag{2.13}$$

which implies that $\|T\| \le \frac{1}{l^2-M}$. Therefore, $r(T) < \|T\| \le \frac{1}{l^2-M}$.

On the other hand, we need prove that the operator T is positive. From (2.11) and (2.12), we only need to prove $(I - T_0B_1 + T_0P)T_0$ is positive. For $h \in C_\omega(\mathbb{R}, K)$, let $v_0 = T_0h(0) \in K$, then v_0 can be regarded as a constant function in $C_\omega(\mathbb{R}, K)$. From (2.7), it follows that

$$T_0h \ge \sigma v_0,\ \ T_0h \le \frac{1}{\sigma}v_0,\ \ h \in C_\omega(\mathbb{R}, P).$$

By the definitions of B_1, P, and $\frac{L^2}{2} \le M < l^2 \le \sigma^2L^2$, for $\forall u \ge 0$, we have

$$(B_1 - P)u = Mu - (L^2 - b(t))u \ge (M + l^2 - L^2)u \ge 0.$$

Thus, we obtain $B_1 - P$ is positive. And positivity of T_0, we get $T_0B_1 - T_0P$ is also positive. Hence, we have

$$(T_0B_1 - T_0P)T_0h \le \frac{M}{\sigma L^2}v_0.$$

Thus, by the assumption of Lemma , we have

$$\begin{aligned} (I - T_0B_1 + T_0P)T_0h &= T_0h - (T_0B_1 - T_0P)T_0h \\ &\ge \frac{1}{\cosh\frac{L\omega}{2}}v_0 - \cosh\frac{L\omega}{2}\cdot\frac{M}{L^2}v_0 \\ &= \Big(\frac{1}{\cosh\frac{L\omega}{2}} - \cosh\frac{L\omega}{2}\cdot\frac{M}{L^2}\Big)v_0 \ge \theta, \end{aligned}$$

which implies $(I - T_0B_1 + T_0P)T_0$ is positive. Thus, $T : C_\omega(\mathbb{R}, E) \to C_\omega(\mathbb{R}, E)$ is a positive operator. □

Lemma 2.2. *Let $\frac{L^2}{2} \le M \le l^2 \le \sigma^2L^2$, if $u \in C^2_\omega(\mathbb{R}, E)$ satisfies*

$$-u''(t) + b(t)u(t) + Mu(t-\tau) \ge \theta,\ \ t \in \mathbb{R},$$

then $u(t) \ge \theta$, $t \in \mathbb{R}$.

Let $\alpha(\cdot)$ denote the Kuratowski measure of noncompactness of the bounded set. For the detail of the definition and properties of the measure of noncompactness, see [1,3,5], For any $D \subset C_\omega(\mathbb{R}, E)$ and $t \in \mathbb{R}$, set $D(t) = \{u(t)|u \in D\} \subset E$. If D is bounded in $C_\omega(\mathbb{R}, E)$, then $D(t)$ is bounded in E and $\alpha(D(t)) \le \alpha(D)$.

Lemma 2.3. *[13] Let E be a Banach space and let $D \subset C(J, E)$ be bounded and equicontinuous, where J is a finite closed interval in $\mathbb{R}$. Then $\alpha(D(t))$ is continuous on J and*

$$\alpha(D) = \max_{t \in J} \alpha(D(t)) = \alpha(D(J)).$$

Lemma 2.4. *[6] Let E be a Banach space, $D = \{u_n\} \subset C(J, E)$ be a bounded and countable set. Then $\alpha(D(t))$ is Lebesgue integrable on J and*

$$\alpha\Big(\Big\{\int_J u_n(s)ds\Big\}\Big) \leq \int_J \alpha(D(t))dt.$$

3. Main results

Definition 3.1. *If a function $v_0 \in C^2_\omega(\mathbb{R}, E)$ and satisfies*

$$-\alpha''(t) + b(t)\alpha(t) \leq g(t)f(t, \alpha(t), \alpha(t-\tau)), \quad t \in \mathbb{R}, \tag{3.1}$$

we call it an ω-periodic lower solution of Eq.(1.1). If the inequality of (3.1) is inverse, we call it an ω-periodic upper solution of the Eq.(1.1).

Theorem 3.1. *Let E be an ordered Banach space, whose positive cone K is normal cone, $f : \mathbb{R} \times E \times E \to E$ be continuous and $f(t, x, y)$ be ω-periodic in t for every $x, y \in E$, and $g(t)$ is continuous ω-periodic positive functions. Assume that the problem (1.1) has lower and upper ω-periodic solutions $v_0, w_0 \in C^2_\omega(R, E)$ with $v_0 \leq w_0$ and the following conditions*

(H1) there exist constants $0 \leq M < l^2 < \sigma^2 L^2$ such that

$$g(t)f(t, x_2, y_2) - g(t)f(t, x_1, y_1) \geq -M(y_2 - y_1)$$

for all $t \in \mathbb{R}, v_0(t) \leq x_1 \leq x_2 \leq w_0(t), v_0(t-\tau) \leq y_1 \leq y_2 \leq w_0(t-\tau)$,

(H2) there exist constants $m, m_1 \geq 0, m + m_1 < \frac{l^2 - M}{2}$ such that

$$\alpha(\{g(t)f(t, u_n(t), u_n(t-\tau)) + Mu_n(t-\tau))\} \leq m\alpha(\{u_n(t)\}) + m_1\alpha(\{u_n(t-\tau)\})$$

for any $t \in \mathbb{R}$ and monotonous sequence $\{u_n\} \subset [v_0, w_0]$

hold, then Eq.(1.1) has minimal and maximal ω-periodic solution $\underline{u}, \overline{u}$ between v_0 and w_0, which can be obtained by monotone iterative sequences starting from v_0 and w_0.

Proof. For any $h \in [v_0, w_0]$, we consider the following equation

$$-u''(t) + b(t)u(t) + Mu(t-\tau) = g(t)f(t, u(t), u(t-\tau)) + Mh(t-\tau), \quad t \in \mathbb{R}. \tag{3.2}$$

From Lemma 2.1, it implies that Eq.(3.2) exists unique ω-periodic solution $u \in C_\omega(\mathbb{R}, E)$, which can be given as

$$\begin{aligned} u(t) = (I + T_0B_1 - T_0P)^{-1} \int_{t-\omega}^{t} &\Psi(t-s)(g(s)f(s, h(s), h(s-\tau)) \\ &+ Mh(s-\tau))ds := (Qh)(t), \ t \in \mathbb{R}. \end{aligned} \tag{3.3}$$

Clearly, the $Q : [v_0, w_0] \to C_\omega(\mathbb{R}, E)$ is continuous. We rewrite Eq.(1.1) in the following form

$$-u''(t) + b(t)u(t) + Mu(t-\tau) = g(t)f(t, u(t), u(t-\tau)) + Mu(t-\tau), \quad t \in \mathbb{R}. \tag{3.4}$$

Hence, we can claim that $u \in [v_0, w_0]$ is the ω-periodic of Eq.(3.4) if and only if u is the fixed point of the operator Q. Now, we will divide into four steps to complete the proof.

Step 1. We will prove that the operator $Q : [v_0, w_0] \to C_\omega(\mathbb{R}, E)$ is equicontinuous. For any $u \in [v_0, w_0]$, by the periodicity of u, we consider it on $[0, \omega]$. Let $0 \le t_1 < t_2 \le \omega$, we obtain

$$\begin{aligned}
(Qu)(t_2) - (Qu)(t_1) &= (I + T_0B_1 - T_0P)^{-1} \int_{t_2-\omega}^{t_2} \Psi(t_2 - s) \\
&\quad \times (g(s)f(s, u(s), u(s-\tau)) + Mu(s-\tau))ds \\
&\quad - (I + T_0B_1 - T_0P)^{-1} \int_{t_2-\omega}^{t_2} \Psi(t_1 - s)(g(s)f(s, u(s), u(s-\tau)) + Mu(s-\tau))ds \\
&= (I + T_0B_1 - T_0P)^{-1} \int_{t_1-\omega}^{t_2} (\Psi(t_2 - s) - \Psi(t_1 - s))(g(s)f(s, u(s), u(s-\tau)) + Mu(s-\tau))ds \\
&\quad - (I + T_0B_1 - T_0P)^{-1} \int_{t_2-\omega}^{t_1-\omega} \Psi(t_1 - s)(g(s)f(s, u(s), u(s-\tau)) + Mu(s-\tau))ds \\
&\quad + (I + T_0B_1 - T_0P)^{-1} \int_{t_2}^{t_1} \Psi(t_2 - s)(g(s)f(s, u(s), u(s-\tau)) + Mu(s-\tau))ds \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

This means that

$$\|(Qu)(t_2) - (Qu)(t_1)\| \le \|I_1\| + \|I_2\| + \|I_3\|. \tag{3.5}$$

Therefore, we only need to check tend to 0 independently of $u \in [v_0, w_0]$ when $t_2 - t_1 \to 0, i = 1, 2, 3$.

For any $u \in [v_0, w_0]$, and by the condition (H1), we have

$$\begin{aligned}
g(t)f(t, v_0(t), v_0(t-\tau)) + Mv_0(t-\tau) &\le g(t)f(t, u(t), u(t-\tau)) + Mu(t-\tau) \\
&\le g(t)f(t, w_0(t), w_0(t-\tau)) + Mw_0(t-\tau), \quad t \in \mathbb{R}.
\end{aligned}$$

And combine with the normality of the cone P, then there exists $M_1 > 0$ such that

$$\|g(t)f(t, u(t), u(t-\tau)) + Mu(t-\tau)\| \le M_1, \quad t \in \mathbb{R}, \ u \in [v_0, w_0].$$

Hence, we get

$$\begin{aligned}
\|I_1\| &\le \|(I + T_0B_1 - T_0P)^{-1}\| \cdot \int_{t_1-\omega}^{t_2} \|(\Psi(t_2 - s) - \Psi(t_1 - s))(g(s)f(s, u(s), u(s-\tau)) + Mu(s-\tau))\|ds \\
&\le \frac{L^2M_1}{l^2 - M} \int_{t_1-\omega}^{t_2} \|(\Psi(t_2 - s) - \Psi(t_1 - s))\|ds \\
&\le \frac{L^2M_1}{l^2 - M} \int_0^{t_1+\omega-t_2} \|(\Psi(t_2 - t_1 + s) - \Psi(s))\|ds \\
&\to 0,
\end{aligned}$$

when $t_2 - t_1 \to 0$.

$$\begin{aligned}
\|I_2\| &\le \|(I + T_0B_1 - T_0P)^{-1}\| \cdot \int_{t_2-\omega}^{t_1-\omega} \|\Psi(t_1 - s)(g(s)f(s, u(s), u(s-\tau)) + Mu(s-\tau))\|ds \\
&\le \frac{L^2M_1}{l^2 - M} \int_{t_1-\omega}^{t_2-\omega} \|\Psi(t_1 - s)\|ds \\
&\to 0,
\end{aligned}$$

when $t_2 - t_1 \to 0$.

$$\begin{aligned}\|I_3\| &\le \|(I + T_0B_1 - T_0P)^{-1}\| \cdot \int_{t_2}^{t_1} \Psi(t_2 - s)(g(s)f(s, u(s), u(s-\tau)) + Mu(s-\tau))ds \\ &\le \frac{L^2 M_1}{l^2 - M} \int_{t_1}^{t_2} \|\Psi(t_2 - s)\| ds \\ &\to 0,\end{aligned}$$

when $t_2 - t_1 \to 0$.

As a result, $\|(Qu)(t_2) - (Qu)(t_1)\|$ tends to 0 independently of $u \in [v_0, w_0]$ as $t_2 - t_1 \to 0$, this means that $Q : [v_0, w_0] \to C_\omega(R, E)$ is equicontinuous.

Step 2. We demonstrate that the operator Q satisfies the following properties.

(i) $v_0 \le Qv_0$, $Qw_0 \le w_0$,

(ii) For any $u_1, u_2 \in [v_0, w_0]$ with $u_1 \le u_2$, $Qu_1 \le Qu_2$.

Let $v_1 = Qv_0, v = v_0 - v_1$, by the definition 3.1, (H1) and (2.2), we have

$$\begin{aligned}-v''(t) + b(t)v(t) + Mv(t-\tau) &= -v_0''(t) + b(t)v_0(t) + Mv_0(t-\tau) + v_1'' - b(t)v_1(t) - Mv_1(t-\tau) \\ &\le g(s)f(s, v_0(s), v_0(s-\tau)) + Mv_0(s-\tau) - g(s)f(s, v_0(s), v_0(s-\tau)) - Mv_0(s-\tau) \\ &= \theta,\end{aligned}$$

by Lemma 2.2, it follows that $v(t) \le \theta$ for $t \in \mathbb{R}$, we have $v_0 \le Qv_0$. Similarly, we can prove that $Qw_0 \le w_0$. Therefore, (i) holds. For any $u_1, u_2 \in [v_0, w_0]$ with $u_1 \le u_2$ and $t \in \mathbb{R}$, by the condition (H1), we obtain

$$g(t)f(t, u_1(t), u_1(t-\tau)) + Mu_1(t-\tau) \le g(t)f(t, u_2(t), u_2(t-\tau)) + Mu_2(t-\tau),$$

by (3.3), it follows that $Qu_1 \le Qu_2$. Thus, (ii) holds.

Step 3. We define two sequences $\{v_n\}$ and $\{w_n\}$ in $[v_0, w_0]$ by the iterative scheme

$$v_n = Qv_{n-1}, \;\; w_n = Qw_{n-1}, \;\; n = 1, 2. \dots. \tag{3.6}$$

Then from the monotonicity of the operator Q, we have

$$v_0 \le v_1 \le v_2 \le \cdots \le v_n \le \cdots \le w_n \le \cdots \le w_2 \le w_1 \le w_0. \tag{3.7}$$

Clearly, $\{v_n\}, \{w_n\} \subset [v_0, w_0]$ are equicontinuous in $\mathbb{R}$. Next, we prove that $\{v_n\}$ and $\{w_n\}$ are convergent in $C_\omega(\mathbb{R}, E)$. Because $\{v_n\}$ is a bounded and countable set, combine Lemma 2.4 with the condition (H2), we have

$$\begin{aligned}&\alpha(\{v_n(t)\}) = \alpha(\{Qv_{n-1}(t)\}) \\ &= \alpha\Big(\Big\{(I + T_0B_1 - T_0P)^{-1} \int_{t-\omega}^{t} \Psi(t-s)(g(s)f(s, v_{n-1}(s), v_{n-1}(s-\tau)) + Mv_{n-1}(s-\tau))ds\Big\}\Big) \\ &\le 2\|(I + T_0B_1 - T_0P)^{-1}\| \cdot \int_{t-\omega}^{t} \Psi(t-s)\alpha(\{g(s)f(s, v_{n-1}(s), v_{n-1}(s-\tau)) + Mv_{n-1}(s-\tau)\})ds \\ &\le 2\|(I + T_0B_1 - T_0P)^{-1}\| \cdot \int_{t-\omega}^{t} \Psi(t-s)(m\alpha(\{v_{n-1}(s)\}) + m_1\alpha(\{v_{n-1}(s-\tau)\}))ds,\end{aligned}$$

and by the periodicity of v_{n-1} and definition of measure of noncompactness, we have $\alpha(\{v_{n-1}(s)\}) = \alpha(\{v_{n-1}(s-\tau)\})$, thus,

$$\begin{aligned}\alpha(\{v_n(t)\}) &\le 2(m+m_1)\cdot\|(I+T_0B_1-T_0P)^{-1}\|\cdot\int_{t-\omega}^{t}\Psi(t-s)\alpha(\{v_{n-1}(s)\}))ds\\ &\le 2(m+m_1)\cdot\|(I+T_0B_1-T_0P)^{-1}\|\cdot\int_0^{\omega}\Psi(s)ds\max_{t\in[0,\omega]}\alpha(\{v_{n-1}(t)\})\\ &\le \frac{2(m+m_1)}{l^2-M}\alpha(\{v_n\}).\end{aligned}$$

Since $\{v_n\}$ is equicontinuous, by Lemma 2.3, we get

$$0 \le \alpha(\{v_n\}) \le \frac{2(m+m_1)}{l^2-M}\alpha(\{v_n\}).$$

when $\frac{2(m+m_1)}{l^2-M} < 1$, hence $\alpha(\{v_n\}) = 0$. Similarly, we prove that $\alpha(\{w_n\}) = 0$. Thus, $\{v_n\}$ and $\{w_n\}$ are relatively compact in $C_\omega(\mathbb{R}, E)$, so there are convergent subsequences in $\{v_n\}$ and $\{w_n\}$, respectively. Combining this with the monotonicity and the normality of the cone P_C, it is easy to see than $\{v_n\}$ and $\{w_n\}$ themselves are convergent,i.e., there are $u, u \in C_\omega(\mathbb{R}, E)$ such that $\lim_{n\to\infty} v_n = \underline{u}$ and $\lim_{n\to\infty} w_n = \overline{u}$.

Taking limit in (3.6), we obtain

$$\underline{u} = Q\underline{u}, \quad \overline{u} = Q\overline{u}. \tag{3.8}$$

Hence $\underline{u}, \overline{u} \in C_\omega(\mathbb{R}, E)$ are fixed points of Q.

Step 4. We will show that the minimal and maximal property of $\underline{u}, \overline{u}$. Assume that $\widetilde{u}$ is a fixed point of Q with $\widetilde{u} \in [v_0, w_0]$. For every $t \in \mathbb{R}$, $v_0 \le \widetilde{u}(t) \le w_0(t)$.

$$v_1(t) = (Qv_0)(t) \le (Q\widetilde{u})(t) = \widetilde{u}(t) \le (Qw_0)(t) = w_1(t), \quad t \in \mathbb{R}. \tag{3.9}$$

Similarly, $v_1(t) \le \widetilde{u}(t) \le w_1(t)$ for $t \in \mathbb{R}$. Thus, we have

$$v_n \le \widetilde{u} \le w_n, \quad n = 1, 2, \ldots. \tag{3.10}$$

Taking limit in (3.10) as $n \to \infty$, we have $\underline{u} \le \widetilde{u} \le \overline{u}$. Thus, $\underline{u}, \overline{u}$ are minimal and maximal ω-periodic solutions of Eq.(1.1), and $\underline{u}, \overline{u}$ can be obtained by the iterative sequences defined in (3.6) starting from v_0 and w_0. □

Next, we discuss the existence of solutions for Eq. (1.1) without the assumption that the lower and upper solutions of Eq. (1.1) exist.

Theorem 3.2. *Let E be an ordered Banach space, whose positive cone P is normal cone, $f : \mathbb{R}\times E\times E \to E$ be continuous and $f(t, x, y)$ be ω-periodic in t for every $x, y \in E$ and $g(t)$ is continuous ω-periodic positive functions. If the following conditions*

(H1)′ there exist constant $0 \le M < l^2 < \sigma^2 L^2$ such that

$$g(t)f(t, x_2, y_2) - g(t)f(t, x_1, y_1) \ge -M(y_2 - y_1)$$

for all $t \in \mathbb{R}, x_i, y_i \in E(i = 1, 2)$ with $x_1 \le x_2, y_1 \le y_2$.

(H2)′ there exist constant $m, m_1 \ge 0$, $m + m_1 < \frac{l^2-M}{2}$ such that

$$\alpha(\{g(t)f(t, u_n(t), u_n(t-\tau))\} + Mu_n(t-\tau)) \le m\alpha(\{u_n(t)\}) + m_1\alpha(\{u_n(t-\tau)\})$$

for any $t \in \mathbb{R}$ and monotonous sequence $\{u_n\} \subset C_\omega(\mathbb{R}, E)$,

(H3) there exist constants $L_1 \ge 0$ with $L_1 < \sigma^2 l^2$ and $h \in C_\omega(\mathbb{R}, E), h(t) \ge \theta$ such that for $u_1, u_2 \in C_\omega(\mathbb{R}, E)$

(i) $g(t)f(t,u,u_1) \leq L_1u_1 + h(t), \ \ u_1 \geq \theta,$

(ii) $g(t)f(t,u,u_1) \geq L_1u_1 - h(t), \ \ u_1 \leq \theta,$

hold, then Eq.(1.1) has minimal and maximal ω-periodic solution $\underline{u}, \overline{u}$, which can be obtained by monotone iterative sequences starting.

Proof. Consider the existence of ω-periodic solution for the linear equation in E

$$-u''(t) + b(t)u(t) = h(t) + L_1u(t-\tau), \ \ t \in \mathbb{R}, \tag{3.11}$$

where $h \in C_\omega(\mathbb{R}, E)$. Define an operator $B_2 : C_\omega(\mathbb{R}, E) \to C_\omega(\mathbb{R}, E)$ as following

$$B_2u(t) = L_1u(t-\tau), \tag{3.12}$$

thus $B_2 : C_\omega(\mathbb{R}, E) \to C_\omega(\mathbb{R}, E)$ is a linear bounded operator and $\|B_2\| \leq L_1$. Similar to the proof of Lemma 2.1, we obtain the ω-periodic solution of the linear Eq. (3.11) is equivalent to the solution of operator equation

$$(I - T_0B_2 - T_0P)u(t) = T_0h(t), \quad t \in \mathbb{R}. \tag{3.13}$$

Therefore, by the condition (H3), we get $\|T_0B_2 + T_0P\| \leq 1$ which implies that $I - T_0B_2 - T_0P$ has a bounded inverse operator $(I-T_0B_2-T_0P)^{-1}$, similar to the proof of Lemma 2.1, we also can prove that $(I-T_0B_2-T_0P)^{-1}$ is positive. Thus, operator equation (3.12) has unique solution $u = (I-T_0B_2-T_0P)^{-1}T_0h$, which is ω-periodic solution of the equation (3.14) and $u(t) \geq \theta$ by $h(t) \geq \theta$ for $t \in \mathbb{R}$. i.e., the Eq. (3.11) has a unique positive solution $\widetilde{u} \in C^2_\omega(, \mathbb{E})$.

Let $v_0 = -\widetilde{u}$ and $w_0 = \widetilde{u}$, combine with the condition (H3), we have

$$-v''(t) + b(t)u(t) = L_1v_0(t-\tau) - h(t) \leq g(t)f(t, v_0(t), v_0(t-\tau)), \ \ t \in \mathbb{R},$$

and

$$-w''(t) + b(t)u(t) = L_1v_0(t-\tau) - h(t) \geq g(t)f(t, w_0(t), w_0(t-\tau)), \ \ t \in \mathbb{R},$$

Hence, we can state that v_0 and w_0 are lower solution and upper solution of Eq. (1.1), respectively. Thus, our conclusion follows from Theorem 3.1. □

4. Applications

In this section, we give an example to illustrate our main results.

Example 4.1. *Consider the differential equation*

$$-u''(t) + \frac{1}{20}(3 + \cos t)u(t) = f(x, t, u(x,t), u(x, t-\tau)), \ \ x \in J, \ t \in \mathbb{R}, \tag{4.1}$$

where $J = [0, \omega], \omega = 2\pi, f : J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous function which is ω-periodic in t, τ is positive constant which denotes the time delay. Here we have $b(t) = \frac{1}{20}(3 + \cos t), \ g(t) = 1$.

Further,

$$L = \sqrt{\max_{0 \leq t \leq \omega} b(t)} = \sqrt{\frac{1}{5}} \in (0, \frac{\pi}{\omega}).$$

We will use our new results to prove that the equation has a positive solution. We have the following results.

Theorem 4.1. *(F0) let $f(x,t,0,0) \geq 0$ for $(x,t) \in J \times \mathbb{R}$ and there exist a function $w = w(x,t) \in C^{1,2}(J \times \mathbb{R})$ which is ω-periodic in t such that*

$$-\frac{\partial^2}{\partial t^2}w(x,t) \geq f(x, t, w(x,t), w(x, t-\tau)), \ \ (x,t) \in J \times \mathbb{R},$$

(F1) there exist constants M satisfying $0 \leq M < l^2 < \sigma^2 L^2$, *such that*

$$f(x,t,y_0,y_1) - f(x,t,z_0,z_1) \geq -M(y_1 - z_1), \quad x \in J, \ t \in \mathbb{R}$$

for $0 \leq z_0 \leq y_0 \leq w(u,t)$ *and* $0 \leq z_1 \leq y_1 \leq w(u,t-\tau)$,

(F2) there exist constants $m, m_1 \geq 0, m + m_1 < \frac{l^2 - M}{2}$ *such that*

$$\alpha(\{f(x,t,u_n(u,t),u_n(x,t-\tau))) + Mu_n(x,t-\tau)\} \leq m\alpha(\{u_n(x,t)\}) + m_1\alpha(\{u_n(x,t-\tau)\})$$

for any $t \in \mathbb{R}$ *and monotonous sequence* $\{u_n\} \subset [0,w]$ *hold, then Eq.(4.1) has minimal and maximal positive* ω*-periodic solution* u, u *between* 0 *and* w*, which can be obtained by monotone iterative sequences starting from* 0 *and* w*, respectively.*

Proof. Let $u(t) = u(\cdot,t)$, and $\widetilde{f}(t,u(t),u(t-\tau)) = f(\cdot,t,u(\cdot,t-\tau))$, then the periodic problem (4.1) can be reformulated as the abstract Eq. (1.1) in E. Condition (F0) implies that $v_0 \equiv 0$ and $w_0 = w(x,t)$ are the lower and upper solutions of the problem (1.1) with $v_0 \leq w_0$. From the assumptions of function f and the conditions (F1) and (F2), it is easy to see that the conditions of Theorem 3.1 hold. By Theorem 3.1, the periodic problem (4.1) has minimal and maximal ω-periodic solution, between v_0 and w_0, which can be obtained by monotone iterative sequences starting from v_0 and w_0, respectively. The positivity is clear. This completes the proof of Theorem.

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