Filomat 36:16 (2022), 5359–5390
https://doi.org/10.2298/FIL2216359M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Hölder and Lipschitz Continuity in Orlicz-Sobolev Classes, Distortion and Harmonic Mappings

Miodrag Mateljević[a], Ruslan Salimov[b], Evgeny Sevost'yanov[c]

[a]*University of Belgrade, Faculty of Mathematics, 16 Studentski trg, P.O. Box 550, 11001 Belgrade, Serbia*
[b]*Institute of Mathematics of the NAS of Ukraine, 3, Tereschenkivska st., 01024 Kiev-4, Ukraine*
[c]*Zhytomyr Ivan Franko State University, 40 Bol'shaya Berdichevskaya Str., 10 008 Zhytomyr, Ukraine;*
Institute of Applied Mathematics and Mechanics of NAS of Ukraine, 1 Dobrovol'skogo Str., 84 100 Slov'yans'k, Ukraine

Abstract. In this article, we consider the Hölder continuity of injective maps in Orlicz-Sobolev classes defined on the unit ball. Under certain conditions on the growth of dilatations, we obtain the Hölder continuity of the indicated class of mappings. In particular, under certain special restrictions, we show that Lipschitz continuity of mappings holds. We also consider Hölder and Lipschitz continuity of harmonic mappings and in particular of harmonic mappings in Orlicz-Sobolev classes. In addition in planar case, we show in some situations that the map is bi-Lipschitzian if Beltrami coefficient is Hölder continuous.

1. Introduction

It is well known that estimates of the distortion of distances of Hölder and Lipschitz type are one of the most important objects of modern analysis that allow a qualitative description of the behavior of mappings. For example, obtaining such estimates can be used to study the local behavior of solutions of the degenerate Beltrami-type equations, see e.g. [48] and [49]. Recall that K-quasiconformal mappings of the unit disk onto itself with the normalization condition $f(0) = 0$ are Hölder continuous with exponent $1/K$ and Hölder constant 16 (see, e.g., [2], [32, Theorem 3.2.II] [45] and [59, Theorem 18.2, Remark 18.4]). Quite simple examples of mappings, such as the quasiconformal homeomorphism $f = z|z|^{1/K-1}$, $f(0) := 0$, show that the Hölder exponent is of optimal order here and, in particular, quasiconformal mappings, generally speaking, are not Lipschitz. It should be noted that quasiconformal mappings can be Lipschitz in a rather wide subclass, however, in this case, rather specific conditions for their dilatation must be satisfied (see, e.g., [20]). Somewhat later, similar results on Hölder property were also established for maps with branching (quasiregular mappings), see, for example, [39, Theorem 3.2] and [51, Theorem 1.1.2]. Subsequently, the corresponding part of theory of mappings has been developed in the direction of weakening the conditions under which Hölder continuity or some of its analogues still holds. In particular, the study of local estimates of the distortion of distances has long been associated with the study of mappings with finite distortion, while Hölder continuity was often replaced by logarithmic distance estimates, see, for example, [15, Theorems 4 and 5], [22, Theorem 11.2.3], [41, Theorem 7.4], [42, Theorem 3.1], [50, Theorem 5.11]

2020 *Mathematics Subject Classification*. Primary 30C65; Secondary 31A15, 31B25
Keywords. Quasiconformal mappings, Hölder and Lipschitz continuity, harmonic mappings
Received: 23 September 2021; Accepted: 12 October 2021
Communicated by Dragan S. Djordjević
Email addresses: miodrag@matf.bg.ac.rs (Miodrag Mateljević), ruslan.salimov1@gmail.com (Ruslan Salimov), esevostyanov2009@gmail.com (Evgeny Sevost'yanov)

and [57, Theorems 1.1.V and 2.1.V]. [1)] In a number of previous publications, see, for example, [1], [18], [31], [30] and [23]–[24] we study the Orlicz-Sobolev classes under the Calderon condition and this article can be consider as continuation of this study. Here and below, we call the requirement

$$\int\limits_{1}^{\infty} \left(\frac{t}{\varphi(t)} \right)^{\frac{1}{n-2}} dt < \infty, \tag{1}$$

Calderon's condition, see [14]. Note that Calderon's article [14] containing this requirement was published in an inaccessible journal and was apparently forgotten. Orlicz-Sobolev classes under the Calderon condition, denoted by $W^{1,\varphi}_{\rm loc}$, are in many respects more general than the mentioned classes of quasiconformal and quasiregular mappings and, in addition, their properties are very similar to Sobolev classes $W^{1,p}_{\rm loc}$, $p > n-1$, which are obtained from the Orlicz-Sobolev classes by choosing the function $\varphi(t) = t^p$, $p > n-1$. In fact, Calderon's condition implies $W^{1,\varphi}_{\rm loc} \subset W^{1,n-1}_{\rm loc}$, $n \geq 3$. As well as in the Sobolev classes, functions in the Orlicz-Sobolev classes under the Calderon condition are differentiable almost everywhere and are absolutely continuous on spheres with respect to the $(n-1)$-dimensional Hausdorff measure, see, for example, [31, Theorem 1, Corollary 4], cf. [58, Lemma 3], [51, Theorem 1.2.II]. This property enables to establish upper bounds for the distortion of the modulus of the families of paths (weighted Poletski inequalities) for $W^{1,\varphi}_{\rm loc}$, and then apply the distance distortion estimates already established previously for the corresponding mappings (see, for example, [41, (7.47)], [50, (4.31)]). In particular, in our recent publication [47] related to the second and third co-authors, we obtained several similar estimates in the space and in planar case. We also note that consideration of the Sobolev classes $W^{1,p}_{\rm loc}$, $p > 1$, and Orlicz-Sobolev classes of this type, in the planar case is not required, but rather simply Sobolev classes, see ibid. This is due, in turn, to the fact that, by Gehring-Lehto theorem, planar homeomorphisms of Sobolev classes $W^{1,1}_{\rm loc}$, without any additional restrictions, are differentiable almost everywhere and absolutely continuous on almost all circles with respect to the linear Lebesgue measure, and this is already quite enough to establish upper bounds for the distortion of the modulus of families of paths (see, for example, [33, Theorem 3.1] and [32, Theorem 3.1]). The reader should be aware that the situation is completely different in space. Namely, for $n \geq 3$, there is a homeomorphism of class $W^{1,n-1}_{\rm loc}((-1,1)^n, \mathbb{R}^n)$ such that both f and f^{-1} are nowhere differentiable. In the survey [16] the author clarifies the regularity assumptions for a map to be differentiable a.e., and gives some some auxiliary results when it is not, using the notion of approximate differentiability. When dealing with mappings of $W^{1,p}$ with $p < n-1$, the notion of differentiability (that fails in this setting) can be replaced by the notion of approximate differentiability in the change of variable formula. However, the condition (N) plays a fundamental role for these mappings. Indeed for such f, the Luzin condition (N) is equivalent to the validity of the area formula. If the homeomorphism f satisfies the natural assumption $f \in W^{1,n}_{\rm loc}(G, \mathbb{R}^n)$, then f satisfies the condition (N). This is due to Reshetnjak[2)], and is a sharp result in the scale of $W^{1,p}(G, \mathbb{R}^n)$-homeomorphisms thanks to an example of Ponomarev[3)] of a $W^{1,p}$-homeomorphisms $f : [0,1]^n \to [0,1]^n, p < n$, violating the Luzin condition (N). Note that estimates of Hölder type have been investigated for inner points (see, e.g., [31, Theorems 7, 8] and [47, Theorems 1.1, 4.1]) and that this article deals with the corresponding estimates at the boundary of the domain.

The method of moduli of families of paths is one of the main research tools (see, for example, [31, Corollary 9], [30, Theorem 2.2], cf. [41, Theorem 7.3] and [50]) in the subject, and distortion estimates are usually proved with moduli techniques.

In this paper we employ the method of the boundary extension of the studied mappings across the boundary of a ball using the inversion with respect to its sphere, and then apply known distortion estimates

[1)] As a rule, Hölder continuity of mappings with finite distortion does not occur, with the exception of very special conditions for dilatation of mappings considered in this manuscript.

[2)] (cf. [29] in [16])

[3)] (cf. [27, 28] in [16])

for the case of interior points. So in the manuscript practically we do not use the modulus technique directly. Although there is a developed theory related to extension theorems for Sobolev spaces it seems that in this context our approach is a novelty. Of course, in addition to the conditions for smoothness of mappings, this approach also requires analytic conditions that limit the growth of their quasiconformality characteristics. The article considers several similar analytic conditions which are independently from each other.

There is a huge literature in the subject so it is possible that we have missed to quote some important papers for which we apologize to the authors in advance.

Throughout this manuscript, unless otherwise specified, D denotes a domain in $\mathbb{R}^n$, $n \geqslant 2$. We assume that the reader is familiar with the definitions of Sobolev classes $W^{1,1}_{\rm loc}$ and some of their basic properties, see, for example, [51, 2.I]. Here only recall if $f : D \to \mathbb{R}^m$ has ACL (absolutely continuous on lines) property on D we write that $f \in ACL(D)$.

We write $f \in W^{1,\varphi}_{\rm loc}(D)$ for a locally integrable vector-function $f = (f_1, \dots, f_m)$ of n real variables $x_1, \dots, x_n$ if $f_i \in W^{1,1}_{\rm loc}$ and

$$\int\limits_{D^*} \varphi\left(|\nabla f(x)|\right)\, dm(x) < \infty \tag{2}$$

for every subdomain D^* with a compact closure, where $|\nabla f(x)| = \sqrt{\sum\limits_{i,j} \left(\frac{\partial f_i}{\partial x_j}\right)^2}$. If additionally $f \in W^{1,1}(D)$ and

$$\int\limits_{D} \varphi\left(|\nabla f(x)|\right)\, dm(x) < \infty\,, \tag{3}$$

we write $f \in W^{1,\varphi}(D)$. For a mapping $f : D \to \mathbb{R}^n$ having partial derivatives almost everywhere in D, we set

$$J(x, f) := \det f'(x), \quad l\left(f'(x)\right) = \min_{h \in \mathbb{R}^n \setminus \{0\}} \frac{|f'(x)h|}{|h|}\,, \tag{4}$$

for the Jacobian and smallest distortion respectively. The inner dilatation of a map f at a point $x \in D$ is defined by the relation

$$K_I(x, f) \quad = \quad \begin{cases} \frac{|J(x,f)|}{l\left(f'(x)\right)^n}, & J(x, f) \neq 0, \\ 1, & f'(x) = 0, \\ \infty, & \text{otherwise} \end{cases} \;. \tag{5}$$

In what follows, we denote by $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$, and $\mathbb{S}^{n-1} = \partial \mathbb{B}^n$, respectively the unit n-dimensional ball and the unit $n-1$-dimensional sphere.

Theorem 1.1. *Let $n \geqslant 3$, and let $\varphi : (0, \infty) \to [0, \infty)$ be a non-decreasing Lebesgue measurable function wich satisfies Calderon's condition* (1). Suppose also that there exist constants $C > 0$ and $T > 0$ such that

$$\varphi(2t) \leqslant C \cdot \varphi(t) \ \forall\ t \geqslant T\,. \tag{6}$$

Let $Q : \mathbb{B}^n \to [0, \infty]$ be integrable function in $\mathbb{B}^n$. Assume that f is a homeomorphism of $\mathbb{B}^n$ onto $\mathbb{B}^n$ such that $f \in W^{1,\varphi}(\mathbb{B}^n)$ and, in addition, $f(0) = 0$. Let, moreover, $K_I(x, f) \leqslant Q(x)$ for a.e. $x \in \mathbb{B}^n$ and, besides that,

$$\sup_{\varepsilon \in (0, \varepsilon_0)} \frac{1}{\Omega_n \varepsilon^n} \int\limits_{\mathbb{B}^n \cap B(\zeta, \varepsilon)} Q(x)\, dm(x) < C \qquad \forall\ \zeta \in \partial \mathbb{B}^n \tag{7}$$

holds for some $\varepsilon_0 > 0$, where Ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Then f has a homeomorphic extension $f : \overline{\mathbb{B}^n} \to \overline{\mathbb{B}^n}$ and, in addition,

$$|f(x_2) - f(x_1)| \leqslant 2\alpha_n \varepsilon_0^{-\alpha} \cdot |x_2 - x_1|^{\alpha} \qquad \forall\ x_1, x_2 \in \partial\mathbb{B}^n : |x_2 - x_1| < \delta_0\,,$$

where $\delta_0 := \min\left\{\frac{1}{2}, \varepsilon_0^2\right\}$, ω_{n-1} is the area of $n-1$-dimensional sphere $\mathbb{S}^{n-1}$ and $\alpha := \left(\frac{\omega_{n-1}\log 2}{\Omega_n(4^n+1)2^{n+1}C}\right)^{1/(n-1)}$.

Note here that $\omega_{n-1} = n\Omega_n$.

In the second part of the paper we deal with harmonic mappings and harmonic quasiconformal (shortly hqc) mappings among the other things. Note that the planar case is very specific and the subject of planar hqc mappings has been intensively studied by the participants of the Belgrade Analysis Seminar (see, for example, [37] and [13] for more details and references cited there). For recent development of the subject in planar and spatial case see [38].

In Section 5 we consider mappings with growth of distortion (dilatations) of bounded mean value. and local spatial version of Privalov's theorem for harmonic functions (Theorem 5.3) which has an independent interest. As application of results obtained in in the first part and spatial version of Privalov's theorem we show global Hölder continuity of mappings in considered class. In particular, under certain special restrictions, we show that Lipschitz continuity of mappings holds.

Proof of Theorem 5.3 is given in Section 6.

Our focus in Section 7, are results related to planar hqc mappings between Lyapunov domains. Kalaj proved that if h is a hqc mapping of the unit disk onto a Lyapunov domain, then h is Lipschitz (see, e.g., [25]). Then in [13] it is proved h is co-Lipschitz. In planar case the condition (52)[4)] provides sufficient conditions for Hölder and Lipschitz continuity. Next we show that in some situation if Beltrami coefficient is Hölder continuous that the map is biLip [5)]. As application we obtain some version of Kellogg's theorem for quasiconformal mappings.

Finally, let us say a few words about the activities that influenced the research contained in this manuscript:

Remark 1.2. *During Belgrade Analysis seminar, winter semester 2019 and 2020, we have considered subject related to Geometric Function Theory (GFT) and qc mappings and have tried to start some projects related to the subject*[6)]*. After writing a final version of this manuscript, in communication with D. Kalaj it becomes clear in particular that the project is also related to some versions of Kellogg and Warschawski theorem for a class of quasiconformal maps, and our attention has been turned first to [26] and later to results obtained in [43] and in [11]; see in particular Theorem 1.3 [11].*

Let $x_0 \in \overline{D}$, $x_0 \neq \infty$,

$$S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}\,, S_i = S(x_0, r_i)\,, \quad i = 1, 2\,,$$

$$A = A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}\,.$$

Let $Q : \mathbb{R}^n \to \mathbb{R}^n$ be a Lebesgue measurable function satisfying the condition $Q(x) \equiv 0$ for $x \in \mathbb{R}^n \setminus D$. The mapping $f : D \to \overline{\mathbb{R}^n}$ is called a *ring Q-mapping at the point* $x_0 \in \overline{D} \setminus \{\infty\}$, if the condition

$$M(f(\Gamma(S_1, S_2, D))) \leqslant \int\limits_{A\cap D} Q(x) \cdot \eta^n(|x - x_0|)\, dm(x) \tag{8}$$

4) see Theorem 4.1 in section 4, p.19.

5) see the section 7 which can be considered as a separate part

6) In particular, the first author has clarified some facts related to Ahlfors book [4] in discussion with V. Božin and M. Arsenović, and E. Sevost'yanov gave several lectures related to a ring Q-homeomorphism and Orlicz-Sobolev clases. In connection with this the first author of this manuscript started two independent project a) with M. Arsenović related to regularity properties of solutions of Beltrami equation, and b) with R. Salimov and E. Sevost'yanov, related Hölder and Lipschitz class in Orlicz-Sobolev clas.

holds for all $0 < r_1 < r_2 < d_0 := \sup\limits_{x\in D} |x - x_0|$ and all Lebesgue measurable functions $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)dr \geqslant 1 \,. \tag{9}$$

The mapping of f is called a *ring Q-mapping in D*, if condition (8) is satisfied at every point $x_0 \in D$, and a *ring Q-mapping in* $\overline{D}$, if the condition (8) holds at every point $x_0 \in \overline{D}$. For the properties of such mappings see [49] and [42].

2. Proof of Theorem 1.1

It is known that if f is K-qc mapping of the unit ball $\mathbb{B}^n$ onto itself then $f \in W^{1,p}(\mathbb{B}^n)$ for some $p > n$ and it satisfies hypothesis of Theorem 1.1 with $\varphi(t) = t^n$. On the other hand, Example 2.1 below shows that there is a map $f \in W^{1,3}(\mathbb{B}^3)$ with unbounded inner distortion K_I satisfying the conditions of Theorem 1.1 with $Q = K_I$. We leave the interested reader to generalize this example.

I. Observe that f is a ring Q-mapping in $\mathbb{B}^n$ with $Q = K_I(x, f)$, where $K_I(x, f)$ is defined in (5), see [30, Theorem 2.2], cf. [31, Corollary 9]. According to Corollary 6.1 in [41], the function Q, which satisfies (7), has a finite mean oscillation at each point $x_0 \in \partial\mathbb{B}^n$. In this case, it follows from [54, Theorem 1] that there is continuous extension $\tilde{f}$ of the mapping f onto $\mathbb{S}^{n-1} = \partial\mathbb{B}^n$. We also note that the map $\tilde{f}$ is a homeomorphism of the unit ball $\overline{\mathbb{B}^n}$ onto itself, see, for example, [56, Theorem 5].

II. Using conformal transformation $\psi(x) = \frac{x}{|x|^2}$, we extend the mapping f homeomorphically onto the whole n-dimensional Euclidean space $\mathbb{R}^n$ as follows:

$$F(x) = \begin{cases} f(x), & |x| < 1\,, \\ \psi(f(\psi(x))), & |x| \geqslant 1\,. \end{cases} \tag{10}$$

Using the condition, $f \in W^{1,\varphi}(\mathbb{B}^n)$, we will show that $F \in W^{1,\varphi}(B(0, R))$ for any $R > 1$ (in particular, since $F = f$ on $\mathbb{B}^n$, the functions $|\nabla F|$ and $\varphi(|\nabla F|)$ are integrable in $\mathbb{B}^n$, but we show more that these functions also integrable in $B(0, R)$, for any $R > 1$). For this, we observe that, by the differentiation rule of a superposition of mappings,

$$F'(x) = \psi'(f(\psi(x)) \circ f'(\psi(x)) \circ \psi'(x)\,. \tag{11}$$

Here we used the fact that homeomorphisms of the Orlicz-Sobolev classes under the Calderon condition are differentiable almost everywhere, see, for example, [31, Theorem 1]. As usual, put

$$\|f'(x)\| = \max_{h\in\mathbb{R}^n\setminus\{0\}} \frac{|f'(x)h|}{|h|}\,.$$

Using direct calculations, we may establish the inequality

$$\|f'(x)\| \leqslant |\nabla f(x)| \leqslant n^{1/2} \cdot \|f'(x)\| \tag{12}$$

at all points $x \in D$ where the map f has formal partial derivatives. Observe that $\|\psi'(x)\| = \frac{1}{|x|^2}$ (see, e.g., [55, paragraph 7]). Recall that for two linear mappings g and h the relation

$$\|g \circ h\| \leqslant \|g\| \cdot \|h\| \tag{13}$$

holds, and here, equality holds as soon as at least one of the mappings is orthogonal (see, e.g., [51, I.4, relation (4.13)]). Since $f(0) = 0$, $f(\psi(y)) \neq 0$ for $1 < |y| \leqslant R$ and for any $R > 1$. Since the map $f(\psi(y))$ is continuous in $1 \leqslant |y| \leqslant R$ and does not vanish, there is $m > 0$ such that

$$|f(\psi(y))| \geqslant m\,, \qquad 1 \leqslant |y| \leqslant R\,. \tag{14}$$

In this case, from (11), (12), (13) and (14), we obtain that

$$\int\limits_{1<|x|<R} |\nabla F(x)|\,dm(x) \leqslant \int\limits_{1<|x|<R} n^{1/2}\cdot \|F'(x)\|\,dm(x)$$

$$= n^{1/2}\cdot \int\limits_{1<|x|<R} \|\psi'(f(\psi(x))\|\cdot\|f'(\psi(x))\|\cdot\|\psi'(x)\|\,dm(x)$$

$$= n^{1/2}\cdot \int\limits_{1<|x|<R} \frac{1}{|f(\psi(x))|^2}\cdot\|f'(\psi(x))\|\cdot\frac{1}{|x|^2}\,dm(x) \leqslant \frac{n^{1/2}}{m^2}\cdot \int\limits_{1<|x|<R} \|f'(\psi(x))\|\,dm(x)$$

$$= \frac{n^{1/2}}{m^2}\cdot \int\limits_{1/R<|y|<1} \frac{\|f'(y)\|}{|y|^{2n}}\,dm(y) \leqslant \frac{n^{1/2}R^{2n}}{m^2}\cdot \int\limits_{1/R<|y|<1} |\nabla f(y)|\,dm(y) < \infty\,. \tag{15}$$

III. Quite similarly, applying the same arguments to the function $\varphi(|\nabla F|)$ instead of $|\nabla F|$, and taking into account relation (6) together with the non-decreasing property of the function φ, we obtain that

$$\int\limits_{1<|x|<R} \varphi(|\nabla F(x)|)\,dm(x) \leqslant C_1\cdot \int\limits_{1<|x|<R} \varphi(\|F'(x)\|)\,dm(x)$$

$$= C_1\cdot \int\limits_{1<|x|<R} \varphi(\|\psi'(f(\psi(x)))\|\cdot\|f'(\psi(x))\|\cdot\|\psi'(x)\|)\,dm(x)$$

$$= C_1\cdot \int\limits_{1<|x|<R} \varphi\left(\frac{1}{|f(\psi(x))|^2}\cdot\|f'(\psi(x))\|\cdot\frac{1}{|x|^2}\right)dm(x) \leqslant C_2\cdot \int\limits_{1<|x|<R} \varphi(\|f'(\psi(x))\|)\,dm(x)$$

$$= C_2\cdot \int\limits_{1/R<|x|<1} \varphi\left(\frac{\|f'(x)\|}{|x|^{2n}}\right)dm(x) \leqslant C_2R^{2n}\cdot \int\limits_{1/R<|x|<1} \varphi(|\nabla f(x)|)\,dm(x) < \infty\,. \tag{16}$$

IV. It follows from (15) and (16) that

$$\int\limits_{B(0,R)} |\nabla F(x)|\,dm(x) < \infty\,, \int\limits_{B(0,R)} \varphi(|\nabla F(x)|)\,dm(x) < \infty\,, \quad R > 1\,. \tag{17}$$

Reasoning in a similar way, we may also obtain similar relations for the inner dilatation of the map F. Indeed, since the inner dilatation does not change under conformal mapping (see, for example, [51, I.4.(4.15)]), we obtain that

$$\int\limits_{B(0,R)} K_I(x,F)\,dm(x) = \int\limits_{\mathbb{B}^n} K_I(x,f)\,dm(x) + \int\limits_{1<|x|<R} K_I(\psi(x),f)\,dm(x)\,.$$

Making a change of variables here, and taking into account that $K_I(x,f)\in L^1(\mathbb{B}^n)$ by the assumption, we obtain that

$$\int\limits_{B(0,R)} K_I(x,F)\,dm(x) = \int\limits_{\mathbb{B}^n} K_I(x,f)\,dm(x) + \int\limits_{1/R<|y|<1} K_I(y,f)\cdot\frac{1}{|y|^{2n}}\,dm(y)$$

$$\leqslant \int\limits_{\mathbb{B}^n} K_I(x,f)\,dm(x) + R^{2n}\cdot \int\limits_{1/R<|y|<1} K_I(y,f)\,dm(y) < \infty. \tag{18}$$

V. Let us check that $F \in ACL(\mathbb{R}^n)$. It is known if $f \in W^{1,1}(\mathbb{B}^n)$, that the unit ball $\mathbb{B}^n$ may be divided in a standard way into no more than a countable number of parallelepipeds I_s, $s \geq 1$, with disjoint interiors, such that F is absolutely continuous on almost all coordinate segments in each I_s,$s \geq 1$. We call a segment coordinate segment if it is parallel to a coordinate axis. Let us prove:

(A) F is absolutely continuous on almost all segments in $\overline{\mathbb{B}^n}$, parallel to the coordinate axes.

It is enough to consider segments r for which F is absolutely continuous (shortly AC) on $r_s := r \cap I_s$ for every $s \geq 1$. Suppose that $r(t) = \{x \in \mathbb{R}^n : x = x_0 + te, t \in [a, b]\}$ is such a segment in $\overline{\mathbb{B}^n}$, where e is some coordinate unit vector, and $x_0 \in \mathbb{B}^n$.

Two cases are possible: when $z_0 := x_0 + be$ belongs to the interior of the ball, and when the same point lies on the unit sphere. Set $\alpha(t) = f(x_0 + te)$. In the first case, there are finite number of integers $s_1, s_2, ..., s_l$ such that $r = \cup_{\nu=1}^{l} r_{s_\nu}$. Hence F is AC on r.

Note also here that by ACL-characterization of the Sobolev classes (see, e.g., [44, Theorems 1.1.2 and 1.1.3]) and by the fact that for a real-valued functions defined on an interval of the real line, absolute continuity may be formulated by the validity of the fundamental theorem of calculus in terms of Lebesgue integration, (see, for example, see [52, Theorem IV.7.4]), we have $\int\limits_a^b \alpha'(t)\,dt = \alpha(b) - \alpha(a)$. Let now $z_0 \in \mathbb{S}^{n-1}$. Then, as it was proved above with respect to the inner points of the ball, for an arbitrary $a < c < b$ we have that

$$\int\limits_a^c \alpha'(t)\,dt = \alpha(c) - \alpha(a)\,. \tag{19}$$

Since it was also proved above, that the map f is a homeomorphism in the closed unit ball $\overline{\mathbb{B}^n}$, the passage to the limit on the right-hand side of (19) as $c \to b$ gives that $\alpha(b) - \alpha(a)$.

Since (19) holds for every subinterval of r, we first conclude that F is AC on r, and (A) follows. Now consider the family $J(B(0, R))$ of all coordinate segments in $B(0, R)$. It follows from the integrability of the gradient of the mapping F on $B(0, R)$ (see (17) and by virtue of Fubini's theorem (see, for example, [52, Theorem III.8.1]) that the derivative of the function α is integrable on almost all segments in $B(0, R)$ parallel to the coordinate axes. Without loss of generality, we may assume that a segment $r(t)$ has exactly this property.

Since the reflection with respect to the unit sphere is C^∞ change of variables, and $f \in W^{1,1}(\mathbb{B}^n)$, we conclude that $F \in W^{1,1}\Big((B(0, R) \setminus \mathbb{B}^n)\Big)$ (see item 1.1.7 [44] and also definitions of Sobolev spaces on manifolds in literature). Similarly as above, we may verify that:

(B) F is absolutely continuous on almost all segments in $\mathbb{R}^n \setminus \mathbb{B}^n$, parallel to the coordinate axes.

Since F is continuous on $\mathbb{R}^n$, this immediately implies that F is absolutely continuous on the same segments in $\mathbb{R}^n$, as required.

VI. Since $F \in ACL(\mathbb{R}^n)$, by (17) $F \in W^{1,\varphi}_{\rm loc}(B(0, R))$ for any $R > 1$. Thus, by (17) and (18), F is a ring Q^*-mapping in $B(0, R)$, where $Q^*(x) = Q(x)$ for $x \in \mathbb{B}^n$ and $Q^*(x) = Q(\psi(x))$ for $x \in B(0, R) \setminus \mathbb{B}^n$ (see, e.g., [30, Theorem 2.2], cf. [31, Corollary 9]).

VII. Let $\zeta_0 \in \mathbb{S}^{n-1}$ and $r_0 > 0$. Notice, that

$$\psi(B_+(\zeta_0, \varepsilon)) \subset B_-(\zeta_0, \varepsilon) \quad \forall\ \varepsilon \in (0, 1)\,, \tag{20}$$

where

$$B_+(\zeta_0, \varepsilon) = \{x \in \mathbb{R}^n : \exists\, e \in \mathbb{S}^{n-1}, t \in [0, \varepsilon) : x = \zeta_0 + te, |x| > 1\} = B(\zeta_0, \varepsilon) \cap (\mathbb{R}^n \setminus \mathbb{B}^n)\,,$$

$$B_-(\zeta_0, \varepsilon) = \{x \in \mathbb{R}^n : \exists\, e \in \mathbb{S}^{n-1}, t \in [0, \varepsilon) : x = \zeta_0 + te, |x| < 1\} = B(\zeta_0, \varepsilon) \cap \mathbb{B}^n\,,$$

and, as above, $\psi(x) = \frac{x}{|x|^2}$. Indeed, for a given $x = \zeta_0 + te \in B_+(\zeta_0, \varepsilon)$, computing the square of the module of the vector by means of the scalar product $(\cdot, \cdot)$, we obtain that

$$|\psi(x) - \zeta_0|^2 = \left|\frac{\zeta_0 + te}{|\zeta_0 + te|^2} - \zeta_0\right|^2 = \frac{1}{|\zeta_0 + te|^2} - \frac{2(1 + t(\zeta_0, e))}{|\zeta_0 + te|^2} + \frac{|\zeta_0 + te|^2}{|\zeta_0 + te|^2}$$

$$= \frac{1 - 2(1 + t(\zeta_0, e)) + 1 + 2t(\zeta_0, e) + t^2}{|\zeta_0 + te|^2} = \frac{t^2}{|\zeta_0 + te|^2} < t^2\,,$$

that is, $|\psi(x) - \zeta_0| < t$, as required.

VIII. Let $0 < r < 1/2$. Now, by (20) and by formula for the change of variable in the integral (see, e.g., [17, Theorem 3.2.5]) we obtain that

$$\int\limits_{B(\zeta_0,r)\cap(\mathbb{R}^n\setminus\overline{\mathbb{B}^n})} Q^*(y)\,dm(y) = \int\limits_{B(\zeta_0,r)\cap(\mathbb{R}^n\setminus\overline{\mathbb{B}^n})} Q(\psi(y))\,dm(y)$$

$$\leqslant \int\limits_{B(\zeta_0,r)\cap\mathbb{B}^n} Q(y)\cdot\frac{1}{|y|^{2n}}\,dm(y)\,. \tag{21}$$

Let $y \in B(\zeta_0, r)\cap\mathbb{B}^n$. Now $y = \zeta_0 + et$, where $e \in \mathbb{S}^{n-1}$ and $0 \leqslant t < r < 1/2$. Hence, by the Cauchy-Bunyakovsky inequality, we have that

$$|y|^2 = |\zeta_0 + et|^2 = 1 + 2t(\zeta_0, e) + t^2 \geqslant 1 - 2t + t^2 = (1 - t)^2 \geqslant 1/4\,. \tag{22}$$

By (21) and (22),

$$\int\limits_{B(\zeta_0,r)\cap(\mathbb{R}^n\setminus\overline{\mathbb{B}^n})} Q^*(y)\,dm(y) \leqslant 4^n\cdot\int\limits_{B(\zeta_0,r)\cap\mathbb{B}^n} Q(y)\,dm(y)\,. \tag{23}$$

It immediately follows from (23) that

$$\int\limits_{B(\zeta_0,r)} Q^*(y)\,dm(y) \leqslant (4^n + 1)\cdot\int\limits_{B(\zeta_0,r)\cap\mathbb{B}^n} Q(y)\,dm(y) < \infty\,, \tag{24}$$

because Q is integrable in $\mathbb{B}^n$ by the assumption.

IX. Denote by F_Q the family of all homeomorphisms of the class $W^{1,\varphi}(\mathbb{B}^n)$ of the unit ball onto itself satisfying the condition $f(0) = 0$, for which $K_I(x, f) \leqslant Q(x)$ a.e. $x \in \mathbb{B}^n$. According to point VI, every mapping F defined by formula (10), where f satisfies the hypothesis of the theorem, belongs to the class F_Q. Note also that all such mappings obviously do not take the values 0 and ∞ in the domain $\mathbb{R}^n \setminus \{0\}$. Let h be a chordal metric in $\overline{\mathbb{R}^n}$,

$$h(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}},\ \ h(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2}\,\sqrt{1 + |y|^2}}\,,\quad x \neq \infty \neq y\,,$$

and let $h(E) := \sup\limits_{x,y\in E} h(x, y)$ be a chordal diameter of a set $E \subset \overline{\mathbb{R}^n}$ (see, e.g., [59, Definition 12.1]). Based on the above formula, $h(\mathbb{R}^n \setminus \{0\}) = 1$. We note that all the statements obtained in the proof of this theorem up to and including point VIII hold. By (24) and (7),

$$\sup\limits_{\varepsilon\in(0,\varepsilon_0)} \frac{1}{\Omega_n\varepsilon^n}\int\limits_{B(\zeta,\varepsilon)} Q^*(x)\,dm(x)$$

$$\leqslant (4^n+1)\cdot \sup_{\varepsilon\in(0,\varepsilon_0)} \frac{1}{\Omega_n\varepsilon^n}\int\limits_{\mathbb{B}^n\cap B(\zeta,\varepsilon)} Q(x)\,dm(x) < (4^n+1)\cdot C\,.$$

By Lemma 3.1 in [47] for $C_* = (4^n+1)\cdot C$ and $\varphi(t)=1$,

$$\int\limits_{A(x_0,\varepsilon,\varepsilon_0)} \frac{Q^*(x)\,dm(x)}{|x-x_0|^n} \leqslant \frac{\Omega_n(4^n+1)2^nC}{\log 2}\left(\log\frac{1}{\varepsilon}\right),\quad \forall\ \varepsilon\in(0,\varepsilon_0)\,,\quad \forall\ x_0\in\partial\mathbb{B}^n\,.$$

Observe that $\frac{\log\frac{1}{\varepsilon}}{\log\left(\frac{\varepsilon_0}{\varepsilon}\right)} = 1+\frac{\log\frac{1}{\varepsilon_0}}{\log\left(\frac{\varepsilon_0}{\varepsilon}\right)} < 2$ for $\varepsilon\in(0,\delta_0)$, where $\delta_0>0$ is the number defined in the conditions of the theorem. Now

$$\left(\log\left(\frac{\varepsilon_0}{\varepsilon}\right)\right)^{-1}\cdot \int\limits_{A(x_0,\varepsilon,\varepsilon_0)} \frac{Q^*(x)\,dm(x)}{|x-x_0|^n}$$

$$\leqslant \frac{\Omega_n(4^n+1)2^nC}{\log 2}\frac{\log\frac{1}{\varepsilon}}{\log\left(\frac{\varepsilon_0}{\varepsilon}\right)} \leqslant \frac{\Omega_n(4^n+1)2^{n+1}C}{\log 2}\,. \tag{25}$$

Applying Lemma 4.9 in [50] for $\psi(t)=1/t$ we obtain by (25) that

$$h(F(x),F(x_0)) \leqslant \alpha_n\left(\frac{|x-x_0|}{\varepsilon_0}\right)^{\alpha}\,,$$

for every $x\in B(x_0,\varepsilon_0)$ and any $x_0\in\mathbb{R}^n\setminus\{0\}$, where $\alpha=\left(\frac{\omega_{n-1}\log 2}{\Omega_n(4^n+1)2^{n+1}C}\right)^{1/(n-1)}$ and α_n is some constant depending only on n. This inequality also remains valid at the origin, since the same arguments and the assertion of Lemma 4.9 in [50] apply to the map $f:\mathbb{B}^n\to\mathbb{B}^n$. Finally, since $h(x,y)\geqslant\frac{|x-y|}{1+r_0^2}$ for $x,y\in\overline{B(0,r_0)}$, and $F|_{\overline{\mathbb{B}^n}}=f$, we obtain that

$$|f(x)-f(x_0)|\leqslant 2\alpha_n\varepsilon_0^{-\alpha}|x-x_0|^{\alpha}\,.$$

Theorem is proved. □

Example 2.1. *We give an example of a map satisfying the conditions and the conclusion of Theorem 1.1 for which the function Q under the conditions of this theorem is not bounded. We consider the infinite partition of the segment* $[0,1]$ *by points* $\left[\frac{k}{k+1},\frac{k+1}{k+2}\right]$, $k=0,1,2,\ldots$ *. We consider the following function* $\beta:[0,1)\to\mathbb{R}$, *defined as follows:*

$$\beta(t)=\begin{cases} 1, & t\in\bigcup\limits_{k\geqslant 1}\left[\frac{k}{k+1},\frac{k+1}{k+2}-2^{-4k-1}\right],\\ 2^k, & t\in\left(\frac{k+1}{k+2}-2^{-4k-1},\frac{k+1}{k+2}\right)\,,\ \textit{for some}\ k>0\end{cases}$$

$k=0,1,2,\ldots$. *We show that for an arbitrary number* $a\in[0,1]$ *the function* β *satisfies the condition*

$$\int\limits_a^1 \beta(t)\,dt \leqslant 2(1-a)\,. \tag{26}$$

In fact, we fix such a number a. *Then there is* $k_0\in\mathbb{N}$ *such that* $a\in\left(\frac{k_0}{k_0+1},\frac{k_0+1}{k_0+2}\right]$. *In this case, we obtain that*

$$\int\limits_a^1\beta(t)\,dt = \int\limits_a^{\frac{k_0+1}{k_0+2}}\beta(t)\,dt + \int\limits_{\frac{k_0+1}{k_0+2}}^1\beta(t)\,dt$$

$$= \frac{k_0+1}{k_0+2} - a + \sum_{k=k_0+1}^{\infty} \int\limits_{\frac{k}{k+1}}^{\frac{k+1}{k+2}-2^{-4k-1}} dt + \sum_{k=k_0+1}^{\infty} \int\limits_{\frac{k+1}{k+2}-2^{-4k-1}}^{\frac{k+1}{k+2}} 2^k\, dt$$

$$\leqslant \int\limits_a^1 dt + \sum_{k=k_0+1}^{\infty} \int\limits_{\frac{k+1}{k+2}-2^{-4k-1}}^{\frac{k+1}{k+2}} 2^k\, dt \leqslant 1 - a + \sum_{k=k_0+1}^{\infty} 2^{-3k-1}$$

$$= 1 - a + \frac{2^{-3k_0-1}}{7} \leqslant 1 - a + 2^{-k_0-2}\,. \tag{27}$$

Observe that the inequality $2^{-k_0-2} \leqslant 1-a$ *is equivalent to* $a \leqslant 1 - 2^{-k_0-2}$. *In turn, according to the choice of* a, $a \leqslant \frac{k_0+1}{k_0+2} = 1 - \frac{1}{k_0+2}$. *However,* $1 - \frac{1}{k_0+2} \leqslant 1 - 2^{-k_0-2}$ *is equivalent to the obvious inequality* $k_0 + 2 \leqslant 2^{k_0+2}$, $k_0 = 0, 1, 2, \ldots$. *It follows from what has been said that*

$$a \leqslant \frac{k_0+1}{k_0+2} = 1 - \frac{1}{k_0+2} \leqslant 1 - 2^{-k_0-2}\,,$$

so that by (27) we obtain that

$$\int\limits_a^1 \beta(t)\, dt \leqslant 2(1-a)\,. \tag{28}$$

The relation (26) is proved. Choosing now $\varepsilon \in (0,1)$ *and setting* $a := 1-\varepsilon$, *from (28) we obtain that*

$$\int\limits_{1-\varepsilon}^1 \beta(t)\, dt \leqslant 2\varepsilon\,. \tag{29}$$

Now put $Q(x) = \beta(|x|)$. *We show now that condition (7) is fulfilled for the indicated function* Q. *For simplicity, we further consider the case* $n = 3$.

Choose an arbitrary point $\zeta_0 \in \mathbb{S}^2$, *and let* $0 < \varepsilon < 1$. *We estimate the integral over the intersection of the ball* $B(\zeta_0, \varepsilon)$ *with* $\mathbb{B}^3$ *using the Fubini theorem and using some geometric considerations. Using the formula* $S = 2\pi r h$ *for the spherical cap lying on the sphere of the radius* r *and of the hight* h, *we may verify that* $\mathcal{H}^2(B(\zeta_0,\varepsilon) \cap \mathbb{S}^2) = \pi\varepsilon^2$, *where* $\mathcal{H}^2$ *denotes* 2*-dimensional Hausdorff measure on* $\mathbb{S}^2$.

Now, By Fubini's theorem (see, for example, [52, Theorem III.8.1]) we will have that

$$\int\limits_{B(\zeta_0,\varepsilon)\cap\mathbb{B}^3} Q(x)\, dm(x) \leqslant \int\limits_{1-\varepsilon}^1 \int\limits_{S(0,r)\cap B(\zeta_0,\varepsilon)} Q(x)\, d\mathcal{H}^2\, dr$$

$$= \int\limits_{1-\varepsilon}^1 \beta(r) \int\limits_{S(0,r)\cap B(\zeta_0,\varepsilon)} d\mathcal{H}^2\, dr \leqslant \int\limits_{1-\varepsilon}^1 \beta(r) \int\limits_{\mathbb{S}^2\cap B(\zeta_0,\varepsilon)} d\mathcal{H}^2\, dr$$

$$\leqslant \pi\varepsilon^2 \int\limits_{1-\varepsilon}^1 \beta(r)\, dr\,. \tag{30}$$

It follows from (29) and (30) that

$$\frac{3}{4\pi\varepsilon^3} \int\limits_{B(\zeta_0,\varepsilon)\cap\mathbb{B}^3} Q(x)\, dm(x) \leqslant \frac{3}{2}\,.$$

Thus, for the function Q, condition (7) is satisfied.

Guided by Proposition 6.15 in [48], by analogy, we construct the desired spatial map as follows:

$$f(x) = \frac{x}{|x|} e^{\int_1^{|x|} (\beta(t)/t)\,dt}\,, \quad f(0) := 0\,.$$

Note that the map f, defined in this way, is a homeomorphism. We verify that all the conditions of Theorem 1.1 are satisfied. Indeed, guided by Proposition 6.3 in [41], we may calculate the tangential, radial, inner dilatations of the map f and the matrix norm of $f'(x)$ using the following formulas:

$$\delta_\tau(x) = \frac{|f(x)|}{|x|} = e^{\int_1^{|x|} (\beta(t)/t)\,dt}\,, \quad \delta_r(x) = \frac{\partial |f(x)|}{\partial |x|} = e^{\int_1^{|x|} (\beta(t)/t)\,dt} \cdot \frac{\beta(|x|)}{|x|}\,,$$

$$\|f'(x)\| = \max\{\delta_\tau, \delta_r\} = e^{\int_1^{|x|} (\beta(t)/t)\,dt} \cdot \frac{\beta(|x|)}{|x|}\,, \quad K_I(x, f) = \beta(|x|)\,.$$

Note that the norm of the map $f'(x)$ is locally bounded in $\mathbb{B}^3 \setminus \{0\}$; therefore, by virtue of inequality (12), all partial derivatives of the mapping that exist almost everywhere are also locally bounded. From this, in particular, it follows that the map f belongs to the class ACL in $\mathbb{B}^3$.

Observe that the function $\varphi(t) = t^3$ satisfies the Calderon condition (1). Let us verify that the map f belongs to the class $W^{1,\varphi}(\mathbb{B}^3)$. Indeed, by Fubini theorem,

$$\int_{\mathbb{B}^3} \|f'(x)\|^3\,dm(x) = \int_{\mathbb{B}^3} e^{3\int_1^{|x|} (\beta(t)/t)\,dt} \cdot \frac{\beta^3(|x|)}{|x|^3}\,dm(x)$$

$$= \int_{0<|x|<1/2} e^{3\int_1^{|x|} (\beta(t)/t)\,dt} \cdot \frac{\beta^3(|x|)}{|x|^3}\,dm(x) + \int_{1/2<|x|<1} e^{3\int_1^{|x|} (\beta(t)/t)\,dt} \cdot \frac{\beta^3(|x|)}{|x|^3}\,dm(x)$$

$$\leqslant \frac{\pi}{6} + 8\pi \int_{1/2}^{1} \beta^3(t)\,dt < \infty\,. \tag{31}$$

In (31), we took into account that $\int_{1/2}^{1} \beta^3(t)\,dt < \infty$. Indeed, by the construction,

$$\int_{1/2}^{1} \beta^3(t)\,dt \leqslant \int_{1/2}^{1} dt + \sum_{k=0}^{\infty} 2^{-4k-1} \cdot 2^{3k} = 1/2 + 1 = 3/2 < \infty\,.$$

Since $f \in ACL$, it follows from (31) that $f \in W^{1,\varphi}(\mathbb{B}^3)$. Using Hölder's inequality, it may also be obtained from inequalities (31) that $f \in W^{1,1}(\mathbb{B}^3)$.

We show that also $Q(x) = K_I(x, f) = \beta(|x|) \in L^1(\mathbb{B}^n)$. In fact,

$$\int_{\mathbb{B}^3} K_I(x, f)\,dm(x) = 4\pi \int_0^1 t^2 \cdot \beta(t)\,dt < \infty$$

by (26). Thus, all the conditions of Theorem 1.1 are satisfied. Note that the map f is even Lipschitz on the unit sphere.

Example 2.2. *For comparison, we will also construct a similar example on the plane. Note that this example is related to Lemma 4.2 and Theorem 4.1 in [47].*

Let β be the function constructed in Example 2.1. Now put $Q(z) = \beta(|z|)$. Choose an arbitrary point $\zeta_0 \in \mathbb{S}^1$, and let $0 < \varepsilon < 1$. We are going to estimate the integral over the intersection of the disk $B(\zeta_0, \varepsilon)$ with $\mathbb{B}^2$ using the Fubini theorem and using some geometric considerations. Set

$$\theta_1 = \inf_{z\in B(\zeta_0,\varepsilon)\cap\mathbb{B}^2} \arg z\,, \quad \theta_2 = \sup_{z\in B(\zeta_0,\varepsilon)\cap\mathbb{B}^2} \arg z\,,$$

$$L(r, \theta_1, \theta_2) = \{z \in S(0, r) : z = re^{i\theta}, \theta_1 < \theta < \theta_2\}\,.$$

By Fubini's theorem (see, for example, [52, Theorem III.8.1]) we will have that

$$\int\limits_{B(\zeta_0,\varepsilon)\cap\mathbb{B}^2} Q(z)\,dm(z) \leqslant \int\limits_{1-\varepsilon}^{1} \int\limits_{S(0,r)\cap B(\zeta_0,\varepsilon)} Q(z)\,|dz|dr$$

$$= \int\limits_{1-\varepsilon}^{1} \beta(r) \int\limits_{L(r,\theta_1,\theta_2)} |dz|dr = \int\limits_{1-\varepsilon}^{1} \beta(r)r(\theta_2-\theta_1)\,dr$$

$$\leqslant \int\limits_{1-\varepsilon}^{1} \beta(r)(\theta_2-\theta_1)\,dr\,. \tag{32}$$

It follows from (32) that

$$\frac{1}{\pi\varepsilon^2} \int\limits_{B(\zeta_0,\varepsilon)\cap\mathbb{B}^2} Q(z)\,dm(z) \leqslant \frac{1}{\pi\varepsilon^2} \int\limits_{1-\varepsilon}^{1} \beta(r)(\theta_2-\theta_1)\,dr\,. \tag{33}$$

Through direct geometric calculations, it can be found that $\theta_2 - \theta_1 = 2\arccos\left(\frac{2-\varepsilon^2}{2}\right) \sim 2\varepsilon$ as $\varepsilon \to 0$. Now, $(\theta_2 - \theta_1)/\varepsilon \leqslant 3$ for sufficiently small $\varepsilon > 0$. Thus, by (29) and (33)

$$\frac{1}{\pi\varepsilon^2} \int\limits_{B(\zeta_0,\varepsilon)\cap\mathbb{B}^2} Q(z)\,dm(z) \leqslant \frac{3}{\pi\varepsilon} \int\limits_{1-\varepsilon}^{1} \beta(r)\,dr \leqslant \frac{6}{\pi} := c\,, \quad 0 < \varepsilon < \varepsilon_0 \tag{34}$$

for some $\varepsilon_0 > 0$. Thus, for the function Q, condition (7) is satisfied. According to Proposition 6.15 in [48], the mapping

$$w = f(re^{i\theta}) = e^{i\theta + \int\limits_1^r (\beta(t)/t)\,dt}$$

is a homeomorphism of the Sobolev class $W^{1,1}_{\rm loc}(\mathbb{C})$ and has a dilatation $K_I(z, f)$ equal to $\beta(|z|)$. Note that this map takes the unit disk onto itself; moreover, $f|_{\mathbb{S}^1}$ is Hölder continuous with an arbitrary exponent on $\mathbb{S}^1$ since $f(z) = z$ at the points $z \in \mathbb{S}^1$.

3. On Hölder type estimates for mappings with a condition on the mean value

Given a Lebesgue measurable function $Q : \mathbb{B}^n \to [0, \infty]$ we set $Q^*(x) = Q(x)$ for $x \in \mathbb{B}^n$ and $Q^*(x) = Q(\psi(x))$ for $x \in B(0, R) \setminus \mathbb{B}^n$, where $\psi(x) = x/|x|^2$. Put

$$q_{x_0}(r) := \frac{1}{\omega_{n-1}r^{n-1}} \int\limits_{|x-x_0|=r} Q(x)\,d\mathcal{H}^{n-1}\,, \tag{35}$$

$$q^*_{x_0}(r) := \frac{1}{\omega_{n-1}r^{n-1}} \int\limits_{|x-x_0|=r} Q^*(x)\, d\mathcal{H}^{n-1}\,, \tag{36}$$

where $r > 0$, ω_{n-1} is the area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. When calculating the value of $q_{x_0}(r)$ in (35), we assume that Q is extended by zero outside the unit ball. We prove one more important result.

Theorem 3.1. *Let $\alpha \in (0, 1]$. Suppose that under the conditions of Theorem 1.1, instead of requirement (7), the relation*

$$\limsup_{t\to 0} \int\limits_{t}^{\varepsilon_0} \left(\alpha - \frac{1}{q^{*\,1/(n-1)}_{x_0}(r)}\right) \cdot \frac{dr}{r} < +\infty \tag{37}$$

holds for some $x_0 \in \overline{\mathbb{B}^n}$ and some $0 < \varepsilon_0 < 1/2$, where $q^*_{x_0}(r)$ is defined in (36), and $K_I(x, f) \leqslant Q(x)$ for a.e. $x \in \mathbb{B}^n$. If f has a homeomorphic extension $f : \overline{\mathbb{B}^n} \to \overline{\mathbb{B}^n}$, then there exists $C > 0$ and $0 < \widetilde{\varepsilon_0} < \varepsilon_0$ depending only on n, x_0 and Q such that

$$|f(x) - f(x_0)| \leqslant C|x - x_0|^{\alpha} \quad \forall\ x \in B(x_0, \widetilde{\varepsilon_0}) \cap \overline{\mathbb{B}^n}\,. \tag{38}$$

Proof. The proof verbatim builds points II-VI of Theorem 1.1, in particular, the possibility of extending the mapping f to the whole space to a mapping F from the same class as the original map. Based on considerations similar to the proof of this theorem, the map F is a ring Q^*-map.

Denote by F_Q the family of all homeomorphisms of the class $W^{1,\varphi}(\mathbb{B}^n)$ of the unit ball onto itself satisfying the condition $f(0) = 0$, for which $K_I(x, f) \leqslant Q(x)$ a.e. $x \in \mathbb{B}^n$. In accordance with the above, every mapping F defined by formula (10), where f is taken from the hypothesis of the theorem, belongs to the class F_Q. Note also that all such mappings obviously do not take the values 0 and ∞ in the domain $\mathbb{R}^n \setminus \{0\}$. Let h be a chordal metric in $\overline{\mathbb{R}^n}$,

$$h(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}},\ \ h(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2}\sqrt{1 + |y|^2}}\,, \quad x \neq \infty \neq y\,,$$

and let $h(E) := \sup\limits_{x,y\in E} h(x, y)$ be a chordal diameter of a set $E \subset \overline{\mathbb{R}^n}$ (see, e.g., [59, Definition 12.1]). Based on the above formula, $h(\mathbb{R}^n \setminus \{0\}) = 1$. Let $\varepsilon_0 > 0$ be the number from the hypothesis of the theorem, then by [50, Theorem 4.16] we have the estimate

$$h(F(x), F(x_0)) \leqslant \alpha_n \cdot \exp\left\{-\int\limits_{|x-x_0|}^{\varepsilon_0} \frac{dr}{r q^{*\,\frac{1}{n-1}}_{x_0}(r)}\right\}, \tag{39}$$

for $x \in B(x_0, \varepsilon_0)$ and any $x_0 \in \mathbb{R}^n \setminus \{0\}$, where α_n depends only on n. Note that a similar estimate holds for the point $x_0 = 0$, since the same reasoning is applicable to the mapping f in the unit ball $\mathbb{B}^n$, in addition, $h(\overline{\mathbb{R}^n} \setminus \mathbb{B}^n) = 1$. Since $h(x, y) \geqslant \frac{|x-y|}{1+r_0^2}$ for $x, y \in \overline{B(0, r_0)}$, and $F|_{\overline{\mathbb{B}^n}} = f$, it follows from (39) that

$$|f(x) - f(x_0)| \leqslant 2\alpha_n \cdot \exp\left\{-\int\limits_{|x-x_0|}^{\varepsilon_0} \frac{dr}{r q^{*\,\frac{1}{n-1}}_{x_0}(r)}\right\} \tag{40}$$

where α_n depends only on n. Observe that

$$\frac{\exp\left\{-\int\limits_{t}^{\varepsilon_0} \frac{dr}{r q^{\frac{1}{n-1}}_{x_0}(r)}\right\}}{t^{\alpha}}$$

$$\frac{\exp\left\{-\int\limits_t^{\varepsilon_0}\frac{dr}{rq_{x_0}^{\frac{1}{n-1}}(r)}\right\}}{\exp\left\{-\alpha\int\limits_t^{1}\frac{dr}{r}\right\}}=\exp\left\{\int\limits_t^{\varepsilon_0}\frac{\alpha dr}{r}-\int\limits_t^{\varepsilon_0}\frac{dr}{rq_{x_0}^{\frac{1}{n-1}}(r)}\right\} \tag{41}$$

$$=\exp\left\{\int\limits_t^{\varepsilon_0}\left(\alpha-\frac{1}{q_{x_0}^{1/(n-1)}(r)}\right)\cdot\frac{dr}{r}\right\}.$$

Dividing the left side of (40) by $|x-x_0|^\alpha$ and taking into account (41), we obtain that

$$\frac{|f(x)-f(x_0)|}{|x-x_0|^\alpha}\leqslant\widetilde{C_n}\cdot\exp\left\{\int\limits_{|x-x_0|}^{\varepsilon_0}\left(\alpha-\frac{1}{q_{x_0}^{*\,1/(n-1)}(r)}\right)\cdot\frac{dr}{r}\right\}, \tag{42}$$

where $\widetilde{C_n}=2\alpha_n$.

By (37) there exists $M_0>0$, depending only on n, α and Q such that

$$\exp\left\{\int\limits_{|x-x_0|}^{\varepsilon_0}\left(\alpha-\frac{1}{q_{x_0}^{*\,1/(n-1)}(r)}\right)\cdot\frac{dr}{r}\right\}\leqslant M_0\qquad\forall\ x\in B(x_0,\widetilde{\varepsilon_0})\setminus\{x_0\} \tag{43}$$

for some $0<\widetilde{\varepsilon_0}<\varepsilon_0$. Now, by (42) and (43) we obtain that

$$|f(x)-f(x_0)|\leqslant C\cdot|x-x_0|^\alpha\quad\forall\ x\in B(x_0,\widetilde{\varepsilon_0}),$$

where $C=\widetilde{C_n}\cdot M_0$. Theorem is proved. □

Corollary 3.2. *Let $\alpha\in(0,1]$ and let $\varphi:(0,\infty)\to[0,\infty)$ be a non-decreasing Lebesgue measurable function with (1). Let $Q:\mathbb{B}^n\to[0,\infty]$ be integrable function in $\mathbb{B}^n$. Suppose also that there exist $C>0$ and $T>0$ such that (6) holds. Assume that f is a homeomorphism of $\mathbb{B}^n$ onto $\mathbb{B}^n$ such that $f\in W^{1,\varphi}(\mathbb{B}^n)$ and, in addition, $f(0)=0$. Let, moreover, $K_I(x,f)\leqslant Q(x)$ for a.e. $x\in\mathbb{B}^n$ and, besides that, the relation (37) holds for any $x_0\in\overline{\mathbb{B}^n}$, some $0<\varepsilon_0<1/2$, where $q^*_{x_0}(r)$ is defined in (36). Then f has a homeomorphic extension $f:\overline{\mathbb{B}^n}\to\overline{\mathbb{B}^n}$. Moreover, for any $x_0\in\overline{\mathbb{B}^n}$ there exists $C>0$ and $0<\widetilde{\varepsilon_0}<\varepsilon_0$ depending only on n, x_0 and Q such that (38) holds.*

Proof. Taking into account (37) and (41), we obtain that

$$\frac{\exp\left\{-\int\limits_t^{\varepsilon_0}\frac{dr}{rq_{x_0}^{\frac{1}{n-1}}(r)}\right\}}{t^\alpha}\leqslant\exp\left\{\int\limits_t^{\varepsilon_0}\left(\alpha-\frac{1}{q_{x_0}^{*\,1/(n-1)}(r)}\right)\cdot\frac{dr}{r}\right\}. \tag{44}$$

It follows from (44) that $\int\limits_t^{\varepsilon_0}\frac{dr}{rq_{x_0}^{\frac{1}{n-1}}(r)}\to\infty$ as $t\to 0$ for any $x_0\in\overline{\mathbb{B}^n}$. Indeed, the function $\alpha(t):=\exp\left\{-\int\limits_t^{\varepsilon_0}\frac{dr}{rq_{x_0}^{\frac{1}{n-1}}(r)}\right\}$ is monotone by t, thus, it has a limit as $t\to 0$. Assume that $\alpha(t)\to A$ and $A\neq 0$, now

$$\frac{\exp\left\{-\int\limits_t^{\varepsilon_0}\frac{dr}{rq_{x_0}^{\frac{1}{n-1}}(r)}\right\}}{t^\alpha}\to\infty$$

as $t\to 0$. Now, by (44) we obtain that $\exp\left\{\int\limits_t^{\varepsilon_0}\left(\alpha-\frac{1}{q^{*\,1/(n-1)}_{x_0}(r)}\right)\cdot\frac{dr}{r}\right\}\to\infty$ as $t\to 0$, that contradicts (37). The contradiction obtained above prove that $\alpha(t)\to 0$, which implies $\int\limits_t^{\varepsilon_0}\frac{dr}{rq_{x_0}^{\frac{1}{n-1}}(r)}\to\infty$ as $t\to 0$, that is desired conclusion.

By [30, Theorem 2.2] the map f is a ring Q-map at each point $x_0\in\overline{\mathbb{B}^n}$ for $Q=K_I(x,f)$. In this case, the possibility of homeomorphic extension of the mapping f onto $\mathbb{S}^{n-1}=\partial\mathbb{B}^n$ follows by [54, Theorem 1]. We also note that the map f extends to a homeomorphism of the unit ball $\overline{\mathbb{B}^n}$ onto itself, see, for example, [56, Lemma 6]. In this case, the desired conclusion follows from Theorem 3.1. □

Example 3.3. *We give an example of a mapping whose characteristic satisfies condition (37) with $\alpha=1$. Of course, an arbitrary conformal mapping is such, since its inner and outher dilatations are equal to 1, in addition, the map itself belongs to the class $W^{1,n}_{\rm loc}$ and, therefore, also belongs to the class $W^{1,\varphi}_{\rm loc}$ for example, for $\varphi=t^p$, $p>n-1$ (see e.g. [51, 3.I]). Since examples of such mappings are elementary, we will not dwell on them. We give an example of a map corresponding to condition (37) with $\alpha=1$, the dilatations of which are not bounded in a neighborhood of the point under consideration. For this purpose, we use the idea used in the construction of Example 2.1. We restrict ourselves to the case $n=3$.*

We consider the infinite partition of the segment $[0,1]$ by points $\left[\frac{1}{k+1},\frac{1}{k}\right]$, $k=1,2,3,\ldots$. We consider the following function $\beta:(0,1]\to\mathbb{R}$, defined as follows:

$$\beta(t)=\left\{\begin{array}{ll}1, & \left[\frac{1}{k+1},\frac{1}{k}-2^{-4k-1}\right]\\ 2^{k-1}, & t\in\left(\frac{1}{k}-2^{-4k-1},\frac{1}{k}\right),\end{array}\right.\tag{45}$$

$k=1,2,3,\ldots$. Setting $Q(x)=\beta(|x|)$, we obtain by (35) that $q_0(r)=q_0^*(r)=\beta(r)$. Set $\varepsilon_0:=1/4$ and $0<a<1$. Let $k_0\in\mathbb{N}$ be a number such that $a\in\left(\frac{1}{k_0+1},\frac{1}{k_0}\right]$. Let us verify the fulfillment of condition (37) for $\alpha=1$ and $\varepsilon_0=1/4$ at $x_0=0$. We obtain that

$$\int\limits_a^{\varepsilon_0}\frac{dr}{rq_0^{1/2}(r)}=\sum\limits_{k=4}^{k_0-1}\int\limits_{\frac{1}{k+1}}^{\frac{1}{k}-2^{-4k-1}}\frac{dr}{r}+\sum\limits_{k=4}^{k_0-1}\int\limits_{\frac{1}{k}-2^{-4k-1}}^{\frac{1}{k}}\frac{2^{(1-k)/2}\,dr}{r}+\int\limits_a^{\frac{1}{k_0}}\frac{dr}{r\beta^{1/2}(r)}$$

$$\geqslant\sum\limits_{k=4}^{k_0-1}\int\limits_{\frac{1}{k+1}}^{\frac{1}{k}-2^{-4k-1}}\frac{dr}{r}\,.\tag{46}$$

Observe that

$$\sum\limits_{k=4}^{k_0-1}\int\limits_{\frac{1}{k+1}}^{\frac{1}{k}-2^{-4k-1}}\frac{dr}{r}=\sum\limits_{k=4}^{k_0-1}\int\limits_{\frac{1}{k+1}}^{\frac{1}{k}}\frac{dr}{r}-\sum\limits_{k=4}^{k_0-1}\int\limits_{\frac{1}{k}-2^{-4k-1}}^{\frac{1}{k}}\frac{dr}{r}$$

$$=\ln\frac{k_0}{4}-\sum\limits_{k=4}^{k_0-1}\int\limits_{\frac{1}{k}-2^{-4k-1}}^{\frac{1}{k}}\frac{dr}{r}\geqslant\ln\frac{k_0}{4}-\sum\limits_{k=4}^{k_0-1}\frac{2^{-4k-1}}{\frac{1}{k}-2^{-4k-1}}\geqslant\ln\frac{k_0}{4}-c\tag{47}$$

for some $0<c<\infty$ because the series $\sum\limits_{k=4}^{\infty}\frac{2^{-4k-1}}{\frac{1}{k}-2^{-4k-1}}=\sum\limits_{k=4}^{\infty}\frac{k\cdot 2^{-4k-1}}{1-k\cdot 2^{-4k-1}}$ converges, for example, on the Cauchy principle. Thus, by (46),

$$\int\limits_a^{\varepsilon_0}\frac{dr}{rq_0^{1/2}(r)}\geqslant\ln\frac{k_0}{4}-c\,.\tag{48}$$

By (48),

$$\exp\left\{-\int\limits_{a}^{\varepsilon_0}\frac{dr}{rq_0^{1/2}(r)}\right\}\leqslant e^c\cdot(4/k_0)=4e^c\cdot\frac{k_0+1}{k_0(k_0+1)}\leqslant\frac{8e^c}{k_0+1}\leqslant 8e^c\cdot a\,, \tag{49}$$

because $a\geqslant 1/(k_0+1)$ *by the choice of* a. *Thus, by (49)*

$$\lim\limits_{a\to+0}\frac{\exp\left\{-\int\limits_{a}^{\varepsilon_0}\frac{dr}{rq_0^{1/2}(r)}\right\}}{a}\leqslant 8e^c<\infty\,, \tag{50}$$

as required. Finally, the fulfillment of relation (37) follows on the basis of (44) and (50). It should also be noted that condition (37) is satisfied in the neighborhood of any point of the unit sphere, since in some of its neighborhood the map f is identical, and the corresponding function Q is 1.

Guided by Proposition 6.15 in [48], by analogy, we construct the desired spatial map as follows:

$$f(x)=\frac{x}{|x|}e^{\int\limits_1^{|x|}(\beta(t)/t)\,dt}\,,\quad f(0):=0\,.$$

Note that the map f, defined in this way, is a homeomorphism. We verify that all the conditions of Theorem 3.1 are satisfied. Indeed, by [41, Proposition 6.3], we may calculate the tangential, radial, inner dilatations of the map f and the matrix norm of $f'(x)$ using the following formulas:

$$\delta_\tau(x)=\frac{|f(x)|}{|x|}=e^{\int\limits_1^{|x|}(\beta(t)/t)\,dt}\,,\quad \delta_r(x)=\frac{\partial|f(x)|}{\partial|x|}=e^{\int\limits_1^{|x|}(\beta(t)/t)\,dt}\cdot\frac{\beta(|x|)}{|x|}\,,$$

$$\|f'(x)\|=\max\{\delta_\tau,\delta_r\}=e^{\int\limits_1^{|x|}(\beta(t)/t)\,dt}\cdot\frac{\beta(|x|)}{|x|}\,,\quad K_I(x,f)=\beta(|x|)\,.$$

Note that the norm of the map $f'(x)$ is locally bounded in $\mathbb{B}^3\setminus\{0\}$; therefore, by virtue of inequality (12), all partial derivatives of the mapping that exist almost everywhere are also locally bounded. From this, in particular, it follows that the map f belongs to the class ACL in $\mathbb{B}^3$.

We now note that the function $\varphi(t)=t^3$ satisfies the Calderon condition (1). Let us verify that the map f belongs to the class $W^{1,\varphi}(\mathbb{B}^3)$. Indeed, by Fubini theorem,

$$\int\limits_{\mathbb{B}^3}\|f'(x)\|^3\,dm(x)=\int\limits_{\mathbb{B}^3}e^{3\int\limits_1^{|x|}(\beta(t)/t)\,dt}\cdot\frac{\beta^3(|x|)}{|x|^3}\,dm(x)$$

$$=4\pi\int\limits_0^1 e^{3\int\limits_1^{r}(\beta(t)/t)\,dt}\cdot\frac{\beta^3(r)}{r}\,dr=4\pi\sum\limits_{k=1}^{\infty}\int\limits_{\frac{1}{k+1}}^{\frac{1}{k}-2^{-4k-1}}r^2\,dr+4\pi\sum\limits_{k=1}^{\infty}\int\limits_{\frac{1}{k}-2^{-4k-1}}^{\frac{1}{k}}e^{3\int\limits_1^{r}(\beta(t)/t)\,dt}\cdot\frac{\beta^3(r)}{r}\,dr$$

$$\leqslant 4\pi\int\limits_0^1 r^2\,dr+4\pi\sum\limits_{k=1}^{\infty}\int\limits_{\frac{1}{k}-2^{-4k-1}}^{\frac{1}{k}}\frac{2^{3k-3}}{r}\,dr$$

$$\leqslant(4/3)\pi+4\pi\sum\limits_{k=1}^{\infty}\frac{2^{3k-3}}{\frac{1}{k}-2^{-4k-1}}\cdot 2^{-4k-1}<\infty\,. \tag{51}$$

Since $f \in ACL$, it follows from (51) that $f \in W^{1,\varphi}(\mathbb{B}^3)$.

We show that also $Q(x) = K_I(x, f) = \beta(|x|) \in L^1(\mathbb{B}^3)$. In fact,

$$\int\limits_{\mathbb{B}^3} K_I(x, f)\, dm(x) = 4\pi \int\limits_0^1 r^2 \cdot \beta(r)\, dr$$

$$= 4\pi \sum_{k=1}^{\infty} \int\limits_{\frac{1}{k+1}}^{\frac{1}{k}-2^{-4k-1}} r^2\, dr + 4\pi \sum_{k=1}^{\infty} \int\limits_{\frac{1}{k}-2^{-4k-1}}^{\frac{1}{k}} r^2 \cdot 2^{k-1}\, dr$$

$$\leqslant (4\pi)/3 + 4\pi \sum_{k=1}^{\infty} 2^{-4k-1} \cdot 2^{k-1} < \infty\, .$$

Thus, all the conditions of Theorem 3.1 are satisfied. Note that the map f is even Lipschitz on the unit sphere, since it identically maps in some neighborhood of it. According to this theorem, the map f is Lipschitz at the point 0, and also on the boundary of the unit ball.

4. An extended version of Theorem 3.1 in planar case

On the plane, Theorem 3.1 looks somewhat simpler; in particular, in approximately the same classes of mappings, the Calderon condition (1) is not required. To state an analogue of this theorem, we introduce the following notations.

Let D be a domain in $\mathbb{C}$. In what follows, a mapping $f : D \to \mathbb{C}$ is assumed to be *sense-preserving*. The following result is fairly close to [47, Theorem 1.1], although here, in contrast to [47], the corresponding property is established at the boundary rather than the inner point of the domain.

Theorem 4.1. *Let $\alpha \in (0, 1]$, and let f be a homeomorphism of $\mathbb{B}^2$ onto $\mathbb{B}^2$ such that $f \in W^{1,1}_{\rm loc}(\mathbb{B}^2)$ and $f(0) = 0$. Let, moreover,*

$$\limsup_{t\to 0} \int\limits_t^{\varepsilon_0} \left(\alpha - \frac{1}{q^*_{z_0}(r)}\right) \cdot \frac{dr}{r} < +\infty \tag{52}$$

for some $0 < \varepsilon_0 < 1/2$ and some $z_0 \in \overline{\mathbb{B}^2}$, where $q^*_{z_0}(r)$ is defined for $Q \in L^1(\mathbb{B}^2)$ in (36), and $K_I(z, f) \leqslant Q(z)$ a.e. in $\mathbb{B}^2$. If f has a homeomorphic extension $f : \overline{\mathbb{B}^2} \to \overline{\mathbb{B}^2}$, then there is $C > 0$ and $0 < \widetilde{\varepsilon_0} < \varepsilon_0$ depending only on z_0 and Q such that

$$|f(z) - f(z_0)| \leqslant C|z - z_0|^{\alpha} \quad \forall\, z \in B(z_0, \widetilde{\varepsilon_0}) \cap \overline{\mathbb{B}^2}\, . \tag{53}$$

Proof. **I.** Using conformal transformation $\psi(z) = \frac{z}{|z|^2}$, we extend the mapping f homeomorphically onto $\mathbb{C}$ as follows:

$$F(z) = \begin{cases} f(z), & |z| < 1\, , \\ \psi(f(\psi(z))), & |z| \geqslant 1\, . \end{cases}$$

As usual, put

$$\|f'(z)\| = \max_{h\in\mathbb{C}\setminus\{0\}} \frac{|f'(z)h|}{|h|}\, .$$

By condition, $f \in W^{1,1}_{\rm loc}(\mathbb{B}^2)$, therefore also $F \in W^{1,1}_{\rm loc}(\mathbb{B}^2)$. We show more, namely, that $F \in W^{1,1}_{\rm loc}(B(0, R))$ for any $R > 1$ (in particular, $\|F'(z)\|$ is not only locally integrable in $\mathbb{B}^2$, but also globally integrable).

II. Since $\|f'(z)\|^2 = K_I(z,f)\cdot|J(z,f)|$ a.e., and, in addition, the inner dilatation of f does not change under conformal mappings (see, e.g., [51, I.4.(4.15)]), by the Hölder inequality we obtain that

$$\int\limits_{\mathbb{B}^2} \|F'(z)\|\,dm(z) \leqslant \left(\int\limits_{\mathbb{B}^2} K_I(z,F)\,dm(z)\right)^{\frac{1}{2}} \cdot \left(\int\limits_{\mathbb{B}^2} |J(z,F)|\,dm(z)\right)^{\frac{1}{2}}. \tag{54}$$

Since the map F is a homeomorphism, by [17, Theorems 3.1.4, 3.1.8 and 3.2.5] we obtain that

$$\int\limits_{\mathbb{B}^2} J(z,F)\,dm(z) \leqslant m(F(\mathbb{B}^2)) = \pi\,. \tag{55}$$

Since $K_I(z,F)\in L^1(\mathbb{B}^2)$, it follows from (54) and (55) that

$$\int\limits_{\mathbb{B}^2} \|F'(z)\|\,dm(z) \leqslant \left(\pi\int\limits_{\mathbb{B}^2} K_I(z,f)\,dm(z)\right)^{\frac{1}{2}} < \infty\,. \tag{56}$$

Reasoning in a similar way, we may also obtain similar relations for the inner dilatation of the map F. Indeed, by (11) and (13), for any $R>1$ we obtain that

$$\int\limits_{B(0,R)} \|F'(z)\|\,dm(z) = \int\limits_{\mathbb{B}^2} \|f'(z)\|\,dm(z) + \int\limits_{1<|z|<R} (f(\psi(z)))^{-2}\cdot\|F'(\psi(z))\|\cdot|\psi(z)|^{-2}\,dm(z)$$

$$\leqslant \int\limits_{\mathbb{B}^2} \|f'(z)\|\,dm(z) + C\cdot\int\limits_{1<|z|<R} \|F'(\psi(z))\|\,dm(z)$$

for some $C>0$. Making a change of variables here, and taking into account that $K_I(z,f)\in L^1(\mathbb{B}^2)$, we obtain that

$$\int\limits_{B(0,R)} \|F'(z)\|\,dm(z) \leqslant \int\limits_{\mathbb{B}^2} \|f'(z)\|\,dm(z) + C\int\limits_{1/R<|y|<1} \|f'(y)\|\cdot\frac{1}{|y|^4}\,dm(y)$$

$$\leqslant \int\limits_{\mathbb{B}^2} \|F'(z)\|\,dm(z) + CR^4\cdot\int\limits_{1/R<|y|<1} \|f'(y)\|\,dm(y) < \infty. \tag{57}$$

III. By virtue of Fubini's theorem and by (see, for example, [52, Theorem III.8.1]) that the derivative of the function $\varphi = f(z_0+te)$, $t\in[a,b]$, $e\in\mathbb{S}^1$, is integrable on almost all segments in $B(0,R)$ parallel to the coordinate axes. In this case, arguing in a similar way to the proof of item V of Theorem 1.1, we may show that $F\in ACL(\mathbb{C})$.

IV. Since $F\in ACL(\mathbb{C})$, by (17) $F\in W^{1,1}_{\rm loc}(B(0,R))$ for any $R>1$. In this case, F is a ring Q^*-map at each point $z_0\in B(0,R)$ with $Q^*=Q$ in $\mathbb{B}^2$ and $Q^*(y)=Q\left(\frac{y}{|y|^2}\right)$ otherwise (see, e.g., [33, Theorem 3.1], cf. [53, Theorem 3.1]).

V. In view of item IV and using again [50, Theorem 4.16], we may show that

$$\frac{|f(z)-f(z_0)|}{|z-z_0|^{\alpha}} \leqslant \widetilde{C}\cdot\exp\left\{\int\limits_{|z-z_0|}^{\varepsilon_0}\left(\alpha-\frac{1}{q^*_{z_0}(r)}\right)\cdot\frac{dr}{r}\right\}. \tag{58}$$

By (52), there exists $M_0 > 0$, depending only on α and Q such that

$$\exp\left\{\int\limits_{|z-z_0|}^{\varepsilon_0}\left(\alpha-\frac{1}{q^*_{z_0}(r)}\right)\cdot\frac{dr}{r}\right\}\leqslant M_0 \qquad \forall\ z\in B(z_0,\widetilde{\varepsilon_0})\setminus\{z_0\}\,. \tag{59}$$

Combining (58) and (59), we arrive at the desired relation (53) with $C=\widetilde{C}\cdot M_0$. $\Box$

In particular, Theorem 4.1 implies the following statement.

Corollary 4.2. *Let $\alpha\in(0,1]$, let $z_0\in\mathbb{S}^1=\partial\mathbb{B}^2$, and let f be a homeomorphism of $\mathbb{B}^2$ onto $\mathbb{B}^2$ such that $f\in W^{1,1}_{\rm loc}(\mathbb{B}^2)$ and $f(0)=0$. Let, moreover, (52) holds for some $0<\varepsilon_0<1/2$ and any $z_0\in\overline{\mathbb{B}^2}$, where $q^*_{z_0}(r)$ is defined for $Q\in L^1(\mathbb{B}^2)$ in (36), and $K_I(z,f)\leqslant Q(z)$ a.e. in $\mathbb{B}^2$. Then f has a homeomorphic extension $f:\overline{\mathbb{B}^2}\rightarrow\overline{\mathbb{B}^2}$. Moreover, for any $z_0\in\overline{\mathbb{B}^2}$ there is $C>0$ and $0<\widetilde{\varepsilon_0}<\varepsilon_0$ depending only on z_0 and Q such that (53) holds.*

The proof of Corollary 4.2 almost literally repeats the proof of Corollary 3.2, and therefore is omitted.

Example 4.3. *We have already constructed an example of a map satisfying the conditions and the conclusion of Theorem 3.1 for $n=3$. For comparison, we will also construct a similar example on the plane. Let β be the function defined in (45). Now put $Q(z)=\beta(|z|)$. Now, by (35) we obtain that $q_0(r)=q_0^*(r)=\beta(r)$. Set $\varepsilon_0:=1/4$ and $0<a<1$. Arguing similarly to (46), we obtain that $\int\limits_a^{\varepsilon_0}\frac{dr}{rq_0(r)}\geqslant\sum\limits_{k=4}^{k_0-1}\int\limits_{\frac{1}{k+1}}^{\frac{1}{k}-2^{-4k-1}}\frac{dr}{r}$. Now, by (47) $\int\limits_a^{\varepsilon_0}\frac{dr}{rq_0(r)}\geqslant\ln\frac{k_0}{4}-c$. Thus, similarly to (48) and (49),*

$$\lim\limits_{a\to+0}\frac{\exp\left\{-\int\limits_a^{\varepsilon_0}\frac{dr}{rq_0(r)}\right\}}{a}\leqslant C<\infty \tag{60}$$

for some $0<C<\infty$. Thus, the condition (37) holds for $\alpha=1$ at 0.

According to Proposition 6.15 in [48], the mapping

$$w=f(z)=\frac{z}{|z|}e^{\int\limits_1^{|z|}(\beta(t)/t)\,dt}$$

is a homeomorphism of the Sobolev class $W^{1,1}_{\rm loc}(\mathbb{C})$ and has a dilatation $K_I(z,f)$ equal to $\beta(|z|)$. Note that this map takes the disk $\mathbb{B}^2$ onto itself and moreover, $f|_{\mathbb{S}^1}\equiv z$. Moreover,

$$\int\limits_{\mathbb{B}^2}K_I(z,f)\,dm(z)=2\pi\int\limits_0^1 r\cdot\beta(r)\,dr$$

$$=2\pi\sum\limits_{k=1}^{\infty}\int\limits_{\frac{1}{k+1}}^{\frac{1}{k}-2^{-4k-1}}r\,dr+2\pi\sum\limits_{k=1}^{\infty}\int\limits_{\frac{1}{k}-2^{-4k-1}}^{\frac{1}{k}}r\cdot2^{k-1}\,dr \tag{61}$$

$$\leqslant\pi+2\pi\sum\limits_{k=1}^{\infty}2^{-4k-1}\cdot2^{k-1}<\infty\,.$$

Reasoning similarly to (55)–(56), we may obtain from (61) that

$$\int_{\mathbb{B}^2} \|f'(z)\|\, dm(z) < \infty\,. \tag{62}$$

By (62), $f \in W^{1,1}(\mathbb{B}^2)$. Thus, the mapping f satisfies all the conditions of Theorem 4.1, and the conclusion of this theorem at the point $z_0 = 0$ is applicable for this mapping with $\alpha = 1$.

5. On Hölder continuity of harmonic mappings on the unit ball in $\mathbb{R}^n$

In this section, we provide initial results related to harmonic functions and plan to publish further results in a future article.

In order to discuss the subject we first need a few basic definitions and results.

Definition 5.1. *Let U be an open subset of $\mathbb{R}^n$. A harmonic function (real valued) is a twice continuously differentiable function $f : U \to \mathbb{R}$ that satisfies Laplace's equation, that is,*

$$\Delta f := \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0$$

everywhere on U. In physics notations often we write $\nabla^2 f$ instead of Δf and this is usually written as $\nabla^2 f = 0$. A function $f = (f_1, f_2, ..., f_m) : U \to \mathbb{R}^m$ is called vector valued harmonic function if f_i, $i = 1, 2, ..., m$, are real valued harmonic functions.

In two dimensions harmonic functions form a useful, strictly larger class of functions including holomorphic functions.

For example, harmonic functions still enjoy a mean-value property, as holomorphic functions do:
The mean value property: If $B(x, r)$ is a ball with center x and radius r which is completely contained in the open set $G \subset \mathbb{R}^n$, then the value $u(x)$ of a harmonic function $u : U \to \mathbb{R}$ at the center of the ball is given by the average value of u on the surface of the ball; this average value is also equal to the average value of u in the interior of the ball.

To get an orientation what we can expect concerning Hölder continuity of functions in Orlicz-Sobolev classes (and the place of these classes with respect to Sobolev classes) it seems useful to have in mind the following classical result.

5.1. Morrey's theorem

Let $G \subset \mathbb{R}^n$ be a bounded open set with C^1 boundary. Assume $n < p < \infty$ and set $\alpha = 1 - n/p > 0$. Then every function $f \in W^{1,p}$ coincides a.e. with a function $\tilde{f} \in C^{0,\alpha}(G)$. Moreover, there exists a constant C such that

$$|\tilde{f}|_{C^{0,\alpha}} \leqslant C|f|_{W^{1,p}} \quad \text{for all} \quad f \in W^{1,p}(G),$$

where $|f|_{W^{1,p}}$ is Sobolev norm of f on G. In statement of Morrey's theorem it is supposed that $W^{1,p}$, $p > n$, i.e., $(h-1)$ (see below Proposition 5.2) holds for particular choice of $\varphi(t) = t^p$, $p > n$. Recall some relations between Sobolev and Orlicz-Sobolev spaces. If $p > n$ and $\liminf_{t\to+\infty} \varphi(t)/t^p > c > 0$, G bounded then $W^{1,\varphi} \subset W^{1,p}$. If in addition $G \subset \mathbb{R}^n$ is a bounded open set with C^1 boundary and $f \in W^{1,\varphi}(G)$, then f is α-Hölder on G, where $\alpha = 1 - n/p > 0$. By this in mind, it seems natural question to consider what is a right version of Morrey's theorem for Orlicz-Sobolev spaces ?

We start with the following proposition which determines the places of Orlicz-Sobolev classes with respect to Sobolev classes.

Proposition 5.2. *Introduce the hypothesis* (h1)*: Let $n \geqslant 3$, $\alpha \in (0,1]$, and let $\varphi : (0,\infty) \to [0,\infty)$ be a non-decreasing Lebesgue measurable function*

(h2)*:*

$$\int_1^\infty \left(\frac{t}{\varphi(t)}\right)^{\frac{1}{n-2}} dt < \infty. \tag{63}$$

Then:

(S1) *$W^{1,p}(D)$, $p > n-1$, is in Orlicz-Sobolev class for $\varphi(t) = t^p$.*

(S2) *Suppose that φ satisfies (h1). If $p > n$ and $\liminf_{t\to+\infty} \varphi(t)/t^p > c > 0$, G bounded then $W^{1,\varphi} \subset W^{1,p}$. If in addition $G \subset \mathbb{R}^n$ is a bounded open set with C^1 boundary and $f \in W^{1,\varphi}(G)$, then f is α-Holder on G, where $\alpha = 1 - n/p > 0$.*

(S3) *Suppose that φ satisfies (h1) and (h2) and D is a bounded domain in $\mathbb{R}^n$, $n \geqslant 3$, and that $f \in W^{1,\varphi}(D)$. Then $W^{1,\varphi}(D) \subset W^{1,n-1}(D)$.*

Proof. (i) is readable. Note that (ii) follows from Morrey's theorem. Let us prove (iii).

Set $A(t) = \left(\frac{t}{\varphi(t)}\right)^{\frac{1}{n-2}}$. Since integral $\int_1^\infty A(u)du < \infty$, then $\int_t^{2t} A(u)du = o(1)$, when $t \to \infty$. Further $A(t) \leqslant A(u)$ for $t \leqslant u$ and therefore $tA(t) \to 0$ and by elementary consideration $\varphi(t) = M(t)t^{n-1}$, where $M(t) \to \infty$ when $t \to \infty$. Hence there is $t_0 > 1$ such that $\varphi(t) \geqslant t^{n-1}$ for $t \geqslant t_0$. Next we conclude $|\nabla f(x)|^{n-1} \leqslant \varphi(|\nabla f(x)|)$ if $|\nabla f(x)| \geqslant t_0$. Hence it is readable $W^{1,\varphi}(D) \subset W^{1,n-1}(D)$. □

Our approach in this section is based on the following results which we call local spatial version of Privalov theorem for harmonic functions and which has an independent interest.

Theorem 5.3. *Suppose that $0 < \alpha < 1$, h is a Euclidean harmonic mapping from $\mathbb{B}^n$, and continuous on $\overline{\mathbb{B}^n}$. Let $x_0 \in \mathbb{S}^{n-1}$ and $|h(x)-h(x_0)| \leqslant M|x-x_0|^\alpha$ for $x \in \mathbb{S}^{n-1}$. Then there is a constant M_n such that $(1-r)^{1-\alpha}|h'(rx_0)| \leqslant M_n$, $0 \leqslant r < 1$.*

For some global results of this type see for example [7] and literature cite there.

As we mention, recall that we can get some versions of the previous theorems in Sections 1-4 for harmonic maps which are immediate corollary of Theorem 5.3:

Corollary 5.4. *If $f \in W^{1,\varphi}(\mathbb{B}^n)$ satisfies the condition of Theorem 1.1 and it is in addition harmonic in the sense of Definition 5.1, then f is α-Höllder on $\mathbb{B}^n$.*

Theorem 5.5. *The following statements are true:*

(i) *If under the condition in statement of Theorem 3.1 (with respect to point x_0) in addition f is harmonic, then there is a constant M_n such that*

(I-1) $(1-r)^{1-\alpha}|f'(rx_0)| \leqslant M_n$, $0 \leqslant r < 1$.

(ii) *If under conditions of Theorem 4.1 (with respect to point z_0) in addition f is harmonic, then (I-1) holds for $n = 2$ with z_0 instead of x_0.*

(iii) *If under the conditions of Corollary 3.2 in addition f is harmonic, then f is α-Hölder on $\overline{\mathbb{B}^n}$.*

(iv) *Under the conditions of Corollary 4.2, in addition f is harmonic, then f is α-Hölder on $\overline{\mathbb{B}^2}$.*

In addition we prove Propositions 5.6 and 5.8.

5.2. Propositions 5.6 and 5.8

Set $Q_\varepsilon(x_0)$ the mean value of Q over ball $B(x_0,\varepsilon)$ and $Q^+(x_0)$ supremum of $Q_\varepsilon(x_0)$ over $0<\varepsilon<\varepsilon_0$.

Kalaj and the first author study mappings in plane and space which satisfy the Poisson differential inequality:

(h3) $|\triangle u| \leqslant a|\nabla u|^2 + b$.

We start with the planar case which has some very specific properties with respect to spatial case. Note that the subject of harmonic quasiconformal (shortly hqc) mappings has been intensively studied by the participants of the Belgrade Analysis Seminar (see for example [13] and [37] for more details). In particular Kalaj proved that if h is a hqc mapping of the unit disk onto a Lyapunov domain, then h is Lipschitz (see [25]). Recently in [13] it is proved h is co-Lipschitz.

Hence

Proposition A. *Suppose $h:\mathbb{U}\to D$ is a hqc homeomorphism, where D is a Lyapunov domain with $C^{1,\mu}$ boundary. Then h is bi-Lipschitz (shortly bi-Lip).*

Here we prove for example the following:

Proposition 5.6. *Let D be a Lyapunov planar domain and $f=g+\overline{h}$ injective harmonic presenting orientation of $\mathbb{D}$ onto D (or more generally C^2 homeomorphism which satisfiies* (h3)*) and either* (i): *(7) holds a.e. on $\mathbb{S}$ with $Q=K_\mu$ and constant $C=K_1$ or* (ii): *$q_z(r)\leqslant c_1$ for almost all $z\in\mathbb{D}$. Then f is bi-Lip.*

Our proof is based on the Hardy spaces theory.

Proof. Since f injective harmonic presenting orientation of $\mathbb{D}$ onto D, $|\mu(z)|<1$ for $z\in\mathbb{D}$, where $\mu=\mu_f$. Let E be the set of points z_0 on $\mathbb{S}$ for which the finite radial limit μ^* exists. From the Hardy spaces theory it is known then the radial limit μ^* exist a.e. on $\mathbb{S}$.

Suppose that (i) holds and denote by $E_0\subset\mathbb{S}$ the set on which (i) holds. Note that the set $\mathbb{S}\setminus(E\cap E_0)$ is of measure 0.

Let $z_0\in E\cap E_0$ and $D(z_0,r)$ is the intersection of disk $B(z_0,r)$ with $\mathbb{D}$. Let $Q(r,z_0)$ be the mean vale of Q over $B(z_0,r)$. Since the angular limit $K_\mu^*(z_0)$ of K_μ also exists at z_0 we conclude that $Q(r,z_0)$ tends to $K_\mu^*(z_0)/2$ from (1.7) that $K_\mu^*(z_0)/2\leqslant K_1$.

Hence if we set $K=2K_1$ and $k=\frac{K-1}{K+1}$, we conclude that $|\mu^*|\leqslant k$ a.e. on $\mathbb{S}$. Next since $|\mu|=|h'/g'|$ and h'/g' is holomorphic function then from Hardy spaces theory $|\mu|\leqslant k$ on $\mathbb{D}$. Hence f is K-qc and therefore by **Proposition A** bi-Lip on $\mathbb{D}$.

If we suppose that (ii) holds on the set $E_1\subset\mathbb{S}$ and let γ_r be the part of the circle $|z-z_0|=r$ in $\mathbb{D}$. If $z_0\in E\cap E_1$ then as the above we first conclude that $q_{z_0}(r)$ tends to $K_\mu^*(z_0)/2$ and from (ii) that $K_\mu^*(z_0)/2\leqslant c_1$. □

Now we will consider a spatial version of Proposition 5.6. We first need some simple properties. Suppose that h is a vector Euclidean harmonic mapping from $\mathbb{B}^n$ into $\mathbb{B}_M$. Then

(i) If $h(0)=0$, then

(1) $|h(x)|\leqslant M_1|x|$.

(ii) h is Lipschitz on any ball $B(0,r_0)$, $r_0\in[0,1)$.

(iii) In particular if, h is harmonic function on some domain $G\subset\mathbb{R}^n$, h is locally Lipschitz at every point $x_0\in G$.

(i) It is clear that there is $M>0$ such that $|h(x)|\leqslant 2M|x|$ for $|x|\geqslant 1/2$. Next on $\mathbb{B}_{1/2}$ partial derivatives of h are bounded and therefore (1) follows.

(ii) partial derivatives of h are bounded $B(0,r_0)$ and hence (ii) follows.

(iii) follows from (ii).

The following lemma shows that the above properties hold in more general setting:

Lemma 5.7. *Let $f : U \to \mathbb{R}^m$ be a differentiable function (where $U \subset \mathbb{R}^n$ is open) and F compact subset of U and suppose that (h6): partial derivatives are bounded on F (in particular (h6) is satisfied if f is C^1 on U). Then f is Lip on F.*

Proof. Note that $d = \operatorname{dist}(F, \partial U) > 0$. Let $x, y \in F$ and set $h = y - x$. If $|h| \leqslant d$, then by the Mean value theorem in several variables one finds points $x + t_i h$ on the line segment $[x, y]$ satisfying $f_i(x + h) - f_i(x) = \nabla f_i(x + t_i h) \cdot h$. By hypothesis (h6) there is a constant $M > 0$ such that $|\nabla f_i(x)| \leqslant M$ on F. Hence f_i is Lip on F if $|y - x| \leqslant d$.

Suppose now that $|y - x| \geqslant d$. Note first since F is compact and f is continuous on U, then there is a constant $M_1 > 0$ such that $|f|$ is bounded on F. Next the ratio of $|fy - fx|$ and $|y - x|$ is bounded by $2M_1 d^{-1}$ and we conclude that therefore f_i is Lip on F and therefore f is Lip on F. □

In order to formulate a spatial version of Proposition 5.6 we need some definitions and a result.

For $x_0 \in \mathbb{R}^n$ and $0 \leqslant r_1 < r_2$ we define $A(x_0, r_1, r_2) = \{x : r_1 < |x - x_0| < r_2\}$ which call a spherical ring. If $x_0 = 0$ we write simply $A(r_1, r_2)$.

Let G be an open subset of $\mathbb{R}^n$ and $f : G \to \mathbb{R}^n$. We say that f has finite distortion if, first of all if $f \in W^{1,1}_{\mathrm{loc}}(G, \mathbb{R}^n)$ and there is a function $K(x) = K(x, f), 1 \leqslant K(x) < \infty$, defined a.e. in G such that

(i) $\|f'(x)\|^n \leqslant K(x) J(x, f)$ a.e. G.

The smallest $K = K_f$ satisfying (i) is called the outer dilatation function of f.

Kalaj and the first author study mappings in plane and space which satisfy the Poisson differential inequality:

(h3) $|\triangle u| \leqslant a|\nabla u|^2 + b$.

In [27] Kalaj proved (see also subsection 6.1, Further results) for a more general result)

Theorem K. *A quasiconformal mapping of the unit ball onto a domain with C^2 smooth boundary, satisfying the Poisson differential inequality, is Lipschitz continuous.*

Proposition 5.8. *Let $G \subset \mathbb{R}^n$ be C^2 domain. If $f : \mathbb{B}^n \xrightarrow{onto} G$ harmonic homeomorphism (or more generally C^2 homeomorphism which satisfies (h3)) and there are a $r_0 \in (0, 1)$ and non negative function Q defined a.e. on the ring $A(r_0, 1)$ such that $K(x, f) \leqslant Q(x)$ for a.e. $x \in \mathbb{B}^n$ and* (h4): *Q^+ is bounded on ring $A(r_0, 1)$, then f is Lipschitz on $\mathbb{B}^n$.*

Proof. The hypothesis (h4) implies that there is $K \geqslant 1$ such that $K_f \leqslant K$ on the ring $A(r_0, 1)$. Hence by Theorem K, f is Lipschitz on $A(r_0, 1)$. Since by Lemma 5.2, f is Lipschitz on $B(0, r_0)$ the result follows. □

5.3. Local spatial version of Privalov theorem for harmonic functions

First we need some definitions and properties of spherical cap. Recall by ω_{n-1} we denote the surface of $n-1$-dimensional sphere $\mathbb{S}^{n-1}$ (pay attention that some authors prefer notation $\mathbb{S}^n$ for $(n-1)$-dimensional sphere and ω_n for its area). Then the surface $(n-1)$-dimensional measure of sphere $S(0, r)$ of radius r is $P(r) = \omega_{n-1} r^{n-1}$.

Definition 5.9. *(Spherical-polar cap). We can define the spherical cap in terms of the so-called contact angle (the angle between the normal to the sphere at the bottom of the cap and the base plane). More precisely, we use the following notations $\hat{x} = x/|x|$ and $\hat{0} = e_1$, $S(\hat{x}, \gamma) = \{y \in \mathbb{S}_n : \langle y, \hat{x}\rangle \geqslant \cos\gamma\}$ for the polar cap with center $\hat{x}$, where γ is the spherical angle of it. In a similar way in planar case we define $C(\hat{x}, \gamma) = \{y \in \mathbb{S}_1 : \langle y, \hat{x}\rangle \geqslant \cos\gamma\}$.*

If f is a function on $\mathbb{S}^{n-1}$ which is constant on $\partial\mathbb{S}^{\varphi} = \{t \in \mathbb{S}^{n-1} : t_n = \cos\varphi\}$ for $0 \leqslant \varphi \leqslant \pi$ we say that f depends only on φ. Let $0 \leqslant \varphi \leqslant \pi$; the surface of spherical (polar) cap $\mathbb{S}^{\varphi}$ of radius φ is

$$A(\varphi) = \omega_{n-1} \int_0^{\sin\varphi} \frac{r^{n-2}}{x_n} dr \,.$$

By change of variables $r = \sin\theta$, $dr = \cos\theta\, d\theta$, $x_n = \cos\theta$ in the previous formula, we find

$$A(\varphi) = \omega_{n-1} \int_0^{\varphi} \sin^{n-2}\theta\, d\theta\,. \tag{64}$$

See also proof of Theorem 2 and formula (11) in Section V in L. Ahlfors book [4].

Proposition 5.10. *If f is a function on $\mathbb{S}^{n-1}$ which depends only on φ, then*
$\int_{\mathbb{S}^{n-1}} f d\sigma = \int_0^{\pi} f(\varphi)\, A'(\varphi)\, d\varphi = \omega_{n-1} \int_0^{\pi} f(\varphi)\, \sin^{n-2}(\varphi)\, d\varphi.$

We only outline a proof. Let $0 = \varphi_0 < \varphi_1 < \varphi_2 < \dots \varphi_n = \pi$, $\varphi_{k-1} < \xi_k < \varphi_k$ and
$S_n = \sum_{k=1}^{n} f(\xi_k)\big(A(\varphi_k) - A(\varphi_{k-1})\big)$. Then $S_n \to \int_{\mathbb{S}^{n-1}} f d\sigma$ and $S_n \to \int_0^{\pi} f(\varphi)\, A'(\varphi)\, d\varphi$ when $n \to \infty$. Hence since $A'(\varphi) = \omega_{n-1} \sin^{n-2}\varphi$ the proof follows.

6. Proof of Theorem 5.3

Before to proceed to the proof we need a few definition and some elementary propererties of spatial harmonic functions.

Definition 6.1. *Recall if $x, y \in \mathbb{R}^n$ by $|x-y|$ we denote Eucledian distance between x and y.*
Further in this definition, we suppose that (i) G is a domain in $\mathbb{R}^n$ and $f : G \to \mathbb{R}^m$.

1. *We say f is locally Hölder (α -Hölder) at $x_0 \in G$ if*
   $Lf_\alpha(x_0) = \limsup_{G \ni x \to x_0} |f(x) - f(x_0)/|x - x_0|^\alpha < \infty$ *when* $G \ni x \to x_0$.
   If $\alpha = 1$ we say that f is locally Lipschitz at x_0. We write $Lf(x_0)$ instead $Lf_\alpha(x_0)$. We can adapt the above definition for $f : \mathbb{S}^{n-1} \to \mathbb{R}^m$.
   We say that f is Hölder continuous, when there are nonnegative real constants C and α, $0 < \alpha \le 1$, such that
   $$|f(x) - f(y)| \le C|x-y|^\alpha$$
   for all x and y in the domain G of f. The number α is called the exponent of the Hölder condition and we also say f is α-Hölder continuous and write $f \in \mathrm{Lip}_\alpha(G, \mathbb{R}^m)$. By $|f|_{C^{0,\alpha}}$ we denote the smallest constant for which the pervious inequality holds and call the Hölder norm of f on G.
2. *If $\alpha = 1$ in the previous inequality, then we say that the function satisfies a Lipschitz condition or it is Lipschitz continuous (shortly Lip) on G with multiplicative constant C. If $m = n$ in (i) and f is homeomorphism and both f and f^{-1} are Lipschitz we say that f is bi-Lipschitz (shortly bi-Lip).*
3. *If the function f and its derivatives up to order $k \in \mathbb{N}$ are bounded on the closure of G and α-Hölder continuous, then we say that f belongs to the Hölder space $C^{k,\alpha}(\overline{G})$.*
4. *Let γ be a closed rectifiable Jordan planar curve of length s_0, G domain enclosed by γ and s an arc length parametar on γ, and $\gamma_0(s)$, $s \in [0, s_0]$, arc length parametarization of γ. We say that G is $C^{k,\alpha}$ domain if γ_0 is $C^{k,\alpha}$ on $[0, s_0]$. In the literature planar domain D is called Lyapunov domain if D has smooth $C^{1,\alpha}$- boundary for some $0 < \alpha < 1$.*

In the proof of next theorem we use the representation of harmonic functions (see formula (65) below) by means the Poisson kernel for harmonic functions on the unit ball $\mathbb{B}^n$ which is given by

$$P(x, \eta) = \frac{1 - |x|^2}{\omega_{n-1}|x - \eta|^n}\,,$$

where $x \in \mathbb{B}^n$ and $\eta \in \mathbb{S}^{n-1}$, and positive Borel measure $d\sigma$ on $\mathbb{S}^{n-1}$. By $d\sigma$ we denote positive Borel measure on $\mathbb{S}^{n-1}$ invariant with respect to orthogonal group $O(n)$ normalized such that $\sigma(S^{n-1}) = 1$.

First recall the statement of Theorem 5.3.

Theorem 6.2. *Suppose that* $0 < \alpha < 1$, h *is a Euclidean harmonic mapping from* $\mathbb{B}^n$ *which is continuous on* $\overline{\mathbb{B}^n}$, *and*
(h1) Let $x_0 \in \mathbb{S}^{n-1}$ *and* $|h(x) - h(x_0)| \leqslant M|x - x_0|^\alpha$ *for* $x \in \mathbb{S}^{n-1}$.
Then there is a constant M_n *such that*
$(1 - r)^{1-\alpha}|h'(rx_0)| \leqslant M_n$, $0 \leqslant r < 1$.

Proof. Let h_b denote the restriction of h on $\mathbb{S}^{n-1}$. Since h is harmonic on $\mathbb{B}^n$ and continuous on $\overline{\mathbb{B}^n}$, then

$$h(x) = \int\limits_{\mathbb{S}^{n-1}} P(x, \eta)h_b(\eta)d\sigma(\eta) \tag{65}$$

for every $x \in \mathbb{B}^n$. Set $d := d(x) = 1-|x|^2$. By computation $\partial_{x_k}P(x,t) = -(\frac{2x_k}{|x-t|^n} + d(x)n\frac{x_k-t_k}{|x-t|^{n+2}})$. Hence if $d \leqslant |x-t|$, then
(1) $|\partial_{x_k}P(x,t)| \leqslant c_1\frac{1}{|x-t|^n}$.

Let $x = re_n$ and θ the angle between t and e_n. Then $s := |x - t|^2 = 1 - 2r\cos\theta + r^2$ depends only on θ for fixed x. Next since $\int\limits_{\mathbb{S}^{n-1}} \partial_k P(x,t)h(e_n)d\sigma(t) = 0$, we find

$$\partial_{x_k}h(x) = \int\limits_{\mathbb{S}^{n-1}} \partial_k P(x,t)\big(h(t) - h(e_n)\big)d\sigma(t)\,. \tag{66}$$

Hence by (1) and the hypothesis $(h1)$, we get

$$|\partial_{x_k}h(x)| \leqslant c_2 \int\limits_{\mathbb{S}^{n-1}} \frac{|e_n - t|^\alpha}{|x - t|^n}d\sigma(t)\,. \tag{67}$$

Therefore the proof of Theorem 6.2 is reduced to the proof of the following proposition. □

Proposition 6.3. *Suppose that* $0 < \alpha < 1$ *and* $x = re_n$, $0 < r < 1$. *Then*

$$I_\alpha(re_n) =: \int\limits_{\mathbb{S}^{n-1}} \frac{|e_n - t|^\alpha}{|x - t|^n}d\sigma(t) \leqslant c \cdot \frac{1}{(1 - r)^{1-\alpha}},$$

where $c = c(\alpha, n)$ *is a positive constant which depends only on n and* α.

Using similar approach if ω is a majorant one can prove

$$I_\omega(re_n) =: \int\limits_{\mathbb{S}^{n-1}} \frac{\omega(|e_n - t|)}{|x - t|^n}d\sigma(t) \leqslant c \cdot \frac{\omega(\delta_r)}{\delta_r}.$$

Proof. We use spherical cups S^θ defined by $t_n > \cos\theta$ and integration with parts. Since for a fixed $\theta \in [0,\pi]$, $|e_n - t| \leqslant \theta$ for $t \in S^\theta$, by an application of Proposition 5.10 to $f(t) = \frac{|e_n-t|^\alpha}{|x-t|^n}$, we get (see also Remark 6.4 below)

$$I_\alpha(re_n) \leqslant c_3 \int_0^\pi \frac{|\theta|^{n-2}|\theta|^\alpha}{((1-r)^2 + \frac{4r}{\pi^2}\theta^2)^{n/2}}\, d\theta < \tag{68}$$

$$c_4 \int_0^\infty \frac{\theta^{\alpha+n-2}}{\left((1-r)^2 + \frac{4r}{\pi^2}\,\theta^2\right)^{n/2}}\, d\theta\,. \tag{69}$$

Next using $(1 + \frac{4r}{\pi^2}u^2)^{-1} \leqslant c_5(1+u^2)^{-1}$ for $\frac{1}{2} \leqslant r < 1$ and the change of variable $\theta = (1-r)u$, we find

$$I_\alpha(re_n) \leqslant c_6(1-r)^{\alpha-1} \int_0^\infty \frac{u^{\alpha+n-2}}{(1+u^2)^{n/2}}\, du\,. \tag{70}$$

Denote by $J(\alpha)$ the last expression on the right hand side of previous formula. Hence since $g(u) = \frac{u^{\alpha+n-2}}{(1+u^2)^{n/2}} \sim u^{\alpha-2}$ for $u \to +\infty$ and by hypothesis $0 < \alpha < 1$ and therefore $\alpha - 2 < -1$, the integral $J(\alpha)$ converges and and therefore

(i) $I_\alpha(re_n) \leqslant c_7(1-r)^{\alpha-1}$ for $\frac{1}{2} \leqslant r < 1$.
$(1-r)^{1-\alpha}A(r)$ is continuous on $[0,1/2]$ and attains a maximum c_8, that is
(ii) $I_\alpha(re_n) \leqslant c_9(1-r)^{\alpha-1}$ for $0 \leqslant r \leqslant \frac{1}{2}$, where $c_9 = c_3c_8$.
Hence from (i) and (ii) with $c = \max\{c_7, c_9\}$ the proof of Proposition follows. □

Combining Proposition 6.3 and (67) we get proof of Theorem.

Remark 6.4. *It is convenient to denote expressions by $A(r)$ and $B(r)$ that appear on the right-hand side in formula (68) and (69) without constants c_3 and c_4 respectively. Note that $A(0)$ is finite and that $B(0) = +\infty$. In order to estimate $A(r)$ we use the change of variable $\theta = (1-r)u$ and therefore the integral $A(r)$ can be transformed to integral over $[0, a(r)]$ with respect to u, where $a(r) = \pi(1-r)^{-1}$. Since $a(r) \to \infty$ if $r \to 1$, it is convenient to estimate integral $A(r)$ by integral $B(r)$ over interval $[0,\infty)$.*

Remark 6.5. *Instead of (ii) we can based the proof of Theorem on the following inequality:*

$$|\partial_{x_k} h(x)| \leqslant c_7 \cdot \frac{1}{(1-r)^{1-\alpha}}$$

for $\frac{1}{2} \leqslant |x| < 1$. Hence since on $\mathbb{B}_{1/2}$ partial derivatives are bounded readably proof of Theorem 6.2 follows. Note that the above proof breaks down for $\alpha = 1$ because $J(1) = \infty$. Moreover, for each $n = 2$, there is a Lipschitz continuous map $f : \mathbb{S}^{n-1} \to \mathbb{R}^n$ such that $u = P[f]$ is not Lipschitz continuous. In planar case, consider $f = u + iv$ such that $zf' = -\log(1-z)$. u'_θ is bounded while its harmonic conjugate ru'_r is not bounded. In spatial case, consider $U(x_1, x_2, ...x_n) = u(x_1 + ix_2, x_3, ...x_n)$.

6.1. Further results

Using an approach as in [6], we can prove further results. Here we only announce the following results:

Theorem 6.6. *Suppose that $f : \mathbb{S}^{n-1} \to \mathbb{R}^n$ is locally Lipschitz* (Lip-1) *at $x_0 \in \mathbb{S}$, $f \in L^\infty(\mathbb{S}^{n-1})$ and $h = P[f]$ is a Euclidean harmonic mapping from $\mathbb{B}^n$.*
Then

S4)

$$|h'(rx_0)T| \leqslant M$$

for every $0 \leqslant r < 1$ *and unit vector* T *which is tangent on* $\mathbb{S}_r^{n-1}$ *at* rx_0*, where* M *depends only on* n*,* $|f|_\infty$ *and* $Lf(x_0)$*.*

If we suppose in addition that h *is K-quasiregular (shortly K-qr) mapping along* $[o, x_0)$*, then*

S5)

$$|h'(rx_0)| \leqslant KM$$

for every $0 \leqslant r < 1$*.*

We can extend our results to class of moduli functions which include $\omega(\delta) = \delta^\alpha$ $(0 < \alpha \leqslant 1)$, so our result generalizes earlier results on Hölder continuity (see [46]) and Lipschitz continuity (see [6]).

In addition, concerning further research we suggest some possibility. Suppose that domains D and Ω are bounded domains in $\mathbb{R}^n$ and its boundaries belong to class $C^{k,\alpha}, 0 \leqslant \alpha \leqslant 1, k \geqslant 2$ (more generally C^2). Suppose further that g and g' are C^1 metric on $\overline{D}$ and $\overline{\Omega}$ respectively. Using inner estimate (cf. Theorem 6.14 [19][10]) we can prove

Theorem MM (Theorem 6.9 [36]). *If* $u : D \to \Omega$ *is a qc* (g, g')*-harmonic map* (*or satisfies* $(h3)$)*, then* u *is Lipschitz on* D*.*

We discussed this result at Workshop on Harmonic Mappings and Hyperbolic Metrics, Chennai, India, Dec. 10-19, 2009, and in [36], where a proof is outlined. For more details see [38].

We now present a few open questions.

Using **Theorem MM** or **Theorem K** we can prove.

Theorem B. *Let* $G \subset \mathbb{R}^n$ *be* C^2 *domain. If* $f : \mathbb{B}^n \xrightarrow{onto} G$ *harmonic homeomorphism and there is a* $r_0 \in (0, 1)$ *such that* $(h4)$*:* Q^+ *is bounded on ring* $A(r_0, 1)$*, then* f *is Lipschitz on* $\mathbb{B}^n$*.*

Question 1. *If we suppose instead of* $(h4)$ *only* $(h5)$*:* Q^+ *is bounded on* $\mathbb{S}^{n-1}$*, whether* f *is Lipschitz on* $\mathbb{B}^n$*?*
What is right version of Theorem 1.1 and 4.1 if in addition it is supposed that f *is harmonic?*

7. On Bi-Lipschitz qc maps

Recall that the condition (52) provides sufficient conditions for Hölder and Lipschitz continuity. In this section we show that in some situation if Beltrami coefficient is Hölder continuous that the map is bi-Lip. In order to discuss the subject we first need some preliminaries.

Definition 7.1. 1. *Let* f *be a complex valued function defined an open set planar set* V*. We use notation* $z = x + iy$ *for complex numbers and for complex partial derivatives* $f_z := (f_x - if_y)/2$ *and* $f_{\overline{z}} := (f_x + if_y)/2$*, where* f_x *and* f_y *are partial derivatives with respect coordinates* x *and* y*. In the literature frequently notation* ∂f *and* $\overline{\partial} f$ *are used instead* f_z *and* $f_{\overline{z}}$ *respectively.*
In this section let Ω *denote a planar domain.*

2. *An equation*

$$\overline{\partial} f = \mu \, \partial f, \tag{71}$$

where μ *is a complex valued measurable function defined a.e. on* Ω *and* $\|\mu\|_\infty < 1$ *is essential supremum with respect to* L^∞*-norm, is called* μ*-Beltrami equation on* Ω*.*

3. *If a homeomorphism* $f : \Omega \xrightarrow{onto} \Omega_* \subset \mathbb{C}$ *satisfies*
   (i) f *is ACL on* Ω*, and*
   (ii) $|f_{\overline{z}}| \leqslant k\,|f_z|$ *almost everywhere in* Ω*, where* $k = \frac{K-1}{K+1} \in [0, 1)$*,*
   we say that f *is a K- quasiconformal (shortly qc); more precisely K-qc in analytic sense.*

The last item 3. of the definition is equivalent to requirement that:
(A) f is homeomorphism and it has locally integrable distributional derivatives which satisfy (ii).

Theorem 7.2. (Existence theorem) *Let μ be a measurable function in a domain Ω with $\|\mu\|_\infty < 1$. Then there is a qc mapping of Ω whose complex dilatation agrees with μ a.e.*

In this setting we say that f is solution of Beltrami equation for μ. The complex dilatation at z_0 is

$$\mu_f = \frac{f_{\bar z}}{f_z}. \tag{72}$$

Frequently the notation Belt(f) is also used instead of μ_f.

Theorem 7.3. *Let f and g be qc map of a domain Ω whose complex dilatations agree a.e. in Ω. Then $f \circ g^{-1}$ is a conformal mapping.*

In order to get a feeling of the subject we first consider some examples.

Let f_0 be a branch of $\sqrt{z}$. Then $\mu_{f_0} = 0$ and f_0 has a singularity at 0 and $\mu_g = 0$ a.e. on $\mathbb{C}$. Next, by Theorem 7.3, g is a Möbius transformation $A = f_0 \circ g^{-1}$ is a conformal mapping. Thus $f_0 = A \circ g$ is a conformal mapping. More precisely if G is a simple connected domain which does not contain 0, $A \circ g$ is a conformal mapping on G.

The following known example shows that qc with continuous Beltrami coefficient are not C^1 in general.

Example 7.4. *[10, 12] Consider $f(z) = -z \ln |z|^2$ for $|z| \leqslant r_0 = e^{-2}$. Then $f : B(0, r_0) \to B(0, 4r_0)$ and*

$$f_{\bar z} = -\frac{z}{\bar z}, f_z = -1 - \log |z|^2, \mu_f = \frac{z}{\bar z(1 + \log |z|^2)}.$$

Hence it is qc with continuous Beltrami coefficient, and $f_{\bar z}$ and f_z are discontinuous at 0, and therefore yet f is not C^1.

Note that in planar case we frequently use notation $\mathbb{S}$ instead of $\mathbb{S}^1$ and $\mathbb{U}$ instead of $\mathbb{B}^2$.

We can modify this example to show that there is $f \in QC(\mathbb{U})$ such that μ_f is continuous on $\mathbb{U}$, but $f_{\bar z}$ and f_z are discontinuous at some point $z_0 \in \mathbb{U}$. We will show that if in addition the second dilatation ν_f is anti holomorphic on $\mathbb{U}$ and f is C^1 up to the boundary, then f is biLipschitz and μ_f continuous up to the boundary.

Example 7.5. *Consider $f_0(z) = \frac{z}{\log |z|^2}$. We check that $(\log |z|^2)_z = 1/z$,*

$$p = \frac{1}{\log |z|^2} - \frac{z}{(\log |z|^2)^2}/z = \frac{1}{\log |z|^2} A(z),$$

where $A(z) = 1 - \frac{1}{\log |z|^2}$. Next

$$q = -\frac{z}{(\log |z|^2)^2} 1/\bar z = -\frac{z}{\bar z} \frac{1}{(\log |z|^2)^2}$$

and therefore

$$\mu_{f_0} = -\frac{z}{\bar z} \frac{1}{\log |z|^2} B(z),$$

where $B = 1/A$. For r_0 small enough f_0 is qc on $B(0, r_0)$, $p(0) = q(0) = \mu_f(0) = 0$, μ_{f_0} and ν_{f_0} are continuous f_0 is C^1, but there is no finite $(f_0^{-1})_x$ at 0. Next if $B = B(z_0, s_0)$ is an arbitrary planar disk using the mapping $f(z) = f_0(\lambda(z - z_0))$ with $\lambda s_0 = r_0$, we conclude that there is a qc C^1 map f on B such that μ_f and ν_f are continuous on B, but f^{-1} has no finite derivatives at $w_0 = f(z_0)$. But note that if both μ_f and ν_f are α Hölder continuous on B, then f and f^{-1} are $C^{1,\alpha}$.

The following example shows that the Beltrami coefficient μ_f of a qc f is uniformly α -Hölder (and therefore f is $C^{1,\alpha}$ up to the boundary) but it does not imply in general that f^{-1} has continuous extension.

Example 7.6. *Let $0 < \alpha < \beta < 1$, $\gamma = \beta - \alpha$ and $0 < k < 1$. Solve equation $f_z = (1-z)^\alpha$ and $f_{\overline{z}} = k(1-\overline{z})^\alpha$. Then check that (i): f is $C^{1,\alpha}$ up to the boundary of the unit disk, but μ_f is discontinuous at 1 if $\alpha = \beta$ and μ_f is γ -Hölder on unit disk $\overline{\mathbb{U}}$ if $\alpha < \beta$. We can write $f = g \circ T$, where $T(z) = 1 - z$, and*

$$g(w) = \frac{w^{\alpha+1}}{\alpha+1} + k\frac{\overline{w}^{\beta+1}}{\beta+1}.$$

Now consider g on $B = B(0,1)$. We are going to show that g has corresponding properties from which (i) follows. Check that $\mu_g = k\frac{\overline{w}^\beta}{w^\alpha}$ is γ -Hölder and $|\mu_g| \leqslant k$ on the closed disk $\overline{B}$. Let ϕ be conformal mapping of $G = g(B)$ onto B with $\phi(0) = 0$ and set $h = \phi \circ g$. Thus h is a qc mapping which maps B onto itself such that $\mu_h = \mu_g$ and $h(0) = 0$. In addition, $\mu_h = \mu_g$ is γ -Hölder on $\overline{B}$, but (ii): partial derivatives of g^{-1} do not have continuous extension to 0. At this point it seems natural to check whether partial derivatives of h^{-1} have continuous extension to $h(0) = 0$; we leave it to the reader and note that Theorem 7.7 below shows that it is the case.
Warning: Note here that $h(B) = B$ is smooth domain and $G = g(B)$ is not smooth (precisely only at a point 0). Therefore there is an essential difference between g and h: g does not satisfy the hypotheses of Theorem 7.7 below and h does it.

Note further that hypothesis that μ is a compactly supported function in Hölder spaces completely changes the situation. Namely, there is a classical result that goes back to Schauder which asserts that f is of class $C^{1,\epsilon}$ provided μ is a compactly supported function in $\mathrm{Lip}_\varepsilon(\mathbb{C},\mathbb{C})$, stated here as (see, for example, Theorem 2.10 and 2.12, Ch II, \$ 5, p.93 in Vekua's book [60] and [10, Chapter 15]):

Theorem S. If μ is a complex valued compactly supported ϵ- Hölder continuous function on $\mathbb{C}$, $0 < \epsilon < 1$, with $|\mu|_\infty < 1$, than principal solution f of μ-Beltrami equation is of class $C^{1,\epsilon}$.

In order to discuss some version of Kellogg and Warschawski theorem for a class of quasiconformal maps we first need some definition and results.

Recall in the literature planar domain D is called Lyapunov domain if D has smooth $C^{1,\alpha}$- boundary for some $0 < \alpha < 1$. We first recall the classical result of Kellogg and Warschawski related to Riemann conformal mapping.

Kellogg's theorem. Let γ be a Jordan curve. By the Riemann mapping theorem there exists a Riemann conformal mapping of the unit disk onto the Jordan domain $G = \mathrm{int}\gamma$. By Caratheodory's theorem it has a continuous extension to the boundary. Moreover, if $\gamma \in C^{n,\alpha}$, $n \in \mathbb{N}$, $0 \leqslant \alpha < 1$, then the Riemann conformal mapping has a $C^{n,\alpha}$ extension to the boundary (this result is known as Kellogg's theorem).

In [26] Kalaj gives some extensions of classical results of Kellogg and Warschawski to a class of quasiconformal (q.c.) mappings. Among the other results the author states the following:

Theorem 7.7. *Suppose that (H_1): f is a q.c. mapping between two planar domains G and G' with smooth $C^{1,\alpha}$ boundaries. Then the following conditions are equivalent:*

(A) *f together with its inverse mapping f^{-1}, is $C^{1,\alpha}$ up to the boundary.*

(B) *the Beltrami coefficient μ_f is uniformly α Hölder continuous* $(0 < \alpha < 1)$.

It our impression that this result can be related with some resent results. For example, we derive a small extension of this result (see Theorem 7.8 below) and we also can get from this result Theorem 1.3 [11].

It is interesting that Theorem 7.7 is related to a result of Mateu, Orobitg and Verdera (Theorem MOV below) proved in [43]:

Theorem MOV. *Principal solution of Beltrami equation with Hölder continuous Beltrami coefficient supported on a Lyapunov domain is bi-Lipschitz.*

Now we are ready to prove the following:

Theorem 7.8. *Let f be a q.c. mapping between two planar domains G and G' with smooth $C^{1,\alpha}$, $0 < \alpha < 1$, boundaries. Then the following conditions are equivalent:*

(A) *f together with its inverse mapping f^{-1}, is $C^{1,\alpha}$ up to the boundary.*
(B) *the Beltrami coefficient μ_f is uniformly α Hölder continuous* $(0 < \alpha < 1)$.
(C) *$f = \phi \circ f_0$, where ϕ is conformal mapping from Lyapunov domain $f_0(G)$ onto G' and f_0 is bi-Lipschitz.*

Proof. Suppose (B). We first prove that (B) implies (C). Using Kellogg's theorem without loss of generality we can reduce the proof to the case $G = \mathbb{B}^2$. In this setting for given $r_0 > 1$ there is Hölder continuous μ_0 supported on $B = B(0, r_0)$, $r_0 > 1$, such that $\mu_0 = \mu_f$ on $\mathbb{B}^2$. Namely extend μ_f to μ by reflection $\mu(z) = \mu_f(Jz)$, where $Jz = 1/\bar{z}$. Next let $\varphi \in C^2_0(B)$, $\varphi = 1$ on $\overline{\mathbb{B}^2}$ and set $\mu_0 = \varphi\mu$. If f_0 is principal solution of μ_0-Beltrami equation, then we have $f = \phi \circ f_0$, where ϕ is conformal mapping from $G_0 := f_0(\mathbb{B}^2)$ onto G'. Since μ_0 is Hölder continuous on $\overline{\mathbb{B}^2}$, by Theorem S we conclude that $p_0 := (f_0)_z$ and $q_0 := (f_0)_{\bar{z}}$ are Hölder continuous on $\overline{\mathbb{B}^2}$ and by Theorem MOV that f_0 is bi-Lipschitz. Hence there are $0 < l_0 < L_0$ such that $l_0 \leqslant |p_0| - |q_0|$ and $|p_0| + |q_0| \leqslant L_0$ on $\overline{\mathbb{B}^2}$. Next by abusing of notation write $f_0'(t)$ instead of $(f_0)_b'(t)$, where $(f_0)_b(t) = f_0(e^{it})$, $0 \leqslant t \leqslant 2\pi$. If $\gamma(t) = f_0(e^{it})$, $0 \leqslant t \leqslant 2\pi$ and s an arc length parametar on γ, then $\gamma'(s) = f_0'(t)/|f_0'(t)|$, where $f_0'(t) = (f_0)_b'(t)$. Since $f_0'(t) = i(p_0 e^{it} - q_0 e^{-it})$, $l_0 \leqslant |f_0'(t)|$ and therefore $\gamma'(s)$ is Hölder continuous on $[0, s_0]$, where s_0 is length of curve γ. Therefore we have proved (C).

Now we prove that (C) implies (A). By Kellogg's theorem ϕ and ϕ^{-1} have a continuous extension to $\overline{G_0}$ and $\overline{G'}$ respectively and therefore ϕ is bi-Lipschitz. Hence from (C) it follows that f is bi-Lipschitz.

Recall that we suppose that $G = \mathbb{B}^2$. Next $p := f_z$ is α- Hölder on $\overline{\mathbb{B}^2}$ and there is $m_0 > 0$ such that $|p| \geqslant m_0$ on $\overline{\mathbb{B}^2}$ and since

$$1/p(z_1) - 1/p(z_2) = \frac{p(z_2) - p(z_1)}{p(z_1)p(z_2)},$$

we get

$$|1/p(z_1) - 1/p(z_2)| \leqslant C|z_2 - z_1|^\alpha / m_0^2$$

and therefore $1/p$ is α- Hölder on $\overline{\mathbb{B}^2}$. Hence also the second dilatation $\nu_f = \mu_f \frac{p}{\bar{p}}$ is α- Hölder up to the boundary of $G = \mathbb{B}^2$. Finally, since f is bi-Lipschitz and $\mu_{f^{-1}} = -\nu_f \circ f^{-1}$, we conclude that $\mu_{f^{-1}}$ is α- Hölder up to the boundary. Hence (A) follows. □

For further discussion we first need the following definition: if $f : G \to G'$ is differentiable at z_0 and $J_f(z_0) \neq 0$ we say that f is regular at z_0.

Remark 7.9. *After writing final version of this manuscript, Kalaj turned our attention on the following results.*

Theorem LK *(Theorem 7.1 [32], p. 232). Let G and G' be domains in $\mathbb{C}$ and $w : G \to G'$ a qc with complex dilatation χ, where $|\chi(z)| \leq k < 1$ a.e. in G. If there is χ_0 such that*

$$\int_{B(z_0,r_0)} \frac{|\chi(z) - \chi_0|}{|z - z_0|^2} dA < +\infty \tag{73}$$

for some $r_0 > 0$, then w is regular at z_0 and $\chi(z_0) = \chi_0$.

If χ satisfies the condition (73) we say that χ satisfies the integral growth condition at z_0, and if χ satisfies the integral growth condition at every point of G we say that χ satisfies the integral growth condition on G.

It is interesting that this result infer simple proof of a few results including, Theorem MOV,and Astala, Prats, and Saksman Theorem 1.3 [11] stated here as

Theorem ASP. *Let $0 < s < 1$, let G be a simply connected, bounded $C^{1,s}$-domain and let $g : G \to G$ be a μ-quasiconformal mapping, with $supp(\mu) \subset \overline{G}$ and $\mu \in C^s(G)$. Then $g \in C^{1,s}(G)$.*

Further from Theorem LK we can also infer the following:

Proposition 7.10. *If $f : \mathbb{C} \to \mathbb{C}$ is a C^1 qc with complex dilatation χ, where $|\chi(z)| \leq k < 1$ a.e. in $\mathbb{C}$, $f(\infty) = \infty$, and χ satisfies the integral growth condition at $\mathbb{C}$, then f is Bi-Lip on every compact subset of $\mathbb{C}$.*

If in addition $supp\chi$ is bounded set then f is Bi-Lip on $\mathbb{C}$.

We can consider Proposition 7.10 as a generalization of Theorem MOV.

Acknowledge . We are indebted to professor Kalaj who turned our attention on Theorem LV and an anonymous referee for careful reading the manuscript and useful suggestion which improved exposition.

References

[1] E. Afanasieva, V. Ryazanov. and R. Salimov, On mappings in the Orlicz-Sobolev classes on Riemannian manifolds, Journal of Mathematical Sciences 181:1 (2012) 1–17.
[2] L. Ahlfors, On quasiconformal mappings, J. Analyse Math. 3 (1954) 1–58.
[3] L. Ahlfors, Möbius transformation in several dimensions, School of mathematics, University of Minnesota, 1981.
[4] L. Ahlfors, Lectures on Quasiconformal mappings with additional chapters by Earle and Kra, Univ. Lectures Series (Providence , R.I.), Shishikura, Hubbard, 2006.
[5] M. Arsenović and V. Manojlović, On the modulus of continuity of harmonic quasiregular mappings on the unit ball in $\mathbb{R}^n$, Filomat 23:3 (2009) 199–202.
[6] M. Arsenović, V. Kojić and M. Mateljević, On Lipschitz continuity of harmonic quasiregular maps on the unit ball in R^n, Ann. Acad. Sci. Fenn. 33 (2008) 315–318.
[7] M. Arsenović, V. Kojić and M. Mateljević, Lipschitz-type spaces and harmonic mappings in the space, Ann. Acad. Sci. Fenn. 35:2 (2010) 379–387.
[8] M. Arsenović and M. Mateljević, On Ahlfors-Beurling Operator, Ukr. Math. Bull 18:3 (2021) 292–302.
[9] M. Arsenović and M. Mateljević, On Holder continuity of Q-homeomorphisms, to appear.
[10] K. Astala, T. Iwaniec and G.J. Martin, Elliptic partial differential equations and quasiconformal mappings in the plane, Princeton University Press, Princeton and Oxford, 2009.
[11] K. Astala, M. Prats, E. Saksman, Global smoothness of quasiconformal mappings in the Triebel-Lizorkin scale, Submitted on 23 Jan 2019, arXiv:1901.07844v1.
[12] B. Bojarski, V. Gutlyanskii, O. Martio, and V. Ryazanov, Infinitesimal geometry of quasiconformal and bi-Lipschitz mappings in the plane, EMS Tracts in Mathematics, vol. 19, European Mathematical Society (EMS), Zürich, 2013.
[13] V. Božin, M. Mateljević, Quasiconformal and HQC Mappings Between Lyapunov Jordan Domains, Ann. Sc. Norm. Super. Pisa Cl. Sci. (V), XXI (2020) 107–132.
[14] A.P. Calderon, On the differentiability of absolutely continuous functions, Riv. Math. Univ. Parma 2 (1951) 203–213.
[15] M. Cristea, Open discrete mappings having local ACL^n inverses, Complex Variables and Elliptic Equations 55:1–3 (2010) 61–90.
[16] L. D'Onofrio, Differentiability versus approximate differentiability, Electronic Journal of Differential Equations, Conference 25 (2018), 77–85, ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
[17] H. Federer, Geometric Measure Theory, Springer, Berlin etc., 1969.
[18] A. Golberg, R. Salimov and E. Sevost'yanov, Normal Families of Discrete Open Mappings with Controlled p-Module, Contemporary Mathematics 667 (2016) 83–103.
[19] D. Gilbarg, N. Trudinger, Elliptic partial Differential Equation of Second Order, Springer Verlag, Berlin, Heidelberg, 2001.
[20] V.Ya. Gutlyanskiĭ and A. Golberg, On Lipschitz continuity of quasiconformal mappings in space, J. d'Anal. Math. 109 (2009) 233–251.
[21] S. Hencl and P. Koskela, Lectures on mappings of finite distortion, Lecture Notes in Mathematics 2096, Springer, 2014.
[22] T. Iwaniec and G. Martin, Geometrical Function Theory and Non-Linear Analysis, Clarendon Press, Oxford, 2001.
[23] D.P. Il'yutko and E.A. Sevost'yanov, Open discrete mappings with unbounded coefficient of quasi-conformality on Riemannian manifolds, Sbornik Mathematics 207:4 (2016) 537–580.
[24] D.P. Il'yutko and E.A. Sevost'yanov, Boundary behaviour of open discrete mappings on Riemannian manifolds, Sbornik Mathematics 209:5 (2018) 605–651.
[25] D. Kalaj, Quasiconformal harmonic mapping between Jordan domains, Math. Z. 260:2 (2008) 237-252.
[26] D. Kalaj, On Kellogg's theorem for quasiconformal mappings, Glasgow Mathematical Journal 54:3 (2012) 599–603.
[27] D. Kalaj, A priori estimate of gradient of a solution to certain differential inequality and quasiregular mappings, Journal d'Anal. Math. 119:1 (2013) 63–88.
[28] D. Kalaj, Corrigendum to the paper David Kalaj: On Kellogg's theorem for quasiconformal mappings, Glasgow Mathematical Journal 54:3 (2012) 599–603.
[29] D. Kalaj and M. Mateljević, Inner estimate and quasiconformal harmonic maps between smooth domains, Journal d'Anal. Math. 100 (2006) 117-132.
[30] D. Kovtonyuk and V. Ryazanov, New modulus estimates in Orlicz-Sobolev classes, Annals of the University of Bucharest (mathematical series) LXIII: 5 (2014) 131–135.
[31] D. Kovtonyuk, V. Ryazanov, R. Salimov and E. Sevost'yanov, Toward the theory of Orlicz-Sobolev classes, St. Petersburg Math. J. 25:6 (2014) 929–963.
[32] O. Lehto and K. Virtanen, Quasiconformal Mappings in the Plane, Springer, New York etc., 1973.
[33] T. Lomako, R. Salimov and E. Sevostyanov, On equicontinuity of solutions to the Beltrami equations, Annals of the University of Bucharest (mathematical series) LIX:2 (2010) 263–274.
[34] M. Mateljevic and M. Vuorinen, On harmonic quasiconformal quasi-isometries, J. Inequal., 2010:1 (2010) 1–19.
[35] M. Mateljević, Distortion of quasiregular mappings and equivalent norms on Lipschitz-type spaces, Abstract and Applied Analysis 2014 (2013) 1–20.

[36] M. Mateljević, The Lower Bound for the Modulus of the Derivatives and Jacobian of Harmonic Injective Mappings, Filomat 29:2 (2015) 221-244.
[37] M. Mateljević, Invertible harmonic and Harmonic quasiconformal mappings, Filomat 29:9 (2015) 1953–1967.
[38] M. Mateljević, Boundary behaviour of partial derivatives for solutions to certain Laplacian-gradient inequalities and spatial qc maps, accepted for publication in Springer Proceedings in Mathematics & Statistics (ISSN: 2194-1009, https://www.springer.com/series/10533), vol.357, Operator Theory and Harmonic Analysis. Vol. 1: New General Trends and Advances of the Theory 2021, 393-418
[39] O. Martio, S. Rickman and J. Väisälä, Distortion and singularities of quasiregular mappings, Ann. Acad. Sci. Fenn. Ser. A1 465 (1970) 1–13.
[40] O. Martio, On harmonic quasiconformal mappings, Ann. Acad. Sci. Fenn., Ser. A I 425 (1968) 3–10.
[41] O. Martio, V. Ryazanov, U. Srebro and E. Yakubov, Moduli in Modern Mapping Theory, Springer Science + Business Media, LLC: New York, 2009.
[42] O. Martio, V. Ryazanov, U. Srebro and E. Yakubov, On Q-homeomorphisms, Ann. Acad. Sci. Fenn. Math. 30:1 (2005) 49–69.
[43] J. Mateu, J. Orobitg and J. Verdera, Extra cancellation of even Calderón-Zygmund operators and quasiconformal mappings, J. Math. Pures Appl. 91:4 (2009) 402–431.
[44] V. Maz'ya, Sobolev classes, Springer, Berlin–New York, 1985.
[45] A. Mori, On an absolute constant in the theory of quasiconformal mappings, J. Math. Soc. Japan 8 (1956) 156–166.
[46] C.A. Nodler and D.M. Oberlin, Moduli of continuity and a Hardy-Littlewood theorem, Lecture Notes in Math., 1351, Springer-Verlag, Berlin etc., 1988.
[47] V. Ryazanov, R. Salimov and E. Sevost'yanov, On the Hölder property of mappings in domains and on boundaries, J. Math. Sci. 246:1 (2020) 60–74.
[48] V. Ryazanov, U. Srebro and E. Yakubov, Plane mappings with dilatation dominated by functions of bounded mean oscillation, Sib. Adv. in Math. 11:2 (2001), 94–130.
[49] V. Ryazanov, U. Srebro and E. Yakubov, On ring solutions of Beltrami equations, J. d'Anal. Math. 96 (2005) 117–150.
[50] V. Ryazanov and E. Sevost'yanov, Toward the theory of ring Q-homeomorphisms, Israel J. Math. 168 (2008), 101–118.
[51] Yu.G. Reshetnyak, Space mappings with bounded distortion, Transl. Math. Monographs 73, AMS, 1989.
[52] S. Saks, Theory of the Integral, Dover, New York, 1964.
[53] R.R. Salimov, On regular homeomorphisms in the plane, Annales Academiæ Scientiarum Fennicæ Mathematica 35 (2010), 285–289.
[54] E.A. Sevost'yanov, On the boundary behavior of open discrete mappings with unbounded characteristic, Ukrainian Math. J. 64:6 (2012) 979–984.
[55] E.A. Sevost'yanov, Towards a theory of removable singularities for maps with unbounded characteristic of quasi-conformity, Izv. Math. 74:1 (2010) 151–165.
[56] E.S. Smolovaya, Boundary behavior of ring Q-homeomorphisms in metric spaces, Ukr. Math. Journ. 62:5 (2010) 785–793.
[57] G.D. Suvorov, Generalized principle of length and area in mapping theory, Naukova Dumka, Kiev, 1985.
[58] J. Väisälä, Two new characterizations for quasiconformality, Ann. Acad. Sci. Fenn. Ser. A 1 Math. 362 (1965) 1–12.
[59] J. Väisälä, Lectures on n-dimensional quasiconformal mappings, Lecture Notes in Math. 229, Springer-Verlag, Berlin etc., 1971.
[60] I.N. Vekua, Generalized Analytic Functions Paperback, Martino Fine Books, 2014.
[61] A. Zygmund, Trigonometrical Series, 2nd ed., Vol. I, II, Cambridge university Press, New York, 1959.