Filomat 36:16 (2022), 5417–5422
https://doi.org/10.2298/FIL2216417L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Commutative Neutrix Convolution Product of Generalized Fresnel Sine Integrals and Applications

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**Abstract.** The generalized Fresnel sine integral $S_k(x)$ and its associated functions $S_{k+}(x)$ and $S_{k-}(x)$ are defined as locally summable functions on the real line. The generalized Fresnel sine integrals have huge applications in physics, specially in optics and electromaghetics. In many diffraction problems the generalized Fresnel integrals plays an important role. In this paper are calculated the commutative neutrix convolutions of the generalized Fresnel sine integral and its associated functions with $x^r, r = 0, 1, 2, \ldots$.

## 1. Introduction and Preliminaries

The generalized Fresnel integrals are transcedent functions, which are named by Augustin-Jean Fresnel who used its in the optics. The generalized Fresnel sine and Fresnel cosine integrals have huge applications in physics, especially in optics and electromaghetics. In many diffraction problems the generalized Fresnel integrals play an important role. Fresnel integrals were first used for calculation of the intensity of electromaghnetic field when the opaque object is shrouded in light, [10]. Later, they were used in construction, engineering, road construction and railways, [11]. In [1], the generalized Fresnel integral semes as a canonical function uniform ray field representation of several high-h-frequency diffraction mechanisms. The generalized Fresnel integral was first introduced in [2] and in [3] its properties were analyzed. The generalized Fresnel integrals are used in vertex problem and many other diffraction problems. They are used also in the simulation for increasing of the electromaghnetic waves, [12]. Its applications becomes more practical if the generalized Fresnel integrals are presented in closed form expressions. Direct numerical calculation of this integral is complicated because of the rapid oscillations of the integrand that occur for certain combinations of the two real arguments, [1]. In [4], the authors introduced two approximations for analytical evaluation of the Fresnel integrals without using of the numerical algorithms.
But, not only the calculation of the generalized Fresnel integrals is complicated. The calculation of many products and convolution products which involving the generelized Fresnel integrals is also complicated in the functional analysis. Because of that, in functional analysis were intoduced new approaches like calculations of convolutions of distributions and products of distributions in neutrix sense introduced by van der Corput in [5],[6], or calculation of products of distribution in Colombeau algebra introduced by

2020 *Mathematics Subject Classification*. Primary 33B10; Secondary 46F10
*Keywords*. neutrix, convolution, neutrix limit, generalized Fresnel sine integral.
Received: 24 September 2021; Accepted: 10 April 2022
Communicated by Hari M. Srivastava
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Colombeau in [7, 8].

The generalized Fresnel integral $S_k(x)$ is defined by:

$$S_k(x) = \int_0^x \sin u^k du, \quad k = 1, 2, \dots,$$

in [9] and its associated functions $S_{k+}(x)$ and $S_{k-}(x)$ are defined by:

$$S_{k+}(x) = H(x)S_k(x), \quad S_{k-}(x) = H(-x)S_k(x),$$

where $H$ is Heaviside's function.
We define the function

$$L_{r,k}(x) = \int_0^x u^r \sin u^k du \tag{1}$$

for $k = 0, 1, 2, \dots$ and $r = 1, 2, \dots$.
Also, we define functions $\sin_+ x^k$ and $\sin_- x^k$ by

$$\sin_+ x^k = H(x)\sin x^k, \quad \sin_- x^k = H(-x)\sin x^k. \tag{2}$$

In mathematical analysis some operations like product of distributions or convolution product of distributions cannot be defined for arbitrary distributions. The convolution product in classical sense is defined by the following definition:

**Definition 1.1.** *Let $f$ and $g$ be two functions. The convolution product $f * g$ is defined by the equation:*

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(t)g(x-t)dt \tag{3}$$

*in all points $x$ for which the integral exists.*

From the definition 1.1, it follows that in classical sense if the convolution product $f * g$ exists, then $g * f$ exists and

$$f * g = g * f, \tag{4}$$

and moreover if $(f * g)'$ exists and $f' * g$ (or $f * g'$) exists and

$$(f * g)' = f' * g \ \ (f * g'). \tag{5}$$

The definition 1.1 cannot be used for arbitrary distributions, because some operations as product, convolution product cannot be defined in general. The following definition which is given in [13] extends the definition 1.1 for distributions $f$ and $g$ in the space $\mathfrak{D}'$.

**Definition 1.2.** *Let $f$ and $g$ be two distributions in $\mathfrak{D}'$. Then the convolution product $f * g$ is defined by the equation*

$$\langle (f * g)(x), \varphi(x) \rangle = \langle f(y), \langle g(x), \varphi(x + y) \rangle \rangle \tag{6}$$

*for arbitray test function $\varphi \in \mathfrak{D}$, when $f$ and $g$ satisfy the following conditions:*

*(1) $f$ and $g$ have bounded support;*
*(2) the supports of $f$ and $g$ are bounded on the same side.*

In [14] are calculated the convolution products of the generalized Fresnel sine integral $S_k(x)$ with the distribution $x^r$.
The definition 1.2 was extended by Fisher in [6] in order to define convolution product to a larger class of distributions. In that way, Fisher has defined the non-commutative neutrix convolution product, in [6]. For that extension, he defined the function $\tau$ in $\mathfrak{D}'$ which satisfies the following properties:

(i) $\tau(x) = \tau(-x)$;
(ii) $0 \le \tau(x) \le 1$;
(iii) $\tau(x) = 1$ for $|x| \le \frac{1}{2}$;
(iv) $\tau(x) = 0$ for $|x| \ge 1$.

The function $\tau_\nu$ is defined for $\nu > 0$ by:

$$\tau_\nu(x) = \begin{cases} 1, & |x| \le \nu \\ \tau\left(\nu^\nu x - \nu^{\nu+1}\right), & x > \nu \\ \tau\left(\nu^\nu x + \nu^{\nu+1}\right), & x < -\nu \end{cases}.$$

The following definition of the non-commutative neutrix convolution was given in [6].

**Definition 1.3.** *Let $f$ and $g$ be distributions in $\mathfrak{D}'$ and let $f_\nu = f\tau_\nu$ for $\nu > 0$. Then the non-commutative neutrix convolution $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_\nu * g\}$, provided the limit $h$ exists in the sense that:*

$$N-\lim_{\nu\to\infty} \langle f_\nu * g, \varphi \rangle = \langle h, \varphi \rangle$$

*for all $\varphi \in \mathfrak{D}$, where N is neutrix, see [5], having domain $N'$ the positive real numbers and negligible functions finite linear sums of the functions*

$$\nu^\lambda \ln^{r-1}\nu, \ln^r \nu, \nu^r \sin \nu^k, \nu^r \cos \nu^k : \ \lambda > 0, r = 1,2,\dots, \ k = 1,2,\dots,$$

*and all functions which converge to zero in normal sense when $\nu$ tends to infinity.*

Any result proved with the original definition of the convolution holds with the new definition of the neutrix convolution product. Actually, the neutrix convolution is a generalization of the convolution, which is shown in [6], i.e.

$$f \circledast g = f * g.$$

The next theorem for commutative neutrix product is given in [15] and it is a generalization of the definition 1.2:

**Theorem 1.4.** *Let $f$ and $g$ be distributions in $\mathfrak{D}'$ satisfying either condition (1) or condition (2) from the definition 1.2. Then the commutative neutrix convolution product $f \boxast g = f * g$.*

The next theorem for commutative neutrix product is proved in [19].

**Theorem 1.5.** *Let $f$ and $g$ be distributions in $\mathfrak{D}'$ and suppose that the neutrix convolution product $f \boxast g$ exists. If $N-\lim_{\nu\to\infty} \left\langle (f\tau_\nu') * g_\nu, \varphi \right\rangle$ exists and equal to $\langle h, \varphi \rangle$ for all test functions $\varphi$ in $\mathfrak{D}$, then $f' \boxast g$ exists and*

$$(f \boxast g)' = f' \boxast g + h. \tag{7}$$

In the next section will be proved theorems and corrolaries for generalized Fresnel sine integral, which are generalization of the results proved in [16].
Also, the set of negligible functions will be extended. In the set will be included all finite linear sums of the functions $\nu^r \sin \nu^k$, $r = 1, 2, ..., k = 0, 1, 2, ....$
In order to prove our results we will use lemma 8, which is proved in [18].

## 2. Main results

We prove the following theorem:

**Theorem 2.1.** *The commutative neutrix convolution* $(\sin_+ x^k) \boxast x^r$ *exists and*

$$(\sin_+ x^k) \boxast x^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} L_{r-i,k} x^i, \tag{8}$$

*for* $r = 0, 1, 2, \dots$ *and* $k = 1, 2, \dots$.

*Proof.* Because we need to calculate commutative neutrix convolution product, according to the definition 1.3 we define the sequence of regular distributions $(\sin_+ x^k)_\nu = (\sin_+ x^k)\tau_\nu(x)$ and $(x^r)_\nu = x^r \tau_\nu(x)$.
Then the convolution $(\sin_+ x^k)_\nu * (x^r)_\nu$ exists.

$$(\sin_+ x^k)_\nu * (x^r)_\nu = \int_0^\nu \sin t^k (x-t)^r \tau_\nu(x-t) dt + \int_\nu^{\nu+\nu^{-\nu}} \sin t^k (x-t)^r \tau_\nu(t)\tau_\nu(x-t) dt = I_1 + I_2. \tag{9}$$

If $0 \le |x| \le \nu$, then for the first integral we have:
$\int_0^\nu \sin t^k (x-t)^r \tau_\nu(x-t) dt = \int_0^\nu \sin t^k \sum_{i=0}^r \binom{r}{i} x^i (-t)^{r-i} \tau_\nu(x-t) dt = \sum_{i=0}^r \binom{r}{i} (-1)^{r-i} L_{r-i,k}(\nu) x^i.$
So for the neutrix limit of this integral, according to lemma 8 in [18], we obtain:

$$N - \lim_{\nu\to\infty} \int_0^\nu \sin t^k (x-t)^r \tau_\nu(x-t) dt = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} L_{r-i,k} x^i. \tag{10}$$

For the second integral from the defined function $\tau_\nu(t)$ and because $|\sin t^k| \le 1$ it follows:
$\left| \int_\nu^{\nu+\nu^{-\nu}} \sin t^k (x-t)^r \tau_\nu(t)\tau_\nu(x-t) dt \right| \le \int_\nu^{\nu+\nu^{-\nu}} (x-t)^r dt \le (\nu + \nu^{-\nu})\nu^{-\nu},$
so for fixed $x$ it follows:

$$\lim_{\nu\to\infty} \int_\nu^{\nu+\nu^{-\nu}} \sin t^k (x-t)^r \tau_\nu(t)\tau_\nu(x-t) dt = 0. \tag{11}$$

So the equation (8) follows from the equations (9), (10) and (11). □

**Corollary 2.2.** *The commutative neutrix convolution* $(\sin_- x^k) \boxast x^r$ *exists and*

$$(\sin_- x^k) \boxast x^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i+1} L_{r-i,k} x^i, \tag{12}$$

*for* $r = 0, 1, 2, \dots$ *and* $k = 1, 2, \dots$.

*Proof.* If in the equation (8) we replace $x$ by $-x$, then we obtain the equation (12). Here, also is noticed that

$$N - \lim_{\nu\to\infty} L_{r,k}(-\nu) = (-1)^{r-1} N - \lim_{\nu\to\infty} L_{r,k}(\nu) = (-1)^{r-1} L_{r,k}. \tag{13}$$

□

**Corollary 2.3.** *The commutative neutrix convolution* $(\sin x^k) \boxast x^r$ *exists and*

$$(\sin x^k) \boxast x^r = 0, \tag{14}$$

*for* $r = 0, 1, 2, \dots$ *and* $k = 1, 2, \dots$.

*Proof.* The equation (14) follows from the equations (12) and (8) and from the equation $\sin x^k = \sin_+ x^k + \sin_- x^k$. □

In the next theorem and two corollaries it wil be calculated the commutative neutrix convolution product of the generalized Fresnel sine integral with the distribution $x^r, r = 0, 1, 2, \ldots$.

**Theorem 2.4.** *The commutative neutrix convolution $S_{k+}(x) \boxast x^r$ exists and*

$$S_{k+}(x) \boxast x^r = \frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} (-1)^{r-i+1} L_{r-i+1,k} x^i, \tag{15}$$

*for $r = 0, 1, 2, \ldots$ and $k = 1, 2, \ldots$.*

*Proof.* In order to calculate commutative neutrix convolution product of the distributions we define the sequence $(S_{k+}(x))_\nu = S_{k+}(x)\tau_\nu(x)$ and $(x^r)_\nu = x^r \tau_\nu(x)$.
Then the convolution $(S_{k+}(x))_\nu * (x^r)_\nu$ exists.

$$(S_{k+}(x))_\nu * (x^r)_\nu = \int_0^\nu S_k(t)(x-t)^r \tau_\nu(x-t)dt + \int_\nu^{\nu+\nu^{-\nu}} S_k(t)t(x-t)^r \tau_\nu(t)\tau_\nu(x-t)dt = I_1 + I_2. \tag{16}$$

If $0 \le |x| \le \nu$, then for the first integral we have:
$\int_0^\nu S_k(t)(x-t)^r\tau_\nu(x-t)dt = \int_0^\nu (x-t)^r \int_0^t \sin u^k du dt = \int_0^\nu \sin u^k \int_u^\nu (x-t)^r dt du = -\frac{1}{r+1}\int_0^\nu \sin u^k \left[(x-\nu)^{r+1} - (x-u)^{r+1}\right] du =$
$= -\frac{1}{r+1}\int_0^\nu \sum_{i=0}^r \left( \begin{array}{c} r+1 \\ i \end{array} \right) x^i \left[(-\nu)^{r-i+1} - (-u)^{r-i+}\right] \sin u^k du.$
So, for the neutrix limit of this integral, according to lemma 8 in [18], we obtain:

$$N - \lim_{\nu\to\infty} \int_0^\nu S_k(t)(x-t)^r\tau_\nu(x-t)dt = -\frac{1}{r+1}\int_0^\nu \sum_{i=0}^r \left( \begin{array}{c} r+1 \\ i \end{array} \right) (-1)^{r+1-i} L_{r+1-i,k} x^i. \tag{17}$$

For each fixed $x$ we have that:
$\left| \int_\nu^{\nu+\nu^{-\nu}} S_k(t)(x-t)^r\tau_\nu(t)\tau_\nu(x-t)dt \right| \le \int_\nu^{\nu+\nu^{-\nu}} |x-t|^r \int_0^t |\sin u^k| du dt \le \int_\nu^{\nu+\nu^{-\nu}} t(t-x)^r dt \le (\nu+\nu^{-\nu})^{r+1}\nu^{-\nu},$
it follows that: $\lim_{\nu\to\infty} S_k(t)(x-t)^r\tau_\nu(t)\tau_\nu(x-t)dt = 0.$

The equation (15) follows from the equations (16), (17) and the last equation. □

**Corollary 2.5.** *The commutative neutrix convolution $S_{k-}(x) \boxast x^r$ exists and*

$$S_{k-}(x) \boxast x^r = \frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} (-1)^{r-i} L_{r-i+1,k} x^i \tag{18}$$

*for $r = 0, 1, 2, \ldots$ and $k = 1, 2, \ldots$.*

*Proof.* The equation (18) follows from the equation (15) by replacing $x$ by $-x$. □

**Corollary 2.6.** *The commutative neutrix convolution $S_k(x) \boxast x^r$ exists and*

$$S_k(x) \boxast x^r = 0 \tag{19}$$

*for $r = 0, 1, 2, \ldots$ and $k = 1, 2, \ldots$.*

*Proof.* The equation (19) follows from the equations (15) and (18) and noting that $S_k(x) = S_{k+} + S_{k-}$. □

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