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Tensor Sum of Infinitesimal Generators

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Abstract. Let $\mathscr{A}$ and $\mathscr{B}$ be C^*-algebras, and let δ be a derivation on the tensor product $\mathscr{A} \otimes \mathscr{B}$ endowed with a uniform cross norm. In this paper, we present a decomposition for δ as $\delta = \Delta \otimes id + id \otimes \nabla$, where id stands for the identity operator and Δ and ∇ are derivations on $\mathscr{A}$ and $\mathscr{B}$, respectively. Moreover, the concept of flow on the tensor product of C^*-algebras and some properties of tensor sum are investigated.

1. Introduction and Preliminaries

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces as state spaces, which are correspond to isolated physical systems $S_{\mathcal{H}}$ and $S_{\mathcal{K}}$, respectively. Then, if we consider the set of these two systems to form one physical system S, then the state space of the global system S is $\mathcal{H} \otimes \mathcal{K}$. Also there is a unique inner product $\langle \cdot, \cdot \rangle$ on algebraic tensor product $\mathcal{H} \otimes \mathcal{K}$ such that $\langle x \otimes y, x' \otimes y' \rangle = \langle x, x' \rangle \langle y, y' \rangle$ for all $x, x' \in \mathcal{H}$ and all $y, y' \in \mathcal{K}$. The norm in $\mathcal{H} \otimes \mathcal{K}$ defined by the inner product is certainly a cross norm, i.e. $\|x \otimes y\| = \|x\|\|y\|$. In this paper we use $\mathcal{H} \otimes \mathcal{K}$ for the *Hilbert space tensor product* of $\mathcal{H}$ and $\mathcal{K}$. Moreover, if $\mathscr{A}$ and $\mathscr{B}$ are two C^*-algebra, then we denote the *spatial tensor product* of $\mathscr{A}$ and $\mathscr{B}$ by $\mathscr{A} \otimes \mathscr{B}$. It is known that if Δ and ∇ are some observables (self-adjoint operators) acting on $\mathcal{H}$ and $\mathcal{K}$, respectively, then the tensor product $\Delta \otimes \nabla$ is an observable in $\mathcal{H} \otimes \mathcal{K}$, which is equal to $(\Delta \otimes id)(id \otimes \nabla)$, where id's stand for the identity operators in $\mathcal{H}$ and $\mathcal{K}$, respectively; see [7, Section 6.3]. In particular, if Δ and ∇ are two angular momentum operators, generators of rotations in different spaces, then the total angular momentum δ, the infinitesimal generator of rotation, is now mad up two parts, namely, $\delta = \Delta \otimes id + id \otimes \nabla$; see [13]. In this paper, we establish such operators and investigate some of its significant properties, and usually called the *tensor sum*. For more information about the tensor sum, the interested reader is referred to [6, 12, 15].

The concept of the *infinitesimal generator* of two-parameter semigroups (flow) has been presented by Hille and Phillips [10], Trotter [14], Abdelaziz [1]. It turns out that the definition given by Trotter and Abdelaziz is the definition of an infinitesimal generator for a section of the semigroup. The definition of infinitesimal generator of two-parameter semigroups gave by Arora [2]. Moreover, a generalization of the above definitions was given by Sarif and Khalil [4].

Let $\mathscr{A}$ be a C^*-algebra and G be a locally compact topological group, and let $\mathbf{Aut}(\mathscr{A})$ be the group of automorphisms on $\mathscr{A}$. A strongly continuous group homomorphism $\alpha : G \longrightarrow \mathbf{Aut}(\mathscr{A})$ is called a *G-flow* over $\mathscr{A}$. If α is a G-flow over the C^*-algebra $\mathscr{A}$, $t \in G$ and $x \in \mathscr{A}$, then we simply denote $\alpha(t)x$ and $\alpha(t)$ by

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$\alpha_t(x)$ and α_t, respectively. To simplify an $\mathbb{R}$-flow is called a flow, whenever $\mathbb{R}$ is the set of real numbers. Moreover, the *infinitesimal generator* of α, denoted by δ_α, is defined by $\delta_\alpha := \lim_{t\to 0} \frac{\alpha_t - id}{t}$. One can easily prove that δ_α is a derivation from $D(\delta_\alpha)$ into $\mathscr{A}$, where $D(\delta_\alpha) = \{x \in \mathscr{A} : \lim_{t\to 0} \frac{\alpha_t(x)-x}{t} \text{ exist}\}$. A subspace E of $D(\delta_\alpha)$ is called a *core* for δ_α if E is dense in $D(\delta_\alpha)$ under the graph norm $|x|_\delta = \|x\| + \|\delta_\alpha(x)\|$. Further information about the generator and properties involving the flow can be found in [5, 9, 11].

In this paper, we give a new finder method of the infinitesimal generator for two parameter semigroups as flow. Section 2 is devoted to establish the concept of flow on the tensor product of C^*-algebras. We show that if $\alpha = \{\alpha_t\}_{t\in G}$ and $\beta = \{\beta_s\}_{s\in H}$ are two families of operators on the C^*-algebras $\mathscr{A}$, $\mathscr{B}$, respectively, then $\alpha \otimes \beta$ is a flow on $\mathscr{A} \otimes \mathscr{B}$ if and only if α, β are two flows on $\mathscr{A}$, $\mathscr{B}$, respectively. The purpose of Section 3 is to study the concepts of tensor sum and the infinitesimal generator on the tensor product of C^*-algebras. We show that if $\mathscr{A}$, $\mathscr{B}$ are C^*-algebras, $\{\alpha_t \otimes \beta_s\}$ is a flow over $\mathscr{A} \otimes \mathscr{B}$ and δ is the infinitesimal generator for $\alpha \otimes \beta$, then

$$\delta(z) = \operatorname{div}(\alpha_t \otimes id, id \otimes \beta_s)\Big|_{(t,s)=(0,0)}(z),$$

for all $z \in \mathscr{A} \otimes \mathscr{B}$. Moreover, if α, β are some flows for the C^*-algebras $\mathscr{A}$, $\mathscr{B}$ res., then it is shown that $\delta_{\alpha\otimes\beta} = \delta_\alpha \otimes id + id \otimes \delta_\beta$. Among the other results of this section, we show that the infinitesimal generator of a flow $\alpha \otimes \beta$ is closed and the domain of a tensor sum is its core. Furthermore, some properties of the tensor sum in the finite-dimensional case are established.

2. Tensor Product of Flows

In this section, we investigate the concept of flow on the tensor product of C^*-algebras. According to the universal property of the tensor product, for every pair of operators α on a C^*-algebra $\mathscr{A}$ and β on a C^*-algebras $\mathscr{B}$, there exists a unique operator $\alpha \otimes \beta$ on $\mathscr{A} \otimes \mathscr{B}$ such that $\alpha \otimes \beta(x \otimes y) = \alpha(x) \otimes \beta(y)$; see [8].

Let $\mathscr{A}$, $\mathscr{B}$ be C^*-algebras, and let G, H be locally compact topological groups with identity elements e_1, e_2, respectively. If $\alpha = \{\alpha_t\}_{t\in G}$, $\beta = \{\beta_s\}_{s\in H}$ are families of operators on $\mathscr{A}$, $\mathscr{B}$, respectively, then the family $\{\alpha_t \otimes \beta_s\}$ is called tensor product (G, H)-*flow* on $\mathscr{A} \otimes \mathscr{B}$, when $\alpha_{e_1} \otimes \beta_{e_2} = id_{\mathscr{A}\otimes\mathscr{B}}$ and $\alpha \otimes \beta := \{\alpha_t \otimes \beta_s\}_{(t,s)\in G\times H}$ is a family of group homomorphisms from $G \times H$ into $\mathbf{Aut}(\mathscr{A} \otimes \mathscr{B})$ such that the map $(t, s) \longmapsto (\alpha_t \otimes \beta_s)z$ is continuous for each $z \in \mathscr{A} \otimes \mathscr{B}$. By a $*$-flow we mean that every $\alpha_t \otimes \beta_s$ is a $*$-map for all $t \in G$ and all $s \in H$.

Lemma 2.1. *Let $\mathscr{A}$, $\mathscr{B}$ be C^*-algebras, and let G, H be two groups. If $\{\alpha_t\}_{t\in G}$, $\{\beta_s\}_{s\in H}$ are families of operators on $\mathscr{A}$, $\mathscr{B}$, then the following conditions are equivalent:*

- *(i)* $\{\alpha_t\}_{t\in G}(\{\beta_s\}_{s\in H})$ is a G-flow (H-flow) on $\mathscr{A}$ ($\mathscr{B}$).
- (ii) $\{\alpha_t \otimes id\}_{t\in G}$ ($\{id \otimes \beta_s\}_{S\in H}$) is a G-flow (H-flow) on $\mathscr{A} \otimes \mathscr{B}$.

Proof. ((i) $\Longrightarrow$ (ii)) Let α be a G-flow on $\mathscr{A}$, and let $x \in \mathscr{A}$. Then we have $\alpha_{e_1} \otimes id = id \otimes id$ and $\alpha_{tu} \otimes id = \alpha_t\alpha_u \otimes id = (\alpha_t \otimes id)(\alpha_u \otimes id)$. Moreover, for any non-zero $x \otimes y \in \mathscr{A} \otimes \mathscr{B}$ we assert that

$$\|(\alpha_t \otimes id)(x \otimes y) - x \otimes y\| = \|\alpha_t(x) - x\|\|y\|. \tag{1}$$

Therefore, the strong continuity of $\alpha_t \otimes id$ follows from (1) and the strong continuity of α_t.

((ii) $\Longrightarrow$ (i)) Suppose that $\alpha_t \otimes id$ is a G-flow on $\mathscr{A} \otimes \mathscr{B}$, and $x \in \mathscr{A}$. Using (1), we conclude the strong continuity of α_t. Moreover, for any non-zero element $y \in \mathscr{B}$ it holds that

$$\|\alpha_{tu}(x) - \alpha_t(x)\alpha_u(x)\|\|y\| = \|(\alpha_{tu} \otimes id - \alpha_t\alpha_u \otimes id)(x \otimes y)\| = 0.$$

Since y is a non-zero element in $\mathscr{B}$, we get $\alpha_{tu} = \alpha_t\alpha_u$. Similarly, we see that β is a H-flow on $\mathscr{B}$ if and only if $id \otimes \beta$, so is. □

Lemma 2.2. *Let $\mathscr{A}$, $\mathscr{B}$ be C^*-algebras and $\{\alpha_t\}_{t\in\mathbb{R}}$, $\{\beta_s\}_{s\in\mathbb{R}}$ be two families of operators on $\mathscr{A}$ and $\mathscr{B}$, respectively. Then $\{\alpha_t \otimes \beta_s\}_{t,s\in\mathbb{R}}$ is strongly continuous if and only if $\{\alpha_t\}$ and $\{\beta_s\}$ are strongly continuous.*

Proof. Suppose that $\{\alpha_t\}$ and $\{\beta_s\}$ are strongly continuous on $\mathscr{A}$, $\mathscr{B}$, respectively. We shall show that $(t,s) \longmapsto (\alpha_t \otimes \beta_s)z$ is continuous for all z in $\mathscr{A} \otimes \mathscr{B}$. Since the algebraic product on $\mathscr{A} \otimes \mathscr{B}$ is continuous, we have

$$\lim_{(t,s)\to(0,0)} \alpha_t \otimes \beta_s(z) = \lim_{t\to 0} (\alpha_t \otimes id)z \lim_{s\to 0} (id \otimes \beta_s)z = (id \otimes id)z.$$

It follows that $\lim_{(t,s)\to(0,0)} \|\alpha_t \otimes \beta_s(z) - z\| = 0$. The converse immediately follows from Lemma 2.1. $\square$

Let $\alpha = \{\alpha_t\}_{t\in G}$, $\beta = \{\beta_s\}_{s\in H}$ be families of operators on C^*-algebras $\mathscr{A}$, $\mathscr{B}$, respectively. Then it is easy to check that $\alpha\otimes\beta$ is a flow on $\mathscr{A}\otimes\mathscr{B}$ if and only if α and β are two flows on $\mathscr{A}$ and $\mathscr{B}$, respectively. Indeed, If $x\otimes y$, $x'\otimes y'$ are in $\mathscr{A}\otimes\mathscr{B}$, then $\alpha\otimes\beta\left[(x\otimes y)(x'\otimes y')\right] = \alpha(xx')\otimes\beta(yy') = (\alpha(x)\otimes\beta(y))(\alpha(x')\otimes\beta(y'))$. Hence, $\alpha\otimes\beta$ is group homomorphism if and only if α, β are. The remainder of the proof is analogous to that of Lemmas 2.1 and 2.2.

Furthermore, if $\alpha\otimes\beta$ is a tensor product $*$-flow on $\mathscr{A}\otimes\mathscr{B}$, then $\|\alpha\otimes\beta\| = 1$. Indeed, Let $z \in \mathscr{A}\otimes\mathscr{B}$. If z is a self-adjoint element, then $sp(\alpha\otimes\beta(z)) \subseteq sp(z)$ and $\|(\alpha\otimes\beta)z\| \le \|z\|$ so $\|\alpha\otimes\beta\| \le 1$. If z is arbitrary, then

$$\begin{aligned}\|(\alpha\otimes\beta)z\|^2 &= \|(\alpha\otimes\beta)z((\alpha\otimes\beta)z)^*\| = \|(\alpha\otimes\beta)z(\alpha\otimes\beta)z^*\| \\ &= \|(\alpha\otimes\beta)zz^*\| \le \|zz^*\| = \|z\|^2.\end{aligned}$$

Hence, $\|\alpha\otimes\beta\| \le 1$. Moreover,

$$1 = \|id\| = \|(\alpha\otimes\beta)(\alpha\otimes\beta)^{-1}\| \le \|\alpha\otimes\beta\|\|(\alpha\otimes\beta)^{-1}\| \le \|\alpha\otimes\beta\|.$$

Consequently, $\|\alpha\otimes\beta\| = 1$.

3. main result

In this section, we discuss about our main theorem and to this end we get the following theorem.

Theorem 3.1. *Let $\mathscr{A}$, $\mathscr{B}$ be C^*-algebras, and let $\alpha\otimes\beta = \{\alpha_t\otimes\beta_s\}$ be a flow on $\mathscr{A}\otimes\mathscr{B}$. If δ is the infinitesimal generator for $\alpha\otimes\beta$, then*

$$\lim_{(t,s)\to(0,0)} \left\| \frac{\alpha_t\otimes\beta_s - id\otimes id}{\|(t,s)\|} - \mathrm{div}(\alpha_t\otimes id, id\otimes\beta_s) \right\| = 0. \tag{2}$$

Proof. We can write

$$\begin{aligned}&\left\|\alpha_t\otimes\beta_s - id\otimes id - \|(t,s)\|\mathrm{div}(\alpha_t\otimes id, id\otimes\beta_s)\right\| \\ &\quad= \left\|\alpha_t\otimes\beta_s - id\otimes id - \|(t,s)\|\frac{\partial}{\partial t}(\alpha_t\otimes id) - \|(t,s)\|\frac{\partial}{\partial s}(id\otimes\beta_s)\right\| \\ &\quad= \left\| s(\alpha_t\otimes id)\left(\frac{id\otimes\beta_s - id\otimes id}{s}\right) - \|(t,s)\|\frac{\partial}{\partial s}(id\otimes\beta_s) \right. \\ &\qquad \left. + t\left(\frac{\alpha_t\otimes id - id\otimes id}{t}\right) - \|(t,s)\|\frac{\partial}{\partial t}(\alpha_t\otimes id)\right\|.\end{aligned}$$

Now, divide both sides by $\|(t,s)\|$, we get

$$\begin{aligned}\left\|\frac{\alpha_t\otimes\beta_s - id\otimes id}{\|(t,s)\|} - \mathrm{div}(\alpha_t\otimes\mathrm{id}, \mathrm{id}\otimes\beta_s)\right\| &\le \left\|\frac{|s|}{\|(t,s)\|}(\alpha_t\otimes id)\frac{id\otimes\beta_s - id\otimes id}{s} - \frac{\partial}{\partial s}(id\otimes\beta_s)\right\| \\ &\quad + \left\|\frac{|t|}{\|(t,s)\|}\frac{\alpha_t\otimes id - id\otimes id}{t} - \frac{\partial}{\partial t}(\alpha_t\otimes id)\right\|.\end{aligned} \tag{3.2}$$

If we take the limit on both sides (3.2) as $(t,s) \longrightarrow (0,0)$, we obtain (2). $\square$

Corollary 3.2. *Let $\mathscr{A}$, $\mathscr{B}$ be C^*-algebras, and let $\{\alpha_t \otimes \beta_s\}$ be a flow on $\mathscr{A} \otimes \mathscr{B}$. If δ is the infinitesimal generator for $\alpha \otimes \beta$, then*

$$\delta(z) = \operatorname{div}(\alpha_t \otimes id, id \otimes \beta_s)\Big|_{(t,s)=(0,0)}(z),$$

for all $z \in \mathscr{A} \otimes \mathscr{B}$.

Corollary 3.3. *If δ is the infinitesimal generator of the flow $\alpha_t \otimes \beta_s$ over the C^*-algebra $\mathscr{A} \otimes \mathscr{B}$, then δ is a derivation from subalgebra $D(\delta)$ into $\mathscr{A} \otimes \mathscr{B}$, where $D(\delta)$ is the set of all elements in $\mathscr{A} \otimes \mathscr{B}$ such that $\alpha_t \otimes \beta_s$ is differentiable at origin.*

Proof. The linearly and Leibniz properties directly follows from Corollary 3.2. □

Theorem 3.4. *If α, β are some flows for the C^*-algebras $\mathscr{A}$, $\mathscr{B}$ res., then $\delta_{\alpha\otimes\beta} = \delta_\alpha \otimes id + id \otimes \delta_\beta$.*

Proof. Let $E = \{a \in \mathscr{A} : \lim_{t\to 0} \frac{\alpha_t(a)-a}{t} \text{ exists}\}$ and $F = \{b \in \mathscr{B} : \lim_{s\to 0} \frac{\beta_s(b)-b}{s} \text{ exists}\}$. Define $\delta_\alpha : E \longrightarrow \mathscr{A}$ by $\delta_\alpha(a) = \lim_{t\to 0} \frac{\alpha_t(a)-a}{t}$ and $\delta_\beta : F \longrightarrow \mathscr{B}$ by $\delta_\beta(b) = \lim_{s\to 0} \frac{\beta_s(b)-b}{s}$. Since $\delta_{\alpha\otimes\beta}$ is infinitesimal generator of the flow $\alpha \otimes \beta$, we get

$$\begin{aligned}
\delta_{\alpha\otimes\beta}(a \otimes b) &= \operatorname{div}(\alpha_t \otimes id, id \otimes \beta)\Big|_{(t,s)=(0,0)}(a \otimes b)\\
&= \frac{\partial}{\partial t}(\alpha_t \otimes id\Big|_{t=0}(a \otimes b) + \frac{\partial}{\partial s}(id \otimes \beta_s\Big|_{s=0}(a \otimes b)\\
&= \lim_{t\to 0} \frac{\alpha_t \otimes id(a \otimes b) - \alpha_0 \otimes id(a \otimes b)}{t} + \lim_{s\to 0} \frac{id \otimes \beta_s(a \otimes b) - id \otimes \beta_0(a \otimes b)}{s}\\
&= \lim_{t\to 0} \frac{\alpha_t(a) - a}{t} \otimes b + a \otimes \lim_{s\to 0} \frac{\beta_s(b) - b}{s} = \delta_\alpha(a) \otimes b + a \otimes \delta_\beta(b).
\end{aligned}$$

□

Example 3.5. *Let $\mathscr{A} = \left\{ f : \mathbb{D} \longrightarrow \mathbb{C}, f(z) = \sum_{n=0}^{+\infty} p_n z^n, \|f\|^2 = \sum_{n=0}^{+\infty} |p_n|^2 < +\infty \right\}$, where $\mathbb{D}$ is an open disc in the complex plan. Define $\alpha_t : \mathscr{A} \longrightarrow \mathscr{A}$, given by*

$$\alpha_t(f) = \sum_{n=0}^{+\infty} p_n(1+n)^{-t}\omega_n, \qquad (t > 0),$$

such that $\omega_n(z) = z^n$, for every $z \in \mathbb{C}$. One can show that α_t is a flow. Moreover, if δ_α is the infinitesimal generator of α, we can write

$$\begin{aligned}
\delta_\alpha(f) &= \lim_{t\to 0} \frac{\alpha_t(f) - f}{t} = \lim_{t\to 0} \frac{\sum_{n=0}^{+\infty} p_n(1+n)^{-t}\omega_n - \sum_{n=0}^{+\infty} p_n\omega_n}{t}\\
&= \lim_{t\to 0} \sum_{n=0}^{+\infty} p_n(1+n)^{-t} \ln \frac{1}{1+n}\omega_n = \sum_{n=0}^{+\infty} p_n \ln \frac{1}{1+n}\omega_n.
\end{aligned}$$

Let $f = \sum_{n=0}^{+\infty} p_n w_n$ and $g = \sum_{n=0}^{+\infty} q_n w_n$. Then we have $f \otimes g = (\sum_{n=0}^{+\infty} p_n w_n)(\sum_{n=0}^{+\infty} q_n w_n) = \sum_{n=0}^{+\infty} r_n w_n$, where $r_n = \sum_{k=0}^{n} p_k q_{n-k}$. Consider another flow β_s on $\mathscr{A}$ with associated infinitesimal generator δ_β. Then $(\delta_\alpha \otimes id + id \otimes \delta_\beta)(f \otimes g) =$

$\sum_{n=0}^{+\infty} r_n \ln \frac{1}{(1+n)^2}\omega_n$. *On the other hand, if* δ *is the infinitesimal generator of* $\alpha_t \otimes \beta_s$*, then*

$$\begin{aligned}\delta_{\alpha\otimes\beta}(f\otimes g) &= \operatorname{div}(\alpha_t \otimes id, id\otimes\beta)\Big|_{(t,s)=(0,0)}(f\otimes g)\\ &= \left(\frac{\partial}{\partial t}\sum_{n=0}^{+\infty} r_n(1+n)^{-t}\omega_n + \frac{\partial}{\partial s}\sum_{n=0}^{+\infty} r_n(1+n)^{-s}\omega_n\right)\Big|_{(t,s)=(0,0)}\\ &= \sum_{n=0}^{+\infty} r_n \ln\frac{1}{1+n}\omega_n + \sum_{n=0}^{+\infty} r_n \ln\frac{1}{1+n}\omega_n = \sum_{n=0}^{+\infty} r_n \ln\frac{1}{(1+n)^2}\omega_n.\end{aligned}$$

This show that $\delta_{\alpha\otimes\beta} = \delta_\alpha \otimes id + id \otimes \delta_\beta$.

We use the symbol $\delta(\alpha_t \otimes \beta_s)$ to mean that $\delta(\alpha_t \otimes \beta_s) = \operatorname{div}(\alpha_t \otimes id, id \otimes \beta_s)$. Corollary 3.2 implies that $\delta(\alpha_t \otimes \beta_s)|_{(0,0)} = \delta_{\alpha\otimes\beta}$.

Proposition 3.6. *Let* $\alpha\otimes\beta$ *be the infinitesimal generator of a derivation over the* C^**-algebra* $\mathscr{A}\otimes\mathscr{B}$*. Then there exist derivations* δ_α *on* $\mathscr{A}$ *and* δ_β *on* $\mathscr{B}$ *such that*

$$\delta(\alpha_t \otimes \beta_s) = \alpha_t\delta_\alpha \otimes id + id \otimes \delta_\beta\beta_s.$$

Proof. *Using Theorem 3.4 there exist derivations* δ_α*,* δ_β *given by* $\delta_\alpha = \lim_{t\to 0}\frac{\alpha_t - id}{t}$ *and* $\delta_\beta = \lim_{s\to 0}\frac{\beta_s - id}{s}$*. Now, we see that*

$$\begin{aligned}\delta(\alpha_t \otimes \beta_s) &= \operatorname{div}(\alpha_t \otimes id, id\otimes\beta_s) = \frac{\partial}{\partial t}(\alpha_t\otimes id) + \frac{\partial}{\partial s}(id\otimes\beta_s)\\ &= \alpha_t\left(\lim_{p\to 0}\frac{\alpha_p - id}{p}\right)\otimes id + id\otimes\left(\lim_{q\to 0}\frac{\beta_q - id}{q}\right)\beta_s\\ &= \alpha_t\delta_\alpha\otimes id + id\otimes\delta_\beta\beta_s.\end{aligned}$$

□

Similarly, we can prove $\delta(\alpha\otimes\beta) = \delta_\alpha\alpha\otimes id + id\otimes\beta\delta_\beta$.

Corollary 3.7. *Let* δ *be infinitesimal generator of the flow* $\alpha\otimes\beta$ *over the* C^**-algebra* $\mathscr{A}\otimes\mathscr{B}$*. Then*

$$\delta\left(\int_0^t\int_0^s \alpha_p\otimes\beta_q\, dq\, dp\right) = s(\alpha_t - id)\otimes id + id\otimes(\beta_s - id)t.$$

Proof. *It follows form The Fubini's theorem that*

$$\begin{aligned}\delta\left(\int_0^t\int_0^s \alpha_p\otimes\beta_q\, dq\ dp\right) &= \int_0^t\int_0^s (\delta_\alpha\alpha_p\otimes id + id\otimes\beta_q\delta_\beta)\, dq\ dp\\ &= s\int_0^t \delta_\alpha\alpha_p\otimes id\ dp + t\int_0^s id\otimes\beta_q\delta_\beta\ dq\\ &= s(\alpha_t\otimes id - id\otimes id) + t(id\otimes\beta_s - id\otimes id)\\ &= s(\alpha_t - id)\otimes id + id\otimes(\beta_s - id)t.\end{aligned}$$

□

Let $\delta : A \longrightarrow B$ be a linear operator between two Banach spaces A and B over the same field of scalars. If $G(\delta)$ is the graph of δ, the set of all pairs (a, b) such that $b = \delta(a)$, then δ is closed if and only if $G(\delta)$ is a closed subset of the Cartesian product space $A \times B$. Moreover, the term $\int_0^t\int_0^s \alpha_p\otimes\beta_q\, dq\ dp$ is usually denoted by $\Phi_{t,s}$ and the term $s(\alpha_t - id)\otimes id + id\otimes(\beta_s - id)t$ is denoted by $\Psi_{t,s}$. With these notations, we have $\delta\Phi_{t,s} = \Psi_{t,s}$.

Proposition 3.8. *If δ is infinitesimal generator of flow $\alpha \otimes \beta$, then δ is closed.*

Proof. We show that $G(\delta)$ is a closed set. Let $z \in \overline{D(\delta)}$. If $\{z_n\} \subseteq D(\delta)$, $z_n \longrightarrow z$ and $\delta(z_n) \longrightarrow w$ implies $z \in D(\delta)$ and $\delta(z) = w$. To this end,

$$\begin{aligned}\delta(z) &= div((\alpha_t \otimes id)z, (id \otimes \beta)z)\Big|_{(t,s)=(0,0)} \\ &= \lim_{n\to\infty} div(\alpha_t \otimes id, id \otimes \beta)\Big|_{(t,s)=(0,0)} z_n = \lim_{n\to\infty} \delta(z_n) = w.\end{aligned}$$

□

A subspace S of $\mathscr{A} \otimes \mathscr{B}$ is said to be *$\alpha\otimes\beta$-invariant* if $(\alpha\otimes\beta)S \subseteq S$. It is easy to check that if $\delta_{\alpha\otimes\beta}$ is infinitesimal generator, then $D(\delta)$ is $\alpha \otimes \beta$-invariant. Moreover, an elaboration of the above arguments shows that if $\Phi_{t,s}$ as in the above, then $\lim_{(t,s)\to(0,0)} \frac{1}{ts}\Phi_{t,s} = id \otimes id$. Furthermore, if δ is a derivation on the C^*-algebra $\mathscr{A} \otimes \mathscr{B}$, then continuity of δ immediately implies that $\lim_{(t,s)\to(0,0)} \frac{1}{ts}\Psi_{t,s} = \delta$.

Theorem 3.9. *Let δ be the infinitesimal generator of $\alpha\otimes\beta$ over $\mathscr{A} \otimes \mathscr{B}$, and let E be an $\alpha\otimes\beta$-invariant dense subspace of $\mathscr{A} \otimes \mathscr{B}$. Then E is a core for δ.*

Proof. Let $z \in D(\delta)$ so that $z \in \bar{D}^{\|\cdot\|}$. Hence, there is a sequence z_n in D such that

$$\begin{aligned}|\Phi(z_n) - \Phi(z)|_\delta &= \|\Phi(z_n) - \Phi(z)\| + \|\delta\Phi(z_n) - \delta\Phi(z)\| \\ &= \|\Phi(z_n - z)\| + \|\Psi(z_n) - \Psi(z)\|.\end{aligned}$$

Since $\{\Phi(z_n)\}$ is a sequence in D, the limit of $\{\Phi(z_n)\}$ is in $\bar{D}$. Moreover,

$$\left|\frac{1}{ts}\Phi(z) - z\right|_\delta = \left\|\frac{1}{ts}\Phi(z) - z\right\| + \left\|\frac{1}{ts}\delta\Phi(z) - \delta(z)\right\| \longrightarrow 0,$$

which implies that $z \in \bar{D}^{|\cdot|_\delta}$. □

Corollary 3.10. *The domain of a tensor sum is its core.*

Proof. It immediately follows from the fact that the domain of a tensor sum is invariant under its flow and it is a dense subspace. □

3.1. More on the properties of Tensor Sum

Definition 3.11. *An operator δ on the tensor product of C^*-algebras $\mathscr{A} \otimes \mathscr{B}$ is called* tensor summable *if there exist two operators Δ, ∇ over $\mathscr{A}$ and $\mathscr{B}$, respectively, such that $\delta = \Delta \otimes id + id \otimes \nabla$ and we write $\delta = \Delta \boxplus \nabla$. Moreover, the* tensor difference *of Δ and ∇, denoted by $\Delta \boxminus \nabla$, is defined by $\Delta \boxminus \nabla := \Delta \otimes id - id \otimes \nabla$.*

Basic operations with tensor sum of operators are summarized as follow. If the notation $\mathbb{F}$ will mean that is the set of all scalars, and the set of all operators on $\mathscr{A}$ is denoted by $Ope(\mathscr{A})$, then for every $\alpha, \beta \in \mathbb{F}$, $\Delta_1, \Delta_2 \in Ope(\mathscr{A})$ and $\nabla_1, \nabla_2 \in Ope(\mathscr{B})$,
(a) $\alpha\beta(\beta^{-1}\Delta_1 \boxplus \nabla_1\alpha^{-1}) = \alpha\Delta_1 \boxplus \nabla_1\beta$ where α, β are non-zero,
(b) $\Delta_1 \boxplus \nabla_1 + \Delta_2 \boxplus \nabla_2 = \Delta_1 \boxplus \nabla_2 + \Delta_2 \boxplus \nabla_1$,
(c) $\alpha(\Delta_1 \boxplus \nabla_1)\beta = \alpha\Delta_1 \boxplus \nabla_1\beta$,
(d) $\Delta_1 \boxplus \Delta_1 = \Delta_1 \boxplus id + id \boxplus \Delta_1 - id \boxplus id$,
(e) $\Delta_1 \boxplus \nabla_1 = \Delta_1 \otimes \nabla_1$ if and only if $\Delta_1 \otimes id$ is a quasi-inverse of $id \otimes \nabla_1$,
(f) $\|\Delta_1 \boxplus \nabla_1\| = \|\Delta_1\|\|\nabla_1\|$ if and only if $\Delta_1 \otimes id$ is a quasi-inverse of $id \otimes \Delta_1$,
(g) If $\Delta \neq \lambda id$ for every non zero scalar $\lambda \in \mathbb{F}$, then $\Delta \boxplus \nabla \neq 0$,
(h) $-(\Delta_1 \boxplus \nabla_1) = -\Delta_1 \boxplus -\nabla_1$,
(k) $(\Delta \boxplus \nabla)(\Delta \boxminus \nabla) = \Delta^2 \boxminus \nabla^2$, (l) If Δ, ∇ are $*$-derivations, then $\Delta \boxminus \nabla$ is a $*$-derivation on $\mathscr{A} \otimes \mathscr{B}$. A similar definition enjoying properties (a)-(l) as above can be stated for the Hilbert space tensor products in the setting of Hilber spaces.

Proposition 3.12. *Let $\mathscr{A}$, $\mathscr{B}$ be unital Banach algebras, and let Δ, ∇ be invertible elements of $\mathscr{A}$ and $\mathscr{B}$, respectively. If $\|\Delta \otimes \nabla\| < 1$, then $\Delta \boxminus \nabla$ is invertible.*

Proof. Invertibility of Δ implies that $\Delta \otimes id$ is invertible. Moreover, we can write

$$\left\|(\Delta \otimes id)^{-1}(id \otimes \nabla)\right\| = \left\|(\Delta^{-1} \otimes id)(id \otimes \nabla)\right\| = \left\|\Delta^{-1} \otimes \nabla\right\| = \|\Delta \otimes \nabla\| < 1.$$

It follows that $1 - (\Delta \otimes id)^{-1}(id \otimes \nabla)$ is invertible. An easy computation shows that

$$(\Delta \otimes id)^{-1}(1 - (\Delta \otimes id)^{-1}(id \otimes \nabla))^{-1}(\Delta \boxminus \nabla) = id \otimes id,$$

which follows that $\Delta \boxminus \nabla$ is left invertible. Similarly we see that $\Delta \boxminus \nabla$ is right invertible. □

So far we have discussed the tensor sum of the operators honest in Leibniz's property on C^*-algebras. Finally, we discuss the properties of tensor sum of operators on Hilbert spaces. Recall that if Δ is an operator on a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, and E is an orthonormal basis for $\mathcal{H}$, then the *trace class norm* is defined by $\|\Delta\|_1 = \sum_{x \in E} \||\Delta|^{1/2}(x)\|^2$. Also, Δ is a *trace-class* operator if $\|\Delta\|_1 < +\infty$. The trace of a trace-class operator Δ is defined by $tr(\Delta) = \sum_{x \in E} \langle \Delta(x), x \rangle$.

Theorem 3.13. *Let Δ and ∇ be trace-class operators on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Then $tr(\Delta \boxplus \nabla) = tr(\Delta) + tr(\nabla)$.*

Proof. Let E, F be orthonormal basis for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Then the set $\{x \otimes y : x \in E, y \in F\}$ is an orthonormal basis for $\mathcal{H} \otimes \mathcal{K}$,

$$\begin{aligned} tr(\Delta \boxplus \nabla) &= tr(\Delta \otimes id + id \otimes \nabla) = tr(\Delta \otimes id) + tr(id \otimes \nabla) \\ &= \sum_{\substack{x \in E \\ y \in F}} \langle \Delta \otimes id(x \otimes y), x \otimes y \rangle + \sum_{\substack{x' \in E \\ y' \in F}} \langle id \otimes \nabla(x' \otimes y'), x' \otimes y' \rangle \\ &= \sum_{\substack{x \in E \\ y \in F}} \langle \Delta x, x \rangle \langle y, y \rangle + \sum_{\substack{x' \in E \\ y' \in F}} \langle x', x' \rangle \langle \nabla y', y' \rangle \\ &= \sum_{x \in E} \langle \Delta x, x \rangle + \sum_{y' \in F} \langle \nabla y', y' \rangle = tr(\Delta) + tr(\nabla). \end{aligned}$$

□

3.2. *Finite dimensional case*

Let A and B be finite dimensional vector spaces over a field $\mathbb{F}$, and let $\{a_i \ : 1 \le i \le n\}$ be a basis of A and $\{b_j \ : 1 \le i \le m\}$ be a basis of B. For $i = 1, ..., n$ and $j = 1, ..., m$ set $a_i \otimes b_j = a_i b_j^t$. Then $\{a_i \otimes b_j \ : 1 \le i \le n, \ 1 \le j \le m\}$ is a basis for the some vector space which is denoted by $A \otimes B$, the dimension of $A \otimes B$ is the product of the dimensions of its factors. Suppose that Δ and ∇ are operators on A, B, respectively. Since $\{a_i \otimes b_j \ : 1 \le i \le n, \ 1 \le j \le m\}$ form a basis for $A \otimes B$ so there exists a unique operator $\Delta \otimes \nabla$ on $A \otimes B$ with $\Delta \otimes \nabla(a_i \otimes b_j) = \Delta(a_i) \otimes \nabla(b_j)$. Secondly, $\Delta \otimes \nabla$ also satisfies $\Delta \otimes \nabla(a \otimes b) = \Delta(a) \otimes \nabla(b)$. Now consider two linear transformations $\Delta : A \longrightarrow A'$ and $\nabla : B \longrightarrow B'$ where A, A', B and B' are finite dimensional vector spaces and let $[\Delta_{ij}]_{m \times n}$ and $[\nabla_{ij}]_{p \times q}$ be matrices of linear transformations Δ and ∇, respectively. Then the Kronecker product of Δ and ∇ is defined as the block matrix $[\Delta_{ij}\nabla]_{mp \times nq}$ and it is denoted by $\Delta \otimes \nabla$ again. See also [3]. Let m, n be two natural numbers. Then the discrete interval of m, n is denoted by $< m, n >$ and define as the set $\{m, m+1, ..., n\}$. The set H_{mn} is defined to be the set

$$\{< 1, n >, < n+1, 2n >, ..., < rn - n + 1, rn >, ..., \\ < sn - n + 1, sn >, ..., < mn - n + 1, mn >\}.$$

If $(\Delta \otimes I)_{ij}$ is the entry in row i and column j of the matrix $\Delta \otimes I$ of order m, then $(\Delta \otimes I)_{ij} = \Delta_{rs}\delta_{pq}$, where $i \in < rn - n + 1, rn >$, $j \in < sn - n + 1, sn >$, and δ_{pq} denotes the so called Kronecker delta with $p, q \in \mathbb{Z}_n$. (Note that $\mathbb{Z}_n$ is called the additive group of integers modulo n, the set of equivalence classes of $\mathbb{Z}$ under congruence modulo n.) Since the class of zero is equals to the class of n so we may assume that $\{\bar{1}, \bar{2}, ..., \bar{n}\}$. Similarly, $(I_m \otimes \nabla)_{ij} = \delta_{rs}\nabla_{pq}$.

Theorem 3.14. *Let D be a matrix of order k. If there exist natural numbers m, $n \geq 2$ such that $k = mn$, and D is partitioned as m^2 blocks of blocks of order n with this property that each block-diagonal are diagonal adaptable and others blocks of D has the form cI_n for some entry c of D. Then there are matrices Δ, ∇ of order m, n, respectively, such that $D = \Delta \boxplus \nabla$.*

Proof. The proof is somewhat numerically but straightforward. Consider two matrices Δ and ∇ of order m, n, respectively. The matrix $\Delta \otimes I_n + I_m \otimes \nabla$ contains $m^2 + n^2$ unknown entries, and each its entry has the form $\Delta_{rs}\delta_{pq} + \delta_{rs}\nabla_{pq}$, where r, s are in the members of H_{mn} and $p, q \in \mathbb{Z}_n$ see the previous argument. The equality $D = \Delta \otimes \nabla$ is obtained if we take $D_{ij} = \Delta_{rs}\delta_{pq} + \delta_{rs}\nabla_{pq}$, where $1 \leq i, j \leq mn$. If $i \neq j$ and D_{ij} lies entirely in one of the diagonal blocks of D then $i, j \in < rn - n + 1, rn >$ for some r and $i \in \bar{p}$, $j \in \bar{q}$ so $p \neq q$. Hence, $D_{ij} = \nabla_{pq}$ for $n^2 - n$ unknown entries of ∇. But if D_{ij} is outside of the diagonal blocks then D_{ij} lies in a block of the form cI_n for some entry c of D. Therefore $i \in < rn - n + 1, rn >$, $j \in < sn - n + 1, sn >$ and $r \neq s$ so that $D_{ij} = \Delta_{rs}\delta_{pq}$ in case $D_{ij} = c$ we have $\delta_{pq} = 1$, then $D_{ij} = \Delta_{rs}$ for $m^2 - m$ unknown entries of Δ, only then solve the linear system contains $m^2 + n^2 - (m^2 - m) - (n^2 - n)$ unknown and mn first order linear equation. Since $m + n \leq mn$ for every natural number opposite of one then it has a solution. □

Example 3.15. *Let*

$$D = \begin{pmatrix} 0 & 3 & 2 & 0 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Then $H_{2,2} = \{< 1, 2 >, < 3, 4 >\}$ and $\mathbb{Z}_2 = \{\bar{1}, \bar{2}\}$. Consider $D_{21} = \Delta_{rs}\delta_{pq} + \delta_{rs}\nabla_{pq}$ so $r, s = 1$, $2 \in \bar{2}$, and $1 \in \bar{1}$. Hence, $\Delta_{21} = 1$. Continuing this fashion we obtain $\Delta = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ and $\nabla = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$. Then $D = \Delta \otimes I_2 + I_2 \otimes \nabla$, where I_2 denotes the identity matrix of order two.

Data availability: All data generated or analyzed during this study are included in this published article.

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