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On Sequential Maxima of Geometric Sample Means, with Extension to the Ruin Probability

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Abstract. In this paper, we consider the ruin probability of risk models with a geometric distribution of claim sizes. Since their probabilities can't be calculated directly, we use exponential distribution to estimate its upper and lower bounds and asymptotic estimates based on the relationship between geometric distribution and exponential distribution. Finally, some numerical simulations are given to prove the superiority of our estimates.

1. Introduction

Recently, risk models have attracted much attention in the insurance businesses, especially the problems associated with the calculation of ruin probabilities. Many scholars have studied the effects of claim time, claim size and initial capital for the ruin probability in risk models. In this article, we study the ruin probability under a certain claim size in the classical discrete-time risk model. We assume that all the processes are defined in a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Taking into account the normal operation of the insurance companies, we assume that the average premium income per unit time should be greater than the average claim amount. For $n \geq 1$, let random variables T_n be the moment when each claim size X_n occurs, take values $\{1, 2, 3, \ldots\}$ and $T_1 \leq T_2 \leq T_3 \leq \cdots$. Suppose $U_{n,c}^X$ as the surplus at the end of nth time epoch, and the expected premium rate for per time unit is a constant c. So the formula

$$U_{n,c}^X = u + cn - S_n, \ n \in \mathbb{N}^+ \tag{1}$$

is a classical discrete-time risk model, where $U_{0,c}^X = u > 0$ is the initial capital. $S_n = \sum_{i=1}^n X_i$ represents the accumulated amount of the claim occurring in the nth time epoch, where $(X_i)_{i\geq 1}$ is an independent identically distributed (i.i.d.) random variable sequence. The ruin probability at the time n is

$$\varphi_n^X(u, c) = \mathbb{P}\Big(\bigcap_{j=1}^{n} (U_{j,c}^X < 0) \,|U_{0,c}^X = u\Big).$$

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Then, we can denote the infinite time ruin probability for the model (1) as

$$\varphi_{\infty}^{X}(u,c) = \mathbb{P}\Big(\bigcap_{j=1}^{\infty}(U_{j,c}^{X} < 0)\,|U_{0,c}^{X} = u\Big).$$

In addition, $\phi_{n}^{X}(u,c) = 1 - \varphi_{n}^{X}(u,c)$ denotes the survival probability at the time n.

In view of Lundberg's work about the basic collective insurance risk model beginning of the last century, many researchers started to study the ruin probability of the risk model. Ignatov and Kaishev (2000), Raducan et al (2016), Asmussen and Rolski (1994) and Sundt and Teugels (1995) have been devoted to finding the upper and lower bounds of the ruin probabilities. Sundt and Teugels (1995) discussed the equations, approximations, and two-sided bounds of the ruin probability in the case with zero initial reserves and the case with exponential claim sizes. Cai (2002) studied the ruin probabilities in two generalized risk models and developed a renewal recursive technique to get generalized Lundberg inequalities for the ruin probabilities. Sattayatham and Klongdee (2013) proved the existence of the minimum initial capital under the given ruin probability α. Lin et al (2015) presented two kinds of methods, the recursive equations and the martingale approach, for minimizing the upper bound of the ruin probability, and also showed the martingale approach is better than the another one. Cheliotis and Papadatos (2019) obtained the ruin probability in a risk-theoretic model by computing the distribution of the maximal average in a sequence of i.i.d. exponential random variables. Fokkink et al (2021) proved an inequality $\mathbb{P}(S_k \geq k) \geq \mathbb{P}(S_{k+1} \geq k+1)$, where $(X_i)_{i\geq 1}$ is a sequence of i.i.d. non-negative integer random variables, $k \geq 1$, and $S_k = \sum_{i=1}^{k} X_i$. Li (2005) gave recursive and explicit formulas for the ruin probability, which includes the surplus before ruin, the expected discounted penalty function at the ruin, and the deficit atruin. Vernic (2015) proved the conjecture in Raducan et al (2015) about the ruin probability in a particular case. We remark that the study of bankruptcy theory has achieved fruitful results in many special risk models with claim sizes, such as exponential distribution (Asmussen (2000), Cai (2002), Sattayatham and Klongdee (2013), Lin et al (2015)), nonhomogeneous Erlang distribution (Asmussen (2000), Raducan et al (2015)) and binomial distribution (Asmussen (2000), Li (2005)).

On the other hand, geometric distribution has been widely studied in discrete online leasing problems (Jun et al (2005)), face detection (Toyama (2004)), vertebrate dispersal distances model (Carroll (1989)) and queuing systems (Evans (1967)). To the best of our knowledge, there is a remarkably small number of papers treating the application of the geometric claim sizes in ruin probability. Since geometric distribution is a very important life distribution, its application is very vital in reliability mathematics. At the same time, in the discrete life model, the geometric distribution acts as the exponential distribution in the continuous risk model. Therefore, the study of ruin probability models with geometric claim sizes is crucial, and it has meaningful theoretical value and practical value. Hence, in this paper, we aim to seek the ruin probability in risk models with geometric claim sizes. For overcoming the difficulty in the calculation of the ruin probability with geometric claim sizes, we use the relationship between geometric distribution and exponential distribution appeared in Steutel and Thiemann (1989).

The remainder of this paper consists of four sections. Stating the preliminary results of geometric distribution in Section 2. Extend to the discrete-time risk model with geometric claim sizes from the preliminary results in Section 3. Some numerical simulations of the main results are presented in Section 4. The proofs of the main results are given in Section 5.

2. Preliminary results

Here and later, N has a geometric distribution with success probability $p \in (0,1)$, i.e. $\mathbb{P}(N = k) = p(1-p)^{k-1}$, for $k = 1, 2, 3, \ldots$. Let $(X_i)_{i\geq 1}$ be an i.i.d. random variable sequence of the geometric distribution with $p = 1/2$. Define $\bar{X}_i := (X_1 + X_2 + \cdots + X_i)/i$ for each $i \in \mathbb{N}^{+}$. $\bar{X}_i$ converges to 2 with probability 1 by the strong law of large numbers. Assume random variables

$$M_n(X) := \max\{\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_n\} \tag{2}$$

for each $n \in \mathbb{N}^+$, $F_n^X(x)$ is the distribution function of $M_n(X)$. For $y \in \mathbb{R}$, let $\lfloor y \rfloor$ be the integer part, i.e. the largest integer not exceeding y, and $y - 1 < \lfloor y \rfloor \le y$. The next proposition gives a more explicit expression of the distribution function of $M_n(X)$.

Proposition 2.1. *For all $x \ge 0$, and $n \in \mathbb{N}^+$, the distribution function of $M_n(X)$ is*

$$F_n^X(x) = 1 - \sum_{i=1}^{n} D_i(x), \tag{3}$$

where

$$D_i(x) = \begin{cases} \left(\frac{1}{2}\right)^{\lfloor x \rfloor}, & \text{if } i = 1, \\ \lfloor x \rfloor \left(\frac{1}{2}\right)^{\lfloor 2x \rfloor}, & \text{if } i = 2, \\ \sum\limits_{k_1=1}^{\lfloor x \rfloor} \cdots \sum\limits_{k_{i-2}=1}^{\lfloor (i-2)x - k_1 - \cdots - k_{i-3} \rfloor} \left(\frac{1}{2}\right)^{\lfloor ix \rfloor} \lfloor (i-1)x - k_1 - \cdots - k_{i-2} \rfloor, & \text{if } i \ge 3. \end{cases} \tag{4}$$

Proof: Letting $F_0^X(x) = 1$, we calculate $F_n^X(x)$ for $x \ge 0$ and $n \ge 3$, that is

$$\begin{aligned} F_n^X(x) &= \mathbb{P}(M_n(X) \le x) \\ &= \sum_{k_1=1}^{\lfloor x \rfloor} \sum_{k_2=1}^{\lfloor 2x - k_1 \rfloor} \cdots \sum_{k_n=1}^{\lfloor nx - k_1 - \cdots - k_{n-1} \rfloor} \left(\frac{1}{2}\right)^{k_1 + k_2 + \cdots + k_n} \\ &= \sum_{k_1=1}^{\lfloor x \rfloor} \sum_{k_2=1}^{\lfloor 2x - k_1 \rfloor} \cdots \sum_{k_{n-1}=1}^{\lfloor (n-1)x - k_1 - \cdots - k_{n-2} \rfloor} \left(\left(\frac{1}{2}\right)^{k_1 + k_2 + \cdots + k_{n-1}} - \left(\frac{1}{2}\right)^{\lfloor nx \rfloor} \right) \\ &= F_{n-1}^X(x) - \sum_{k_1=1}^{\lfloor x \rfloor} \cdots \sum_{k_{n-2}=1}^{\lfloor (n-2)x - k_1 - \cdots - k_{n-3} \rfloor} \lfloor (n-1)x - k_1 - \cdots - k_{n-2} \rfloor \left(\frac{1}{2}\right)^{\lfloor nx \rfloor} \\ &= F_{n-1}^X(x) - D_n(x), \end{aligned}$$

where

$$D_n(x) = \sum_{k_1=1}^{\lfloor x \rfloor} \cdots \sum_{k_{n-2}=1}^{\lfloor (n-2)x - k_1 - \cdots - k_{n-3} \rfloor} \lfloor (n-1)x - k_1 - \cdots - k_{n-2} \rfloor \left(\frac{1}{2}\right)^{\lfloor nx \rfloor}.$$

Particularly, $F_1^X(x) = 1 - (\frac{1}{2})^{\lfloor x \rfloor}$ and $F_2^X(x) = 1 - (\frac{1}{2})^{\lfloor x \rfloor} - \lfloor x \rfloor \left(\frac{1}{2}\right)^{\lfloor 2x \rfloor}$. By recursive method, the explicit expression of $F_n^X(x)$ is

$$F_n^X(x) = 1 - \sum_{i=1}^{n} D_i(x) \ \text{ for all } x \ge 0,$$

where $D_i(x)$ is (4). The proof is completed. □

Define $M_\infty(X) := \sup_{i \in \mathbb{N}^+} \bar{X}_i$, the random sequence $(M_n(X))_{n \ge 1}$ is increasing and $M_n(X) \to M_\infty(X)$ a.s. For all $x \in [0, \infty)$, the distribution function $F_\infty^X(x)$ of $M_\infty(X)$ can be written as

$$F_\infty^X(x) = \mathbb{P}(M_\infty(X) \le x) = \mathbb{P}\left\{ \bigcap_{n=1}^{\infty} (M_n(X) \le x) \right\} = \lim_{n \to \infty} F_n^X(x). \tag{5}$$

Then $F_\infty^X(x) = 1 - \sum_{i=1}^{\infty} D_i(x)$, where $D_i(x)$ is defined by (4). In most cases, however, the calculation of $F_\infty^X(x)$ is difficult, since $(\lfloor ix \rfloor)_{i \ge 1}$ in (3) can't be calculated directly. Fortunately, we obtain the mutual transformation relationship between exponential distribution and geometric distribution by the following fact, which is derived from Steutel and Thiemann (1989). For the integrity of our paper, here we have to restate it and add a simple proof.

Lemma 2.1. *If N has a geometric distribution, i.e.*

$$\mathbb{P}(N = k) = p(1-p)^{k-1}, \ (k = 1, 2, \ldots, \ 0 < p < 1),$$

then

$$N \overset{d}{=} \lfloor Y + 1 \rfloor,$$

where Y has an exponential distribution with density

$$f_Y(y) = \beta e^{-\beta y}, \ (y > 0, \ \beta = -\log(1-p)).$$

Proof:

$$\begin{aligned}
\mathbb{P}(\lfloor Y + 1 \rfloor = k) &= \mathbb{P}(k \le Y + 1 < k + 1) \\
&= \mathbb{P}(Y + 1 < k + 1) - P(Y + 1 < k) \\
&= (1 - e^{-\beta k}) - (1 - e^{-\beta(k-1)}) \\
&= e^{-\beta(k-1)}(1 - e^{-\beta}).
\end{aligned}$$

Then $\lfloor Y + 1 \rfloor$ is geometric distribution with $p = 1 - e^{-\beta}$. This implies that $N \overset{d}{=} \lfloor Y + 1 \rfloor$, i.e. N and $\lfloor Y + 1 \rfloor$ have the same distribution function and the proof is completed. □

Letting $\beta = \log 2$ in Lemma 2.1, it is easy to get $F_{Y+1}(y) = 1 - (\frac{1}{2})^{y-1}$ for all $y \ge 1$. $\{Y + 1\}$ is the fractional part of $Y + 1$, then $(1 - \{Y + 1\}) \in (0, 1]$. Combining $\lfloor Y + 1 \rfloor = Y + 1 - \{Y + 1\}$, there is a key idea that we use $Y + \frac{1}{2}$ instead of $\lfloor Y + 1 \rfloor$ to estimate (3).

Let $(Y_i)_{i \ge 1}$ be independent copies of Y. For $i \in \mathbb{N}^+$, define $Z_i := Y_i + 1$ and $W_i := Y_i + 1/2$. Assume that $\bar{Y}_i := (Y_1 + Y_2 + \cdots + Y_i)/i$, $\bar{Z}_i := (Z_1 + Z_2 + \cdots + Z_i)/i$ and $\bar{W}_i := (W_1 + W_2 + \cdots + W_i)/i$. For each $n \in \mathbb{N}^+$, we define these random variables as follows,

$$\begin{aligned}
M_n(Y) &: = \max\{\bar{Y}_1, \bar{Y}_2, \ldots, \bar{Y}_n\}, \\
M_n(Z) &: = \max\{\bar{Z}_1, \bar{Z}_2, \ldots, \bar{Z}_n\}, \\
M_n(W) &: = \max\{\bar{W}_1, \bar{W}_2, \ldots, \bar{W}_n\}.
\end{aligned} \tag{6}$$

$F_n^Y(x)$, $F_n^Z(x)$ and $F_n^W(x)$ are the distribution functions of $M_n(Y)$, $M_n(Z)$ and $M_n(W)$, respectively. The following proposition derives the important relationship among $F_n^X(x)$, $F_n^Y(x)$, $F_n^Z(x)$ and $F_n^W(x)$ for all $n \in \mathbb{N}^+$ and $x \in [0, \infty)$.

Proposition 2.2. *For $x \in [0, \infty)$, $n \in \mathbb{N}^+$, we have*

$$F_n^Z(x) \le F_n^X(x) \le F_n^Y(x), \tag{7}$$

and

$$F_n^Z(x) \le F_n^W(x) \le F_n^Y(x). \tag{8}$$

Proof: First of all, it is clear that $\lfloor Y_i + 1 \rfloor \overset{d}{=} X_i$ for all $i \in \mathbb{N}^+$ by Lemma 2.1. Then, for $x \in [0, \infty)$,

$$\mathbb{P}(X_i \le x) = \mathbb{P}(\lfloor Y_i + 1 \rfloor \le x).$$

Note that $(Y_i)_{i \ge 1}$ and $(X_i)_{i \ge 1}$ are independent copies of Y and X, respectively. Therefore, $(X_1, X_2, \ldots, X_n)$ and $(\lfloor Y_1 + 1 \rfloor, \lfloor Y_2 + 1 \rfloor, \ldots, \lfloor Y_n + 1 \rfloor)$ have the same joint distribution function for $n \in \mathbb{N}^+$. Furthermore,

$$\max\left\{\lfloor Y_1 + 1 \rfloor, \ldots, \frac{1}{n}\sum_{i=1}^{n} \lfloor Y_i + 1 \rfloor\right\} \overset{d}{=} \max\{\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_n\}. \tag{9}$$

Secondly, since $Y_i < \lfloor Y_i + 1 \rfloor \le Y_i + 1$ for all $i \le n$, then

$$\max\{\bar{Y}_1, \bar{Y}_2, \dots, \bar{Y}_n\} < \max\left\{\lfloor Y_1 + 1 \rfloor, \dots, \frac{1}{n}\sum_{i=1}^{n}\lfloor Y_i + 1 \rfloor\right\} \le \max\{\bar{Z}_1, \bar{Z}_2, \dots, \bar{Z}_n\}. \tag{10}$$

Therefore, based on (9) and (10), we have

$$\mathbb{P}(M_n(Z) \le x) \le \mathbb{P}(M_n(X) \le x) \le \mathbb{P}(M_n(Y) \le x),$$

i.e. (7) is established.

Similarly, since $Y_i < Y_i + 1/2 < Y_i + 1$, for $n \in \mathbb{N}^+$, we have

$$\max\{\bar{Y}_1, \bar{Y}_2, \dots, \bar{Y}_n\} < \max\left\{\left\lfloor Y_1 + \frac{1}{2}\right\rfloor, \dots, \frac{1}{n}\sum_{i=1}^{n}\left\lfloor Y_i + \frac{1}{2}\right\rfloor\right\} < \max\{\bar{Z}_1, \bar{Z}_2, \dots, \bar{Z}_n\}.$$

Hence, $\mathbb{P}(M_n(Z) \le x) \le \mathbb{P}(M_n(W) \le x) \le \mathbb{P}(M_n(Y) \le x)$ is established. That is (8). The proof is completed. □

It is easy to verify that $M_n(Y) \to M_\infty(Y)$ a.s., $M_n(Z) \to M_\infty(Z)$ a.s. and $M_n(W) \to M_\infty(W)$ a.s.. Combining (5), we have

$$F_\infty^Y(x) = \mathbb{P}(M_\infty(Y) \le x) = \mathbb{P}\left\{\bigcap_{n=1}^{\infty}(M_n(Y) \le x)\right\} = \lim_{n\to\infty} F_n^Y(x),$$

$$F_\infty^Z(x) = \mathbb{P}(M_\infty(Z) \le x) = \mathbb{P}\left\{\bigcap_{n=1}^{\infty}(M_n(Z) \le x)\right\} = \lim_{n\to\infty} F_n^Z(x),$$

and

$$F_\infty^W(x) = \mathbb{P}(M_\infty(W) \le x) = \mathbb{P}\left\{\bigcap_{n=1}^{\infty}(M_n(W) \le x)\right\} = \lim_{n\to\infty} F_n^W(x).$$

Therefore, according to Proposition 2.2, it is clear that $F_\infty^Z(x) \le F_\infty^W(x) \le F_\infty^Y(x)$ for all $x \in [0, \infty)$.

Next, we mainly calculate the upper bound $F_\infty^Y(x)$, the lower bound $F_\infty^Z(x)$ and the asymptotic estimates $F_\infty^W(x)$ of $F_\infty^X(x)$. In the process of calculating the estimates of $F_\infty^X(x)$, we need the following key lemma about $V_n(x,t)$, which is derived from Cheliotis and Papadatos (2019).

Lemma 2.2. *For $x \ge 0$, $x + t \ge 0$, and $n \in \mathbb{N}^+$, define*

$$K_n(x,t) := \{(y_1, y_2, \dots, y_n) \in \mathbb{R}_+^n : \ y_1 + \cdots + y_i \le ix + t \ \textit{for all} \ \ i = 1, 2, \dots, n\}.$$

Then,

$$V_n(x,t) := \mathrm{Vol}(K_n(x,t)) = \frac{1}{n!}(x+t)((n+1)x+t)^{n-1}, \ n = 1, 2, \dots,$$

in particular, $\mathrm{Vol}(K_n(x,0)) = \frac{1}{n!}(n+1)^{n-1}x^n$ *and* $\mathrm{Vol}(K_0(x,0)) = 1$.

Based on Proposition 2.2, Lemma 2.1 and Lemma 2.2, we give the upper bound $F_\infty^Y(x)$, the lower bound $F_\infty^Z(x)$, and the asymptotic estimates $F_\infty^W(x)$ of $F_\infty^X(x)$ in the following theorem.

Theorem 2.1. (i) *For all $x > \frac{1}{\log 2}$, the upper bound of $F_\infty^X(x)$ is*

$$F_\infty^Y(x) = 1 - \sum_{k=1}^{\infty}\frac{(k\log 2)^{k-1}}{k!}\left(\frac{1}{2^x}\right)^k x^{k-1}.$$

$F^Y_\infty(x)$ *is positive on* $(\frac{1}{\log 2}, \infty)$*, and* $F^Y_\infty(x) = 0$ *for all* $x \in [0, \frac{1}{\log 2}]$.

(ii) *For all* $x > 1 + \frac{1}{\log 2}$*, the lower bound of* $F^X_\infty(x)$ *is*

$$F^Z_\infty(x) = 1 - \sum_{k=1}^{\infty} \frac{(k\log 2)^{k-1}}{k!} \Big(\frac{1}{2^{x-1}}\Big)^k (x-1)^{k-1}.$$

$F^Z_\infty(x)$ *is positive on* $(1 + \frac{1}{\log 2}, \infty)$*, and* $F^Z_\infty(x) = 0$ *for all* $x \in [0, 1 + \frac{1}{\log 2}]$.

(iii) *For all* $x > \frac{1}{2} + \frac{1}{\log 2}$*, the asymptotic estimates of* $F^X_\infty(x)$ *is*

$$F^W_\infty(x) = 1 - \sum_{k=1}^{\infty} \frac{(k\log 2)^{k-1}}{k!} \Big(\frac{1}{2^{x-\frac{1}{2}}}\Big)^k \Big(x - \frac{1}{2}\Big)^{k-1}.$$

$F^W_\infty(x)$ *is positive on* $(\frac{1}{2} + \frac{1}{\log 2}, \infty)$*, and* $F^W_\infty(x) = 0$ *for all* $x \in [0, \frac{1}{2} + \frac{1}{\log 2}]$.

In the following corollary, we give the inverse functions of the three distribution functions in Theorem 2.1.

Corollary 2.1. (i) *For* $x \in (\frac{1}{\log 2}, \infty)$*, the inverse function of* $F^Y_\infty(x)$ *is*

$$F^{Y\,-1}_\infty(\alpha) = \frac{-\log_2(1-\alpha)}{\alpha} \quad \textit{for all } \alpha \in (0,1).$$

(ii) *For* $x \in (1 + \frac{1}{\log 2}, \infty)$*, the inverse function of* $F^Z_\infty(x)$ *is*

$$F^{Z\,-1}_\infty(\alpha) = 1 + \frac{-\log_2(1-\alpha)}{\alpha} \quad \textit{for all } \alpha \in (0,1).$$

(iii) *For* $x \in (\frac{1}{2} + \frac{1}{\log 2}, \infty)$*, the inverse function of* $F^W_\infty(x)$ *is*

$$F^{W-1}_\infty(\alpha) = \frac{1}{2} + \frac{-\log_2(1-\alpha)}{\alpha} \quad \textit{for all } \alpha \in (0,1).$$

3. Extend to the ruin probability from the preliminary results

In Section 2, we get the upper bound $F^Y_\infty(x)$, the lower bound $F^Z_\infty(x)$ and the asymptotic estimates $F^W_\infty(x)$ of $F^X_\infty(x)$ by the relationships among X, Y, Z and W. Next, combining preliminary results, we expand them to the classical discrete-time risk model with geometric distribution claim sizes.

3.1. *The ruin probability of classical discrete-time risk model*

According to (1), we easily get the survival probability at the time n. That is

$$\begin{aligned}
\phi^X_n(u,c) &= \mathbb{P}(U^X_{1,c} > 0, U^X_{2,c} > 0, \dots, U^X_{n,c} > 0) \\
&= \mathbb{P}(u + c - S_1 > 0, u + 2c - S_2 > 0, \dots, u + nc - S_n > 0) \\
&= \mathbb{P}(S_1 < u + c, S_2 < u + 2c, \dots, S_n < u + nc) \\
&= \mathbb{P}\Big(S_1 < c\Big(\frac{u}{c} + 1\Big), S_2 < c\Big(\frac{u}{c} + 2\Big), \dots, S_n < c\Big(\frac{u}{c} + n\Big)\Big) \\
&= \mathbb{P}\Big(\frac{S_1}{(\frac{u}{c} + 1)} < c, \frac{S_2}{(\frac{u}{c} + 2)} < c, \dots, \frac{S_n}{(\frac{u}{c} + n)} < c\Big) \\
&= 1 - \varphi^X_n(u,c).
\end{aligned} \tag{11}$$

Meanwhile, we extend (2) based on (11) as follows. For all $n \in \mathbb{N}^+$ and $x \in [0, \infty)$, define

$$M_{n,\lambda}(X) := \max\left\{\frac{X_1}{1+\lambda}, \frac{X_1+X_2}{2+\lambda}, \ldots, \frac{X_1+X_2+\cdots+X_n}{n+\lambda}\right\}. \tag{12}$$

$F^X_{n,\lambda}(x)$ is the distribution function of $M_{n,\lambda}(X)$. Let $F^X_{0,\lambda}(x) = 1$ and $\lambda \geq 0$. For all $x \geq 0$ and $n \geq 3$, we have

$$F^X_{n,\lambda}(x) = 1 - \sum_{i=1}^{n} D_{i,\lambda}(x),$$

where

$$D_{i,\lambda}(x) = \begin{cases} \left(\frac{1}{2}\right)^{\lfloor(1+\lambda)x\rfloor}, & \text{if } i = 1, \\ \lfloor(1+\lambda)x\rfloor\left(\frac{1}{2}\right)^{\lfloor(2+\lambda)x\rfloor}, & \text{if } i = 2, \\ \sum\limits_{k_1=1}^{\lfloor(1+\lambda)x\rfloor} \cdots \sum\limits_{k_{i-2}=1}^{\lfloor(i-2+\lambda)x-k_1-\cdots-k_{i-3}\rfloor} \left(\frac{1}{2}\right)^{\lfloor(i+\lambda)x\rfloor} \lfloor(i-1+\lambda)x-k_1-\cdots-k_{i-2}\rfloor, & \text{if } i \geq 3. \end{cases} \tag{13}$$

Particularly, $F^X_{1,\lambda}(x) = 1 - \left(\frac{1}{2}\right)^{\lfloor(1+\lambda)x\rfloor}$ and $F^X_{2,\lambda}(x) = 1 - \left(\frac{1}{2}\right)^{\lfloor(1+\lambda)x\rfloor} - \lfloor(1+\lambda)x\rfloor\left(\frac{1}{2}\right)^{\lfloor(2+\lambda)x\rfloor}$. Then, we have $\varphi^X_{n,\lambda}(x) = 1 - F^X_{n,\lambda}(x)$. In addition, we define $M_{\infty,\lambda}(X) := \lim_{n\to\infty} M_{n,\lambda}(X)$, and $F^X_{\infty,\lambda}(x)$ as the distribution function of $M_{\infty,\lambda}(X)$. We can get $F^X_{\infty,\lambda}(x) = \lim_{n\to\infty} F^X_{n,\lambda}(x)$. Hence, $F^X_{\infty,\lambda}(x) = 1 - \sum_{i=1}^{\infty} D_{i,\lambda}(x)$, where $D_{i,\lambda}(x)$ is defined by (13).

Similarly, we define

$$\begin{aligned} M_{n,\lambda}(Y) &:= \max\left\{\frac{Y_1}{1+\lambda}, \frac{Y_1+Y_2}{2+\lambda}, \ldots, \frac{Y_1+Y_2+\cdots+Y_n}{n+\lambda}\right\}, \\ M_{n,\lambda}(Z) &:= \max\left\{\frac{Z_1}{1+\lambda}, \frac{Z_1+Z_2}{2+\lambda}, \ldots, \frac{Z_1+Z_2+\cdots+Z_n}{n+\lambda}\right\}, \\ M_{n,\lambda}(W) &:= \max\left\{\frac{W_1}{1+\lambda}, \frac{W_1+W_2}{2+\lambda}, \ldots, \frac{W_1+W_2+\cdots+W_n}{n+\lambda}\right\}. \end{aligned} \tag{14}$$

$F^Y_{n,\lambda}(x)$, $F^Z_{n,\lambda}(x)$ and $F^W_{n,\lambda}(x)$ are the distribution functions of $M_{n,\lambda}(Y)$, $M_{n,\lambda}(Z)$ and $M_{n,\lambda}(W)$, respectively. Similar to the proof of Proposition 2.2, it is easy to verify that

$$F^Z_{n,\lambda}(x) \leq F^X_{n,\lambda}(x) \leq F^Y_{n,\lambda}(x), \tag{15}$$

and

$$F^Z_{n,\lambda}(x) \leq F^W_{n,\lambda}(x) \leq F^Y_{n,\lambda}(x).$$

Let $M_{\infty,\lambda}(Y) := \lim_{n\to\infty} M_{n,\lambda}(Y)$, $M_{\infty,\lambda}(Z) := \lim_{n\to\infty} M_{n,\lambda}(Z)$ and $M_{\infty,\lambda}(W) := \lim_{n\to\infty} M_{n,\lambda}(W)$. $F^Y_{\infty,\lambda}(x)$, $F^Z_{\infty,\lambda}(x)$ and $F^W_{\infty,\lambda}(x)$ as the distribution functions of $M_{\infty,\lambda}(Y)$, $M_{\infty,\lambda}(Z)$ and $M_{\infty,\lambda}(W)$, respectively. Therefore, we have an important theorem of $F^X_{\infty,\lambda}(x)$ as follows.

Theorem 3.1. (i) *The distribution function $F^X_{\infty,\lambda}(x)$ has the upper bound in $x \in (\frac{1}{\log 2}, \infty)$, that is*

$$F^Y_{\infty,\lambda}(x) = 1 - (1+\lambda)\sum_{k=1}^{\infty} \frac{(k+\lambda)^{k-2}(\log 2)^{k-1}}{(k-1)!}\left(\frac{1}{2^x}\right)^{k+\lambda} x^{k-1}.$$

(ii) *The distribution function $F^X_{\infty,\lambda}(x)$ has the lower bound in $x \in (1 + \frac{1}{\log 2}, \infty)$, that is*

$$F^Z_{\infty,\lambda}(x) = 1 - \sum_{k=1}^{\infty} \frac{(x-1+\lambda x)(kx-k+\lambda x)^{k-2}}{(k-1)!}\left(\frac{1}{2}\right)^{(k+\lambda)x-k} (\log 2)^{k-1}.$$

(iii) *The distribution function* $F^X_{\infty,\lambda}(x)$ *has the asymptotic estimates in* $x \in (\frac{1}{2} + \frac{1}{\log 2}, \infty)$, *that is*

$$F^W_{\infty,\lambda}(x) = 1 - \sum_{k=1}^{\infty} \frac{(x - \frac{1}{2} + \lambda x)(kx - \frac{1}{2}k + \lambda x)^{k-2}}{(k-1)!} \Big(\frac{1}{2}\Big)^{(k+\lambda)x - \frac{1}{2}k} (\log 2)^{k-1}.$$

Remark 3.1. *According to Corollary 2.1, note that* $F^Y_{\infty,\lambda}(x)$ *also has an inverse function for* $x \in (\frac{1}{\log 2}, \infty)$. *Since, the distribution function* $F^Y_{\infty,\lambda}(x)$ *equals*

$$F^Y_{\infty,\lambda}(x) = \begin{cases} 1 - \frac{t(x)}{x}\Big(\frac{1}{2}\Big)^{-\lambda(t(x)-x)}, & \text{if } x > \frac{1}{\log 2}, \\ 0, & \text{if } 0 \le x \le \frac{1}{\log 2}, \end{cases}$$

where $t(x)$ *depends on* x. *Therefore, for* $x \in (\frac{1}{\log 2}, \infty)$, *the inverse function of* $F^Y_{\infty,\lambda}(x)$ *is*

$${F^Y_{\infty,\lambda}}^{-1}(\alpha) = \frac{-\log_2(1-\alpha)}{(1+\lambda)(1-(1-\alpha)^{\frac{1}{\lambda+1}})} \text{ for all } \alpha \in (0,1).$$

Unfortunately, we can't get ${F^Z_{\infty,\lambda}}^{-1}(\alpha)$ *and* ${F^W_{\infty,\lambda}}^{-1}(\alpha)$ *with our method, which is also our future research direction.*

Next, let $\varphi^Z_{\infty,\lambda}(x)$ and $\varphi^Y_{\infty,\lambda}(x)$ be the upper and lower bounds of $\varphi^X_{\infty,\lambda}(x) = 1 - F^X_{\infty,\lambda}(x)$, respectively. Combining (15) and Theorem 3.1, it is easy to obtain that

$$\varphi^Y_{\infty,\lambda}(x) = 1 - F^Y_{\infty,\lambda}(x) \le \varphi^X_{\infty,\lambda}(x) \le 1 - F^Z_{\infty,\lambda}(x) = \varphi^Z_{\infty,\lambda}(x).$$

Whereas, the numerical simulation of $\varphi^X_{\infty,\lambda}(x)$ is difficult. Fortunately, we can estimate the range of $(\varphi^X_{\infty,\lambda}(x) - \varphi^X_{n,\lambda}(x))_{n\ge 1}$ according to the following lemma of $\Gamma(k+1)$, the lemma appeared in Lu and Wang (2013) .

Lemma 3.1. (i) *For every* $m \le 5$, *there exists* m_1 *depending on* m, *such that for every* $k \ge m_1$, *it holds:*

$$\Gamma(k+1) < \sqrt{2\pi k}\Big(\frac{k}{e}\Big)^k \Big(1 + \frac{m}{12k} + \frac{m^2}{288k^2}\Big)^{1/m}.$$

(ii) *For every* $m \ge 6$, *there exists* m_2 *depending on* m, *such that for every* $k \ge m_2$, *it holds:*

$$\Gamma(k+1) > \sqrt{2\pi k}\Big(\frac{k}{e}\Big)^k \Big(1 + \frac{m}{12k} + \frac{m^2}{288k^2}\Big)^{1/m}.$$

We next give an important theorem of the range of $(\varphi^X_{\infty,\lambda}(x) - \varphi^X_{n,\lambda}(x))_{n\ge 1}$ according to Lemma 3.1.

Theorem 3.2. *For* $x \ge 3$ *and* $\lambda \ge 0$, *let* $\frac{\lambda}{n+1} \le \frac{1}{25}$ *and* $c(x) = \frac{e \log 2^x}{2^x}$, *the error between* $\varphi^X_{\infty,\lambda}(x)$ *and* $\varphi^X_{n,\lambda}(x)$ *is*

$$F^Z_{n,\lambda}(x) - F^Y_{n,\lambda}(x) + \frac{288(1+\lambda)c^n(x)(n \log \frac{1}{c(x)} - 1.5)}{313\sqrt{2\pi}2^{\lambda x + x} n^{\frac{5}{2}} (\log c(x))^2} \le \varphi^X_{\infty,\lambda}(x) - \varphi^X_{n,\lambda}(x) \le F^Y_{n,\lambda}(x) - F^Z_{n,\lambda}(x) + \frac{e^2(x - 1 + \lambda x)\log 2}{2^{\lambda x + 2x - 3}\sqrt{2n\pi}}.$$

3.2. The ruin probability with the constant interest rate

In this subsection, we consider the classical risk model with a constant interest rate $\gamma > 0$. Then, the new discrete-time risk model is

$$U^X_{n,c,\gamma} = (1+\gamma)U^X_{n-1,c,\gamma} + c(1+\gamma) - X_n, \ n \in \mathbb{N}^+. \tag{16}$$

Through a simple integration, we get that (16) is equivalent to

$$U^X_{n,c,\gamma} = u(1+\gamma)^n + \sum_{i=1}^{n} c(1+\gamma)^i - \sum_{i=1}^{n} X_i(1+\gamma)^{n-i},\ n \in \mathbb{N}^+.$$

The ruin probability at the time n is

$$\varphi^X_n(u,c,\gamma) = \mathbb{P}\Big(\bigcap_{j=1}^{n}(U^X_{j,c,\gamma} < 0)\,|U^X_{0,c,\gamma} = u\Big).$$

Next, we give the upper and lower bounds of $\varphi^X_n(u,c,\gamma)$ following the proof of Theorem 2.1.

According to Proposition 2.2, for $y \geq 0$, and $n \in \mathbb{N}^+$, the lower bound of $\varphi^X_n(u,c,\gamma)$ can be written as

$$\varphi^Y_n(u,c,\gamma) = \mathbb{P}\Big(\bigcap_{j=1}^{n}(U^Y_{j,c,\gamma} < 0)\,|U^Y_{0,c,\gamma} = u\Big),$$

where

$$U^Y_{j,c,\gamma} = u(1+\gamma)^j + \sum_{i=1}^{j} c(1+\gamma)^i - \sum_{i=1}^{j} Y_i(1+\gamma)^{j-i}.$$

Then, we calculate the ruin probability $\varphi^Y_n(u,c,\gamma)$. That is

$$\begin{aligned}\varphi^Y_n(u,c,\gamma) =& 1 - \mathbb{P}(Y_1 \leq (u+c)(1+\gamma), Y_2 + Y_1(1+\gamma) \leq u(1+\gamma)^2 + c(1+\gamma)^2 + c(1+\gamma),\\ &\ldots, \sum_{i=1}^{n} Y_i(1+\gamma)^{n-i} \leq u(1+\gamma)^n + \sum_{i=1}^{n} c(1+\gamma)^i)\\ =&\varphi^Y_{n-1}(u,c,\gamma) + E^Y_n(u,c,\gamma),\end{aligned}$$

where $E^Y_1(u,c,\gamma) = (\frac{1}{2})^{(u+c)(1+\gamma)}$, and when $n \geq 2$,

$$\begin{aligned}&E^Y_n(u,c,\gamma)\\ &= (\log 2)^{n-1}\Big(\frac{1}{2}\Big)^{u(1+\gamma)^n + \sum\limits_{i=1}^{n} c(1+\gamma)^i} \int_0^{(u+c)(1+\gamma)} \cdots \int_0^{u(1+\gamma)^{n-1} + \sum\limits_{i=1}^{n-1} c(1+\gamma)^i - \sum\limits_{i=1}^{n-2} y_i(1+\gamma)^{n-1-i}} \Big(\frac{1}{2}\Big)^{\sum\limits_{i=1}^{n-1} y_i(1-(1+\gamma)^{n-i})} dy_{n-1}\cdots dy_1.\end{aligned}$$

Since $\varphi^Y_\infty(u,c,\gamma) = \lim_{n\to\infty} \varphi^Y_n(u,c,\gamma)$. Then

$$\varphi^Y_\infty(u,c,\gamma) = \sum_{n=1}^{\infty} E^Y_n(u,c,\gamma). \tag{17}$$

On the other hand, we give the upper bound of $\varphi^X_n(u,c,\gamma)$. That is $\varphi^Z_n(u,c,\gamma) = P(\bigcap_{j=1}^{n}(U^Z_{j,c,\gamma} < 0)\,|U^Z_{0,c,\gamma} = u)$, where $U^Z_{j,c,\gamma} = u(1+\gamma)^j + \sum_{i=1}^{j} c(1+\gamma)^i - \sum_{i=1}^{j} Z_i(1+\gamma)^{j-i}$. We have

$$\begin{aligned}\varphi^Z_n(u,c,\gamma) =& 1 - \mathbb{P}(Z_1 \leq (u+c)(1+\gamma), \ldots, \sum_{i=1}^{n} Z_i(1+\gamma)^{n-i} \leq u(1+\gamma)^n + \sum_{i=1}^{n} c(1+\gamma)^i)\\ =& 1 - \mathbb{P}(Y_1 + 1 \leq (u+c)(1+\gamma), \ldots, \sum_{i=1}^{n}(Y_i+1)(1+\gamma)^{n-i} \leq u(1+\gamma)^n + \sum_{i=1}^{n} c(1+\gamma)^i)\\ =&\varphi^Z_{n-1}(u,c,\gamma) + E^Z_n(u,c,\gamma),\end{aligned}$$

where $E_1^Z(u,c,\gamma) = (\frac{1}{2})^{(u+c)(1+\gamma)-1}$, and when $n \geq 2$,

$$E_n^Z(u,c,\gamma) = (\log 2)^{n-1}\left(\frac{1}{2}\right)^{u(1+\gamma)^n+\sum_{i=1}^{n} c(1+\gamma)^i}$$

$$\times \int_0^{(u+c)(1+\gamma)-1} \cdots \int_0^{u(1+\gamma)^{n-1}+\sum_{i=1}^{n-1} c(1+\gamma)^i-\sum_{i=1}^{n-2}(y_i+1)(1+\gamma)^{n-1-i}-1} \left(\frac{1}{2}\right)^{\sum_{i=1}^{n-1} y_i-\sum_{i=1}^{n-1}(y_i+1)(1+\gamma)^{n-i}-1} dy_{n-1}\cdots dy_1.$$

Therefore,

$$\varphi_\infty^Z(u,c,\gamma) = \lim_{n\to\infty} \varphi_n^Z(u,c,\gamma) = \sum_{n=1}^{\infty} E_n^Z(u,c,\gamma). \tag{18}$$

Since (17) and (18) are very complicated, we can't obtain more detailed results. Meanwhile, it has an important reference value for the following numerical simulation.

4. Numerical Simulations

We here illustrate the accuracy of the main results through the approach of numerical simulation. Let $\varphi_{n,\lambda}^Y(x) = 1 - F_{n,\lambda}^Y(x)$ and $\varphi_{n,\lambda}^Z(x) = 1 - F_{n,\lambda}^Z(x)$. This section consists of four parts. Simulating the upper bound $F_n^Y(x)$, the lower bound $F_n^Z(x)$ and the asymptotic estimates $F_n^W(x)$ of $F_n^X(x)$, and using numerical simulation to examine the accuracy of the asymptotic estimates in Subsection 4.1. Giving the comparison of $\varphi_{n,\lambda}^W(x)$ and $\varphi_{n,\lambda}^*(x)$, and giving the plot of $(\varphi_{\infty,0}^Z(x), \varphi_{\infty,0}^W(x), \varphi_{\infty,0}^Y(x))$ in Subsection 4.2. Giving numerical illustration of the ruin probability $\varphi_n^Y(u,c,\gamma)$ and $\varphi_n^Z(u,c,\gamma)$ in a specific situation in Subsection 4.3. Giving numerical illustration of the error between $\varphi_n^X(x,\lambda)$ and $\varphi_\infty^X(x,\lambda)$ in Subsection 4.4.

4.1. The estimates of ($F_n^Y(x)$, $F_n^X(x)$, $F_n^W(x)$, $F_n^Z(x)$) and the accuracy of the asymptotic estimates

4.1.1. The estimates of ($F_n^Y(x)$, $F_n^X(x)$, $F_n^W(x)$, $F_n^Z(x)$)

For numerical illustration about the estimations of $F_n^X(x)$, we generate 10000 samples for random variables X and Y in the R environment, respectively. It is easy to obtain Z and W based on the relationship among Y, Z and W. Meanwhile, we visualize Theorem 2.1 by the Monte Carlo method under the case of $n = 10, 20, 30, 40$. Figure 1 presents the simulation of ($F_n^Y(x)$, $F_n^X(x)$, $F_n^W(x)$, $F_n^Z(x)$). As is seen, $F_n^Y(x)$ is always above $F_n^X(x)$, and $F_n^Z(x)$ is below $F_n^X(x)$ for $x \in [0,\infty)$. As x increases, they tend to overlap. In addition, Figure 1 also shows that $F_n^W(x)$ and $F_n^X(x)$ are relatively close. In summary, this confirms the feasibility of Theorem 2.1.

4.1.2. The accuracy of the asymptotic estimates

In this subsection, we evaluate the approximating performance of $F_n^W(x)$ to $F_n^X(x)$ by their absolute error and relative error. We here consider $n = 1, 5, 10, 20, 30, 40$ in Theorem 2.1. It is clear that the choice of x is crucial for the results, the absolute and relative errors of $F_n^W(x)$ and $F_n^X(x)$, such that we consider $x = 2.0, 2.5, \ldots, 6.0$. One sees that the absolute errors fluctuate in 0.0070($x = 2.5$), 0.0570($x = 3.0$), 0.0005($x = 3.5$), 0.0030($x = 4.0$), 0.0010($x = 4.5$), 0.0140($x = 5.0$), 0.0009($x = 5.5$) and 0.0060($x = 6.0$) from Table 1. In addition, the relative errors fluctuate in 0.0010($x = 2.5$), 0.0700($x = 3.0$), 0.0070($x = 3.5$), 0.0300($x = 4.0$), 0.0020($x = 4.5$), 0.0140($x = 5.0$), 0.0009($x = 5.5$) and 0.0060($x = 6.0$) based on Table 2. Then $F_n^W(x)$ can be used to estimate $F_n^X(x)$ within the allowable range of error. It confirms the effectiveness of the asymptotic estimates.

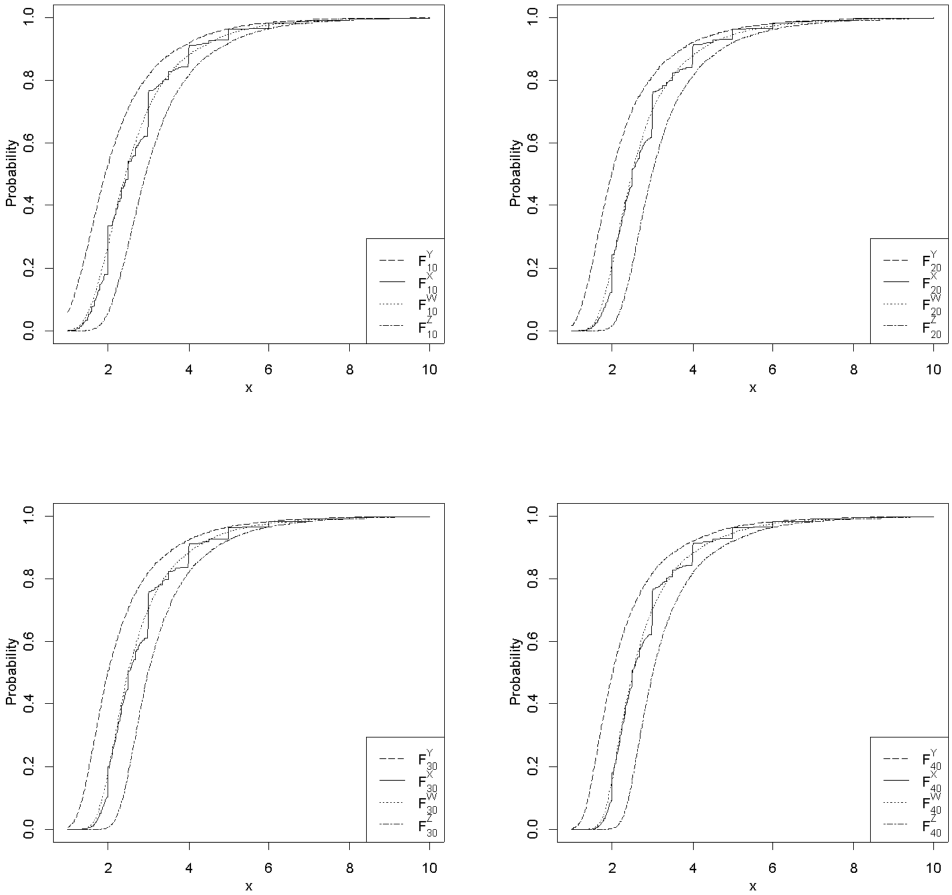

Figure 1: Trend for $(F_n^Y(x), F_n^X(x), F_n^W(x), F_n^Z(x))(n = 10, 20, 30, 40)$

Table 1: Approximate of absolute errors between $F_n^W(x)$ and $F_n^X(x)$

	$x = 2.5$	$x = 3.0$	$x = 3.5$	$x = 4.0$	$x = 4.5$	$x = 5.0$	$x = 5.5$	$x = 6.0$
$n = 1$	0.0008	0.0523	0.0003	0.0272	0.0013	0.0142	0.0012	0.0076
$n = 5$	0.0116	0.0573	0.0050	0.0289	0.0018	0.0142	0.0008	0.0067
$n = 10$	0.0070	0.0530	0.0045	0.0286	0.0010	0.0133	0.0002	0.0058
$n = 20$	0.0088	0.0607	0.0075	0.0310	0.0032	0.0142	0.0007	0.0065
$n = 30$	0.0037	0.0580	0.0062	0.0301	0.0015	0.0132	0.0001	0.0065
$n = 40$	0.0059	0.0587	0.0076	0.0303	0.0040	0.0153	0.0015	0.0070

Table 2: Approximate of relative errors between $F_n^W(x)$ and $F_n^X(x)$

	$x = 2.5$	$x = 3.0$	$x = 3.5$	$x = 4.0$	$x = 4.5$	$x = 5.0$	$x = 5.5$	$x = 6.0$
$n = 1$	0.0010	0.0599	0.0003	0.0291	0.0014	0.0146	0.0013	0.0077
$n = 5$	0.0199	0.0733	0.0060	0.0317	0.0019	0.0148	0.0008	0.0068
$n = 10$	0.0131	0.0693	0.0055	0.0314	0.0011	0.0139	0.0002	0.0059
$n = 20$	0.0169	0.0795	0.0091	0.0340	0.0035	0.0147	0.0007	0.0066
$n = 30$	0.0074	0.0760	0.0076	0.0330	0.0017	0.0138	0.0000	0.0067
$n = 40$	0.0116	0.0768	0.0091	0.0333	0.0043	0.0159	0.0016	0.0071

4.2. The simulations of the ruin probability

4.2.1. The comparisons of $\varphi^W_{n,\lambda}(x)$ and $\varphi^*_{n,\lambda}(x)$

Based on the effectiveness of the asymptotic estimates of $\varphi^X_{n,\lambda}(x)$, we further evaluate $\varphi^W_{n,\lambda}(x) = 1 - F^W_{n,\lambda}(x)$ and $\varphi^*_{n,\lambda}(x) = (\varphi^Z_{n,\lambda}(x) + \varphi^Y_{n,\lambda}(x))/2$. Let $c = 2 + \theta > 0$, where θ is the safe loading of the insurance company. Under the case of $n = 30$, we simulate $(\varphi^Z_{30,\lambda}(x), \varphi^X_{30,\lambda}(x), \varphi^W_{30,\lambda}(x), \varphi^*_{30,\lambda}(x), \varphi^Y_{30,\lambda}(x))$ by taking the parameters $\theta = 0.25, 0.50, 1.00$ and $u = 2, 5, 8, 10$ to compare the approximating performance in the R environment. Table 3 presents that the errors between $\varphi^X_{30,\lambda}(x)$ and $\varphi^W_{30,\lambda}(x)$ are very small. Combining Figure 1 and Table 3, we note that $\varphi^W_{30,\lambda}(x)$ is the particularly satisfactory estimate of $\varphi^X_{30,\lambda}(x)$ as c increases. On the other hand, the errors between $\varphi^W_{30,\lambda}(x)$ and $\varphi^X_{30,\lambda}(x)$ are small when compared to the error between $\varphi^*_{30,\lambda}(x)$ and $\varphi^X_{30,\lambda}(x)$. It shows that the estimation of $\varphi^W_{30,\lambda}(x)$ is better than the another one. In conclusion, the numerical simulation confirms the effectiveness of the asymptotic estimates $\varphi^W_{30,\lambda}(x)$.

Table 3: The simulation of $(\varphi^Z_{30,\lambda}(x), \varphi^X_{30,\lambda}(x), \varphi^W_{30,\lambda}(x), \varphi^*_{30,\lambda}(x), \varphi^Y_{30,\lambda}(x))$

	$u = 2$			$u = 5$		
	$c = 2.25$	$c = 2.50$	$c = 3.00$	$c = 2.25$	$c = 2.50$	$c = 3.00$
$\varphi^Z_{30,\lambda}(x)$	0.84592	0.61202	0.24579	0.66415	0.37308	0.08410
$\varphi^X_{30,\lambda}(x)$	0.41587	0.24968	0.08905	0.19938	0.09023	0.02050
$\varphi^W_{30,\lambda}(x)$	0.39731	0.24557	0.11006	0.18149	0.08513	0.02510
$\varphi^*_{30,\lambda}(x)$	0.50380	0.36119	0.15177	0.35487	0.19943	0.04744
$\varphi^Y_{30,\lambda}(x)$	0.16168	0.11035	0.05774	0.04559	0.02577	0.01077
	$u = 8$			$u = 10$		
	$c = 2.25$	$c = 2.50$	$c = 3.00$	$c = 2.25$	$c = 2.50$	$c = 3.00$
$\varphi^Z_{30,\lambda}(x)$	0.47887	0.21154	0.02879	0.37044	0.13978	0.01382
$\varphi_{30,\lambda}(x)$	0.09130	0.03272	0.00522	0.05263	0.01690	0.00189
$\varphi^W_{30,\lambda}(x)$	0.07965	0.03005	0.00592	0.04609	0.01482	0.00247
$\varphi^*_{30,\lambda}(x)$	0.24592	0.10900	0.01550	0.18792	0.07100	0.00728
$\varphi^Y_{30,\lambda}(x)$	0.01297	0.00646	0.00221	0.00539	0.00222	0.00074

4.2.2. The approximation of $(\varphi^Z_{\infty,0}(x), \varphi^W_{\infty,0}(x), \varphi^Y_{\infty,0}(x))$

Based on the limitation of the R software, it is difficult to present the relations among $\varphi^Z_{\infty,\lambda}(x)$, $\varphi^W_{\infty,\lambda}(x)$ and $\varphi^Y_{\infty,\lambda}(x)$. Therefore, under the case of $\lambda = 0$, we give the following plot of $(\varphi^Z_{\infty,0}(x), \varphi^W_{\infty,0}(x), \varphi^Y_{\infty,0}(x))$ based on their expressions in Theorem 2.1 in Python. From Figure 2, we give the two-side bounds and the asymptotic estimates of $\varphi^X_{\infty,0}(x)$. The result further proves the rationality of Subsection 4.2.1.

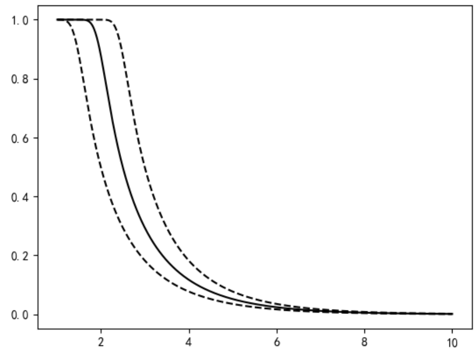

Figure 2: The trend of $(\varphi^Z_{\infty,0}(x), \varphi^W_{\infty,0}(x), \varphi^Y_{\infty,0}(x))$

4.3. The numerical illustrations of $\varphi^Y_n(u, c, \gamma)$ *and* $\varphi^Z_n(u, c, \gamma)$

In this subsection, we confirm the effectiveness of (17) and (18) under the case of $\gamma = 1.75\%$, $u = 2, 5, 8, 10$ and $c = 2.25, 2.50$ in Table 4. For this table, we note that $\varphi^Y_n(u, c, \gamma)$ and $\varphi^Z_n(u, c, \gamma)$ gradually decrease as u increases for $c = 2.25$ and $c = 2.50$. On the other hand, $\varphi^Z_n(u, c, \gamma) - \varphi^Y_n(u, c, \gamma) \geq 0$ and the error between $\varphi^Z_n(u, c, \gamma)$ and $\varphi^Y_n(u, c, \gamma)$ gradually decrease for $n = 1, 2, 3, 4, 5$. The results of this numerical simulation prove the accuracy of theoretical in Subsection 3.2 about the ruin probability with the constant interest rate γ.

Table 4: The numerical illustrations of $\varphi^Y_n(u, c, \gamma)$ and $\varphi^Z_n(u, c, \gamma)$

	$c = 2.25$				$c = 2.50$			
	$u = 2$	$u = 5$	$u = 8$	$u = 10$	$u = 2$	$u = 5$	$u = 8$	$u = 10$
$\varphi^Z_1(u, c, \gamma)$	0.0998	0.0120	0.0015	0.0004	0.0837	0.0101	0.0012	0.0004
$\varphi^Y_1(u, c, \gamma)$	0.0499	0.0060	0.0007	0.0002	0.0418	0.0050	0.0006	0.0002
$\varphi^Z_2(u, c, \gamma)$	0.1921	0.0330	0.0051	0.0014	0.1534	0.0254	0.0039	0.0014
$\varphi^Y_2(u, c, \gamma)$	0.0797	0.0120	0.0017	0.0005	0.0640	0.0094	0.0013	0.0005
$\varphi^Z_3(u, c, \gamma)$	0.2682	0.0588	0.0111	0.0034	0.2077	0.0425	0.0077	0.0034
$\varphi^Y_3(u, c, \gamma)$	0.0981	0.0169	0.0027	0.0008	0.0764	0.0126	0.0020	0.0008
$\varphi^Z_4(u, c, \gamma)$	0.3306	0.0868	0.0191	0.0065	0.2505	0.0597	0.0124	0.0065
$\varphi^Y_4(u, c, \gamma)$	0.1200	0.0207	0.0037	0.0011	0.0837	0.0148	0.0025	0.0011
$\varphi^Z_5(u, c, \gamma)$	0.3823	0.1153	0.0287	0.0106	0.2851	0.0763	0.0176	0.0106
$\varphi^Y_5(u, c, \gamma)$	0.1180	0.0236	0.0044	0.0014	0.0883	0.0163	0.0029	0.0014

4.4. *The errors between $\varphi^X_{n,\lambda}(x)$ and $\varphi^X_{\infty,\lambda}(x)$*

In this subsection, we illustrate the errors between $\varphi^X_{n,\lambda}(x)$ and $\varphi^X_{\infty,\lambda}(x)$ according to Theorem 3.2. Let $L_n(x) := F^Z_{n,\lambda}(x) - F^Y_{n,\lambda}(x) + (288(1+\lambda)c^n(x)(n\log\frac{1}{c(x)} - 1.5))/(313\sqrt{2\pi}2^{\lambda x+x}n^{\frac{5}{2}}(\log c(x))^2)$ and $H_n(x) := F^Y_{n,\lambda}(x) - F^Z_{n,\lambda}(x) + (e^2(x-1+\lambda x)\log 2)/(2^{\lambda x+2x-3}\sqrt{2n\pi})$. Furthermore, the results are shown in Figure 3 under the case of $\lambda = 1$ and $x = 5, 10$. One sees that the values of $H_n(x)$ getting closer to the values of $L_n(x)$ as $x \to 10$. The results of the errors between $\varphi^X_{n,\lambda}(x)$ and $\varphi^X_{\infty,\lambda}(x)$ show that $\varphi^X_{n,\lambda}(x)$ can be used to estimate $\varphi^X_{\infty,\lambda}(x)$, and it is effective.

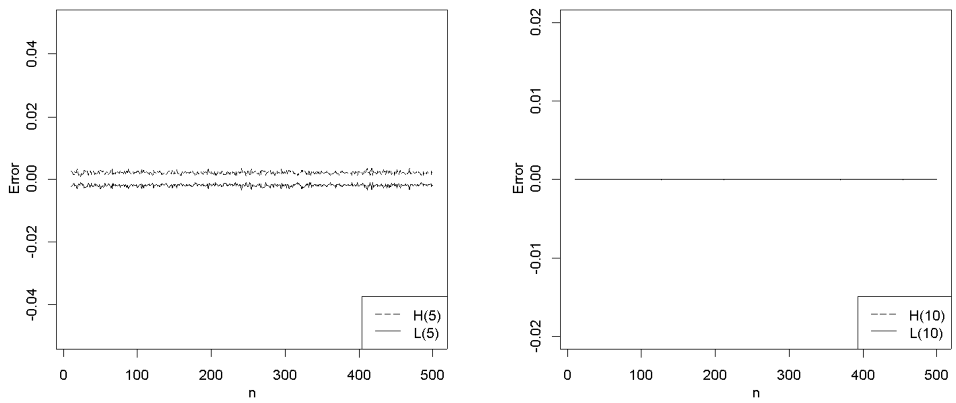

Figure 3: Trend for the errors between $\varphi^X_{n,\lambda}(x)$ and $\varphi^X_{n,\lambda}(x)$(x=5,10).

5. The proofs of theorems and corollary

5.1. *The proof of Theorem 2.1*

Firstly, we give the detailed calculation procedure of $F^Y_n(x)$ as follows. For all $x > \frac{1}{\log 2}$, we have $F^Y_1(x) = 1 - (\frac{1}{2})^x$ and define $F^Y_0(x) = 1$. Then

$$\begin{aligned} F^Y_n(x) &= \mathbb{P}(M_n(Y) \le x) \\ &= \mathbb{P}(\bar{Y}_1 \le x, \bar{Y}_2 \le x, \ldots, \bar{Y}_n \le x) \\ &= \int_0^x \cdots \int_0^{nx-y_1-\cdots-y_{n-1}} (\log 2)^n \Big(\frac{1}{2}\Big)^{y_1+y_2+\cdots+y_n} dy_n \cdots dy_1 \\ &= F^Y_{n-1}(x) - (\log 2)^{n-1}\mathrm{Vol}(K_{n-1}(x,0))\Big(\frac{1}{2}\Big)^{nx} \end{aligned}$$

for $n \ge 2$ and $x > \frac{1}{\log 2}$, where $\mathrm{Vol}(K_{n-1}(x,0)) = \frac{1}{(n-1)!}n^{n-2}x^{n-1}$ from Lemma 2.2. Therefore, for all $x > \frac{1}{\log 2}$ and $n \in \mathbb{N}^+$, the explicit form of $F^Y_n(x)$ is

$$F^Y_n(x) = 1 - \sum_{k=1}^{n} \frac{(k\log 2)^{k-1}}{k!}\Big(\frac{1}{2^x}\Big)^k x^{k-1}. \tag{19}$$

Then, $F^Y_\infty(x) = 1 - \sum_{k=1}^\infty \frac{(k\log 2)^{k-1}}{k!}\Big(\frac{1}{2^x}\Big)^k x^{k-1}$ for all $x > \frac{1}{\log 2}$. By the strong law of large numbers, we can get that $F^Y_\infty(x) = 0$ for all $x \in [0, \frac{1}{\log 2}]$, and $F^Y_\infty(x)$ is positive on $(\frac{1}{\log 2}, \infty)$. Since the left derivative and the right derivative of $F^Y_\infty(x)$ are not equal at $x = \frac{1}{\log 2}$, $F^Y_\infty(x)$ is differentiable on $[0,\infty)\backslash\{\frac{1}{\log 2}\}$.

In a similar way, for all $x > 1 + \frac{1}{\log 2}$, defining $F^Z_0(x) = 1$, we have

$$\begin{aligned} F^Z_n(x) &= \mathbb{P}(M_n(Z) \le x) \\ &= \mathbb{P}(\bar{Z}_1 \le x, \bar{Z}_2 \le x, \dots, \bar{Z}_n \le x) \\ &= \mathbb{P}(\bar{Y}_1 \le x-1, \bar{Y}_2 \le x-1, \dots, \bar{Y}_n \le x-1) \\ &= F^Y_n(x-1) \\ &= 1 - \sum_{k=1}^{n} \frac{(k\log 2)^{k-1}}{k!}\Big(\frac{1}{2^{x-1}}\Big)^k (x-1)^{k-1}, \end{aligned}$$

when $n \ge 1$. Therefore, $F^Z_\infty(x) = 1 - \sum_{k=1}^{\infty} \frac{(k\log 2)^{k-1}}{k!}\Big(\frac{1}{2^{x-1}}\Big)^k (x-1)^{k-1}$ for all $x > 1 + \frac{1}{\log 2}$. $F^Z_\infty(x) = 0$ is established for all $x \in [0, 1+\frac{1}{\log 2}]$, and $F^Z_\infty(x)$ is positive on $(1+\frac{1}{\log 2}, \infty)$. $F^Z_\infty(x)$ is differentiable in $[0,\infty)\backslash\{1+\frac{1}{\log 2}\}$.

Furthermore, for all $x > \frac{1}{2} + \frac{1}{\log 2}$, defining $F^W_0(x) = 1$, we have

$$F^W_n(x) = F^Y_n\Big(x - \frac{1}{2}\Big) = 1 - \sum_{k=1}^{n} \frac{(k\log 2)^{k-1}}{k!}\Big(\frac{1}{2^{x-\frac{1}{2}}}\Big)^k \Big(x - \frac{1}{2}\Big)^{k-1},$$

when $n \ge 1$. Therefore, $F^W_\infty(x) = 1 - \sum_{k=1}^{\infty} \frac{(k\log 2)^{k-1}}{k!}(\frac{1}{2^{x-\frac{1}{2}}})^k (x-\frac{1}{2})^{k-1}$ for all $x \in (\frac{1}{2} + \frac{1}{\log 2}, \infty)$. Meanwhile, $F^W_\infty(x) = 0$ for all $x \in [0, \frac{1}{2}+\frac{1}{\log 2}]$ and $F^W_\infty(x)$ is positive on $(\frac{1}{2}+\frac{1}{\log 2}, \infty)$. $F^W_\infty(x)$ is differentiable in $[0,\infty)\backslash\{\frac{1}{2}+\frac{1}{\log 2}\}$. The proof is completed. □

5.2. *The proof of Corollary 2.1*

Based on (19), $F^Y_\infty(x) = 0$ for all $x \in [0, 1/\log 2]$ and $F^Y_\infty(x)$ is continuous in $[1/\log 2, \infty)$. for all $x \in [0, 1/\log 2]$, we have

$$\sum_{k=1}^{\infty} \frac{k^{k-1}}{k!}\Big(\frac{\log 2^x}{2^x}\Big)^k = x\log 2. \tag{20}$$

Let $g(x) := \frac{\log 2^x}{2^x}$ and $h(g(x)) := \sum_{k=1}^{\infty} \frac{k^{k-1} g^k(x)}{k!}$. Furthermore $h(g(x)) = x\log 2$. Note that $g(x)$ is strictly monotone increasing in $[0, 1/\log 2]$ and strictly monotone decreasing in $[1/\log 2, \infty)$, and it is easy to verify that $g(0) = 0$, $g(1/\log 2) = (1/2)^{1/\log 2}$ and $g(\infty) = 0$. There exists a unique inverse function $h^{-1}(x)$ such that $h^{-1}(x\log 2) = g(x)$ for all $x \in [0, 1/\log 2]$. On the other hand, for any $x \in (1/\log 2, \infty)$, there exists $t_1(x) \in [0, 1/\log 2]$, $\frac{\log 2^x}{2^x} = \frac{\log 2^{t_1(x)}}{2^{t_1(x)}}$ is holding. We have $h^{-1}(t_1(x)\log 2) = \frac{\log 2^x}{2^x}$, then

$$t_1(x)\log 2 = h\Big(\frac{\log 2^x}{2^x}\Big). \tag{21}$$

Hence, the piecewise function $F^Y_\infty(x)$ is

$$F^Y_\infty(x) = \begin{cases} 1 - \frac{t_1(x)}{x}, & \text{if } x > \frac{1}{\log 2}, \\ 0, & \text{if } 0 \le x \le \frac{1}{\log 2}. \end{cases} \tag{22}$$

Now, for any fixed $\alpha \in (0,1)$, $F^Y_\infty(x) = \alpha$. Then, we can get $x - t_1(x) = \alpha x$, i.e. $t_1(x) = (1-\alpha)x$. Consequently,

$$\frac{1}{1-\alpha} = \frac{x}{t_1(x)} = 2^{x - t_1(x)} = 2^{\alpha x}. \tag{23}$$

Therefore, $x = (-\log_2(1-\alpha))/\alpha$. i.e.

$$F_{\infty}^{Y\,-1}(\alpha) = \frac{-\log_2(1-\alpha)}{\alpha} \text{ for all } \alpha \in (0,1). \tag{24}$$

In a similar way from (20) to (24), there exists $t_2(x)$ depending on x, such that $F_{\infty}^{Z}(x)$ can be written as

$$F_{\infty}^{Z}(x) = \begin{cases} 1 - \frac{t_2(x)-1}{x-1}, & \text{if } x > \frac{1}{\log 2} + 1, \\ 0, & \text{if } 0 \le x \le \frac{1}{\log 2} + 1, \end{cases}$$

and the inverse function of $F_{\infty}^{Z}(x)$ is

$$F_{\infty}^{Z\,-1}(\alpha) = \frac{-\log_2(1-\alpha)}{\alpha} + 1 \text{ for all } \alpha \in (0,1).$$

Next, it is easy to obtain that

$$F_{\infty}^{W}(x) = \begin{cases} 1 - \frac{t_3(x)-\frac{1}{2}}{x-\frac{1}{2}}, & \text{if } x > \frac{1}{\log 2} + \frac{1}{2}, \\ 0, & \text{if } 0 \le x \le \frac{1}{\log 2} + \frac{1}{2}, \end{cases}$$

where $t_3(x)$ depends on x, and the inverse function of $F_{\infty}^{W}(x)$ is

$$F_{\infty}^{W-1}(\alpha) = \frac{-\log_2(1-\alpha)}{\alpha} + \frac{1}{2} \text{ for all } \alpha \in (0,1).$$

The proof is completed. □

5.3. The proof of Theorem 3.1

According to the same calculation from (12) to (14), for all $\lambda \ge 0$, $n \in \mathbb{N}^+$ and $x \in [0,\infty)$, we can obtain $F_{n,\lambda}^{Y}(x)$, $F_{n,\lambda}^{Z}(x)$ and $F_{n,\lambda}^{W}(x)$, recursively. We define $F_{0,\lambda}^{Y}(x) = 1$, $F_{0,\lambda}^{Z}(x) = 1$ and $F_{0,\lambda}^{W}(x) = 1$. For $n \ge 1$ and $x \in [0,\infty)$, we have

$$\begin{aligned} F_{n,\lambda}^{Y}(x) &= \mathbb{P}(M_{n,\lambda}(Y) \le x) \\ &= \mathbb{P}(Y_1 \le (1+\lambda)x, \dots, Y_1 + Y_2 + \cdots + Y_n \le (n+\lambda)x) \\ &= F_{n-1}^{Y}(x) - (\log 2)^{n-1}\mathrm{Vol}(K_{n-1}(x,\lambda x))\left(\frac{1}{2}\right)^{(n+\lambda)x} \\ &= 1 - (1+\lambda)\sum_{k=1}^{n} \frac{k(k+\lambda)^{k-2}}{k!}\left(\frac{1}{2}\right)^{(k+\lambda)x}(\log 2^x)^{k-1}, \end{aligned}$$

$$\begin{aligned} F_{n,\lambda}^{Z}(x) &= \mathbb{P}(M_{n,\lambda}(Z) \le x) \\ &= \mathbb{P}\left(\frac{Z_1}{1+\lambda} \le x, \frac{Z_1+Z_2}{2+\lambda} \le x, \dots, \frac{Z_1+Z_2+\cdots+Z_n}{n+\lambda} \le x\right) \\ &= \mathbb{P}(Y_1 \le (x-1) + \lambda x, \dots, Y_1 + Y_2 + \cdots + Y_n \le n(x-1) + \lambda x) \\ &= 1 - \sum_{k=1}^{n} \frac{k(x-1+\lambda x)(k(x-1)+\lambda x)^{k-2}}{k!}\left(\frac{1}{2}\right)^{(k+\lambda)x-k}(\log 2)^{k-1} \end{aligned}$$

and

$$\begin{aligned} F^W_{n,\lambda}(x) &= \mathbb{P}(M_{n,\lambda}(W) \le x) \\ &= \mathbb{P}\Big(\frac{W_1}{1+\lambda} \le x, \frac{W_1+W_2}{2+\lambda} \le x, \dots, \frac{W_1+W_2+\cdots+W_n}{n+\lambda} \le x\Big) \\ &= \mathbb{P}\Big(Y_1 \le \Big(x-\frac{1}{2}\Big)+\lambda x, \dots, Y_1+Y_2+\cdots+Y_n \le n\Big(x-\frac{1}{2}\Big)+\lambda x\Big) \\ &= 1-\sum_{k=1}^{n} \frac{k(x-\frac{1}{2}+\lambda x)(k(x-\frac{1}{2})+\lambda x)^{k-2}}{k!}\Big(\frac{1}{2}\Big)^{(k+\lambda)x-\frac{1}{2}k}(\log 2)^{k-1}. \end{aligned}$$

Then Theorem 3.1 is established. The proof is completed. □

5.4. *The proof of Theorem 3.2*

According to Theorem 3.1, we have $1-F^Y_{n,\lambda}(x) \le \varphi^X_{n,\lambda}(x) \le 1-F^Z_{n,\lambda}(x)$ and $1-F^Y_{\infty,\lambda}(x) \le \varphi^X_{\infty,\lambda}(x) \le 1-F^Z_{\infty,\lambda}(x)$ for $x \in [0,\infty)$. Then, it is easy to obtain that the errors between $\varphi^X_{\infty,\lambda}(x)$ and $\varphi^X_{n,\lambda}(x)$ has the following relationship. That is

$$F^Z_{n,\lambda}(x) - F^Y_{\infty,\lambda}(x) \le \varphi^X_{\infty,\lambda}(x) - \varphi^X_{n,\lambda}(x) \le F^Y_{n,\lambda}(x) - F^Z_{\infty,\lambda}(x). \tag{25}$$

For the convenience of calculation, we rearrange the left and right sides of the above formula (25). They are

$$F^Y_{n,\lambda}(x) - F^Z_{\infty,\lambda}(x) = (F^Y_{n,\lambda}(x) - F^Z_{n,\lambda}(x)) + (F^Z_{n,\lambda}(x) - F^Z_{\infty,\lambda}(x)), \tag{26}$$

and

$$F^Z_{n,\lambda}(x) - F^Y_{\infty,\lambda}(x) = (F^Z_{n,\lambda}(x) - F^Y_{n,\lambda}(x)) + (F^Y_{n,\lambda}(x) - F^Y_{\infty,\lambda}(x)). \tag{27}$$

Firstly, based on (ii) in Lemma 3.1, we get the upper bound of $F^Z_{n,\lambda}(x) - F^Z_{\infty,\lambda}(x)$ in (26). Given $x \ge 3$ and $\lambda \ge 0$, there exists a number $\varepsilon_k > 0$ such that $\varepsilon_k(k(x-1)) = \lambda x$ is established for $k \ge n+1$. Meanwhile, $\lim_{k\to\infty} \varepsilon_k = 0$. Then

$$\begin{aligned} F^Z_{n,\lambda}(x) - F^Z_{\infty,\lambda}(x) &= \frac{x-1+\lambda x}{2^{\lambda x+x-1}} \sum_{k=n+1}^{\infty} \frac{k(kx-k+\lambda x)^{k-2}}{k!}\Big(\frac{\log 2}{2^{x-1}}\Big)^{k-1} \\ &= \frac{x-1+\lambda x}{2^{\lambda x+x-1}} \sum_{k=n+1}^{\infty} \frac{k((1+\varepsilon_k)(x-1)k)^{k-2}}{k!}\Big(\frac{\log 2}{2^{x-1}}\Big)^{k-1} \\ &\le \frac{e^2(x-1+\lambda x)\log 2}{2^{\lambda x+2x-2}\sqrt{2\pi}} \sum_{k=n+1}^{\infty} \frac{1}{k^{\frac{3}{2}}}\Big(\frac{(1+\varepsilon_k)e\log 2^{x-1}}{2^{x-1}}\Big)^{k-2} \\ &= \frac{e^2(x-1+\lambda x)\log 2}{2^{\lambda x+2x-2}\sqrt{2\pi}} \sum_{k=n+1}^{\infty} \frac{d_k^{k-2}}{k^{\frac{3}{2}}} \\ &\le \frac{e^2(x-1+\lambda x)\log 2}{2^{\lambda x+2x-3}\sqrt{2n\pi}} \end{aligned} \tag{28}$$

where $d_k = \frac{(1+\varepsilon_k)e\log 2^{x-1}}{2^{x-1}}$. Note that, for $\frac{\lambda}{n+1} \le \frac{1}{25}$, $0 \le d_k \le 1$ and $\sum_{k=n+1}^{\infty} d_k^{k-2}k^{-\frac{3}{2}} \le \sum_{k=n+1}^{\infty} (\frac{1}{k})^{\frac{3}{2}} \le \frac{2}{\sqrt{n}}$.

On the other hand, according to (i) in Lemma 3.1, we calculate the lower bound of $F^Y_{n,\lambda}(x) - F^Y_{\infty,\lambda}(x)$ which is appeared in (27). That is

$$
\begin{aligned}
F^{Y}_{n,\lambda}(x) - F^{Y}_{\infty,\lambda}(x) &= \frac{(1+\lambda)}{2^{\lambda x+x}} \sum_{k=n+1}^{\infty} \frac{(k+\lambda)^{k-2}}{(k-1)!}\left(\frac{\log 2^x}{2^x}\right)^{k-1} \\
&\geq \frac{(1+\lambda)}{2^{\lambda x+x}} \sum_{k=n}^{\infty} \frac{k^{k-1}}{k!}\left(\frac{\log 2^x}{2^x}\right)^{k} \\
&\geq \frac{(1+\lambda)}{\sqrt{2\pi}(1+\frac{1}{12}+\frac{1}{288})2^{\lambda x+x}} \sum_{k=n}^{\infty} \frac{c^k(x)}{k^{\frac{3}{2}}} \\
&\geq \frac{288(1+\lambda)}{313\sqrt{2\pi}2^{\lambda x+x}} \int_n^{\infty} \frac{c^y(x)}{y^{\frac{3}{2}}} dy \\
&\geq \frac{288(1+\lambda)c^n(x)(n\log\frac{1}{c(x)} - 1.5)}{313\sqrt{2\pi}2^{\lambda x+x} n^{\frac{5}{2}} (\log c(x))^2}
\end{aligned} \tag{29}
$$

where $c(x) = \frac{e\log 2^x}{2^x} \in [0,1)$ for $x \in [3,\infty)$. Combining (28) and (29), Theorem 3.2 is established. The proof is completed. □

Data availability statement All data generated during the current study are available from the corresponding author on reasonable request.

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