Filomat 36:17 (2022), 5895–5912
https://doi.org/10.2298/FIL2217895Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Statistical Convergence of Szász-Mirakjan-Kantorovich-Type Operators and their Bivariate Extension

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Abstract. Our main aim is to investigate the approximation properties of the summation integral type operators in a statistical sense. In this regard, we prove the statistical convergence theorem using well known Korovkin theorem and the degree of approximation is determined. Also using the weight function, the weighted statistical convergence theorem with the help of the Korovkin theorem is obtained. The statistical rate of convergence in the terms of modulus of continuity and function belonging to the Lipschitz class is determined. To support the convergence results of the proposed operators to the function, graphical representations take place and a comparison is shown with Szász-Mirakjan-Kantorovich operators through examples. The last section deals with, a bivariate extension of the proposed operators to determine the approximation of the function of two variables, additionally, the rate of convergence is estimated as well as the convergence of the bivariate operators is shown by graphical representations.

1. Introduction

Approximating the high-order polynomial curves and curved surfaces with the low-order ones plays an important role in data compression, data transmission, data exchange, etc., in geometric modeling tasks. Approximation theory basically deals with the approximation of functions by simpler functions or more easily calculated functions. Broadly it is divided into theoretical and constructive approximation. Weierstrass approximation theorem [1] was first developed in regard to the approximation of function and constructive proof of this theorem was given by S.N. Bernstein by constructing polynomials using probabilistic interpolation, thus these polynomials are said to be Bernstein polynomials. Similarly, computer-aided geometric design (CAGD) is a discipline that deals with computational aspects of geometric objects. And the cause of that it plays an important role in the mathematical development of curves and surfaces such that they become compatible with computers.

2020 *Mathematics Subject Classification*. Primary 41A25; Secondary 41A36
Keywords. Szász-Mirakjan operators, Szász-Mirakjan-Kantorovich operators, Korovkin-type approximation results, statistical convergence, modulus of continuity, weighted modulus of continuity
Received: 11 November 2021; Revised: 04 April 2022; Accepted: 14 April 2022
Communicated by Eberhard Malkowsky
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In recent years, the research on Bézier curve approximation with degree reduction attracts lots of attention. Current algorithms for Bézier Curve degree reduction can be summarized into two categories. The first category is based on the base function conversion. For instance, in reference (Lu, and Wang [2]) proposes to utilize the Chebyshev Polynomials for base function conversion. Sánchez-Reyes [3] uses the S power base to accurately represent the Bernstein base function. Several famous programs are utilized with the help of Bernstein polynomials like font imaging systems such as postscript, Adobe's illustrator, and flash to form Bézier curves [4]. Moving forward in the theory of approximation, one more property has been established, and that is statistical study. The present article deals with the properties of linear positive operators in a statistical sense.

First of all, Fast [5] introduced statistical convergence and further investigated by Steinhaus [6], in that order, Schoenberg [7] reintroduced as well as gave some basics properties and studied the summability theory of the statistical convergence. Nowadays, statistical convergence has become an area, which is broad and also very active, even though it has been introduced over fifty years ago and is being used very frequently in many areas, we refer to some citations as [8–13]. Also, this area is being concerned to investigate the approximation properties of quantum calculus. Some statistical approximation properties have been determined by researchers in their research articles [14–23] and [24–26, 28–31]. In 2003, Duman [32], studied the A-statistical convergence of the linear positive operators for function belonging to the space of all 2π-periodic and continuous functions on the whole real line, where A represents a non-negative regular summability matrix. Since, statistical type study has vast applications in several fields of research like as functional analysis, real analysis, etc. and of course, is being done in the theory of approximation as well.

So, the main objective of this paper is to investigate the statistical convergence properties of the sequence of linear positive operators in the theory of approximation. Here the Korovkin theory is considered to deal with the approximation of function g by the operators $\{\tilde{S}^*_{n,a}(g;x)\}$ [33] for the summation integral type operators, which are as follows.

$$\tilde{S}^*_{n,a}(g;x) = n\sum_{k=0}^{\infty} s^a_n(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}} g(u)\,du, \quad \forall\, x \in [0,\infty), n \in \mathbb{N}, \tag{1}$$

where $s^a_n(x) = \frac{x\log a}{\left(-1+a^{\frac{1}{n}}\right)n}$, $g \in C[0,\infty)$. The authors studied the summation integral type operators (Szász-Mirakjan-Kantorovich type operators) along with their rate of convergence in the sense of local approximation results with the help of modulus of smoothness, second-order modulus of continuity, Peetre's K-functional, and functions belonging to the Lipschitz class. Further, for computing the order of approximation of the operators, we discuss the weighted approximation properties by using the weighted modulus of continuity and prove the theorem. The operators (1) are a generalized version of the Szász-Mirakjan type operators, defined in [34].

Here it is considered some Lemmas those will be useful in the further study of the theorem. Consider the function $e_i = x^i$, $i = 0\cdots 4$, we have

Lemma 1.1. *[33]* For each $x \in [0,\infty)$ and $a > 1$ fixed, we have

1. $\tilde{S}^*_{n,a}(e_0;x) = 1$,
2. $\tilde{S}^*_{n,a}(e_1;x) = \dfrac{1}{2n} + \dfrac{x\log a}{\left(-1+a^{\frac{1}{n}}\right)n}$,
3. $\tilde{S}^*_{n,a}(e_2;x) = \dfrac{1}{3n^2} + \dfrac{2x\log a}{\left(-1+a^{\frac{1}{n}}\right)n^2} + \dfrac{x^2(\log a)^2}{\left(-1+a^{\frac{1}{n}}\right)^2 n^2}$,

4. $\tilde{S}^*_{n,a}(e_3;x) = \dfrac{1}{4n^3} + \dfrac{7}{2}\dfrac{x\log a}{\left(-1+a^{\frac{1}{n}}\right)n^3} + \dfrac{9}{2}\dfrac{x^2(\log a)^2}{\left(-1+a^{\frac{1}{n}}\right)^2 n^3} + \dfrac{x^3(\log a)^3}{\left(-1+a^{\frac{1}{n}}\right)^3 n^3}.$

To find central moments for the defined operators, we have

Lemma 1.2. *[33]* For each $x \geq 0$, we have

1. $\tilde{S}^*_{n,a}(\xi_x(u);x) = -\dfrac{(-1+2nx)}{2n} + \dfrac{x\log a}{n(-1+a^{\frac{1}{n}})},$
2. $\tilde{S}^*_{n,a}(\xi_x^2(u);x) = \dfrac{(1-3nx+3n^2x^2)}{3n^2} - \dfrac{2(-1+a^{\frac{1}{n}})(-1+nx)x\log a}{\left(-1+a^{\frac{1}{n}}\right)^2 n^2} + \dfrac{x^2(\log a)^2}{\left(-1+a^{\frac{1}{n}}\right)^2 n^2},$
3. $\tilde{S}^*_{n,a}(\xi_x^3(u);x) = -\dfrac{(-1+4nx-6n^2x^2+4n^3x^3)}{4n^3} + \dfrac{x(7-12nx+6n^2x^2)\log a}{2\left(-1+a^{\frac{1}{n}}\right)n^3}$
$\qquad - \dfrac{3x^2(-3+2nx)(\log a)^2 + 4x^3(\log a)^3}{2\left(-1+a^{\frac{1}{n}}\right)^2 n^3},$
4. $\tilde{S}^*_{n,a}(\xi_x^4(u);x) = \dfrac{1}{5\left(-1+a^{\frac{1}{n}}\right)^4 n^4}\Bigg(\left(-1+a^{\frac{1}{n}}\right)^4(1-5nx+10n^2x^2-10n^3x^3+5n^4x^4)$
$\qquad -10\left(-1+a^{\frac{1}{n}}\right)^3 x(-3+7nx-6n^2x^2+2n^3x^3)\log a$
$\qquad +15\left(-1+a^{\frac{1}{n}}\right)^2 x^2(5-6nx+2n^2x^2)(\log a)^2$
$\qquad -20\left(-1+a^{\frac{1}{n}}\right)x^3(-2+nx)(\log a)^3 + 5x^4(\log a)^4\Bigg),$

where $\xi_x(t) = (u-x)^i$, $i = 1,2,3\cdots$

2. Korovkin and Weierstrass type statistical theorem

If $\{O_n(g;x)\}$ is a sequence of linear positive operators such that the sequence $\{O_n(1;x)\}$, $\{O_n(t;x)\}$, $\{O_n(t^2;x)\}$ converge uniformly to 1, x, x^2 respectively in the defined interval $[a,b]$, then it implies that the sequence $\{O_n(g;x)\}$ converges to the function g uniformly provided g is bounded and continuous in the interval $[a,b]$.

Before proceeding to statistical convergence, here a brief concept of statistical convergence is considered.

Definition 2.1. *Consider a set $\mathcal{K} \subseteq \mathbb{N}$, such that $K_n = \{k \in \mathcal{K} : k \leq n\}$, where $n \in \mathbb{N}$. Then the natural density $d(\mathcal{K})$ of a set $\mathcal{K}$ is defined as*

$$\lim_{n\to\infty} \frac{1}{n}|K_n|, \tag{2}$$

provided the limit exists. Here $|K_n|$ represents the cardinality of the set K_n.

Definition 2.2. *Let $p \in \mathbb{R}$, a sequence $\{x_k\}$ is said to be convergent statistically to p, if for each $\epsilon > 0$, we have*

$$d(\{k \leq n : |x_k - p| \geq \epsilon\}) = 0, \tag{3}$$

i.e.,

$$\lim_n \frac{1}{n}|\{k \leq n : |x_k - p| \geq \epsilon\}| = 0. \tag{4}$$

By using the properties of statistical convergence, here we shall prove the Korovkin theorem and the Weierstrass type approximation theorem.

In [35], it is proved that the classical Korovkin theorem, according to that theorem, let $A_n(g;x)$ be linear positive operators defined on the set of all continuous and bounded function $C_B[a,b]$ to $C[a,b]$, be set of all continuous function defined on $[a,b]$ for which, the conditions

$$\lim_n \|A_n(e_i;x) - e_i\|_{C[a,b]} = 0, \qquad \text{where } e_i = x^i, \ i = 0,\ 1,\ 2,$$

satisfy, then for any function $g \in C[a,b]$,

$$\lim_n \|S_n(g;x) - g(x)\|_{C[a,b]} = 0, \quad \text{as } n \to \infty.$$

Theorem 2.3. *[36] Let P_n be a sequence of positive linear operators defined on $C_B[a,b]$ to $C[a,b]$ and if it satisfies the conditions*

$$st - \lim_n \|P_n(e_i;x) - e_i\|_{C[a,b]} = 0, \qquad \textit{where } \ i = 0,1,2,$$

then for each function $g \in C_{\mathcal{B}}[a,b]$, we have

$$st - \lim_n \|P_n(g;x) - g(x)\|_{C[a,b]} = 0.$$

With these correlations, we have

Theorem 2.4. *Let $\{\tilde{S}^*_{n,a}\}$ be the sequence of linear positive operators defined by (1). Then for every $g \in C_B[0,l]$, $l > 0$, we have*

$$st - \lim_n \|\tilde{S}^*_{n,a}(g;x) - g(x)\| = 0,$$

where $C_B[0,l]$ is the space of all continuous and bounded function defined on $[0,l]$ with the norm

$$\|g\| = \sup_{0 \le x \le l} |g(x)|.$$

Proof. By (1) of Lemma 1.1, we easily get

$$st - \lim_n \|\tilde{S}^*_{n,a}(e_0;x) - e_0\| = 0 \tag{5}$$

Now by (2) of Lemma 1.1, we have

$$\begin{aligned} |S_n(e_1;x) - x\| &= \|\frac{1}{2n} + \frac{x}{\left(-1 + a^{\frac{1}{n}}\right)n} \log a - x\| \\ &\le \left|\frac{1}{2n}\right| + \left|\left\{\frac{1}{\left(-1 + a^{\frac{1}{n}}\right)n} \log a - 1\right\} l\right|, \end{aligned}$$

define the sets, for any $\epsilon > 0$ as:

$$O = \{n : \|\tilde{S}^*_{n,a}(e_1;x) - x\| \ge \epsilon\}$$

and

$$\begin{aligned} O' &= \left\{n : \frac{1}{2n} \ge \frac{\epsilon}{2}\right\} \\ O'' &= \left\{n : \left(\frac{\log a}{\left(-1 + a^{\frac{1}{n}}\right)n} - 1\right) l \ge \frac{\epsilon}{2}\right\}, \end{aligned}$$

here it can be observed that $O \subseteq O' \cup O''$ and it can be expressed as

$$d\left\{n \leq k : \|\tilde{S}^*_{n,a}(e_1;x) - x\| \geq \epsilon\right\} \quad \leq \quad d\left\{n \leq k : \frac{1}{2n} \geq \frac{\epsilon}{2}\right\} + d\left(n \leq k : \left(\frac{\log a}{\left(-1 + a^{\frac{1}{n}}\right)n} - 1\right) l \geq \frac{\epsilon}{2}\right). \tag{6}$$

But since

$$st - \lim_n \left(\frac{1}{2n}\right) = 0 \qquad \text{and} \;\; st - \lim_n \left(\frac{\log a}{\left(-1 + a^{\frac{1}{n}}\right)n} - 1\right) = 0,$$

hence, by Inequality 6, it follows

$$st - \lim_n \|S_n(e_1;x) - x\| = 0. \tag{7}$$

Similarly,

$$\begin{aligned}
\|\tilde{S}^*_{n,a}(e_2;x) - e_2\| &= \|\frac{1}{3n^2} + \frac{2x\log a}{\left(-1 + a^{\frac{1}{n}}\right)n^2} + \frac{x^2(\log a)^2}{\left(-1 + a^{\frac{1}{n}}\right)^2 n^2} - x^2\| \\
&\leq \left|\frac{1}{3n^2}\right| + \left|\frac{2\log a}{\left(-1 + a^{\frac{1}{n}}\right)n^2}\right| l + \left\|\left(\frac{(\log a)^2}{\left(-1 + a^{\frac{1}{n}}\right)^2 n^2} - 1\right)\right\| l^2 \\
&\leq m^2\left(\frac{1}{3n^2} + \frac{2\log a}{\left(-1 + a^{\frac{1}{n}}\right)n^2} + \Big(\frac{(\log a)^2}{\left(-1 + a^{\frac{1}{n}}\right)^2 n^2} - 1\Big)\right),
\end{aligned}$$

where $m^2 = \max\{1,\ l,\ l^2\}$, i.e.,

$$\|\tilde{S}^*_{n,a}(e_2;x) - e_2\| \leq m^2\left\{\frac{1}{3n^2} + \frac{2\log a}{\left(-1 + a^{\frac{1}{n}}\right)n^2} + \left\{\frac{1}{\left(-1 + a^{\frac{1}{n}}\right)^2 n^2}(\log a)^2 - 1\right\}\right\}. \tag{8}$$

Again by defining the following sets for any $\epsilon > 0$, one can get

$$P = \{n : \|\tilde{S}^*_{n,a}(e_2;x) - e_2\| \geq \epsilon\} \tag{9}$$

$$P_1 = \{n : \frac{1}{3n^2} \geq \frac{\epsilon}{3m^2}\}, \tag{10}$$

$$P_2 = \{n : \frac{2\log a}{\left(-1 + a^{\frac{1}{n}}\right)n^2} \geq \frac{\epsilon}{3m^2}\} \tag{11}$$

$$P_3 = \left\{n : \left\{\left(\frac{\log a}{\left(-1 + a^{\frac{1}{n}}\right)n}\right)^2 - 1\right\} \geq \frac{\epsilon}{3m^2}\right\}, \tag{12}$$

where $P \subseteq P_1 \cup P_2 \cup P_3$, and it gives

$$\begin{aligned}
d\{n \leq k : \|\tilde{S}^*_{n,a}(e_2;x) - e_2\| \geq \epsilon\} \leq d\left\{n \leq k : \frac{1}{3n^2} \geq \frac{\epsilon}{3m^2}\right\} + d\left\{n \leq k : \frac{2\log a}{\left(-1 + a^{\frac{1}{n}}\right)n^2} \geq \frac{\epsilon}{3m^2}\right\} \\
+ d\left\{n \leq k : \left(\left(\frac{\log a}{\left(-1 + a^{\frac{1}{n}}\right)n}\right)^2 - 1\right) \geq \frac{\epsilon}{3m^2}\right\}.
\end{aligned} \tag{13}$$

Hence

$$st - \lim \alpha_n = 0 = st - \lim \beta_n = st - \lim \gamma_n, \tag{14}$$

where

$$\alpha_n = \frac{1}{3n^2}, \ \ \beta_n = \frac{2\log a}{\left(-1 + a^{\frac{1}{n}}\right)n^2}, \ \ \gamma_n = \left(\left(\frac{\log a}{\left(-1 + a^{\frac{1}{n}}\right)n}\right)^2 - 1\right).$$

So by 13 and 14, we have

$$st - \lim \|\tilde{S}^*_{n,a}(e_2; x) - e_2\| = 0 \tag{15}$$

Hence proof is completed. □

Now, there is an example that satisfies Theorem 2.4, but not the classical Korovkin theorem.

Example 2.5. *Consider a sequence of linear positive operators $T_n(g; x)$ which are defined on $C_B[0, l]$ by $T_n(g; x) = (1 + u_n)\tilde{S}^*_{n,a}$, where $\tilde{S}^*_{n,a}$ be the sequence positive linear operators and u_n is unbounded statistically convergent sequence.*

Since $\tilde{S}^*_{n,a}$ is statistically convergent and also u_n is statistically convergent but not convergent so one can observe that the sequence T_n satisfies the Theorem 2.4, but not the classical Korovkin theorem.

Definition 2.6. *Let ξ_k be a sequence that converges statistically to ξ, having degree $\beta \in (0, 1)$, if for each $\epsilon > 0$, we have*

$$\lim_n \frac{\{k \leq n : |\xi_k - \xi| \geq \epsilon\}}{n^{1-\beta}} = 0$$

In this case, we can write

$$\xi_k - \xi = st - o(k^{-\beta}), \quad k \to \infty.$$

Theorem 2.7. *Let $\{\tilde{S}^*_{n,a}\}$ be a sequence defined by (1) that satisfies the conditions*

$$st - \lim_{n\to\infty} \|\tilde{S}^*_{n,a}(e_0; x) - e_0\| = st - o(n^{-\zeta_1}), \tag{16}$$

$$st - \lim_{n\to\infty} \|\tilde{S}^*_{n,a}(e_1; x) - e_1\| = st - o(n^{-\zeta_2}), \tag{17}$$

$$st - \lim_{n\to\infty} \|\tilde{S}^*_{n,a}(e_2; x) - e_2\| = st - o(n^{-\zeta_3}), \tag{18}$$

as $n \to \infty$. Then for each $g \in C_B[0, l]$, we have

$$st - \lim_{n\to\infty} \tilde{S}^*_{n,a}(g; x) - g(x)\| = st - o(n^{-\zeta}), \quad \text{as } n \to \infty,$$

where $\zeta = \min\{\zeta_1, \zeta_2, \zeta_3\}$.

Proof. One can write the inequality (13) of Theorem 2.4 as:

$$\frac{|\{n \leq p : \|\tilde{S}^*_{n,a}(g; x) - g(x)\| \geq \epsilon\}|}{p^{1-\zeta}} \leq \frac{\left|\left\{n \leq p : \frac{1}{3n^2} \geq \frac{\epsilon}{3m^2}\right\}\right|}{p^{1-\zeta_1}}\frac{p^{1-\zeta_1}}{p^{1-\zeta}} + \frac{\left|\left\{n \leq p : \frac{2\log a}{\left(-1+a^{\frac{1}{n}}\right)n^2} \geq \frac{\epsilon}{3m^2}\right\}\right|}{p^{1-\zeta_2}}\frac{p^{1-\zeta_2}}{p^{1-\zeta}}$$
$$+ \frac{\left|\left\{n \leq p : \left(\left(\frac{\log a}{\left(-1+a^{\frac{1}{n}}\right)n}\right)^2 - 1\right) \geq \frac{e}{3m^2}\right\}\right|}{p^{1-\zeta_3}}\frac{p^{1-\zeta_3}}{p^{1-\zeta}}.$$

By letting $\zeta = \min\{\zeta_1, \zeta_2, \zeta_3\}$ and as $n \to \infty$, the desired result can be achieved. □

3. Weighted Statistical Convergence

Next, we introduce the convergence properties of the proposed operators (1) using Korovkin type theorem , recall from [37, 38], here the weight function is $w(x) = 1 + \gamma^2(x)$, where $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is an unbounded strictly increasing continuous function for which, there exists a constant $M > 0$ and $\alpha \in (0, 1]$ such that:

$$|x - y| \leq M|\gamma(x) - \gamma(y)|, \quad \forall\, x, y \geq 0.$$

By letting the weighted function $w(x) = 1 + x^2$, consider some class of functions spaces $B_w[0, \infty)$ be the space defined by:

$$\begin{aligned}
B_w[0,\infty) &= \left\{ g : [0,\infty) \to \mathbb{R} \mid |g(x)| \leq M_g w(x) \text{ with } \|g\|_w = \sup_{x\geq 0} \frac{g(x)}{w(x)} \right\} \\
C_w[0,\infty) &= \{ g \in B_w[0,\infty),\ g \text{ is continuous}\}, \\
C_w^k[0,\infty) &= \left\{ g \in C_w[0,\infty),\ \lim_{x\to\infty} \frac{g(x)}{w(x)} = k_g < +\infty \right\},
\end{aligned}$$

M_g and k_g both are constants and depend on g. We can move towards the main theorem by considering the functions from defined spaces.

Theorem 3.1. *Let $\tilde{S}^*_{n,a}g$ be a linear positive operators defined by (1). Then for each $g \in C_w^k[0, \infty)$, we have*

$$st - \lim_{n\to\infty} \|\tilde{S}^*_{n,a}(g; x) - g(x)\|_w = 0.$$

Proof. Using the Lemma 1.1, we have $\tilde{S}^*_{n,a}(e_0; x) = 1$, and then it is obvious that

$$\|\tilde{S}^*_{n,a}(e_0; x) - 1\|_w = 0.$$

Now we have

$$\begin{aligned}
\|\tilde{S}^*_{n,a}(e_1; x) - e_1)\|_w &= \sup_{x\geq 0} \left(\frac{1}{2n} + \left\{ \frac{1}{\left(-1 + a^{\frac{1}{n}}\right)n} \log a - 1 \right\} x \right) \frac{1}{1+x^2} \\
&= \sup_{x\geq 0} \left(\frac{1}{2n}\frac{1}{1+x^2} + \frac{x\log a}{\left(-1 + a^{\frac{1}{n}}\right)n}\frac{1}{1+x^2} - \frac{x}{1+x^2} \right) \\
&= \sup_{x\geq 0} \left(\frac{1}{2n}\frac{1}{1+x^2} + \left(\frac{\log a}{\left(-1 + a^{\frac{1}{n}}\right)n} - 1 \right) \frac{x}{1+x^2} \right) \\
&\leq \left(\frac{1}{2n} + \left(\frac{\log a}{\left(-1 + a^{\frac{1}{n}}\right)n} - 1 \right) \right).
\end{aligned}$$

Now for any $\epsilon > 0$, on defining the following sets:

$$\begin{aligned}
P &= \left\{ n : \|\tilde{S}^*_{n,a}(e_1; x) - e_1)\|_w \geq \epsilon \right\} \\
P' &= \left\{ n : \frac{1}{2n} \geq \frac{\epsilon}{2} \right\} \\
P'' &= \left\{ n : \left(\frac{\log a}{\left(-1 + a^{\frac{1}{n}}\right)n} - 1 \right) \geq \frac{\epsilon}{2} \right\},
\end{aligned}$$

where $P \subseteq P' \cup P''$, it follows

$$d\left\{n \leq m : \|\tilde{S}^*_{n,a}(e_1;x) - e_1)\|_w \geq \epsilon\right\} \leq d\left\{n \leq m : \frac{1}{2n} \geq \frac{\epsilon}{2}\right\} + d\left\{n \leq m : \left(\frac{\log a}{\left(-1 + a^{\frac{1}{n}}\right)n} - 1\right) \geq \frac{\epsilon}{2}\right\}. \quad (19)$$

Right hand side of inequality (19) is statistically convergent, hence

$$st - \lim_{n} \|\tilde{S}^*_{n,a}(e_1;x) - e_1)\|_w = 0. \quad (20)$$

Similarly,

$$\begin{aligned} \|\tilde{S}^*_{n,a}(e_2;x) - e_2\|_w &= \sup_{x \geq 0} \left\{\frac{1}{3n^2} + \frac{2x \log a}{\left\{-1 + a^{\frac{1}{n}}\right\}n^2} + \frac{x^2 (\log a)^2}{\left\{-1 + a^{\frac{1}{n}}\right\}^2 n^2} - x^2\right\} \frac{1}{1+x^2} \\ &\leq \left\{\frac{1}{3n^2} + \frac{2 \log a}{\left(-1 + a^{\frac{1}{n}}\right)n^2} + \left(\frac{(\log a)^2}{\left(-1 + a^{\frac{1}{n}}\right)^2 n^2} - 1\right)\right\}. \end{aligned}$$

Similarly, for any $\epsilon > 0$, again define the following sets

$$\begin{aligned} H &= \left\{n : \|\tilde{S}^*_{n,a}(e_2;x) - e_2\|_w \geq \epsilon\right\} \\ H' &= \left\{n : \frac{1}{3n^2} \geq \frac{\epsilon}{3}\right\} \\ H'' &= \left\{n : \frac{2 \log a}{\left(-1 + a^{\frac{1}{n}}\right)n^2} \geq \frac{\epsilon}{3}\right\} \\ H''' &= \left\{n : \left(\frac{(\log a)^2}{\left(-1 + a^{\frac{1}{n}}\right)^2 n^2} - 1\right) \geq \frac{\epsilon}{3}\right\}, \end{aligned}$$

where $H \subseteq H' \cup H'' \cup H'''$, it follows

$$d\{n \leq m : \|\tilde{S}^*_{n,a}(e_2;x) - e_2\|_w \geq \epsilon\} = 0 \quad (21)$$

$$d\left\{n \leq m : \frac{2 \log a}{\left(-1 + a^{\frac{1}{n}}\right)n^2} \geq \frac{\epsilon}{3}\right\} = 0 \quad (22)$$

$$d\left\{n \leq m : \left(\frac{(\log a)^2}{\left(-1 + a^{\frac{1}{n}}\right)^2 n^2} - 1\right) \geq \frac{\epsilon}{3}\right\} = 0. \quad (23)$$

By relations (21-23), it yields:

$$st - \lim_{n \to \infty} \|\tilde{S}^*_{n,a}(e_2;x) - e_2\|_w = 0. \quad (24)$$

Hence

$$|\tilde{S}^*_{n,a}(g;x) - g(x)\|_w \leq \|\tilde{S}^*_{n,a}(e_0;x) - e_0)\|_w + \|\tilde{S}^*_{n,a}(e_1;x) - e_1)\|_w + \|\tilde{S}^*_{n,a}(e_2;x) - e_2)\|_w, \quad (25)$$

and we get

$$\begin{aligned} st - \lim_{n \to \infty} \|\tilde{S}^*_{n,a}(g;x) - g(x)\|_w &\leq st - \lim_{n \to \infty} \|\tilde{S}^*_{n,a}(e_0;x) - e_0)\|_w + st - \lim_{n \to \infty} \|\tilde{S}^*_{n,a}(e_1;x) - e_1)\|_w \quad (26) \\ &+ st - \lim_{n \to \infty} \|\tilde{S}^*_{n,a}(e_2;x) - e_2)\|_w, \end{aligned}$$

which implies that

$$st-\lim_{n\to\infty}\|\tilde{S}^*_{n,a}(g;x)-g(x)\|_w=0.$$

Hence proved. □

4. Rate of Statistical Convergence

In this section, we shall introduce the order of approximation of the operators by means of the modulus of continuity and function belonging to the Lipschitz class.

Let $g\in C_B[0,\infty)$, the space of all continuous and bounded functions defined on the interval $[0,\infty)$ and for any $x\geq 0$, the modulus of continuity of g is defined by

$$\omega(g;\delta)=\sup_{|u-x|\leq\delta}|g(u)-g(x)|,\qquad u\in[0,\infty),$$

and for any $\delta>0$ and each $x,\ u\in[0,\infty)$, we have

$$|g(u)-g(x)|\leq\omega(g;\delta)\left(\frac{|u-x|}{\delta}+1\right)\tag{27}$$

Next theorem deals with error estimation using the modulus of continuity:

Theorem 4.1. *Let $g\in C_B[0,\infty)$ be a non-decreasing function. Then we have*

$$|\tilde{S}^*_{n,a}(g;x)-g(x)|\leq 2\omega\left(g;\sqrt{\delta_{n,a}}\right),\qquad x\geq 0,$$

where

$$\delta_{n,a}=\left(\frac{(1-3nx+3n^2x^2)}{3n^2}-\frac{2(-1+a^{\frac{1}{n}})(-1+nx)x\log a}{\left(-1+a^{\frac{1}{n}}\right)^2n^2}+\frac{x^2(\log a)^2}{\left(-1+a^{\frac{1}{n}}\right)^2n^2}\right).$$

Proof. With the linearity and positivity properties of the defined operators (1), it can be expressed as

$$\begin{aligned}|\tilde{S}^*_{n,a}(g;x)-g(x)| &\leq \tilde{S}^*_{n,a}(|g(u)-g(x)|;x)\\ &= n\sum_{k=0}^{\infty}s^a_n(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}}|g(u)-g(x)|\,du.\end{aligned}\tag{28}$$

By using property (27) in above inequality, we can write:

$$\begin{aligned}|\tilde{S}^*_{n,a}(g;x)-g(x)| &\leq n\sum_{k=0}^{\infty}s^a_n(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}}\omega(g;\delta)\left(\frac{|u-x|}{\delta}+1\right)du\\ &= \omega(g;\delta)\Big\{1+\frac{n}{\delta}\sum_{k=0}^{\infty}s^a_n(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}}|u-x|\,du\Big\}\\ &\leq \omega(g;\delta)\Big\{1+\frac{n}{\delta}\Big(\Big(\sum_{k=0}^{\infty}s^a_n(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}}(u-x)^2\,du\Big)^{\frac{1}{2}}\\ &= \omega(g;\delta)\Big\{1+\frac{1}{\delta}\sqrt{\tilde{S}^*_{n,a}(\xi^2_x(u);x)}\Big\}.\end{aligned}$$

Now, we choose $\delta = \delta_{n,a}$, where

$$\delta_{n,a} = \left(\frac{(1-3nx+3n^2x^2)}{3n^2} - \frac{2(-1+a^{\frac{1}{n}})(-1+nx)x\log a}{\left(-1+a^{\frac{1}{n}}\right)^2 n^2} + \frac{x^2(\log a)^2}{\left(-1+a^{\frac{1}{n}}\right)^2 n^2}\right). \quad (29)$$

Hence, the required result can be obtained. □

Remark 4.2. *With the help of (29), one can get*

$$st - \lim_n \delta_{n,a} = 0, \quad (30)$$

and by (27), it can be obtained

$$st - \lim_n \omega(g; \delta_{n,a}) = 0, \quad (31)$$

and hence the pointwise rate of convergence of the operators $\tilde{S}^*_{n,a}(g;x)$ can be determined.

Theorem 4.3. *[33] For $g \in C_B[0,\infty)$ and if $g \in Lip_{\mathcal{M}}(\alpha)$, $\alpha \in (0,1]$ holds, that is the inequality*

$$|g(u) - g(x)| \leq \mathcal{M}|u-x|^{\alpha}, \quad u, x \in [0,\infty), \text{ where } \mathcal{M} \text{ is a positive constant},$$

then for every $x \geq 0$, we have

$$|\tilde{S}^*_{n,a}(g;x) - g(x)| \leq \mathcal{M}\delta_{n,a}^{\frac{\alpha}{2}},$$

where $\delta_{n,a} = \tilde{S}^*_{n,a}((u-x)^2; x)$.

Proof. Since, we have $g \in C_B[0,\infty) \cap \mathrm{Lip}_{\mathcal{M}}(\alpha)$, so

$$\begin{aligned} |\tilde{S}^*_{n,a}(g;x) - g(x)| &\leq \tilde{S}^*_{n,a}(|g(u)-g(x)|; x) \\ &\leq M\tilde{S}^*_{n,a}(|u-x|^{\alpha}; x) = \mathcal{M}\Big(n\sum_{k=0}^{\infty} s_n^a(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} |u-x|^{\alpha}\, du\Big). \end{aligned}$$

Now, by applying Hölder inequality with $\mathsf{p} = \frac{2}{\alpha}$ and $\mathsf{q} = \frac{2}{2-\alpha}$, we have

$$\begin{aligned} |\tilde{S}^*_{n,a}(g;x) - g(x)| &\leq \mathcal{M}\left(n\sum_{k=0}^{\infty} s_n^a(x)\Big\{\int_{\frac{k}{n}}^{\frac{k+1}{n}} (u-x)^2\, du,\Big\}^{\frac{\alpha}{2}}\right) \leq \mathcal{M}(\tilde{S}^*_{n,a}(\xi_x^2(u); x))^{\frac{\alpha}{2}} \\ &= \mathcal{M}\delta_{n,a}^{\frac{\alpha}{2}}. \end{aligned}$$

Hence proved. □

Remark 4.4. *Similarly, by equation (29), we can justify*

$$st - \lim_n \delta_{n,a} = 0, \quad (32)$$

and it can be seen that the rate of statistical convergence of the operators 1 to $g(x)$ is estimated by means of a function belonging to the Lipschitz class.

5. Graphical approach

Based upon the defined operators (1), we will show the convergence rate by some graphical representations.

Example 5.1. *Consider the function* $f(x) = e^{-2x}$ *and choose the values of* $n = 5, 10$ *for which the corresponding operators are* $\tilde{S}^*_{5,a}(f;x), \tilde{S}^*_{10,a}(f;x)$ *respectively. We can observe that the error is minimum for large value of n and it can be seen by set of Figures 1.*

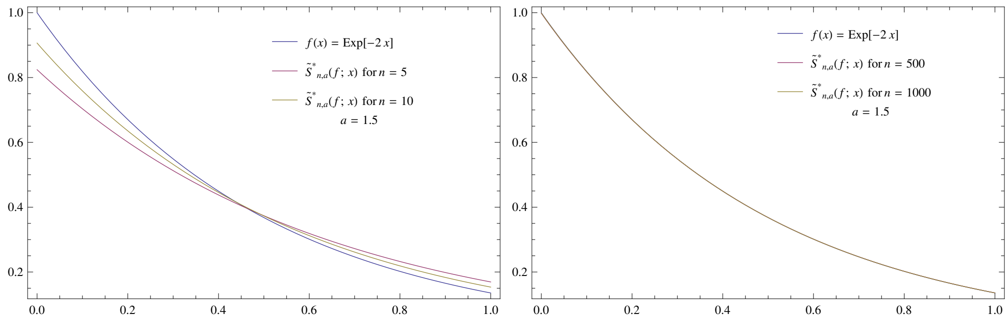

Figure 1: Convergence of operators $\tilde{S}^*_{n,a}(f;x)$ to $f(x)$

In fact, in Figure 1, as the value of n increases, the operator $\tilde{S}^*_{n,a}(f;x)$ approaches towards the function $f(x) = e^{-2x}$ keeping $a = 1.5$ fixed.

Example 5.2. *Consider the function* $f(x) = x$ *and choose the value* $n = 100, 500$*, the convergence can be seen by graphical representation, given by Figure 2.*

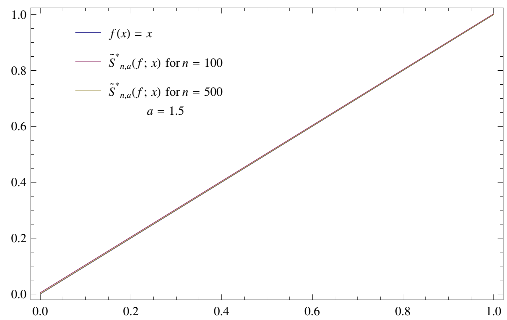

Figure 2: Convergence of operators $\tilde{S}^*_{n,a}(f;x)$ to $f(x)$

Example 5.3. *For the convergence of the proposed operators (1) to the function* $f(x) = \left(x - \frac{1}{2}\right)\left(x - \frac{1}{3}\right)\left(x - \frac{1}{4}\right)$*, choose* $n = 15, 30, 500, 1000$ *and then the errors can be observed by the given Figures 3.*

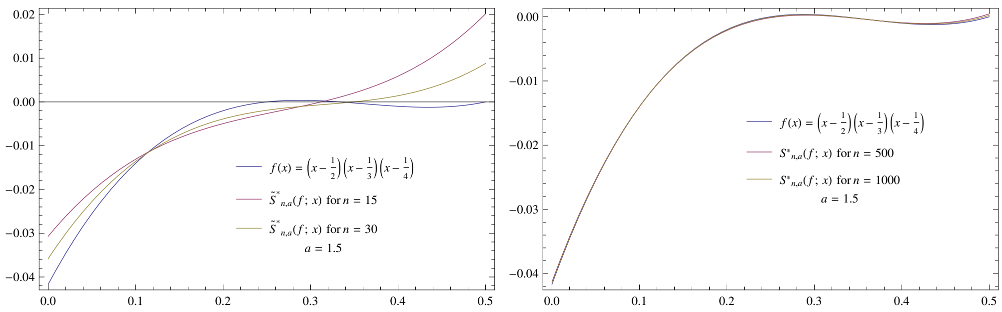

Figure 3: Convergence of operators $\tilde{S}^*_{n,a}(f;x)$ to $f(x)$

Concluding Remark: We have seen the approach of the proposed operators by all the above figures, for large value of n, the approximation is good. Moreover, one can observe by Figure 2, as the value of n is increased, the operators $\tilde{S}^*_{n,a}(f;x)$ converge to the function $f(x) = x$ and in Figure 3, it can be seen that the operator $\tilde{S}^*_{n,a}(t;x)$ converges to the function $f(x) = \left(x-\frac{1}{2}\right)\left(x-\frac{1}{3}\right)\left(x-\frac{1}{4}\right)$ for large value of n.

5.1. *A comparison with Szász-Mirakjan-Kantorovich operators*

In 1983, V. Totik [39] introduced the Kantorovich variant of the Szász-Mirakjan operators in L^p-space for $p > 1$, which is as follows:

$$K_n(g;x) = ne^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \int_{\frac{k}{n}}^{\frac{k+1}{n}} g(u)\, du. \tag{33}$$

Now we shall show a comparison of the proposed operators (1) with the operators (33) by graphical representation.

In Figure 4, one can see that the said operators have a better rate of convergence as compared to the operators (33).

Example 5.4. *For the same degree of approximation of the operators $\tilde{S}^*_{n,a}(f;x)$ and $K_n(f;x)$ to the function $f(x) = x^3$, comparison is shown by Figure 4.*

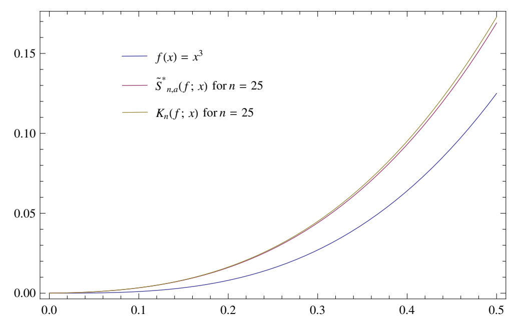

Figure 4: Comparison of operators $\tilde{S}^*_{n,a}(f;x)$ and $K_n(f;x)$

Example 5.5. *Comparison of convergence can be seen for the operators $\tilde{S}^*_{n,a}(f;x)$ and $K_n(f;x)$ to the function in the given Figures 5.*

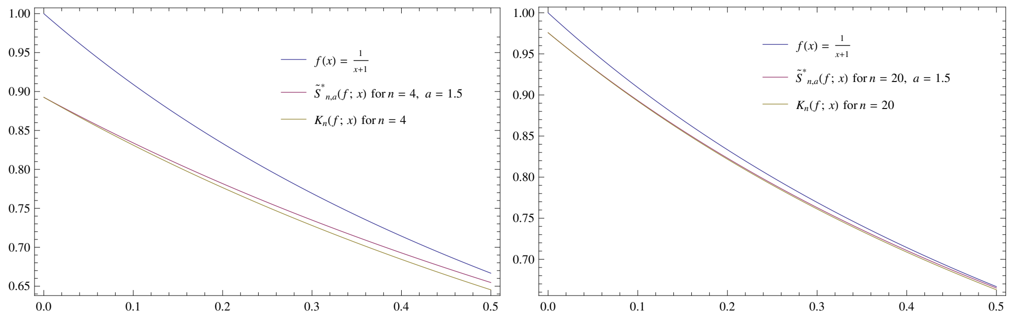

Figure 5: Comparison of operators $\tilde{S}^*_{n,a}(f;x)$ and $K_n(f;x)$

So, from Figure 5, it can be observed that the summation-integral-type operators (1) are approaching more faster than Szász-Mirakjan-Kantorovich operators to the function $f(x) = \frac{1}{1+x}$.

6. An extension in sense of bivariate operators

To discuss the approximation of the function of two variables, we generalize the defined operators $\tilde{S}^*_{n,a}$ as an extension into bivariate operators in the space of integral functions to investigate the rate of convergence with the help of statistical convergence. Let $g : C[0,\infty)\times C[0,\infty) \to C[0,\infty)\times C[0,\infty)$, we define the operators with one parameter as follows:

$$Y^*_{m,m,a}(g;x,y) = m^2 \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} s^a_{m,m}(x,y) \int_{\frac{k_2}{m}}^{\frac{k_2+1}{m}} \int_{\frac{k_1}{m}}^{\frac{k_1+1}{m}} g(u,v)\, du\, dv, \tag{34}$$

where $s^a_{m,m}(x,y) = a^{\left(\frac{-x-y}{-1+a^{\frac{1}{m}}}\right)} \frac{x^{k_1} y^{k_2} (\log a)^{k_1+k_2}}{(-1+a^{\frac{1}{m}})^{k_1+k_2} k_1! k_2!} = s^a_m(x) \times s^a_m(y)$.

Define a function $e_{i,j} = x^i y^j$, for all $x, y \geq 0$, where $i, j \in \mathbb{N} \cup \{0\}$ for the following lemma.

Lemma 6.1. *For all $x,\ y \geq 0$, bivariate operators (34), satisfy the following equalities:*

1. $Y^*_{m,m,a}(e_{00};x,y) = 1$
2. $Y^*_{m,m,a}(e_{11};x,y) = \dfrac{1}{4(-1+a^{\frac{1}{m}})m^2}\left\{\left(-1+a^{\frac{1}{n}}+2x\log a\right)\left(-1+a^{\frac{1}{n}}+2y\log a\right)\right\}$
3. $Y^*_{m,m,a}(e_{22};x,y) = \dfrac{1}{9(-1+a^{\frac{1}{m}})^4 m^4}\left\{\left(\left(-1+a^{\frac{1}{m}}\right)^2+6(-1+a^{\frac{1}{m}})x\log+3x^2(\log)^2\right)\left(\left(-1+a^{\frac{1}{m}}\right)^2+6(-1+a^{\frac{1}{m}})y\log+3y^2(\log)^2\right)\right\}$
4. $Y^*_{m,m,a}(e_{33};x,y) = \dfrac{1}{16\left(-1+a^{\frac{1}{m}}\right)^6 m^6}\left\{\left(\left(-1+a^{\frac{1}{m}}\right)^3+14\left(-1+a^{\frac{1}{m}}\right)^2 x\log a+18\left(-1+a^{\frac{1}{m}}\right)x^2(\log a)^2 +4x^3(\log a)^3\right)\left(\left(-1+a^{\frac{1}{m}}\right)^3+14\left(-1+a^{\frac{1}{m}}\right)^2 y\log a+18\left(-1+a^{\frac{1}{m}}\right)y^2(\log a)^2 +4y^3(\log a)^3\right)\right\}.$

Proof. To prove (1) of Lemma 6.1, we put $g(x,y) = e_{00} = 1$ in bivariate operators (34), we have

$$\begin{aligned}
1.\ Y^*_{m,m,a}(e_{00};x,y) &= m^2 \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} s^a_{m,m}(x,y) \int_{\frac{k_2}{m}}^{\frac{k_2+1}{m}} \int_{\frac{k_1}{m}}^{\frac{k_1+1}{m}} du\, dv \\
&= \left(\sum_{k_1=0}^{\infty} s^a_m(x)\right)\left(\sum_{k_2=0}^{\infty} s^a_m(y)\right) \\
&= 1. \\
2.\ Y^*_{m,m,a}(e_{11};x,y) &= m^2 \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} s^a_{m,m}(x,y) \int_{\frac{k_2}{m}}^{\frac{k_2+1}{m}} \int_{\frac{k_1}{m}}^{\frac{k_1+1}{m}} uv\, du\, dv \\
&= \frac{m^2}{4}\Big(\sum_{k_1=0}^{\infty} s^a_m(x)\left(\frac{1}{m^2}+\frac{2k_1}{m}\right)\Big)\Big(\sum_{k_2=0}^{\infty} s^a_m(y)\left(\frac{1}{m^2}+\frac{2k_2}{m}\right)\Big) \\
&= \frac{1}{4(-1+a^{\frac{1}{m}})m^2}\Big\{\left(-1+a^{\frac{1}{m}}+2x\log a\right)\left(-1+a^{\frac{1}{m}}+2y\log a\right)\Big\}.
\end{aligned}$$

Similarly, it can be proved the other equalities. □

Lemma 6.2. *For all $x, y \geq 0$ and $m \in \mathbb{N}$, we have*

1. $Y^*_{m,m,a}((u-x);x,y) = -\dfrac{(-1+2mx)}{2m} + \dfrac{x\log a}{m(-1+a^{\frac{1}{m}})}$,
2. $Y^*_{m,m,a}((v-y);x,y) = -\dfrac{(-1+2ny)}{2n} + \dfrac{y\log a}{n(-1+a^{\frac{1}{n}})}$,
3. $Y^*_{m,m,a}((u-x)^2;x,y) = \dfrac{(1-3mx+3m^2x^2)}{3m^2} - \dfrac{2(-1+a^{\frac{1}{m}})(-1+mx)x\log a}{\left(-1+a^{\frac{1}{m}}\right)^2 m^2} + \dfrac{x^2(\log a)^2}{\left(-1+a^{\frac{1}{m}}\right)^2 m^2}$,
4. $Y^*_{m,m,a}((v-y)^2;x,y) = \dfrac{(1-3ny+3n^2y^2)}{3n^2} - \dfrac{2(-1+a^{\frac{1}{n}})(-1+ny)y\log a}{\left(-1+a^{\frac{1}{n}}\right)^2 n^2} + \dfrac{y^2(\log a)^2}{\left(-1+a^{\frac{1}{n}}\right)^2 n^2}$.

Proof. One can easily prove, all equalities with the help of properties, which are proved in [33]. So we omit the proof. □

7. Rate of convergence of bivariate operators

In this section, we find rate of convergence of the bivariate operators (34), for function of two variables. Now, we define the supremum norm, by letting $X = [0,\infty)\times[0,\infty)$, we have

$$\|g\| = \sup_{x,y\in X}|g(x,y)|, \quad g \in C_B(X).$$

Consider the modulus of continuity $\omega(g;\delta_1,\delta_2)$ for the bivariate operators (34), where $\delta_1,\delta_2 > 0$, and is defined by:

$$\omega(g;\delta_1,\delta_2) = \{\sup|g(u,v)-g(x,y)| : (u,v),\ (x,y)\in X,\ \text{and } |u-x|\leq\delta_1,\ |v-y|\leq\delta_2\}. \tag{35}$$

Lemma 7.1. *If $g \in C_B(X)$, then for $\delta_1,\ \delta_2 > 0$, we have the following properties of modulus of continuity:*
1. For given function g, $\omega(g;\delta_1,\delta_2) \to 0$ as $\delta_1,\delta_2 \to 0$.
2. $|g(u,v)-g(x,y)| \leq \omega(g;\delta_1,\delta_2)\left(1+\frac{|u-x|}{\delta_1}\right)\left(1+\frac{|v-y|}{\delta_2}\right)$.
Fore more details, see [40].

Theorem 7.2. *If $g \in C_B(X)$ and $x, y \in [0,\infty)$, then we have*

$$|Y^*_{m,m,a}(g;x,y)-g(x,y)| \leq 4\omega(g;\sqrt{\delta_{m,a}},\sqrt{\delta'_{m,a}}), \tag{36}$$

where

$$\delta_{m,a} = \left(\frac{(1-3mx+3m^2x^2)}{3m^2} - \frac{2(-1+a^{\frac{1}{m}})(-1+mx)x\log a}{\left(-1+a^{\frac{1}{m}}\right)^2 m^2} + \frac{x^2(\log a)^2}{\left(-1+a^{\frac{1}{m}}\right)^2 m^2}\right), \tag{37}$$

$$\delta'_{m,a} = \left(\frac{(1-3my+3m^2y^2)}{3m^2} - \frac{2(-1+a^{\frac{1}{m}})(-1+my)y\log a}{\left(-1+a^{\frac{1}{m}}\right)^2 m^2} + \frac{y^2(\log a)^2}{\left(-1+a^{\frac{1}{m}}\right)^2 m^2}\right). \tag{38}$$

Proof. By using the linearity and the positivity of the defined operators $Y^*_{m,m,a}$ (34) and applying on (2) of the Lemma 7.1, then for any $\delta_1, \delta_2 > 0$, we have

$$\begin{aligned}
&|Y^*_{m,m,a}(g;x,y)-g(x,y)| \\
&\qquad \leq Y^*_{m,m,a}(|g(t,s)-g(x,y)|;x,y) \\
&\qquad \leq \omega(g;\delta_1,\delta_2)\left(1+\frac{1}{\delta_1}Y^*_{m,m,a}(|u-x|;x,y)\right)\times\left(1+\frac{1}{\delta_2}Y^*_{m,m,a}(|v-y|;x,y)\right) \\
&\qquad \leq \omega(g;\delta_1,\delta_2)\left(1+\frac{1}{\delta_1}\left(Y^*_{m,m,a}((u-x)^2;x,y)\right)^{\frac{1}{2}}\right)\times\left(1+\frac{1}{\delta_2}\left(Y^*_{m,m,a}((v-y)^2;x,y)\right)^{\frac{1}{2}}\right)
\end{aligned}$$

(using the Cauchy-Schwartz inequality).

Next one step will complete the proof. □

At last, we shall see the rate of convergence of the bivariate operators (34) in the sense of functions belonging to the Lipschitz class $\mathrm{Lip}_{\mathcal{M}}(\alpha_1,\alpha_2)$, where $\alpha_1,\alpha_2 \in (0,1]$ and $\mathcal{M} \geq 0$ is any constant, and Lipschitz class is defined by:

$$|g(u,v)-g(x,y)| \leq \mathcal{M}|u-x|^{\alpha_1}|v-y|^{\alpha_2}, \quad \forall\, x,y,u,v \in [0,\infty). \tag{39}$$

Our next approach is to prove the theorem for finding the rate of convergence, when the function is belonging to the Lipschitz class.

Theorem 7.3. *If $g \in Lip_{\mathcal{M}}(\alpha_1,\alpha_2)$, then for each $g \in C_B(X)$, we have*

$$|Y^*_{m,m,a}(g;x,y)-g(x,y)| \leq \mathcal{M}\delta_{m,a}^{\frac{\alpha_1}{2}}\delta'^{\frac{\alpha_2}{2}}_{m,a},$$

where $\delta_{m,a}$ and $\delta'_{m,a}$ are defined by (37) and (38) respectively.

Proof. Since defined bivariate operators $Y^*_{m,m,a}(g;x,y)$ are linear positive and also $g \in \mathrm{Lip}_{\mathcal{M}}(\alpha_1,\alpha_2)$, where $\alpha_1,\ \alpha_2 \in (0,1]$, then we have

$$\begin{aligned}
|Y^*_{m,m,a}(g;x,y)-g(x,y)| &\leq Y^*_{m,m,a}(|g(t,s)-g(x,y)|;x,y)| \\
&\leq Y^*_{m,m,a}(\mathcal{M}|u-x|^{\alpha_1}|v-y|^{\alpha_2};x,y)| \\
&= \mathcal{M}Y^*_{m,m,a}(|u-x|^{\alpha_1};x,y)|\times Y^*_{m,m,a}(|v-y|^{\alpha_2};x,y)|
\end{aligned}$$

Applying the Hölder inequality with $p' = \frac{2}{\alpha_1}$, $q' = \frac{2}{2-\alpha_1}$ and $p'' = \frac{2}{\alpha_2}$, $q'' = \frac{2}{2-\alpha_2}$, we have

$$|Y^*_{m,m,a}(g;x,y) - g(x,y)| \leq \mathcal{M}(Y^*_{m,m,a}(u-x)^2;x,y))^{\frac{\alpha_1}{2}} \times (Y^*_{m,m,a}(v-y)^2;x,y))^{\frac{\alpha_2}{2}}$$
$$= \mathcal{M}\delta_{m,a}^{\frac{\alpha_1}{2}}\delta'^{\frac{\alpha_2}{2}}_{m,a}.$$

Thus, the proof is completed. □

7.1. Graphical approach of bivariate operators

Now we shall see that, the convergence of the bivaiate operators (34) to the function $g(x,y)$ by graphical representation.

Example 7.4. *Let $g \in C(X)$ and choose $m = 5$ 10, 20, $a = 3$ (fixed), the convergence of $Y^*_{m,m,a}(g;x,y)$ to the function $g(x,y)$ (blue) takes place and is illustrated in Figure 6. For different values of m, the corresponding operators $Y^*_{5,5,a}(g;x,y)$, $Y^*_{10,10,a}(g;x,y)$ and $Y^*_{20,20,a}(g;x,y)$ represent red, green and magenta colors respectively.*

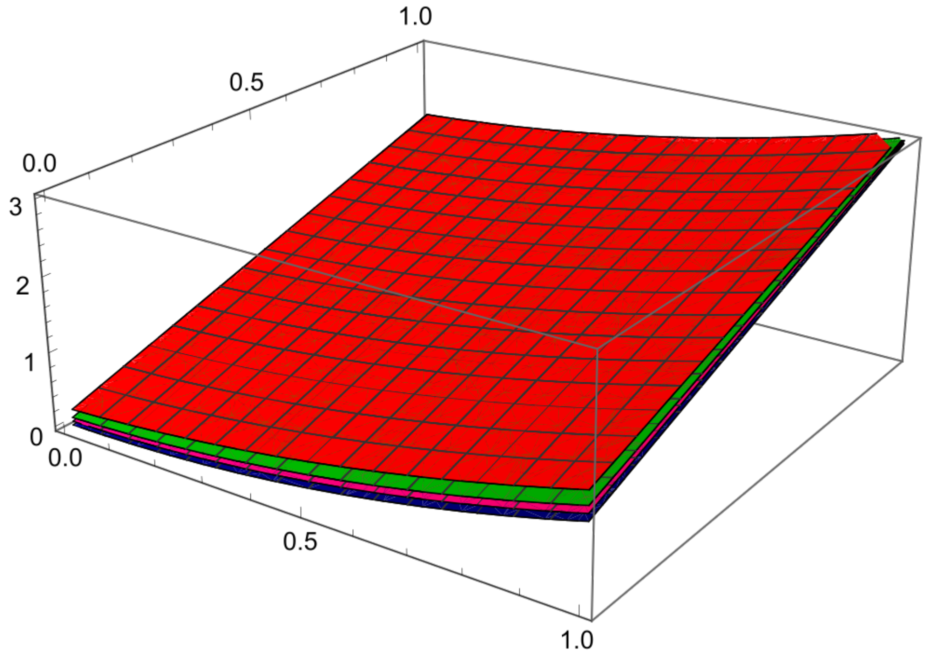

Figure 6: Convergence of operators $Y^*_{m,m,a}(g;x)$ to $g(x,y)$

Again for the same function $g(x,y)$, but for large value of $m = 100, 500$, the corresponding operators are $Y^*_{100,100,a}(g;x,y)$ (red) and $Y^*_{500,500,a}(g;x,y)$ (green), which almost overlap to the function $g(x,y)$ (blue), and that is illustrated in Figure 7.

Concluding remark: The convergence of the bivariate operators $Y^*_{m,m,a}(g;x)$ to the function $g(x,y)$ is taking place as if we increase the value of m, i.e., for the large value of m, the bivariate operators converge to the function.

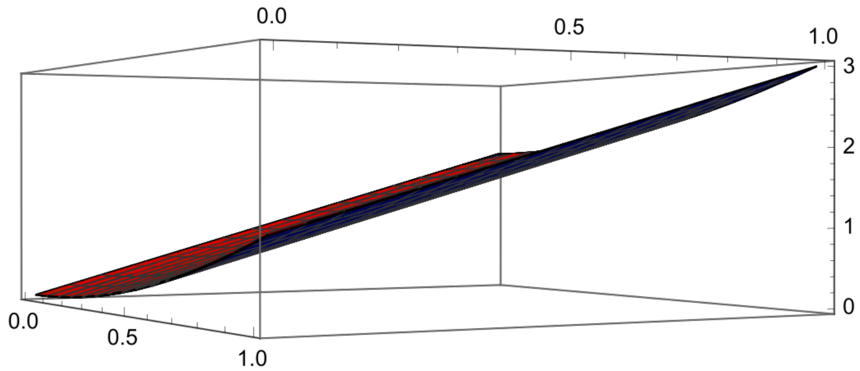

Figure 7: Convergence of operators $Y^*_{m,m,a}(g;x)$ to $g(x,y)$

Conclusion: Convergence of the proposed operators 1 via statistical sense and order of approximation have been determined, moreover; weighted statistical convergence properties and the rate of statistical convergence have been investigated in some sense of approximation results with the help of modulus of continuity. To support the approximation results, the graphical representations took place and along with these, a comparison has been shown for the proposed operators (1). An extension is given to determine the rate of convergence in bivariate sense as well as some graphical analysis has been given.

Conflicts of interest/Competing interests: The article is free from conflict of interest.

Availability of data and material: Not Applicable.

Code availability (software application or custom code): Not Applicable.

Authors' contributions: All the authors have equal contribution in the article.

Ethics approval: Not Applicable.

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