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Attractivity and Ulam-Hyers Stability Results for Fractional Delay Differential Equations

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Abstract. In this paper, we look into some results for the global attractivity and Ulam stability of solutions for fractional delay differential equations via Hilfer-Hadamard fractional derivative. The results are obtained by using Krasnoselskii's fixed point theorem and Banach contraction principle.

1. Introduction

The topic of fractional differential equations (FDEs) has increased significance and attractiveness by reason of its applications in extensive fields of engineering and science; see the monographs of of Abbas et al. [2], Kilbas et al. [17], Miller and Ross [21]. FDEs involving the Riemann-Liouville fractional derivative or the Caputo fractional derivative have been focused more and more attention [5, 6, 17, 19, 31, 33, 34].

Hilfer [11] suggested a general operator for fractional derivative, called "Hilfer fractional derivative", which merges Caputo and Riemann-Liouville fractional derivatives. The Hilfer fractional derivative is performed, for example, in the hypothetical simulation of dielectric relaxation in glass forming materials. In [1], S. Abbas et al. proved some results concerning the existence of solutions and weak solutions for some classes of Hadamard and Hilfer fractional differential equations. Recently, the authors studied some existence and Ulam-Hyers-Rassias stability results for a class of functional differential equations involving the Hilfer-Hadamard fractional derivative in [3] and the existence and Ulam stability for Hilfer type FDEs were investigated in [4]. In [29], D. Vivek et al. analyzed existence and Ulam stability results for pantograph equation via Hilfer fractional derivative. For detail study on Hilfer-Hadamard fractional derivative, we refer to [16, 24]. Generally, some works were studied on the existence and uniqueness of solutions for FDEs; see [2–4, 8, 17, 21, 31] and references therein. On the other hand, the stability of fractional order model remains an open problem [27] .The most important cause is that, even if Lyapunov's direct method as well Lyapunov's second method offers a method to analyze the stability of integer-order differential equations, unfortunately it is not acceptable to analyze the stability of fractional-order ones because of the lack of geometry analysis of the fractional derivatives. In integer-order differential equations and in the

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case where it is not easy to make an appropriate Lyapunov function, fixed point theorems are frequently considered in stability. By making an appropriate compact set, the investigate to the attractivity of FDEs is competently changed into the argument about the existence of a fixed point for corresponding FDEs. In [10], Fulai Chen et al. presented some results for the global attractivity of solutions for FDEs involving Riemann-Liouville fractional calculus. Recently, the question of attractivity of solutions for FDEs in abstract space was initiated in [32].

The stability of functional equations was firstly raised by Ulam in 1940 in a converse given at Wisconsin University. The problem posed by Ulam was the following: "Under what conditions does there exist an additive mapping near an approximately additive mapping?" (for more details see [28]). The first respond to the issue of Ulam was given by Hyers in 1941 in the case of Banach spaces in [13]. In 1978, Rassias [25] provided a remarkable generalization of the Ulam-Hyers stability (U-H stability) of mappings in view of variables. The stability properties of all kinds of equations have focused the attention of several mathematicians. Especially, the U-H stability and Ulam-Hyers-Rassias stability(U-H-Rassias stability) have been taken up by some mathematicians and the study of this area has developed to be one of the essential topicss in the mathematical analysis area. For more facts on the recently advanced on the U-H stability and U-H-Rassias stability of differential equations, one can see the monographs of Hyers [12] and Jung [14] and the research papers of Jung [15], Miura et al. [22, 23], Rus [26] and Lungu and Popa [20].

In the present paper, we will attempt to derive some sufficient conditions on existence and globally attractivity results for fractional delay differential equations (FDDEs) via Hilfer-Hadamard fractional derivative. Then, we use the concept of U-H stability which suitable to describe the characteristic of FDDEs.

2. Prerequisites

Let us remember the following space and results. For more details, see[24, 30].

Let $+\infty \geq T > a > -\infty$ and $C[a,T]$ be the Banach space of all continuous functions from $[a,T]$ into $\mathbb{R}$ with the norm $\|x\|_C = \max\{|x(t)| : t \in [a,T]\}$. For $0 \leq \gamma < 1$, we denote the space $C_{\gamma,\ln}[a,T]$ as

$$C_{\gamma,\ln}[a,T] := \left\{ f(t) : (a,T] \to \mathbb{R} | \left(\ln \frac{t}{a}\right)^{\gamma} f(t) \in C[a,T] \right\},$$

where $C_{\gamma,\ln}[a,T]$ is the weighted space of all continuous functions f on the finite interval $[a,T]$.

Obviously, $C_{\gamma,\ln}[a,T]$ is the Banach space with the norm

$$\|f\|_{C_{\gamma,\ln}} = \left\| \left(\ln \frac{t}{a}\right)^{\gamma} f(t) \right\|_C.$$

Meanwhile, $C^n_{\gamma,\ln}[a,T] := \left\{ f \in C^{n-1}(a,T] : f^{(n)} \in C_{\gamma,\ln}[a,T] \right\}$ is the Banach space with the norm

$$\|f\|^n_{C_{\gamma,\ln}} = \sum_{i=0}^{n-1} \left\|f^k\right\|_C + \left\|f^{(n)}\right\|_{C_{\gamma,\ln}}, \quad n \in \mathbb{N}.$$

Moreover, $C^0_{\gamma,\ln}[a,T] := C_{\gamma,\ln}[a,T]$.

In order to solve our problem, the following spaces are presented.

$$C^{\alpha,\beta}_{1-\gamma,\ln} = \left\{ f \in C_{1-\gamma,\ln}[a,T],\ {}_H D^{\alpha,\beta}_{a^+} f \in C_{1-\gamma,\ln}[a,T] \right\}$$

and

$$C^{\gamma}_{1-\gamma,\ln} = \left\{ f \in C_{1-\gamma,\ln}[a,T],\ {}_H D^{\gamma}_{a^+} f \in C_{1-\gamma,\ln}[a,T] \right\}.$$

It is obvious that

$$C^{\gamma}_{1-\gamma}[a,T] \subset C^{\alpha,\beta}_{1-\gamma,\ln}[a,T].$$

We require the basic definitions of fractional Hadamard derivative, which are generally used in the development of the paper.

Definition 2.1. *The gamma function is intrinsically tied to fractional calculus. The simplest interpretation of the gamma function is simply the generalization of the factorial for all real numbers. The definition of the gamma function* $\Gamma(z)$ *is given below:*

$$\Gamma(z) = \int_0^\infty e^\tau \tau^{z-1} d\tau.$$

Definition 2.2. *The Hadamard fractional integral of order* $\alpha \in \mathbb{R}^+$ *of function* $f(t)$, *for all* $t > a$, *is defined by*

$${}_H I^\alpha_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s) \frac{ds}{s},$$

where $\Gamma(\cdot)$ *is the Euler Gamma function.*

Definition 2.3. *The Hadamard derivative of order* $\alpha \in [n-, n)$, $n \in \mathbb{Z}^+$ *of function* $f(t)$ *is given as follows*

$${}_H D^\alpha_{a^+} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(t \frac{d}{dt}\right)^n \int_a^t \left(\ln \frac{t}{s}\right)^{n-\alpha-1} f(s) \frac{ds}{s}.$$

Definition 2.4. *(Hilfer-Hadamard fractional derivative) The left-sided Hilfer-Hadamard fractional derivative of order* $0 < \alpha < 1$ *and* $0 \leq \beta \leq 1$ *of function* $f(t)$ *is defined by*

$${}_H D^{\alpha,\beta}_{a^+} f(t) = \left({}_H I^{\beta(1-\alpha)}_{a^+} D \left({}_H I^{(1-\beta)(1-\alpha)}_{a^+} f\right)\right)(t), \quad D = \frac{d}{dt}.$$

This innovative fractional derivative possibly viewed as interpolating the Hadamard fractional derivative and the Caputo-Hadamard fractional derivative.

Remark 2.5. *It is clear that;*

- *The operator* ${}_H D^{\alpha,\beta}_{a^+}$ *also can be written as*

$${}_H D^{\alpha,\beta}_{a^+} = {}_H I^{\beta(1-\alpha)}_{a^+} D \, {}_H I^{(1-\beta)(1-\alpha)}_{a^+} = {}_H I^{\beta(1-\beta)}_{a^+} \, {}_H D^\gamma_{a^+}, \quad \gamma = \alpha + \beta - \alpha\beta.$$

- *Let* $\beta = 0$, *the left-sided Hadamard fractional derivative can be expressed as* ${}_H D^\alpha_{a^+} := {}_H D^{\alpha,0}_{a^+}$.
- *Let* $\beta = 1$, *the left-sided Caputo-Hadamard fractional derivative can be expressed as* ${}^C_H D^\alpha_{a^+} := {}_H I^{1-\alpha}_{a^+} D$.

The advance of a fixed point theorem of Krasnoselskii [18] as a result of Burton [9], will be employed in the proof.

Theorem 2.6. *[18] Let S be a nonempty, closed, convex and bounded subset of the Banach space X and let* $A : X \to X$ *and* $B : S \to X$ *be two operators such that*

(a) A is contraction with constant $L < 1$,

(b) B is continuous,BS resides in a compact subset of X,

(c) $[x = Ax + By, \ y \in S] \Rightarrow x \in S$.

Then the operator equation $Ax + By = x$ *has a solution in S.*

Inspired by the papers [3, 4, 7, 10], we revise the existence and stability of Hilfer-Hadamard type delay differential equations with nonlocal condition of the form

$$ {}_{H}D^{\alpha,\beta}_{a^+}x(t) = f(t, x(\sigma_1(t)), x(\sigma_2(t)), ..., x(\sigma_n(t))), \quad t > a, \tag{1}$$

$$ {}_{H}I^{1-\gamma}_{a^+}x(t)|_{t=a} + g(t_1, t_2, ..., t_p, x(\cdot)) = x_0, \tag{2}$$

where ${}_{H}D^{\alpha,\beta}_{a^+}$ is Hilfer-Hadamard fractional derivative of order α and type β, $0 < \alpha < 1$, $0 \le \beta \le 1$, $a \le t_0 < t_1 < ... < t_p \le \infty$, and let $J := (a, \infty)$, $f : J \times \mathbb{R}^n \to \mathbb{R}$, $g : J \times \mathbb{R} \to \mathbb{R}$, $\sigma_i : J \times J$ $(i = 1, ..., n)$ are given continuous functions. ${}_{H}I^{1-\gamma}_{a^+}$ is the left-sided mixed Hadamard integral of order $1-\gamma$. For brevity of notations, we shall take ${}_{H}D^{\alpha,\beta}_{a^+}$ as $D^{\alpha,\beta}$ and ${}_{H}I^{1-\gamma}_{a^+}$ as $I^{1-\gamma}$.

It is simple to see problems (1)-(2) are equivalent to the following integral system

$$\begin{aligned} x(t) = &\frac{\left(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))\right)}{\Gamma(\gamma)}\left(\ln\frac{t}{a}\right)^{\gamma-1} \\ &+ \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1} f(s, x(\sigma_1(s)), x(\sigma_2(s)), ..., x(\sigma_n(s)))\frac{ds}{s}. \end{aligned} \tag{3}$$

3. Attractivity

In this part, we examine the attractivity of solutions for Hilfer-Hadamard type delay differential equations with nonlocal condition.

Definition 3.1. *The zero solution $x(t)$ of the problem* (1)-(2) *is globally attractivity if every solution of* (1)-(2) *tends to zero as $t \to \infty$.*

Define the operators

$$\begin{aligned} Px(t) = &\frac{\left(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))\right)}{\Gamma(\gamma)}\left(\ln\frac{t}{a}\right)^{\gamma-1} \\ &+ \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1} f(s, x(\sigma_1(s)), x(\sigma_2(s)), ..., x(\sigma_n(s)))\frac{ds}{s}, \end{aligned} \tag{4}$$

$$\begin{aligned} Ax(t) &= \frac{\left(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))\right)}{\Gamma(\gamma)}\left(\ln\frac{t}{a}\right)^{\gamma-1}, \\ By(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1} f(s, y(\sigma_1(s)), y(\sigma_2(s)), ..., y(\sigma_n(s)))\frac{ds}{s}, \ t > a. \end{aligned}$$

It is obvious that $x(t)$ is a solution of (1) if it is a fixed point of the operator P, and the operator A is contraction with constant 0.

We list the following conditions:

(C1) f is Lebesgue measurable with respect to t on $[a, \infty)$, and there exists a constant $\alpha_1 \in (0, \alpha)$ such that $\int_a^t \left|f(t, x(\sigma_1(t)), x(\sigma_2(t)), ..., x(\sigma_n(t)))\right|^{\frac{1}{\alpha_1}} \frac{dt}{t} < \infty$ for all $h < \infty$, and f is continuous with respect to x on $[a, \infty)$.

(C2) $\left|f(t, x(\sigma_1(t)), x(\sigma_2(t)), ..., x(\sigma_n(t)))\right| \le L\left(\ln\frac{t}{a}\right)^{-\beta_1}$ for $t \in (a, \infty)$ and $x(t) \in C_{1-\gamma,\ln}[J, \mathbb{R}]$, $L \ge 0$ and $\alpha < \beta_1 < 1$.

(C3) $\left|f(t, x(\sigma_1(t)), x(\sigma_2(t)), ..., x(\sigma_n(t)))\right| \le L_1\left(\ln\frac{t}{a}\right)^{-\beta_2}|x(t)|$ for $t \in (a, \infty)$ and $x(t) \in C_{1-\gamma,\ln}[J, \mathbb{R}]$, $L_1 \ge 0$ and $\alpha < \beta_2 < \frac{1}{2}(1 + \alpha)$.

(C4) $\left|f(t,x(\sigma_1(t)),x(\sigma_2(t)),...,x(\sigma_n(t)))-f(t,y(\sigma_1(t)),y(\sigma_2(t)),...,y(\sigma_n(t)))\right| \le L_1\left(\ln\frac{t}{a}\right)^{-\beta_2}\left|x(t)-y(t)\right|$ for $t\in(a,\infty)$ and $x(t),y(t)\in C_{1-\gamma,\ln}[J,\mathbb{R}]$, $L_1\ge 0$ and $\alpha<\beta_2<\frac{1}{2}(1+\alpha)$, $f(t,0,...,0)\equiv 0$.

(C5) $\left|f(t,x(\sigma_1(t)),x(\sigma_2(t)),...,x(\sigma_n(t)))\right| \le K\left(\ln\frac{t}{a}\right)^{-\beta_3}|x(t)|^{\eta}$ for $t\in(a,\infty)$ and $x(t)\in C_{1-\gamma,\ln}[J,\mathbb{R}]$, $K\ge 0,\eta\ge 0$ and $\alpha<\beta_3<\frac{2+\eta\alpha}{2+\eta}$.

(C6) $\left|f(t,x(\sigma_1(t)),x(\sigma_2(t)),...,x(\sigma_n(t)))\right| \le K\left(\ln\frac{t}{a}\right)^{-\beta_3}|x(t)|^{\eta}$ for $t\in(a,\infty)$ and $x(t)\in C_{1-\gamma,\ln}[J,\mathbb{R}]$, $K\ge 0,\eta>0$ and $\alpha-(\eta-1)(1-\alpha)<\beta_3<\alpha$.

(C7) $\left|g(t_1,t_2,...,t_p,(\cdot))\right| = \max_{x\in C_{1-\gamma,\ln}}[J,\mathbb{R}] = G$ for $t\in(a,\infty)$ and $G>0$.

Lemma 3.2. *Assume that the function f satisfies condtions (C1),(C2) and (C7). Then the operator B is continuous and BS_1 resides in a compact subset of $\mathbb{R}$ for $t\ge a+T_1$, where*

$$S_1=\left\{y(t)|y(t)\in C_{1-\gamma,\ln}[J,\mathbb{R}],\ \left|y(t)\right|\ge\left(\ln\frac{t}{a}\right)^{-\gamma_1}\ \textit{for}\ t\ge t_0+T_1\right\},$$

$\gamma_1=\frac{1}{2}(\beta_1-\alpha)$, and T_1 satisfies that

$$\frac{(|x_0|+G)}{\Gamma(\gamma)}(\ln T_1)^{\frac{1}{2}(\gamma-1)}+\frac{L\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}(\ln T_1)^{-\frac{1}{2}(\beta_1-\alpha)}\le 1. \tag{5}$$

Proof. Initially, we prove that B maps S_1 in S_1 for $t\ge a+T_1$.
Since the above condition of S_1, it is simple to know that S_1 is a closed, bounded and convex subset of $\mathbb{R}$. Applying condition (C2), for $t\ge a$, we get

$$\begin{aligned}\left|By(t)\right| &\le \frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\left|f(s,y(\sigma_1(s)),y(\sigma_2(s)),...,y(\sigma_n(s)))\right|\frac{ds}{s}\\ &\le \frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-1}L\left(\ln\frac{s}{a}\right)^{-\beta_1}\frac{ds}{s}\\ &\le \frac{L\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}\left(\ln\frac{t}{a}\right)^{-(\beta_1-\alpha)},\end{aligned}$$

and the only constraint for the above inequality is the integrability of $\left(\ln\frac{s}{a}\right)^{-\beta_1}$, namely $\beta_1<1$.

For $t\ge a+T_1$, inequality (5) and $\beta_1>\alpha$ yeilds that

$$\frac{L\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}\left(\ln\frac{t}{a}\right)^{-\frac{1}{2}(\beta_1-\alpha)}\le\frac{L\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}(\ln T_1)^{-\frac{1}{2}(\beta_1-\alpha)}\le 1.$$

Then, for $t\ge a+T_1$, we have

$$\begin{aligned}\left|By(t)\right| &\le \left[\frac{M\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}\left(\ln\frac{t}{a}\right)^{-\frac{1}{2}(\beta_1-\alpha)}\right]\left(\ln\frac{t}{a}\right)^{-\frac{1}{2}(\beta_1-\alpha)}\\ &\le\left(\ln\frac{t}{a}\right)^{-\gamma_1},\end{aligned}$$

which implies that $BS_1\subset S_1$ for $t\ge a+T_1$.
After that we prove that B is continuous.

For any $y_m(t)$, $y(t)\in S_1$, $m=1,2,...$ with $\lim_{m\to\infty}\left|y_m(t)-y(t)\right|=0$, we get $\lim_{m\to\infty}y_m(t)=y(t)$ and $\lim_{m\to\infty}f(t,y_m(\sigma_1(t)),y_m(\sigma_2(t)),...,y_m(\sigma_n(t)))=f(t,y(\sigma_1(t)),y(\sigma_2(t)),...,y(\sigma_n(t)))$ for $t\ge a+T_1$.

Let $\epsilon > 0$ be given, fix $T > a + T_1$ such that

$$\frac{L\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}\left(\ln\frac{T}{a}\right)^{-(\beta_1-\alpha)} < \frac{\epsilon}{2}.$$

Let $v = \frac{\alpha-1}{1-\alpha_1}$, then $1 + v > 0$ since $\alpha_1 \in (0, \alpha)$. For $a + T_1 \le t \le T$, we have

$$\begin{aligned}
&\left|By_m(t) - By(t)\right| \\
&\le \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\left|f(s, y_m(\sigma_1(s)), y_m(\sigma_2(s)), ..., y_m(\sigma_n(s))) - f(s, y(\sigma_1(s)), y(\sigma_2(s)), ..., y(\sigma_n(s)))\right|\frac{ds}{s} \\
&\le \frac{1}{\Gamma(\alpha)}\left\{\int_a^t \left[\left(\ln\frac{t}{s}\right)\right]^{\frac{1}{1-\alpha_1}}\frac{ds}{s}\right\}^{1-\alpha_1} \\
&\quad\times\left[\int_a^t \left|f(s, y_m(\sigma_1(s)), y_m(\sigma_2(s)), ..., y_m(\sigma_n(s))) - f(s, y(\sigma_1(s)), y(\sigma_2(s)), ..., y(\sigma_n(s)))\right|^{\frac{1}{\alpha_1}}\frac{ds}{s}\right]^{\alpha_1} \\
&\le \frac{1}{\Gamma(\alpha)}\left(\frac{1}{1+v}\left(\ln\frac{T}{a}\right)^{1+v}\right)^{1-\alpha_1} \\
&\quad\times\left[\int_a^t \left|f(s, y_m(\sigma_1(s)), y_m(\sigma_2(s)), ..., y_m(\sigma_n(s))) - f(s, y(\sigma_1(s)), y(\sigma_2(s)), ..., y(\sigma_n(s)))\right|^{\frac{1}{\alpha_1}}\frac{ds}{s}\right]^{\alpha_1} \\
&\le \frac{1}{\Gamma(\alpha)}\left(\frac{1}{1+v}\left(\ln\frac{T}{a}\right)^{1+v}\right)^{1-\alpha_1} \\
&\times\left(\ln\frac{T}{a}\right)^{\alpha_1}\left\|f(s, y_m(\cdot), y_m(\cdot), ..., y_m(\cdot)) - f(s, y(\cdot), y(\cdot), ..., y(\cdot)\right\| \\
&\to 0 \quad \text{as} \quad m \to 0.
\end{aligned}$$

For $t > T$, we have

$$\begin{aligned}
&\left|By_m(t) - By(t)\right| \\
&\le \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\left|f(s, y_m(\sigma_1(s)), y_m(\sigma_2(s)), ..., y_m(\sigma_n(s))) - f(s, y(\sigma_1(s)), y(\sigma_2(s)), ..., y(\sigma_n(s)))\right|\frac{ds}{s} \\
&\le \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\left|f(s, y_m(\sigma_1(s)), y_m(\sigma_2(s)), ..., y_m(\sigma_n(s)))\right| \\
&\qquad + \left|f(s, y(\sigma_1(s)), y(\sigma_2(s)), ..., y(\sigma_n(s)))\right|\frac{ds}{s} \\
&\le \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\left[2L\left(\ln\frac{s}{a}\right)^{-\beta_1}\right]\frac{ds}{s} \\
&\le \frac{2L\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}\left(\ln\frac{t}{a}\right)^{-\beta_1-\alpha} \\
&\le \frac{2L\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}\left(\ln\frac{T}{a}\right)^{-\beta_1-\alpha} \\
&\le \epsilon.
\end{aligned}$$

Then, for $t \ge a + T_1$, it is clear that

$$\left|By_m(t) - By(t)\right| \to 0 \quad \text{as} \quad m \to \infty$$

which implies that B is continuous.
In the end, we prove that BS_1 is equicontinuous.

Let $\epsilon > 0$ be given. Since $\lim_{t\to\infty}\left(\ln\frac{t}{a}\right)^{-\gamma_1} = 0$, there is a $T' > a + T_1$ such that $\left(\ln\frac{t}{a}\right)^{-\gamma_1} < \frac{\epsilon}{2}$ for $t > T'$. Let $t_1, t_2 \geq a+T_1$ and $t_2 > t_1$. If $t_1, t_2 \in [a+T_1, T']$, $\int_a^{T'} \left|f(t, x(\sigma_1(s)), x(\sigma_2(s)), ..., x(\sigma_n(s)))\right|^{\frac{1}{\alpha_1}} \frac{ds}{s}$ exists by condition (C1), then

$$
\begin{aligned}
&|Bx(t_2) - Bx(t_1)| \\
&= \left|\frac{1}{\Gamma(\alpha)}\int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{\alpha-1} f(s, x(\sigma_1(s)), x(\sigma_2(s)), ..., x(\sigma_n(s)))\frac{ds}{s} - \right.\\
&\quad \left.\frac{1}{\Gamma(\alpha)}\int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{\alpha-1} f(s, x(\sigma_1(s)), x(\sigma_2(s)), ..., x(\sigma_n(s)))\frac{ds}{s}\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_a^{t_1}\left[\left(\ln\frac{t_1}{s}\right)^{\alpha-1} - \left(\ln\frac{t_2}{s}\right)^{\alpha-1}\right]\left|f(s, x(\sigma_1(s)), x(\sigma_2(s)), ..., x(\sigma_n(s)))\right|\frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{\alpha-1}\left|f(s, x(\sigma_1(s)), x(\sigma_2(s)), ..., x(\sigma_n(s)))\right|\frac{ds}{s} \\
&\leq \frac{1}{\Gamma(\alpha)}\left\{\int_a^{t_1}\left[\left(\ln\frac{t_1}{s}\right)^{\alpha-1} - \left(\ln\frac{t_2}{s}\right)^{\alpha-1}\right]^{\frac{1}{1-\alpha_1}}\frac{ds}{s}\right\}^{1-\alpha_1}\left[\int_a^{t_1}\left|f(s, x(\sigma_1(s)), x(\sigma_2(s)), ..., x(\sigma_n(s)))\right|^{\frac{1}{\alpha_1}}\frac{ds}{s}\right]^{\alpha_1} \\
&\quad + \frac{1}{\Gamma(\alpha)}\left[\int_{t_1}^{t_2}\left(\ln\frac{t_2}{s}\right)^{\frac{\alpha-1}{1-\alpha_1}}\frac{ds}{s}\right]^{1-\alpha_1}\left[\int_a^{t_2}\left|f(s, x(\sigma_1(s)), x(\sigma_2(s)), ..., x(\sigma_n(s)))\right|^{\frac{1}{\alpha_1}}\frac{ds}{s}\right]^{\alpha_1} \\
&\leq \frac{1}{\Gamma(\alpha)}\left(\frac{1}{1+v}\right)^{1-\alpha_1}\left[\int_a^{T'}\left|f(s, x(\sigma_1(s)), x(\sigma_2(s)), ..., x(\sigma_n(s)))\right|^{\frac{1}{\alpha_1}}\frac{ds}{s}\right]^{\alpha_1}\left(\ln\frac{t_2}{t_1}\right)^{\alpha-\alpha_1} \\
&\to 0 \quad \text{as } t_2 \to t_1.
\end{aligned}
$$

If $t_1, t_2 > T'$, then we have

$$
\begin{aligned}
|Bx(t_2) - Bx(t_1)| &\leq \frac{1}{\Gamma(\alpha)}\int_a^{t_2}\left(\ln\frac{t_2}{s}\right)^{\alpha-1}\left|f(s, x(\sigma_1(s)), x(\sigma_2(s)), ..., x(\sigma_n(s)))\right|\frac{ds}{s} \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_a^{t_1}\left(\ln\frac{t_1}{s}\right)^{\alpha-1}\left|f(s, x(\sigma_1(s)), x(\sigma_2(s)), ..., x(\sigma_n(s)))\right|\frac{ds}{s} \\
&\leq \left(\ln\frac{t_2}{a}\right)^{-\gamma_1} + \left(\ln\frac{t_1}{a}\right)^{-\gamma_1} \leq \epsilon \quad \text{as } t_2 \to t_1.
\end{aligned}
$$

If $a + T_1 \leq t_1 < T' < t_2$, note that if $t_2 \to t_1$, $t_2 \to T'$ and $T' \to t_1$, as said by the above discussion, we have

$$
|Bx(t_2) - Bx(t_1)| \leq \left|Bx(t_2) - Bx(T')\right| + \left|Bx(T') - Bx(t_1)\right| \to 0 \quad \text{as} \quad t_2 \to t_1.
$$

Hence, it is obvious that $|Bx(t_2) - Bx(t_1)| \to 0$ as $t_2 \to t_1$. Therefore BS_1 is equicontinuous; BS_1 is included in a compact subset of $\mathbb{R}$ for $t \geq a + T_1$. □

Lemma 3.3. *Assume that conditions (C1), (C2) and (C7) hold, then a solution of* (1)-(2) *is in* S_1 *for* $t \geq a + T_1$.

Proof. Note that if $x(t)$ is a fixed point of P then it is a solution of (1)-(2). To verify this, it remains to prove that, for fixed $y \in S_1$ and for all $x \in C[J, \mathbb{R}]$, $x = Ax + By \Rightarrow x \in S_1$ holds. If $x = Ax + By$, apply condition (C2), we have

$$
\begin{aligned}
|x(t)| &\leq |Ax(t)| + \left|By(t)\right| \\
&\leq \frac{(|x_0| + G)}{\Gamma(\gamma)}\left(\ln\frac{t}{a}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\left|f(s, y(\sigma_1(s)), y(\sigma_2(s)), ..., y(\sigma_n(s)))\right|\frac{ds}{s} \\
&\leq \frac{(|x_0| + G)}{\Gamma(\gamma)}\left(\ln\frac{t}{a}\right)^{\gamma-1} + \frac{L\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}\left(\ln\frac{t}{a}\right)^{-(\beta_1-\alpha)}.
\end{aligned}
$$

For $t \geq a + T_1$, from inequality (5) and $0 < \alpha < \beta_1 < 1$, we have

$$\begin{aligned}
&\frac{(|x_0| + G)}{\Gamma(\gamma)} \left(\ln \frac{t}{a}\right)^{\frac{1}{2}(\gamma-1)} + \frac{L\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)} \left(\ln \frac{t}{a}\right)^{-\frac{1}{2}(\beta_1-\alpha)} \\
&\leq \frac{(|x_0| + G)}{\Gamma(\gamma)} (\ln T_1)^{\frac{1}{2}(\gamma-1)} + \frac{L\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)} (\ln T_1)^{-\frac{1}{2}(\beta_1-\alpha)} \\
&\leq 1.
\end{aligned}$$

Then, for $t \geq a + T_1$, we have

$$\begin{aligned}
|x(t)| &\leq \left[\frac{(|x_0| + G)}{\Gamma(\gamma)} \left(\ln \frac{t}{a}\right)^{\frac{1}{2}(\gamma-1)} + \frac{L\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)} \left(\ln \frac{t}{a}\right)^{-\frac{1}{2}(\beta_1-\alpha)}\right] \left(\ln \frac{t}{a}\right)^{-\gamma_1} \\
&\leq \left(\ln \frac{t}{a}\right)^{-\gamma_1},
\end{aligned}$$

which implies that $x(t) \in S_1$ for $t \geq a + T_1$. Along with Theorem 2.6 and Lemma 3.2, there exists a $y \in S_1$ such that $y = Ay + By$, i.e., P has a fixed point in S_1 which is solution of (1)-(2) for $t \geq a + T_1$. □

Theorem 3.4. *Assume that conditions (C1), (C2) and (C7) hold, then the zero solution of* (1)-(2) *is globally attractive.*

Proof. By Lemma 3.2, for $t \geq a + T_1$, the solution of (1)-(2) exists and is in S_1. All function in S_1 tend to 0 as $t \to \infty$, then the solution of (1)-(2) tends to zero as $t \to \infty$. □

Theorem 3.5. *Assume that the function f satisfies conditions (C1) and (C3). Then the zero solution of* (1)-(2) *is globally attractive.*

Proof. Let $S_2 = \left\{ y(t) | y(t) \in C_{1-\gamma,\ln}[J, \mathbb{R}],\ |y(t)| \leq \left(\ln \frac{t}{a}\right)^{-\gamma_2} \text{ for } t \geq a + T_2 \right\}$, where, $\gamma_2 = \frac{1}{2}(1-\alpha)$, and T_2 satisfies that

$$\frac{(|x_0| + G)}{\Gamma(\gamma)} (\ln T_2)^{\frac{1}{2}(\gamma-1)} + \frac{L_1\Gamma(1-\beta_1-\gamma_2)}{\Gamma(1+\alpha-\beta_2-\gamma_2)} (\ln T_2)^{-(\beta_2-\alpha)} \leq 1. \tag{6}$$

We first show that if, for fixed $y \in S_2$ and for all $x \in \mathbb{R}$, $x = Ax + By \Rightarrow x \in S_2$ holds. If $x = Ax + By$, from condtion (C2) we have

$$\begin{aligned}
|x(t)| &\leq |Ax(t)| + |By(t)| \\
&\leq \frac{(|x_0| + G)}{\Gamma(\gamma)} \left(\ln \frac{t}{a}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \left| f(s, y(\sigma_1(s)), y(\sigma_2(s)), ..., y(\sigma_n(s))) \right| \frac{ds}{s} \\
&\leq \frac{(|x_0| + G)}{\Gamma(\gamma)} \left(\ln \frac{t}{a}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} L_1 \left(\ln \frac{s}{a}\right)^{-\beta_2} |y(s)| \frac{ds}{s} \\
&\leq \frac{(|x_0| + G)}{\Gamma(\gamma)} \left(\ln \frac{t}{a}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} L_1 \left(\ln \frac{s}{a}\right)^{-\beta_2-\gamma_2} \frac{ds}{s} \\
&\leq \frac{(|x_0| + G)}{\Gamma(\gamma)} \left(\ln \frac{t}{a}\right)^{\gamma-1} + \frac{L_1\Gamma(1-\beta_2-\gamma_2)}{\Gamma(1+\alpha-\beta_2-\gamma_2)} \left(\ln \frac{t}{a}\right)^{-(\beta_2+\gamma_2-\alpha)}.
\end{aligned} \tag{7}$$

Note that $\left(\ln \frac{s}{a}\right)^{-\beta_2-\gamma_2}$ in (7) is integrable by means of $\beta_2 < \frac{1}{2}(1+\alpha)$ and $\gamma_2 = \frac{1}{2}(1-\alpha)$.

For $t \geq a + T_2$, from equity (6) and $0 < \alpha < \beta_2 < \frac{1}{2}(1+\alpha) < 1$, we have

$$\begin{aligned}
&\frac{(|x_0| + G)}{\Gamma(\gamma)} \left(\ln \frac{t}{a}\right)^{\frac{1}{2}(\gamma-1)} + \frac{L_1\Gamma(1-\beta_2-\gamma_2)}{\Gamma(1+\alpha-\beta_2-\gamma_2)} \left(\ln \frac{t}{a}\right)^{-(\beta_2-\alpha)} \\
&\leq \frac{(|x_0| + G)}{\Gamma(\gamma)} (\ln T_2)^{\frac{1}{2}(\gamma-1)} + \frac{L_1\Gamma(1-\beta_2-\gamma_2)}{\Gamma(1+\alpha-\beta_2-\gamma_2)} (\ln T_2)^{-(\beta_2-\alpha)} \\
&\leq 1.
\end{aligned}$$

Thus, for $t \geq a + T_2$, we have

$$|x(t)| \leq \left[\frac{(|x_0| + G)}{\Gamma(\gamma)} \left(\ln \frac{t}{a}\right)^{\frac{1}{2}(\gamma-1)} + \frac{L_1 \Gamma(1-\beta_2-\gamma_2)}{\Gamma(1+\alpha-\beta_2-\gamma_2)} \left(\ln \frac{t}{a}\right)^{-(\beta_2-\alpha)} \right] \left(\ln \frac{t}{a}\right)^{-\gamma_2}$$
$$\leq \left(\ln \frac{t}{a}\right)^{-\gamma_2}, \tag{8}$$

which implies that $x(t) \in S_2$ for $t \geq a + T_2$. For now, merging (7)-(8) also implies that $\left|By(t)\right| \leq \left(\ln \frac{t}{a}\right)^{-\gamma_2}$ which leads to $BS_2 \subset S_2$ for $t \geq a + T_2$.

Comparable to the proof of Lemma 3.2, it is clear that the operator B is continuous and BS_2 resides in a compact subset of $\mathbb{R}$ for $t \geq a + T_2$. By Theorem2.6, the revision of Krasnoselskii's theorem, there exists a $y \in S_2$ such that $y = Ay + By$, i.e., P has fixed point in S_2, which is a solution of (1)-(2). Moreover, all functions in S_2 tend to 0 as $t \to \infty$, then the solution of (1)-(2) tends to zero as $t \to \infty$, which shows that the zero solution of (1)-(2) is globally attractive. □

Corollary 3.6. *Assume that the function f satisfies (C1) and (C4). Then, the zero solution of* (1)-(2) *is globally attractivity.*

Proof. From condition (C2), we have

$$\left|f(t, x(\sigma_1(t)), x(\sigma_2(t)), ..., x(\sigma_n(t)))\right| = \left|f(t, x(\sigma_1(t)), x(\sigma_2(t)), ..., x(\sigma_n(t))) - f(t, 0, ..., 0)\right|$$
$$\leq L_1 \left(\ln \frac{t}{a}\right)^{-\beta_2} |x - 0|$$
$$= L_1 \left(\ln \frac{t}{a}\right)^{-\beta_2} |x(t)|, \tag{9}$$

which implies that condition (C3) holds. The global attractivity result can directly be obtained by Theorem 3.4. □

Theorem 3.7. *Assume that the function f satisfies conditions (C1), (C5) and (C7). Then the zero solution of* (1)-(2) *is globally attractive.*

Proof. Let $S_3 = \left\{y(t) | y(t) \in C_{1-\gamma,\ln}[J, \mathbb{R}], \ \left|y(t)\right| \leq \left(\ln \frac{t}{a}\right)^{-\gamma_3} \text{ for } t \geq a + T_3\right\}$, where $\gamma_3 = \frac{1}{2}(\beta_3 - \alpha)$, and $T_3 \geq 1$ and satisfies

$$\frac{(|x_0| + G)}{\Gamma(\gamma)} (\ln T_3)^{\frac{1}{2}(\gamma-1)} + \frac{K\Gamma(1-\beta_3-\gamma_3\eta)}{\Gamma(1+\alpha-\beta_3-\gamma_3\eta)} (\ln T_3)^{-\frac{1}{2}(\beta_3-\alpha)} \leq 1. \tag{10}$$

First, we show that condition (c) of Theorem 2.6 holds. If $x = Ax + By$, from condition (C5), we have

$$|x(t)| \leq |Ax(t)| + \left|By(t)\right|$$
$$\leq \frac{(|x_0| + G)}{\Gamma(\gamma)} \left(\ln \frac{t}{a}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \left|f(s, y(\sigma_1(s)), y(\sigma_2(s)), ..., y(\sigma_n(s)))\right| \frac{ds}{s}$$
$$\leq \frac{(|x_0| + G)}{\Gamma(\gamma)} \left(\ln \frac{t}{a}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} K \left(\ln \frac{s}{a}\right)^{-\beta_3} \left|y(s)\right| \frac{ds}{s}$$
$$\leq \frac{(|x_0| + G)}{\Gamma(\gamma)} \left(\ln \frac{t}{a}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} K \left(\ln \frac{s}{a}\right)^{-\beta_3-\gamma_3\eta} \frac{ds}{s}$$
$$\leq \frac{(|x_0| + G)}{\Gamma(\gamma)} \left(\ln \frac{t}{a}\right)^{\gamma-1} + \frac{K\Gamma(1-\beta_3-\gamma_3\eta)}{\Gamma(1+\alpha-\beta_3-\gamma_3\eta)} \left(\ln \frac{t}{a}\right)^{-(\beta_3+\gamma_3\eta-\alpha)}. \tag{11}$$

Note that $\left(\ln \frac{s}{a}\right)^{-\beta_3-\gamma_3\eta}$ in (11) is integrable because $\beta_3 < \frac{2+\eta\alpha}{2+\eta}$ and $\gamma_3 = \frac{1}{2}(\beta_3 - \alpha)$ leads to $\beta_3 + \gamma_3\eta < 1$.

For $t \geq a + T_3$, from inequity (10) and $0 < \alpha < \beta_3 < \frac{2+\eta\alpha}{2+\eta} < 1$, we have

$$\frac{(|x_0| + G)}{\Gamma(\gamma)}\left(\ln\frac{t}{a}\right)^{\frac{1}{2}(\gamma-1)} + \frac{K\Gamma(1-\beta_3-\gamma_3\eta)}{\Gamma(1+\alpha-\beta_3-\gamma_3\eta)}\left(\ln\frac{t}{a}\right)^{-\frac{1}{2}(\beta_3-\alpha)}$$
$$\frac{(|x_0| + G)}{\Gamma(\gamma)}(\ln T_3)^{\frac{1}{2}(\gamma-1)} + \frac{K\Gamma(1-\beta_3-\gamma_3\eta)}{\Gamma(1+\alpha-\beta_3-\gamma_3\eta)}(\ln T_3)^{-\frac{1}{2}(\beta_3-\alpha)}$$
$$\leq 1.$$

Thus, for $t \geq a + T_3$.

$$\begin{aligned}
|x(t)| &\leq \frac{(|x_0| + G)}{\Gamma(\gamma)}\left(\ln\frac{t}{a}\right)^{\gamma-1} + \frac{K\Gamma(1-\beta_3-\gamma_3\eta)}{\Gamma(1+\alpha-\beta_3-\gamma_3\eta)}\left(\ln\frac{t}{a}\right)^{-(\beta_3+\gamma_3\eta-\alpha)}\\
&\leq \frac{(|x_0| + G)}{\Gamma(\gamma)}\left(\ln\frac{t}{a}\right)^{\gamma-1} + \frac{K\Gamma(1-\beta_3-\gamma_3\eta)}{\Gamma(1+\alpha-\beta_3-\gamma_3\eta)}\left(\ln\frac{t}{a}\right)^{-(\beta_3-\alpha)}\\
&\leq \left[\frac{(|x_0| + G)}{\Gamma(\gamma)}\left(\ln\frac{t}{a}\right)^{\gamma-1} + \frac{K\Gamma(1-\beta_3-\gamma_3\eta)}{\Gamma(1+\alpha-\beta_3-\gamma_3\eta)}\left(\ln\frac{t}{a}\right)^{-\frac{1}{2}(\beta_3-\alpha)}\right]\left(\ln\frac{t}{a}\right)^{-\gamma_3}\\
&\leq \left(\ln\frac{t}{a}\right)^{-\gamma_3},
\end{aligned} \tag{12}$$

which implies that $x(t) \in S_3$ for $t \geq a + T_3$, meanwhile, combining (11) and (12) also implies that $\left|By(t)\right| \leq \left(\ln\frac{t}{a}\right)^{-\gamma_3}$ which leads to that $BS_3 \subset S_3$, for $t \geq a + T_3$.

The remain part of the proof is similar to that Theorem 3.4, and we skip it. □

Starting the proof of Theorem 3.5, we determine that the term $\gamma_3\eta$ is really out of work in the proof, the attractivity result may be accomplished if we consider a weaker condition than condition (C5). Then it follows the next theorem.

Theorem 3.8. *Assume that the function f satisfies condition (C1),(C6) and (C7). Then the zero solution of* (1)-(2) *is globally attractive.*

Proof. Let Let $S_3' = \left\{y(t)|y(t) \in C_{1-\gamma,\ln}[J,\mathbb{R}],\ \left|y(t)\right| \leq \left(\ln\frac{t}{a}\right)^{-\gamma_3'} \text{ for } t \geq a + T_3'\right\}$, where $\frac{1}{\eta-1}(\alpha-\beta_3) < \gamma_3' < 1-\alpha$, and T_3' and satisfies that

$$\frac{(|x_0| + G)}{\Gamma(\gamma)}(\ln T_3')^{\gamma-1+\gamma_3'} + \frac{K\Gamma(1-\beta_3-\gamma_3'\eta)}{\Gamma(1+\alpha-\beta_3-\gamma_3'\eta)}(\ln T_3')^{-\beta_3+\alpha-(\eta-1)\gamma_3'} \leq 1. \tag{13}$$

Since $\eta > 1$, for $t \geq a + T_3'$, similar to (12) we have

$$\begin{aligned}
|x(t)| &\leq \frac{(|x_0| + G)}{\Gamma(\gamma)}\left(\ln\frac{t}{a}\right)^{\gamma-1} + \frac{K\Gamma(1-\beta_3-\gamma_3'\eta)}{\Gamma(1+\alpha-\beta_3-\gamma_3'\eta)}\left(\ln\frac{t}{a}\right)^{-(\beta_3+\gamma_3'\eta-\alpha)}\\
&\leq \left[\frac{(|x_0| + G)}{\Gamma(\gamma)}\left(\ln\frac{t}{a}\right)^{\gamma-1+\gamma_3'} + \frac{K\Gamma(1-\beta_3-\gamma_3'\eta)}{\Gamma(1+\alpha-\beta_3-\gamma_3'\eta)}\left(\ln\frac{t}{a}\right)^{-\beta_3+\alpha-(\eta-1)\gamma_3'}\right]\left(\ln\frac{t}{a}\right)^{-\gamma_3'}\\
&\leq \left[\frac{(|x_0| + G)}{\Gamma(\gamma)}\left(\ln T_3'\right)^{\gamma-1+\gamma_3'} + \frac{K\Gamma(1-\beta_3-\gamma_3'\eta)}{\Gamma(1+\alpha-\beta_3-\gamma_3'\eta)}\left(\ln T_3'\right)^{-\beta_3+\alpha-(\eta-1)\gamma_3'}\right]\left(\ln\frac{t}{a}\right)^{-\gamma_3'}\\
&\leq \left(\ln\frac{t}{a}\right)^{-\gamma_3'},
\end{aligned}$$

which implies that $x(t) \in S_3'$ for $t \geq a + T_3'$.

The residual part of the proof is related to that of Theorem 3.7, and we skip it. □

Remark 3.9. *β_3 in condition (C6) satisfies that $\alpha-(\eta-1)(1-\alpha) < \beta_3 < \alpha$, then $-\beta_3$ may be positive if $1 < \eta \leq \frac{1}{1-\alpha}$.*

4. Stability analysis

Motivated by the famous Ulam's type stability theories of ordinary differential equations and FDEs [8], we recall U-H stability, generalized U-H stability, U-H-Rassias stability and generalized U-H-Rassias stability to explain our current problem.

Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$, ϵ is a positive real number and let $J' = (a, b]$, $f : J' \times \mathbb{R}^n \to \mathbb{R}$ be a function such that $f \in C_{1-\gamma,\ln}[J', \mathbb{R}]$ for any $x \in \mathbb{R}$ and $\varphi : (a, b] \to \mathbb{R}^+$ be a continuous function. We consider the following FDDEs with Hilfer-Hadamard fractional derivative

$$ {}_H D_{1^+}^{\alpha,\beta} x(t) = f(t, x(\sigma_1(t)), x(\sigma_2(t)), ..., x(\sigma_n(t))), \quad t \in J', \tag{14}$$

and the following inequalities:

$$ \left| {}_H D_{1^+}^{\alpha,\beta} z(t) - f(t, z(\sigma_1(t)), z(\sigma_2(t)), ..., z(\sigma_n(t))) \right| \leq \epsilon, \qquad t \in (a, b]; \tag{15}$$

$$ \left| {}_H D_{1^+}^{\alpha,\beta} z(t) - f(t, z(\sigma_1(t)), z(\sigma_2(t)), ..., z(\sigma_n(t))) \right| \leq \varphi(t), \qquad t \in (a, b]; \tag{16}$$

$$ \left| {}_H D_{1^+}^{\alpha,\beta} z(t) - f(t, z(\sigma_1(t)), z(\sigma_2(t)), ..., z(\sigma_n(t))) \right| \leq \epsilon\varphi(t), \qquad t \in (a, b]. \tag{17}$$

Definition 4.1. *The equation* (14) *is U-H stable if there exists a real number* $c_f > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $z \in C^{\gamma}_{1-\gamma,\ln}[J', \mathbb{R}]$ *of inequality* (**??**) *there exists a solution* $x \in C^{\gamma}_{1-\gamma,\ln}[J', \mathbb{R}]$ *of the equation* (14) *with*

$$ |z(t) - x(t)| \leq c_f \epsilon, \quad t \in J. $$

Definition 4.2. *The equation* (14) *is generalized U-H stable if there exists* $\theta_f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\theta_f(0) = 0$ *such that for each solution* $z \in C^{\gamma}_{1-\gamma,\ln}[J', \mathbb{R}]$ *of the inequality* (**??**) *there exists a solution* $x \in C^{\gamma}_{1-\gamma,\ln}[J', \mathbb{R}]$ *of the equation* (14) *with*

$$ |z(t) - x(t)| \leq \theta_f(\epsilon), \quad t \in J. $$

Definition 4.3. *The equation* (14) *is U-H-Rassias stable with respect to* φ *if there exists* $c_{f,\varphi} > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $z \in C^{\gamma}_{1-\gamma,\ln}[J', \mathbb{R}]$ *of the inequality* (**??**) *there exists a solution* $x \in C^{\gamma}_{1-\gamma,\ln}[J', \mathbb{R}]$ *of the equation* (14) *with*

$$ |z(t) - x(t)| \leq c_{f,\varphi} \epsilon \varphi(t), \quad t \in J. $$

Definition 4.4. *The equation* (14) *is generalized U-H-Rassias stable with respect to* φ *if there exists* $c_{f,\varphi} > 0$ *such that for each solution* $z \in C^{\gamma}_{1-\gamma,\ln}[J', \mathbb{R}]$ *of the inequality* (15) *there exists a solution* $x \in C^{\gamma}_{1-\gamma,\ln}[J', \mathbb{R}]$ *of the equation* (14) *with*

$$ |z(t) - x(t)| \leq c_{f,\varphi} \varphi(t), \quad t \in J. $$

Remark 4.5. *A function* $z \in C^{\gamma}_{1-\gamma,\ln}[J', \mathbb{R}]$ *is a solution of* (**??**) *if and only if there exists a function* $u \in C^{\gamma}_{1-\gamma,\ln}[J', \mathbb{R}]$ *(which depends on z) such that*

(a) $|u(t)| \leq \epsilon, \quad t \in J'$;

(b) ${}_H D_{1^+}^{\alpha,\beta} z(t) = f(t, z(\sigma_1(t)), z(\sigma_2(t)), ..., z(\sigma_n(t))) + u(t), \quad t \in J'$.

One can have similar remarks for the inequalities (**??**) and (15).

So, the Ulam stabilities of FDDEs with Hilfer-Hadamard fractional derivative are some special types of data dependence of the solutions of FDDEs with Hilfer-Hadamard fractional derivative.

Remark 4.6. *Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$, if $z \in C^{\gamma}_{1-\gamma,\ln}[J', \mathbb{R}]$ is a solution of the inequality (**??**) then z is a solution of the following inequality*

$$\Bigg| z(t) - \frac{\left(z_0 - g(t_1, t_2, ..., t_p, z(\cdot))\right)}{\Gamma(\gamma)} \left(\ln \frac{t}{a}\right)^{\gamma-1} - \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s, z(\sigma_1(s)), z(\sigma_2(s)), ..., z(\sigma_n(s))) \frac{ds}{s} \Bigg| \leq \frac{\epsilon}{\Gamma(\alpha+1)} \left(\ln \frac{t}{a}\right)^{\alpha},$$

for $t \in J$, where ${}_H I^{1-\gamma}_{a^+} z(t)|_{t=a} = z_0 - g(t_1, t_2, ..., t_p, z(\cdot))$.

Indeed, by Remark 4.5, we have that

$$D^{\alpha,\beta} z(t) = f(t, z(\sigma_1(t)), z(\sigma_2(t)), ..., z(\sigma_n(t))) + u(t), \quad t \in J'.$$

This implies that

$$z(t) = \frac{\left(z_0 - g(t_1, t_2, ..., t_p, z(\cdot))\right)}{\Gamma(\gamma)} \left(\ln \frac{t}{a}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} u(s) \frac{ds}{s}, + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s, z(\sigma_1(s)), z(\sigma_2(s)), ..., z(\sigma_n(s))) \frac{ds}{s}, \ t \in J.$$

From this it follows that

$$\begin{aligned} &\Bigg| z(t) - \frac{\left(z_0 - g(t_1, t_2, ..., t_p, z(\cdot))\right)}{\Gamma(\gamma)} \left(\ln \frac{t}{a}\right)^{\gamma-1} \\ &- \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s, z(\sigma_1(s)), z(\sigma_2(s)), ..., z(\sigma_n(s))) \frac{ds}{s} \Bigg| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} |u(s)| \frac{ds}{s} \\ &\leq \frac{\epsilon}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\ &\leq \frac{\epsilon}{\Gamma(\alpha+1)} \left(\ln \frac{t}{a}\right)^{\alpha} \end{aligned}$$

*For the moment, we have the following remarks for the solutions of the Hilfer-Hadamard fractional type inequalities (**??**) and (15).*

Remark 4.7. *Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$, if $z \in C^{\gamma}_{1-\gamma,\ln}[J', \mathbb{R}]$ is a solution of the inequality (**??**) then z is a solution of the following integral inequality*

$$\begin{aligned} &\Bigg| z(t) - \frac{\left(z_0 - g(t_1, t_2, ..., t_p, z(\cdot))\right)}{\Gamma(\gamma)} \left(\ln \frac{t}{a}\right)^{\gamma-1} \\ &- \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s, z(\sigma_1(s)), z(\sigma_2(s)), ..., z(\sigma_n(s))) \frac{ds}{s} \Bigg| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \varphi(s) \frac{ds}{s}, \quad t \in J'. \end{aligned}$$

Remark 4.8. *Let* $0 < \alpha < 1, 0 \leq \beta \leq 1$, *if* $z \in C^{\gamma}_{1-\gamma,\ln}[J', \mathbb{R}]$ *is a solution of the inequality* (15) *then* z *is a solution of the following integral inequality*

$$\begin{aligned}&\Bigg|z(t) - \frac{\left(z_0 - g(t_1, t_2, ..., t_p, z(\cdot))\right)}{\Gamma(\gamma)}\left(\ln\frac{t}{a}\right)^{\gamma-1}\\ &- \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1} f(s, z(\sigma_1(s)), z(\sigma_2(s)), ..., z(\sigma_n(s)))\frac{ds}{s}\Bigg|\\ &\leq \frac{\epsilon}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \varphi(s)\frac{ds}{s}, \quad t \in J.\end{aligned}$$

The following lemma plays an imperative role to attain Ulam stability results.

Lemma 4.9. *[30] Let* $y, w : J \to [1, +\infty)$ *be continuous functions. If* w *is nondecreasing and there are constants* $k \geq 0$ *and* $0 < \alpha < 1$ *such that*

$$y(t) \leq w(t) + k\int_1^t \left(\ln\frac{t}{s}\right)^{\alpha-1} y(s)\frac{ds}{s}, \quad t \in J,$$

then

$$y(t) \leq w(t) + \int_1^t \left[\sum_{n=1}^{\infty} \frac{(k\Gamma(\alpha))^n}{\Gamma(n\alpha)}\left(\ln\frac{t}{s}\right)^{n\alpha-1} w(s)\right]\frac{ds}{s}, \quad t \in J.$$

Remark 4.10. *Under the assumptions of Lemma 4.9, let* $w(t)$ *be a nondecreasing function on* J. *Then we have*

$$y(t) \leq w(t)E_{\alpha,1}\left(k\Gamma(\alpha)(\ln t)^{\alpha}\right),$$

where $E_{\alpha,1}$ *is the Mittag-leffler function defined by*

$$E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \quad z \in \mathbb{C}.$$

Lemma 4.11. *Assume the following assumptions hold.*

(A1) $f : (a, b] \times \mathbb{R}^n \to \mathbb{R}$ *is a function such that* $f \in C_{1-\gamma,\ln}[J', \mathbb{R}]$ *for any* $x \in \mathbb{R}$.

(A2) There exists a constant $L^* > 0$ *such that*

$$\begin{aligned}&|f(t_1, x_1, x_2, ..., x_n) - f(t_2, y_1, y_2, ..., y_n)|\\ &\leq L^*\left\{|t_1 - t_2| + \left|x_1 - y_1\right| + \left|x_2 - y_2\right| + ... + \left|x_n - y_n\right|\right\}\end{aligned}$$

for $t_1, t_2 \in (a, b]$ *and all* $x_i, y_i \in \mathbb{R}$, $(i = 1, ..., n)$.

(A3) $g : J \times \mathbb{R} \to \mathbb{R}$ *and there exists a constant* $q > 0$ *such that*

$$\left|g(t_1, t_2, ..., t_p, y(\cdot)) - g(t_1, t_2, ..., t_p, \overline{y}(\cdot))\right| \leq q\left\|y - \overline{y}\right\|_{C_{1-\gamma,\ln}}$$

for any $y, \overline{y} \in C_{1-\gamma,\ln}[J', \mathbb{R}]$.

If

$$\left(\frac{q}{\Gamma(\gamma)} + \frac{nL^*\Gamma(\gamma)}{\Gamma(\alpha + \gamma)}\left(\ln\frac{b}{a}\right)^{\alpha}\right) < 1 \tag{18}$$

then problem (1)-(2) *has a unique solution.*

Proof. Consider the operator $P : C_{1-\gamma,\ln}[J', \mathbb{R}] \to C_{1-\gamma,\ln}[J', \mathbb{R}]$.

$$(Px)(t) = \frac{\left(x_0 - g(t_1, t_2, ..., t_p, x(\cdot))\right)}{\Gamma(\gamma)}\left(\ln\frac{t}{a}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1} f(s, x(\sigma_1(s)), x(\sigma_2(s)), ..., x(\sigma_n(s)))\frac{ds}{s}. \tag{19}$$

It is clear that the fixed points of P are solutions of problem (1)-(2).
Let $x, y \in C_{1-\gamma,\ln}[J', \mathbb{R}]$, we have for any $t \in J$

$$\begin{aligned}
&\left|((Px)(t) - (Py)(t))\left(\ln\frac{t}{a}\right)^{1-\gamma}\right| \\
&\leq \frac{1}{\Gamma(\gamma)}\left|g(t_1, t_2, ..., t_p, x(\cdot)) - g(t_1, t_2, ..., t_p, y(\cdot))\right| \\
&\quad + \left(\ln\frac{t}{a}\right)^{1-\gamma}\frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1}\left|f(s, x(\sigma_1(s)), x(\sigma_2(s)), ..., x(\sigma_n(s)))\right. \\
&\left.- f(s, y(\sigma_1(s)), y(\sigma_2(s)), ..., y(\sigma_n(s)))\right|\frac{ds}{s} \\
&\leq \frac{q}{\Gamma(\gamma)}\left\|x - y\right\|_{C_{1-\gamma,\ln}} + \frac{nL^*}{\Gamma(\alpha)}\left(\ln\frac{t}{a}\right)^{1-\gamma}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1}(\ln s)^{\gamma-1}(\ln s)^{1-\gamma}\left|x(s) - y(s)\right|\frac{ds}{s} \\
&\leq \frac{q}{\Gamma(\gamma)}\left\|x - y\right\|_{C_{1-\gamma,\ln}} + \frac{nL^*}{\Gamma(\alpha)}\left(\ln\frac{t}{a}\right)^{1-\gamma}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1}(\ln s)^{\gamma-1} d\ln s\left\|x - y\right\|_{C_{1-\gamma,\ln}} \\
&\leq \left(\frac{q}{\Gamma(\gamma)} + \frac{nL^*\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\ln\frac{t}{a}\right)^{\alpha}\right)\left\|x - y\right\|_{C_{1-\gamma,\ln}},
\end{aligned}$$

Thus,

$$\left\|Px - Py\right\|_{C_{1-\gamma,\ln}} \leq \left(\frac{q}{\Gamma(\gamma)} + \frac{nL^*\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\ln\frac{b}{a}\right)^{\alpha}\right)\left\|x - y\right\|_{C_{1-\gamma,\ln}}.$$

From (18), it follows that P has a unique fixed point which is solution of equations (1)-(2). □

Theorem 4.12. *Assume that the assumptions (A1)-(A2) and* (18) *hold. Then, the equation* (14) *is U-H stable.*

Proof. Let $z \in C^{\gamma}_{1-\gamma,\ln}[J', \mathbb{R}]$ be a solution of the inequality (**??**). Denote by x the unique solution of the Hilfer-Hadamard fractional type delay differential equation

$$\begin{cases} {}_H D^{\alpha,\beta}_{a^+} x(t) = f(t, x(\sigma_1(t)), x(\sigma_2(t)), ..., x(\sigma_n(t))), \quad \text{for all } t \in (1, b], \\ {}_H I^{1-\gamma}_{a^+} x(t)|_{t=a} = {}_H I^{1-\gamma}_{a^+} z(t)|_{t=a} = z_0 - g(t_1, t_2, ..., t_p, z(\cdot)). \end{cases} \tag{20}$$

We have that

$$x(t) = \frac{\left(z_0 - g(t_1, t_2, ..., t_p, z(\cdot))\right)}{\Gamma(\gamma)}\left(\ln\frac{t}{a}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1} f(s, x(\sigma_1(s)), x(\sigma_2(s)), ..., x(\sigma_n(s)))\frac{ds}{s}.$$

By differential inequality (**??**), we have:

$$\begin{aligned}
&\left| z(t) - \frac{\left(z_0 - g(t_1, t_2, ..., t_p, z(\cdot))\right)}{\Gamma(\gamma)} \left(\ln \frac{t}{a}\right)^{\gamma-1} \right.\\
&\left. - \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s, z(\sigma_1(s)), z(\sigma_2(s)), ..., z(\sigma_n(s))) \frac{ds}{s} \right|\\
&\leq \frac{\epsilon}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s}\\
&\leq \frac{\epsilon}{\Gamma(\alpha+1)} \left(\ln \frac{t}{a}\right)^{\alpha},
\end{aligned}$$

for all $t \in J$. From above it follows:

$$\begin{aligned}
&|z(t) - x(t)|\\
&= \left| z(t) - \frac{\left(z_0 - g(t_1, t_2, ..., t_p, z(\cdot))\right)}{\Gamma(\gamma)} \left(\ln \frac{t}{a}\right)^{\gamma-1} \right.\\
&\left. - \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s, x(\sigma_1(s)), x(\sigma_2(s)), ..., x(\sigma_n(s))) \frac{ds}{s} \right|\\
&\leq \left| z(t) - \frac{\left(z_0 - g(t_1, t_2, ..., t_p, z(\cdot))\right)}{\Gamma(\gamma)} \left(\ln \frac{t}{a}\right)^{\gamma-1} \right.\\
&\left. - \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s, z(\sigma_1(s)), z(\sigma_2(s)), ..., z(\sigma_n(s))) \frac{ds}{s} \right|.\\
&\quad + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s, z(\sigma_1(s)), z(\sigma_2(s)), ..., z(\sigma_n(s))) \frac{ds}{s}\\
&\quad \left. - \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s, x(\sigma_1(s)), x(\sigma_2(s)), ..., x(\sigma_n(s))) \frac{ds}{s} \right|\\
&\leq \left| z(t) - \frac{\left(z_0 - g(t_1, t_2, ..., t_p, z(\cdot))\right)}{\Gamma(\gamma)} \left(\ln \frac{t}{a}\right)^{\gamma-1} \right.\\
&\left. - \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s, z(\sigma_1(s)), z(\sigma_2(s)), ..., z(\sigma_n(s))) \frac{ds}{s} \right|\\
&\quad + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \Big| f(s, z(\sigma_1(s)), z(\sigma_2(s)), ..., z(\sigma_n(s)))\\
&- f(s, x(\sigma_1(s)), x(\sigma_2(s)), ..., x(\sigma_n(s))) \Big| \frac{ds}{s}\\
&\leq \frac{\epsilon}{\Gamma(\alpha+1)} \left(\ln \frac{t}{a}\right)^{\alpha} + \frac{nL^*}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} |z(s) - x(s)| \frac{ds}{s}.
\end{aligned}$$

By Lemma 4.9 and Remark 4.10, for all $t \in J$, we have that:

$$\begin{aligned}
|z(t) - x(t)| &\leq \frac{\left(\ln \frac{t}{a}\right)^{\alpha} E_{\alpha,1}\left(nL^* \left(\ln \frac{t}{a}\right)^{\alpha}\right) \epsilon}{\Gamma(\alpha+1)}\\
&\leq \frac{\left(\ln \frac{b}{a}\right)^{\alpha} E_{\alpha,1}\left(nL^* \left(\ln \frac{b}{a}\right)^{\alpha}\right) \epsilon}{\Gamma(\alpha+1)} := c_f \epsilon.
\end{aligned}$$

Thus, the equation (14) is U-H stable. This completes the proof. □

Theorem 4.13. *Assume that (A1)-(A2),* (18) *and*

(A4) $\varphi \in C(J, \mathbb{R}_+)$ *is continuous, nondecreasing, and there exists* $\lambda_\varphi > 0$ *such that*

$$\frac{1}{\Gamma(\alpha)} \int_1^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \varphi(s) \frac{ds}{s} \leq \lambda_\varphi \varphi(t), \quad \text{for each } t \in J',$$

are satisfied. Then, the equation (14) *is U-H-Rassias stable.*

Proof. Let $z \in C^{\gamma}_{1-\gamma,\ln}[J', \mathbb{R}]$ be a solution of the inequality (15). Denote by x the unique solution of the Hilfer-Hadamard fractional type delay differential equation (20). We have that

$$x(t) = \frac{\left(z_0 - g(t_1, t_2, ..., t_p, z(\cdot))\right)}{\Gamma(\gamma)} \left(\ln \frac{t}{a}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s, x(\sigma_1(s)), x(\sigma_2(s)), ..., x(\sigma_n(s))) \frac{ds}{s},$$

for all $t \in J'$.
By differential inequality (15), we have:

$$\left| z(t) - \frac{\left(z_0 - g(t_1, t_2, ..., t_p, z(\cdot))\right)}{\Gamma(\gamma)} \left(\ln \frac{t}{a}\right)^{\gamma-1} - \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s, z(\sigma_1(s)), z(\sigma_2(s)), ..., z(\sigma_n(s))) \frac{ds}{s} \right|$$
$$\leq \epsilon \lambda_\varphi \varphi(t), \quad t \in J'.$$

From above it follows

$$|z(t) - x(t)| \leq \epsilon \lambda_\varphi \varphi(t) + \frac{nL^*}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} |z(s) - x(s)| \frac{ds}{s}.$$

By Lemma 4.9 and Remark 4.10, we have that:

$$|z(t) - x(t)| \leq \epsilon \lambda_\varphi \varphi(t) E_{\alpha,1}\left(nL^* \left(\ln \frac{b}{a}\right)^{\alpha}\right) := c_f \epsilon \varphi(t), \quad t \in J'.$$

Thus, the equation (14) is U-H-Rassias stable. □

Remark 4.14. *One can extend the above results to case of equation* (14) *with* $b = +\infty$.

5. An example

In this part, we present an example to exemplify our main results. Consider the following Hilfer-Hadamard fractional type delay differential equation

$$_H D^{\alpha,\beta}_{1^+} x(t) = a(x(t-8) - x(t-10)), \quad t \in (1, e], \tag{21}$$

$$_H I^{1-\gamma}_{1^+} x(1) + \sum_{i=1}^{n} c_i x(t_i) = 1, \tag{22}$$

and the following inequality

$$\left| _H D^{\alpha,\beta}_{1^+} z(t) - f(t, z(\sigma_1(t)), z(\sigma_2(t))) \right| \leq \epsilon, \quad t \in (1, e],$$

where $1 < t_1 < t_2 < ... < t_n < e$ and $c_i = 1, ..., n$ are positive constants with

$$\sum_{i=1}^{n} c_i \leq \frac{1}{3}.$$

Set $f(t, x(\sigma_1(t)), x(\sigma_2(t))) = a(x(\sigma_1(t)) - x(\sigma_2(t))),\ \forall\, t, \sigma_1, \sigma_2 \in J := [1, e], \quad x \in \mathbb{R}$.
Let $a = \frac{1}{8}$ and choose $\alpha = \frac{1}{2}, \beta = \frac{1}{2}$ and $\gamma = \frac{3}{4}$.

Denote $\sigma_1(\cdot) = -8, \sigma_2 = -10$, $f(\cdot, \sigma_1(x(\cdot)), \sigma_2(x(\cdot))) = \frac{1}{8}(x(\cdot - 8) - x(\cdot - 10))$.
Clearly, the function f is continuous. For each $x, \overline{x} \in \mathbb{R}$ and $\sigma_1, \sigma_2, t \in J$:

$$\left|f(t, x(\sigma_1(t)), x(\sigma_2(t))) - f(t, \overline{x}(\sigma_1(t)), \overline{x}(\sigma_2(t)))\right| \leq \frac{1}{4}\left|x - \overline{x}\right|.$$

On the other hand, we have

$$\begin{aligned}\left|g(x) - g(\overline{x})\right| &\leq \left|\sum_{i=1}^{n} c_i x - \sum_{i=1}^{n} c_i \overline{x}\right| \\ &\leq \sum_{i=1}^{n} c_i \left|x - \overline{x}\right| \\ &\leq \frac{1}{3}\left|x - \overline{x}\right|.\end{aligned}$$

Hence conditions (A2) and (A3) are satisfied with $n = 2, L^* = \frac{1}{8}$ and $q = \frac{1}{3}$.
Thus,

$$\left(\frac{q}{\Gamma(\gamma)} + \frac{nL^*\Gamma(\gamma)}{\Gamma(\alpha + \gamma)}(\ln b)^{\alpha}\right) < 1.$$

It follows from Lemma 4.11 that the problem (**??**)-(21) has a unique solution. Now all the assumptions in Theorem 4.12 are satisfied, the equation (**??**) is U-H stable on J.
Let $\varphi(t) = (\ln t)^{\frac{1}{2}}$. We have

$$\begin{aligned}I^{\alpha}\varphi(t) &= \frac{1}{\Gamma(\frac{1}{2})}\int_1^t \left(\ln\frac{t}{s}\right)^{\frac{1}{2}-1} (\ln t)^{\frac{1}{2}}\frac{ds}{s} \\ &\leq \frac{1}{\Gamma(\frac{1}{2})}\int_1^t \left(\ln\frac{t}{s}\right)^{\frac{1}{2}-1} \frac{ds}{s} \\ &= \frac{2\varphi(t)}{\sqrt{\pi}}.\end{aligned}$$

Thus

$${}_HI^{\alpha}_{1^+}\varphi(t) \leq \frac{2}{\sqrt{\pi}}(\ln t)^{\frac{1}{2}} := \lambda_{\varphi}\varphi(t).$$

Thus condition (A4) is satisfied with $\varphi(t) = (\ln t)^{\frac{1}{2}}$ and $\lambda_{\varphi} = \frac{2}{\sqrt{\pi}}$. It follows from Lemma 4.11 that the problem (**??**)-(21) has a unique solution and from Theorem 4.13, the equation (**??**) is U-H-Rassias stable.

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Author contributions

All of the authors equally contributed to the conception and development of this manuscript.

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