Filomat 36:17 (2022), 5957–5972
https://doi.org/10.2298/FIL2217957P

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Riemannian Poisson Warped Product Space

Buddhadev Pal[a], Pankaj Kumar[a]

[a]*Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi-221005, India*

Abstract. A formal treatment of Killing 1-form and 2-Killing 1-form on Riemannian Poisson manifold, Riemannian Poisson warped product space are presented. In this way, we obtain Bochner type results on compact Riemannian Poisson manifold, compact Riemannian Poisson warped product space for Killing 1-form and 2-Killing 1-form. Finally, we give the characterization of a 2-Killing 1-form on $(\mathbb{R}^2, g, \Pi)$.

1. Introduction

To provide the example of Riemannian spaces having negative curvature Bishop and O'Neill [1] introduced the notion of warped space. From then on original and generalized forms of warped product spaces have been widely discussed by both mathematicians and physicists [2–9].

Let $(\tilde{M}_1, \tilde{g}_1)$ and $(\tilde{M}_2, \tilde{g}_2)$ are two pseudo-Riemannian manifolds with positive smooth function f on $\tilde{M}_1$. Let $\pi_1 : \tilde{M}_1 \times \tilde{M}_2 \to \tilde{M}_1$ and $\pi_2 : \tilde{M}_1 \times \tilde{M}_2 \to \tilde{M}_2$ are the projections. The warped product $\tilde{M} = \tilde{M}_1 \times_f \tilde{M}_2$ is the product manifold $\tilde{M}_1 \times \tilde{M}_2$ endowed with the metric tensor

$$\tilde{g}^f = \pi_1^*(\tilde{g}_1) + (f \circ \pi_1)^2 \pi_2^*(\tilde{g}_2),$$

called warped product and the ordered-pair $(\tilde{M}, \tilde{g}^f)$ known as warped product space. Here $\tilde{M}_1$, $\tilde{M}_2$ and f are respectively known as base space, fiber space and warping function of the warped product space $(\tilde{M}, \tilde{g}^f)$ and * stand for pull-back operator.

Killing vector fields are the relevant object for the geometry specially in pseudo-Riemannian geometry where mathematicians characterized the existence of Killing vector fields. Killing vector fields are also studied by many physicists in the prospective of general relativity in which these are expounded in the term of symmetry. Bochner [10–12], studied in detail Killing vector fields and provided various remarkable results. K. Yano [13, 14], consider a compact orientable Riemannian spaces with boundary and generalized the Bochner technique to study Killing vector fields on it. S. Yorozu [15, 16], discussed the non-existence of Killing vector fields on complete Riemannian spaces and also did the same for non-compact Riemannian spaces with boundary. Generalized forms of Killing vector fields like conformal vector fields, 2-Killing vector fields have been investigated in [17–22], for ambient spaces. T. Opera [23], introduced the perception of 2-Killing vector fields and provided the relation between curvature, monotone vector fields and 2-Killing

2020 *Mathematics Subject Classification*. Primary 53C15; Secondary 53D17.
Keywords. Warped product, Killing 1-forms, Levi-Civita contravariant connection, Poisson structure, Riemannian Poisson manifold.
Received: 17 November 2021; Accepted: 14 May 2022
Communicated by Dragan S. Djordjević
The Second author is supported by UGC JRF of India, Ref. No: 1269/(SC)(CSIR-UGC NET DEC. 2016).
Email addresses: pal.buddha@gmail.com (Buddhadev Pal), pankaj.kumar14@bhu.ac.in (Pankaj Kumar)

vector fields on Riemannian spaces. Moreover, characterized the 2-Killing vector field on $\mathbb{R}^n$. S. Shenawy and B. *Ünal* [24], provided some results of 2-Killing vector field for warped product space and apply these results to characterize it on some famous warped space time model. Z. Erjavec [25], currently characterized proper conformal Killing vector fields and determine some proper 2-Killing vector fields in Sol space.

Poisson [26], introduced a bracket as a tool for classical dynamics and Lie [27], explored the geometry of this bracket. In [28, 29], authors adopted the Poisson structure and provided the notion of Poisson manifold. The geometric notions like connection, curvatures, metric etc., were discussed in [30–33], on Poisson manifold. In [34], authors formulated several concepts on product manifold like product Poisson tensor and product Riemannian metric . In [35], authors discussed the some geometric notions like contravariant Levi-Civita connection, Riemann and Ricci curvatures on the product of two pseudo-Riemannian spaces which is associated with the product Poisson structure, warped bivector field.

The aim of this article is to provide the notions of Killing 1-form and 2-Killing form and try to study these two notions on Riemannian Poisson manifold and Riemannian Poisson warped product space.

The outline of this article is as follows. In Section 2, we look back on some classical notions like cometric, curvatures, contravariant Levi-Civita connection on Poisson manifold and give the definition of Riemannian Poisson manifold $(\tilde{M}, g, \Pi)$. Moreover, we provide the explicit form of cometric g^f and contravariant Levi-Civita connection $\mathcal{D}$ on $(M = \tilde{M}_1 \times_f \tilde{M}_2, g^f)$. In Section 3, we characterize the Killing 1-form on Riemannian Poisson manifold and Riemannian Poisson warped product space $(M = \tilde{M}_1 \times_f \tilde{M}_2, g^f, \Pi)$. Moreover, we introduce the concept of parallel 1-form and provide Bochner type results on compact Riemannian Poisson manifold and Riemannian Poisson warped product space in Theorems 3.10, 3.11. In Section 4, we study 2-Killing 1-form and characterize 2-Killing 1-form on $\mathbb{R}^2$ in Theorem 4.8.

2. Preliminaries

2.1. Geometric structure on Poisson manifold

Lots of basic terms and consequence related to Poisson manifold presented in [28]. Let $\tilde{M}$ be a manifold. A Lie bracket map $\{.,.\} : C^\infty(\tilde{M}) \times C^\infty(\tilde{M}) \to C^\infty(\tilde{M})$ is said to be Poisson bracket on $\tilde{M}$ if it follows the Leibniz identity i.e.,

$$\{\phi_1, \phi_2\phi_3\} = \{\phi_1, \phi_2\}\phi_3 + \phi_2\{\phi_1, \phi_3\}, \qquad \forall\ \phi_1, \phi_2, \phi_3 \in C^\infty(\tilde{M}).$$

The pair $(\tilde{M}, \{.,.\})$ is said to be Poisson manifold.
Let $(\tilde{M}, \{.,.\})$ be Poisson manifold and $\phi_1 \in C^\infty(\tilde{M})$ then we can find a unique vector field X_{ϕ_1} on $\tilde{M}$ associate to ϕ_1 such that

$$X_{\phi_1}(\phi_2) = \{\phi_2, \phi_1\}, \qquad \forall\ \phi_2 \in C^\infty(\tilde{M}).$$

The vector field X_{ϕ_1} is said to be Hamiltonian vector field of the function ϕ_1. If

$$X_{\phi_1}(\phi_2) = 0, \qquad \forall\ \phi_2 \in C^\infty(\tilde{M}),$$

then $\phi_1 \in C^\infty(\tilde{M})$ is called Casimir function. The Leibniz identity also guarantee the existence of a bivector field $\Pi \in \mathfrak{X}^2(\tilde{M}) = \Gamma(\Lambda^2 T\tilde{M})$ such that

$$\{\phi_1, \phi_2\} = \Pi(d\phi_1, d\phi_2), \qquad \forall\ \phi_1, \phi_2 \in C^\infty(\tilde{M}).$$

A bracket $[.,.]_S$ on $(\tilde{M}, \{.,.\})$ considered to be Schouten bracket associated with bivector field Π if

$$\frac{1}{2}[\Pi, \Pi]_S(d\phi_1, d\phi_2, d\phi_3) = \{\{\phi_1, \phi_2\}, \phi_3\} + \{\{\phi_2, \phi_3\}, \phi_1\} + \{\{\phi_3, \phi_1\}, \psi_2\}.$$

A bivector field Π on $\tilde{M}$ is called Poisson tensor if $[\Pi, \Pi]_S = 0$.

Remark 2.1. *Many authors assume $(\tilde{M}, \Pi)$ as a Poisson manifold when Π is a Poisson tensor. Here, we consider same notion.*

Let us assume that if $\tilde{M}$ is a manifold with a bivector field Π then there is a natural homomorphism $\sharp_\Pi : T^*\tilde{M} \to T\tilde{M}$ corresponding to Π given by

$$\eta(\sharp_\Pi(\omega)) = \Pi(\omega, \eta), \qquad \forall \ \omega, \eta \in T^*\tilde{M},$$

called sharp map(anchor map).

If Π is a bivector field on $\tilde{M}$, then it give rise to a bracket $[.,.]_\Pi$ on smooth 1-forms $\Gamma(T^*\tilde{M})$ is said to be Koszual bracket defined by

$$[\omega, \eta]_\Pi = \mathcal{L}_{\sharp_\Pi(\omega)}\eta - \mathcal{L}_{\sharp_\Pi(\eta)}\omega - d(\Pi(\omega, \eta)).$$

Let $(\tilde{M}, \Pi)$ be a Poisson manifold where Π is a Poisson tensor on $\tilde{M}$ then Koszual bracket $[.,.]_\Pi$ convert into usual Lie bracket.

If $\sharp_\Pi$ is the sharp map on Poisson manifold $(\tilde{M}, \Pi)$, then there is a Lie algebra homomorphism $\sharp_\Pi : \Gamma(T^*\tilde{M}) \to \Gamma(T\tilde{M})$, such that

$$\sharp_\Pi([\omega, \eta]_\Pi) = [\sharp_\Pi(\omega), \sharp_\Pi(\eta)],$$

where $[.,.]$ is the usual Lie bracket on $\Gamma(T\tilde{M})$.

Let $(\tilde{M}, \Pi)$ be a Poisson manifold. In [30], authors introduced the concept of contravariant connection $\mathcal{D}$ on $\tilde{M}$. The torsion and curvature tensors corresponding to this connection $\mathcal{D}$ are given by

$$\begin{aligned} \mathcal{T}(\omega, \eta) &= \mathcal{D}_\omega\eta - \mathcal{D}_\eta\omega - [\omega, \eta]_\Pi, \\ \mathcal{R}(\omega, \eta)\gamma &= \mathcal{D}_\omega\mathcal{D}_\eta\gamma - \mathcal{D}_\eta\mathcal{D}_\omega\gamma - \mathcal{D}_{[\omega,\eta]_\Pi}\gamma, \end{aligned}$$

where $\mathcal{T}$ is $(2, 1)$-type tensor and $\mathcal{R}$ is $(3, 1)$-type tensor. Here $\mathcal{D}$ is said to be torsion-free if $\mathcal{T} = 0$ and flat if $\mathcal{R} = 0$.

Let $(M, \tilde{g})$ be a pseudo-Riemannian manifold. The bundle isomorphism $\flat_{\tilde{g}} : TM \to T^*M$ is a map such that $X \mapsto \tilde{g}(X, .)$ and its inverse map

$$\begin{aligned} \sharp_{\tilde{g}} :& T^*M \to TM \\ &\omega \mapsto \sharp_{\tilde{g}}(\omega) \end{aligned}$$

such that $\omega(X) = \tilde{g}(\sharp_{\tilde{g}}(\omega), X)$. The metric g on the cotangent bundle T^*M is defined by

$$g(\omega, \eta) = \tilde{g}(\sharp_{\tilde{g}}(\omega), \sharp_{\tilde{g}}(\eta)).$$

This metric g is said to be cometric of the metric $\tilde{g}$.

Let $(\tilde{M}, \Pi)$ be a Poisson manifold and g is cometric then there exists a unique contravariant connection $\mathcal{D}$ on $\tilde{M}$ characterized by

$$\begin{aligned} 2g(\mathcal{D}_\omega\eta, \gamma) &= \sharp_\Pi(\omega)g(\eta, \gamma) + \sharp_\Pi(\eta)g(\omega, \gamma) - \sharp_\Pi(\gamma)g(\omega, \eta) \\ &\quad + g([\omega, \eta]_\Pi, \gamma) - g([\eta, \gamma]_\Pi, \omega) + g([\gamma, \omega]_\Pi, \eta), \end{aligned} \tag{1}$$

for any $\omega, \eta, \gamma \in \Omega^1(\tilde{M})$, and follows the following two conditions

(i). $\mathcal{D}_\omega\eta - \mathcal{D}_\eta\omega = [\omega, \eta]_\Pi$(Torsion-free),

(ii). $\sharp_\Pi(\omega)g(\eta, \gamma) = g(\mathcal{D}_\omega\eta, \gamma) + g(\eta, \mathcal{D}_\omega\gamma)$(Metric condition),

for any $\omega, \eta, \gamma \in \Omega^1(\tilde{M})$. Contravariant connection $\mathcal{D}$ with properties (*i*) and (*ii*) is said to be contravariant Levi-Civita connection associated to pair (g, Π) on $\tilde{M}$.

Let $(\tilde{M}, \Pi)$ be a n-dimensional Poisson manifold with connection $\mathcal{D}$ and p is any point on $\tilde{M}$. The Ricci curvature Ric_p and scalar curvature at p corresponding to the local orthonormal coframe $\{\theta_1, ..., \theta_n\}$ of $T^*_p\tilde{M}$, given by

$$Ric_p(\omega, \eta) = \sum_{i=1}^{n} g_p(\mathcal{R}_p(\omega, \theta_i)\theta_i, \eta), \tag{2}$$

$$S_p = \sum_{i=1}^{n} Ric_p(\theta_i, \theta_i), \tag{3}$$

for any $\omega, \eta \in T^*_p\tilde{M}$.

Let $(\tilde{M}, \Pi)$ be a Poisson manifold with connection $\mathcal{D}$ and $f \in C^\infty(\tilde{M})$ then $\mathcal{D}f = df \circ \sharp_\Pi \in \mathfrak{X}^1(\tilde{M})$, defined by

$$(\mathcal{D}f)(\omega) = \mathcal{D}_\omega f = \sharp_\Pi(\omega)(f) = df(\sharp_\Pi(\omega)),$$

for any $\omega \in \Omega^1(\tilde{M})$.

Let $(\tilde{M}, \Pi)$ be a Poisson manifold with connection $\mathcal{D}$ satisfies $\mathcal{D}\Pi = 0$ i.e.,

$$\sharp_\Pi(\omega)\Pi(\eta, \gamma) - \Pi(\mathcal{D}_\omega\eta, \gamma) - \Pi(\eta, \mathcal{D}_\omega\gamma) = 0,$$

for any $\omega, \eta, \gamma \in \Omega^1(M)$, then triplet $(\tilde{M}, g, \Pi)$ called Riemannian Poisson manifold.

Let $(\tilde{M}, \Pi)$ be a Poisson manifold with cometric g, then field endomorphism

$$J : T^*\tilde{M} \to T^*\tilde{M}$$

provides

$$\Pi(\omega, \eta) = g(J\omega, \eta) = -g(\omega, J\eta),$$

for any $\omega, \eta \in T^*\tilde{M}$.

Let $(\tilde{M}, g, \Pi)$ be a Riemannian Poisson manifold and J is a field endomorphism on $\tilde{M}$ then $\mathcal{D}J = 0$ i.e.,

$$\mathcal{D}_\omega(J\eta) = J\mathcal{D}_\omega\eta,$$

for any $\omega, \eta \in T^*\tilde{M}$.

2.2. *Cometric and contravariant Levi-Civita connection on warped product space*

The explicit form of the warped metric

$$\tilde{g}^f = \pi^*(\tilde{g}_1) + (f^h)^2\sigma^*(\tilde{g}_2),$$

on $(\tilde{M}_1, \tilde{g}_1)$ and $(\tilde{M}_2, \tilde{g}_2)$ is given by

$$\begin{cases} \tilde{g}^f(X_1^h, Y_1^h) = \tilde{g}_1(X_1, Y_1)^h, \\ \tilde{g}^f(X_1^h, Y_2^v) = \tilde{g}_1(X_2^v, Y_1^h) = 0, \\ \tilde{g}^f(X_2^v, Y_2^v) = (f^h)^2\tilde{g}_2(X_2, Y_2)^v, \end{cases} \tag{4}$$

for any $X_1, Y_1 \in \Gamma(T\tilde{M}_1)$ and $X_2, Y_2 \in \Gamma(T\tilde{M}_2)$. Here $f \circ \pi = f^h$ is horizontal lift of f from $\tilde{M}_1$ to $\tilde{M}_1 \times \tilde{M}_2$. For more detail of horizontal and vertical lifts on product space see in ([34, 36, 37]).

As a consequence of the Proposition 3.3 of ([36],p. 23), one has the following proposition which provides explicit form to the cometric

$$g^f = g_1^h + \frac{1}{(f^h)^2}g_2^v,$$

of warped metric $\tilde{g}^f$.

Proposition 2.2. *Let two pseudo-Riemannian manifolds be $(\tilde{M}_1, \tilde{g}_1)$ and $(\tilde{M}_2, \tilde{g}_2)$ and a smooth function be $f : \tilde{M}_1 \to \mathbb{R}^+$. Then cometric g^f of the metric $\tilde{g}^f$ is explicitly can be written as*

$$\begin{cases} g^f(\omega_1^h, \eta_1^h) = g_1(\omega_1, \eta_1)^h, \\ g^f(\omega_1^h, \eta_2^v) = g_1(\omega_2^v, \eta_1^h) = 0, \\ g^f(\omega_2^v, \eta_2^v) = \frac{1}{(f^h)^2} g_2(\omega_2, \eta_2)^v, \end{cases} \tag{5}$$

for any $\omega_1, \eta_1 \in \Gamma(T^*\tilde{M}_1)$ and $\omega_2, \eta_2 \in \Gamma(T^*\tilde{M}_2)$. Where g_1 and g_2 are the cometric of the metric $\tilde{g}_1$ and $\tilde{g}_2$ respectively. The ordered pair $(\tilde{M} = \tilde{M}_1 \times_f \tilde{M}_2, g^f)$ is said to be contravariant warped product space of warped space $(\tilde{M} = \tilde{M}_1 \times_f \tilde{M}_2, \tilde{g}^f)$.

Contravariant Levi-Civita connection $\mathcal{D}$ associated with pair (g^f, Π) (where $g^f = g_1^h + \frac{1}{(f^h)^2} g_2^v$ and $\Pi = \Pi_1 + \Pi_2$) on $(\tilde{M} = \tilde{M}_1 \times_f \tilde{M}_2, g^f)$ is given by proposition:

Proposition 2.3. *For any $\omega_1, \eta_1 \in \Gamma(T^*\tilde{M}_1)$ and $\omega_2, \eta_2 \in \Gamma(T^*\tilde{M}_2)$, we have*

(i). $\mathcal{D}_{\omega_1^h} \beta_1^h = (\mathcal{D}^1_{\omega_1} \eta_1)^h,$

(ii). $\mathcal{D}_{\omega_2^v} \eta_2^v = (\mathcal{D}^2_{\omega_2} \eta_2)^v - \dfrac{1}{(f^h)^3} g_2(\omega_2, \eta_2)^v (J_1 df)^h,$

(iii). $\mathcal{D}_{\omega_1^h} \eta_2^v = \mathcal{D}_{\eta_2^v} \omega_1^h = \dfrac{1}{f^h} g_1(J_1 df, \omega_1)^h \eta_2^v.$

If we assume that, $\Pi = \Pi_1 + \Pi_2$ in Theorem 5.2 of ([35],p. 294) then we conclude that:

Theorem 2.4. *Let f be a Casimir function. Then both $(\tilde{M}_1, g_1, \Pi_1)$ and $(\tilde{M}_2, g_2, \Pi_2)$ are Riemannian Poisson manifolds if and only if the triplet $(\tilde{M} = \tilde{M}_1 \times_f \tilde{M}_2, g^f, \Pi)$ is a Riemannian Poisson warped product space.*

3. Killing 1-form

Let $(\tilde{M}, g, \Pi)$ be a Riemannian Poisson manifold. In ([30],p. 5), author define the Lie derivative on the space of k-vector fields $\mathfrak{X}^k(\tilde{M}) = \Gamma(\wedge^k T\tilde{M})$. Let $T \in \mathfrak{X}^k(\tilde{M})$, then the Lie derivative of T in the direction of 1-form $\alpha \in \Omega^1(\tilde{M})$ is a map $\mathcal{L}_\alpha : \mathfrak{X}^k(\tilde{M}) \to \mathfrak{X}^k(\tilde{M})$ such that

$$(\mathcal{L}_\alpha T)(\alpha_1, ..., \alpha_k) = \sharp_\Pi(\alpha)(T(\alpha_1, ..., \alpha_k)) - \sum_{i=1}^{k} T(\alpha_1, ..., [\alpha, \alpha_i]_\Pi, ..., \alpha_k), \tag{6}$$

where $\alpha_1, ..., \alpha_k \in \Omega^1(M)$.

A 1-form $\eta \in \Omega^1(\tilde{M})$ on $(\tilde{M}, g, \Pi)$ is said to be Killing 1-form corresponding to the cometric g if

$$\mathcal{L}_\eta g = 0,$$

where $\mathcal{L}_\eta$ is Lie derivative on $\tilde{M}$ with respect to 1-form η.

In the following two propositions, we will find the expression of Lie derivative $\mathcal{L}_\eta$ with respect to cometric g^f on contravariant warped product space $(\tilde{M} = \tilde{M}_1 \times_f \tilde{M}_2, g^f)$ and Riemannian Poisson warped product space $(M = \tilde{M}_1 \times_f \tilde{M}_2, g^f, \Pi)$. Here we will consider $\eta = \eta_1^h + \eta_2^v$, $\alpha = \alpha_1^h + \alpha_2^v$ and $\beta = \beta_1^h + \beta_2^v$.

Proposition 3.1. *Let $(\tilde{M} = \tilde{M}_1 \times_f \tilde{M}_2, g^f)$ be a contravariant warped product space and $\mathcal{D}$ is the contravariant Levi-Civita connection associated with pair (g^f, Π) on $\tilde{M}$. Then for any $\eta \in \Omega^1(\tilde{M})$, we have*

$$\begin{aligned}(\mathcal{L}_\eta g^f)(\alpha, \beta) = &\Big[(\mathcal{L}^1_{\eta_1} g_1)(\alpha_1, \beta_1)\Big]^h \\ &+ \frac{1}{(f^h)^2}\Big[(\mathcal{L}^2_{\eta_2} g_2)(\alpha_2, \beta_2)\Big]^v + \Big(\frac{g_1(J_1 df, \eta_1)}{f^3}\Big)^h g_2(\alpha_2, \beta_2)^v,\end{aligned}$$

for any $\alpha, \beta \in \Omega^1(\tilde{M})$.

Proof. From equation (6), we conclude that

$$\begin{aligned}(\mathcal{L}_\eta g^f)(\alpha,\beta) &= \sharp_\Pi(\eta)(g^f(\alpha,\beta)) - g^f([\eta,\alpha]_\Pi,\beta) - g^f(\alpha,[\eta,\beta]_\Pi)\\
&= \Big[\sharp_{\Pi_1}(\eta_1)(g_1(\alpha_1,\beta_1))\Big]^h + \Big[\sharp_{\Pi_1}(\eta_1)\Big]^h(\frac{1}{(f^h)^2}g_2(\alpha_2,\beta_2)^v)\\
&+ \frac{1}{(f^h)^2}\Big[\sharp_{\Pi_2}(\eta_2)(g_2(\alpha_2,\beta_2))\Big]^v + \Big[g_1([\eta_1,\alpha_1]_{\Pi_1},\beta_1)\Big]^h\\
&+ \frac{1}{(f^h)^2}\Big[g_2([\eta_2,\alpha_2]_{\Pi_2},\beta_2)\Big]^v + \Big[g_1(\alpha_1,[\eta_1,\beta_1]_{\Pi_1})\Big]^h\\
&+ \frac{1}{(f^h)^2}\Big[g_2(\alpha_2,[\eta_2,\beta_2]_{\Pi_2})\Big]^v\\
&= \Big[(\mathcal{L}^1_{\eta_1}g_1)(\alpha_1,\beta_1)\Big]^h\\
&+ \frac{1}{(f^h)^2}\Big[(\mathcal{L}^2_{\eta_2}g_2)(\alpha_2,\beta_2)\Big]^v + \Big[\sharp_{\Pi_1}(\eta_1)\Big]^h(\frac{1}{(f^h)^2}g_2(\alpha_2,\beta_2)^v).\end{aligned}$$

Since,

$$\Big[\sharp_{\Pi_1}(\eta_1)\Big]^h(\frac{1}{(f^h)^2}g_2(\alpha_2,\beta_2)^v) = \Big(\frac{g_1(J_1 df,\eta_1)}{f^3}\Big)^h g_2(\alpha_2,\beta_2)^v.$$

Thus the result follows. □

Proposition 3.2. *Let $(\tilde M = \tilde M_1 \times_f \tilde M_2, g^f, \Pi)$ be a Riemannian Poisson warped product space and f is a Casimir function on $\tilde M_1$. Then for any $\eta \in \Omega^1(\tilde M)$, we have*

$$(\mathcal{L}_\eta g^f)(\alpha,\beta) = \Big[(\mathcal{L}^1_{\eta_1}g_1)(\alpha_1,\beta_1)\Big]^h + \frac{1}{(f^h)^2}\Big[(\mathcal{L}^2_{\eta_2}g_2)(\alpha_2,\beta_2)\Big]^v,$$

for any $\alpha,\beta \in \Omega^1(\tilde M)$.

Proof. As, f is Casimir function if and only if $J_1 df = 0$. After applying this criterion in Proposition 3.1 provides the result. □

The following proposition is a another characterization of Killing 1-form.

Proposition 3.3. *Let $(\tilde M, g, \Pi)$ be a Riemannian Poisson manifold. A 1-form $\eta \in \Omega^1(\tilde M)$ is a Killing 1-form if and only if*

$$g(\mathcal{D}_\alpha \eta, \alpha) = 0, \tag{7}$$

for any 1-form $\alpha \in \Omega^1(\tilde M)$.

Proof. Since $\eta \in \Omega^1(\tilde M)$ and $\mathcal{D}$ is the contravariant Levi-Civita connection, then

$$(\mathcal{L}_\eta g)(\alpha,\beta) = g(\mathcal{D}_\alpha\eta,\beta) + g(\alpha,\mathcal{D}_\beta\eta), \tag{8}$$

for any $\alpha,\beta \in \Omega^1(\tilde M)$. Putting $\alpha = \beta$ in (8), we have

$$(\mathcal{L}_\eta g)(\alpha,\alpha) = 2g(\mathcal{D}_\alpha\eta,\alpha),$$

for any $\alpha \in \Omega^1(M)$. Thus the result follows. □

In the preceding two propositions, we will provide a result on contravariant warped product space $(\tilde M = \tilde M_1 \times_f \tilde M_2, g^f)$ and Riemannian Poisson warped product space $(\tilde M = \tilde M_1 \times_f \tilde M_2, g^f, \Pi)$, which are helpful to describe the Killing 1-form. Here we will consider $\eta = \eta_1^h + \eta_2^v$ *and* $\alpha = \alpha_1^h + \alpha_2^v$.

Proposition 3.4. *Let $(\tilde{M} = \tilde{M}_1 \times_f \tilde{M}_2, g^f)$ be a contravariant warped product space and $\mathcal{D}$ is the contravariant Levi-Civita connection associated with pair (g^f, Π) on $\tilde{M}$. Then for any $\eta, \alpha \in \Omega^1(\tilde{M})$, we have*

$$g^f(\mathcal{D}_\alpha \eta, \alpha) = g_1(\mathcal{D}^1_{\alpha_1}\eta_1, \alpha_1)^h + \Big(\frac{g_1(J_1 df, \eta_1)}{f^3}\Big)^h (\|\alpha_2\|_2^2)^v + \frac{1}{(f^h)^2} g_2(\mathcal{D}^2_{\alpha_2}\eta_2, \alpha_2)^v.$$

Proof. From Proposition 2.3, for any $\eta, \alpha \in \Omega^1(M)$, we have

$$\begin{aligned}\mathcal{D}_\alpha \eta = &(\mathcal{D}^1_{\alpha_1}\eta_1)^h + \Big(\frac{g_1(J_1 df, \alpha_1)}{f}\Big)^h \eta_2^v + \Big(\frac{g_1(J_1 df, \eta_1)}{f}\Big)^h \alpha_2^v \\ &- \frac{1}{(f^h)^3} g_2(\alpha_2, \eta_2)^v (J_1 df)^h + (\mathcal{D}^2_{\alpha_2}\eta_2)^v. \qquad (9)\end{aligned}$$

Since $g^f(\mathcal{D}_\alpha \eta, \alpha) = g^f(\mathcal{D}_\alpha \eta, \alpha_1^h) + g^f(\mathcal{D}_\alpha \eta, \alpha_2^v)$, thus from (5) and (9), the result follows. □

Proposition 3.5. *Let $(\tilde{M} = \tilde{M}_1 \times_f \tilde{M}_2, g^f, \Pi)$ be a Riemannian Poisson warped product space and f is a Casimir function on $\tilde{M}_1$. Then for any $\eta, \alpha \in \Omega^1(M)$, we have*

$$g^f(\mathcal{D}_\alpha \eta, \alpha) = g_1(\mathcal{D}^1_{\alpha_1}\eta_1, \alpha_1)^h + \frac{1}{(f^h)^2} g_2(\mathcal{D}^2_{\alpha_2}\eta_2, \alpha_2)^v. \qquad (10)$$

Proof. As, f is Casimir function if and only if $J_1 df = 0$. After applying this criterion in Proposition 3.4, provides the result. □

In the following theorem, we have to prove the necessary and sufficient conditions for Killing 1-form on Riemannian Poisson warped product space.

Theorem 3.6. *Let $(\tilde{M} = \tilde{M}_1 \times_f \tilde{M}_2, g^f, \Pi)$ be a Riemannian Poisson warped product space and f is a Casimir function on $\tilde{M}_1$. Then 1-form $\eta \in \Omega^1(M)$ is Killing 1-form if and only if the following conditions holds:*
***(1)**. η_1 is a Killing 1-form on $\tilde{M}_1$.*
***(2)**. η_2 is a Killing 1-form on $\tilde{M}_2$.*

Proof. The if" part is obvious. For the "only if part", let $\eta \in \Omega^1(\tilde{M})$ is Killing 1-form. Putting $\eta = \eta_1^h$ and $\eta = \eta_2^v$ in (10), provide (1) and (2) respectively. □

3.1. Parallel 1-form

Let $(\tilde{M}^n, \Pi)$ be the n-dimensional Poisson manifold and $\mathcal{D}$ is the contravariant Levi-Civita connection associated to (g, Π), then

(i) In ([35],eqn. 5), authors provided contravariant derivative of a multivector field P of degree r i.e., $P \in \mathfrak{X}^r(\tilde{M}) = \Gamma(\Lambda^r T\tilde{M})$ with respect to 1-form $\alpha \in \Omega^1(\tilde{M})$, given by

$$(\mathcal{D}_\alpha P)(\alpha_1, ..., \alpha_r) = \sharp_\Pi(\alpha)(P(\alpha_1, ..., \alpha_r)) - \sum_{i=1}^{r} Q\Big(\alpha_1, ..., \mathcal{D}_\alpha \alpha_i, ..., \alpha_r\Big), \qquad (11)$$

where $\alpha_1, ..., \alpha_r \in \Omega^1(\tilde{M})$.

(ii) Let Q be any tensor field on $\tilde{M}$. In ([33],p. 9), author provided contravariant Laplacian operator corresponding to $\mathcal{D}$ over Q by

$$\Delta^{\mathcal{D}}(Q) := -\sum_{i=1}^{n} \mathcal{D}^2_{\theta_i, \theta_i} Q, \qquad (12)$$

where $\{\theta_1, ..., \theta_n\}$ is any local coframe field on $\tilde{M}$, and

$$\mathcal{D}^2_{\alpha,\beta} = \mathcal{D}_\alpha \mathcal{D}_\beta - \mathcal{D}_{\mathcal{D}_\alpha \beta},$$

is the second order contravariant derivative.

Definition 3.7. *Let $\mathcal{D}$ be the contravariant Levi-Civita connection associated to (Π, g) on Poisson manifold $(\tilde{M}, \Pi)$. A tensor field S is said to be parallel with respect to contravariant Levi-Civita connection $\mathcal{D}$ if*

$$\mathcal{D}S = 0. \tag{13}$$

Remark 3.8. *If we take $S = g$, then it is always parallel.*

From Corollary 4.2, Lemma 4.3 and Corollary 4.7 of [33], we conclude the following lemma. This will be useful later on.

Lemma 3.9. *Let $(\tilde{M}^n, g, \Pi)$ be a compact Riemannian Poisson manifold and a smooth function f on $\tilde{M}$ satisfies $\Delta^{\mathcal{D}}(f) \geq 0$, then $\Delta^{\mathcal{D}}(f) = 0$.*

Bochner [10], provided a result for compact oriented Riemannian manifold $\tilde{M}$, that if Ricci curvature of $\tilde{M}$ is non-positive then every Killing vector field on $\tilde{M}$ is parallel. Later H. H. Wu studied this result in detail (see, [38],p. 324). Now we will prove similar result for Killing 1-form on compact Riemannian Poisson manifold.

Theorem 3.10. *Let η is a Killing 1-form on n-dimensional compact Riemannian Poisson manifold $(\tilde{M}^n, g, \Pi)$ with vanishing $\mathcal{D}_\eta \eta$. If $Ric(\eta, \eta) \leq 0$, then η is parallel.*

Proof. Since η is a Killing 1-form, equation (8) implies that

$$g(\mathcal{D}_\alpha \eta, \beta) + g(\mathcal{D}_\beta \eta, \alpha) = 0, \tag{14}$$

for any $\alpha, \beta \in \Omega^1(\tilde{M})$. Let $\{\theta_1, ..., \theta_n\}$ is any local coframe field on M, then from (12), we have

$$\begin{aligned}\Delta^{\mathcal{D}}\Big(-\frac{1}{2}|\eta|^2\Big) &= \sum_{i=1}^{n}\{\mathcal{D}_{\theta_i}(\mathcal{D}_{\theta_i} g(\eta, \eta)) - \mathcal{D}_{\mathcal{D}_{\theta_i}\theta_i}(g(\eta, \eta))\}\\ &= \sum_{i=1}^{n}\{\mathcal{D}_{\theta_i}(g(\mathcal{D}_{\theta_i}\eta, \eta)) - g(\mathcal{D}_{\mathcal{D}_{\theta_i}\theta_i}\eta, \eta)\}\\ &= |\mathcal{D}\eta|^2 - g(\Delta^{\mathcal{D}}(\eta), \eta),\end{aligned} \tag{15}$$

where $|\mathcal{D}\eta|^2 = \sum_{i=1}^{n} g(\mathcal{D}_{\theta_i}\eta, \mathcal{D}_{\theta_i}\eta)$. Now we will calculate the second term of (15). For any $i \in \{1, ..., n\}$, we have

$$g(\mathcal{D}^2_{\theta_i,\theta_i}\eta, \eta) = g(\mathcal{D}_{\theta_i}\mathcal{D}_{\theta_i}\eta, \eta) - g(\mathcal{D}_{\mathcal{D}_{\theta_i}\theta_i}\eta, \eta). \tag{16}$$

The second term of L. H. S. of the above equation equal to $-g(\mathcal{D}_\eta \eta, \mathcal{D}_{\theta_i}\theta_i)$ by (14), and vanishes as $\mathcal{D}_\eta \eta$. Hence (16), conclude that

$$\begin{aligned}g(\mathcal{D}^2_{\theta_i,\theta_i}\eta, \eta) &= g(\mathcal{D}_{\theta_i}\mathcal{D}_{\theta_i}\eta, \eta)\\ &= g(\mathcal{D}_{\theta_i}\mathcal{D}_\eta\theta_i, \eta) + g(\mathcal{D}_{\theta_i}[\theta_i, \eta]_\Pi, \eta)\\ &= g(\mathcal{D}_{\theta_i}\mathcal{D}_\eta\theta_i, \eta) + \sharp_\Pi(\theta_i) g([\theta_i, \eta]_\Pi, \eta) - g([\theta_i, \eta]_\Pi, \mathcal{D}_{\theta_i}\eta).\end{aligned} \tag{17}$$

Since, α is Killing 1-form therefore

$$g([\theta_i, \eta]_\Pi, \eta) = -\sharp_\Pi(\eta) g(\theta_i, \eta) \tag{18}$$

and

$$\begin{aligned}g([\theta_i, \eta]_\Pi, \mathcal{D}_{\theta_i}\eta) &\overset{(14)}{=} -g(\theta_i, \mathcal{D}_{[\theta_i,\eta]_\Pi}\eta)\\ &= -\sharp_\Pi([\theta_i, \eta]_\Pi) g(\eta, \theta_i) + g(\eta, \mathcal{D}_{[\theta_i,\eta]_\Pi}\theta_i).\end{aligned} \tag{19}$$

After using (18) and (19) in (17), we obtain

$$\begin{aligned} g(\mathcal{D}^2_{\theta_i,\theta_i}\eta,\eta) &= g(\mathcal{D}_{\theta_i}\mathcal{D}_\eta\theta_i,\eta) + \{-\sharp_\Pi(\theta_i)\sharp_\Pi(\eta) + \sharp_\Pi([\theta_i,\eta]_\Pi)\}g(\eta,\theta_i) \\ &\quad - g(\mathcal{D}_{[\theta_i,\eta]_\Pi}\theta_i,\eta) \\ &\overset{(2.1)}{=} g(\mathcal{D}_{\theta_i}\mathcal{D}_\eta\theta_i,\eta) - \sharp_\Pi(\eta)\sharp_\Pi(\theta_i)g(\eta,\theta_i) - g(\mathcal{D}_{[\theta_i,\eta]_\Pi}\theta_i,\eta). \end{aligned} \tag{20}$$

The second term of (20), follows by vanishing of $\mathcal{D}_\eta\eta$

$$\begin{aligned} \sharp_\Pi(\eta)\sharp_\Pi(\theta_i)g(\eta,\theta_i) &= \sharp_\Pi(\eta)\{g(\mathcal{D}_{\theta_i}\eta,\theta_i) + g(\eta,\mathcal{D}_{\theta_i}\theta_i)\} \\ &\overset{(14)}{=} \sharp_\Pi(\eta)g(\eta,\mathcal{D}_{\theta_i}\theta_i) \\ &= g(\mathcal{D}_\eta\mathcal{D}_{\theta_i}\theta_i,\eta). \end{aligned} \tag{21}$$

Using equation (21) in (20), yields

$$g(\mathcal{D}^2_{\theta_i,\eta_i}\eta,\eta) = g(\mathcal{R}(\theta_i,\eta)\theta_i,\eta).$$

After taking summation both sides of the above equation conclude that

$$g(\Delta^{\mathcal{D}}(\eta),\eta) = Ric(\eta,\eta). \tag{22}$$

Now using (22) in (15), we have

$$\Delta^{\mathcal{D}}\Big(-\frac{1}{2}|\eta|^2\Big) = |\mathcal{D}\eta|^2 - Ric(\eta,\eta). \tag{23}$$

Since $Ric(\eta,\eta) \leq 0$ then the right hand side of (23) is non-negative and hence vanishes by Lemma 3.9. It conclude that $|\mathcal{D}\eta|^2 = 0$. This is equivalent to η being parallel. □

In the following theorem, we will prove the above result for compact Riemannian Poisson warped product space.

Theorem 3.11. *Let $(\tilde{M} = \tilde{M}_1 \times_f \tilde{M}_2, g^f, \Pi)$ be a compact Riemannian Poisson warped product space and f is a Casimir function on $\tilde{M}_1$ also let 1-form $\eta = \eta_1^h + \eta_2^v \in \Omega^1(\tilde{M})$. Then*
(1). *$\eta = \eta_1^h + \eta_2^v$ is parallel if the 1-form η_i is a Killing 1-form, $Ric_i(\eta_i,\eta_i) \leq 0$ and $\mathcal{D}^i_{\eta_i}\eta_i$ vanishes, $i = 1,2$.*
(2). *$\eta = \eta_1^h$ is parallel if the 1-form η_1 is a Killing 1-form, $Ric_1(\eta_1,\eta_1) \leq 0$ and $\mathcal{D}^1_{\eta_1}\eta_1$ vanishes.*
(3). *$\eta = \eta_2^v$ is parallel if the 1-form η_2 is a Killing 1-form, $Ric_2(\eta_2,\eta_2) \leq 0$ and $\mathcal{D}^2_{\eta_2}\eta_2$ vanishes.*

Proof. Let U_1 and U_2 are two open subset of $\tilde{M}_1$ and $\tilde{M}_2$ respectively. Assume that $\{dx_1,...,dx_{n_1}\}$ is a local g_1-coframe on U_1 and $\{dy_1,...,dy_{n_2}\}$ is a local g_2-coframe on U_2, then

$$\{dx_1^h,...,dx_{n_1}^h, f^h dy_1^v,...,f^h dy_{n_2}^v\}$$

is a local g^f-coframe on open subset $U_1 \times U_2$ of $\tilde{M}_1 \times \tilde{M}_2$. Thus for any 1-forms $\eta \in \Omega^1(\tilde{M})$, we have

$$|\mathcal{D}\eta|^2 = \sum_{i=1}^{n_1} g^f(\mathcal{D}_{dx_i^h}\eta,\mathcal{D}_{dx_i^h}\eta) + (f^h)^2\sum_{j=1}^{n_2} g^f(\mathcal{D}_{dy_j^v}\eta,\mathcal{D}_{dy_j^v}\eta). \tag{24}$$

Using the condition of Casimir function f in (9) the first term of (24), is given by

$$\begin{aligned} \sum_{i=1}^{n_1} g^f(\mathcal{D}_{dx_i^h}\eta,\mathcal{D}_{dx_i^h}\eta) &= \sum_{i=1}^{n_1} g^f(\mathcal{D}_{dx_i^h}\eta_1^h,\mathcal{D}_{dx_i^h}\eta_1^h) \\ &= \sum_{i=1}^{n_1} g^f((\mathcal{D}^1_{dx_i}\eta_1)^h,(\mathcal{D}^1_{dx_i}\eta_1)^h) \\ &\overset{(5)}{=} \sum_{i=1}^{n_1} g_1(\mathcal{D}^1_{dx_i}\eta_1,\mathcal{D}^1_{dx_i}\eta_1)^h \\ &= (|\mathcal{D}^1\eta_1|^2)^h, \end{aligned} \tag{25}$$

and the second term of (24), is given by

$$
\begin{aligned}
(f^h)^2 \sum_{j=1}^{n_2} g^f(\mathcal{D}_{dy_j^v}\eta, \mathcal{D}_{dy_j^v}\eta) &= (f^h)^2 \sum_{j=1}^{n_2} g^f(\mathcal{D}_{dy_j^v}\eta_2^v, \mathcal{D}_{dy_j^v}\eta_2^v) \\
&= (f^h)^2 \sum_{j=1}^{n_2} g^f((\mathcal{D}^2_{dy_j}\eta_2)^v, (\mathcal{D}^2_{dy_j}\eta_2)^v) \\
&\overset{(5)}{=} \sum_{j=1}^{n_2} g_2(\mathcal{D}^2_{dy_j}\eta_2, \mathcal{D}^2_{dy_j}\eta_2)^v \\
&= (|\mathcal{D}^2\eta_2|^2)^v.
\end{aligned} \tag{26}
$$

After using (25) and (26) in (24), provide that

$$
|\mathcal{D}\eta|^2 = (|\mathcal{D}^1\eta_1|^2)^h + (|\mathcal{D}^2\eta_2|^2)^v. \tag{27}
$$

Thus from Theorem 3.10 and equation (27), follows the result. □

4. 2-Killing 1-form

A 1-form $\eta \in \Omega^1(\tilde{M})$ on a Riemannian Poisson manifold $(\tilde{M}, g, \Pi)$ is said to be 2-Killing 1-form with corresponding to the metric g if

$$
\mathcal{L}_\eta \mathcal{L}_\eta g = 0, \tag{28}
$$

where $\mathcal{L}_\eta$ is the Lie derivative on $\tilde{M}$ corresponding to 1-form η.
The following proposition is alike to the Proposition 3.1 of ([24],p. 6).

Proposition 4.1. *Let $(\tilde{M}, g, \Pi)$ be a Riemannian Poisson manifold and 1-form $\eta \in \Omega^1(\tilde{M})$. Then*

$$
\begin{aligned}
(\mathcal{L}_\eta \mathcal{L}_\eta g)(\alpha, \beta) &= g(\mathcal{D}_\eta \mathcal{D}_\alpha \eta - \mathcal{D}_{[\eta,\alpha]_\Pi}\eta, \beta) \\
&\quad + g(\mathcal{D}_\eta \mathcal{D}_\beta \eta - \mathcal{D}_{[\eta,\beta]_\Pi}\eta, \alpha) + 2g(\mathcal{D}_\alpha \eta, \mathcal{D}_\beta \eta),
\end{aligned} \tag{29}
$$

for any $\alpha, \beta \in \Omega^1(\tilde{M})$.

The following proposition is helpful to describe the definition of 2-Killing 1-form on the Riemannian Poisson manifold.

Proposition 4.2. *Let $(\tilde{M}, g, \Pi)$ be a Riemannian Poisson manifold and 1-form $\eta \in \Omega^1(\tilde{M})$. Then η is 2-Killing 1-form if and only if*

$$
\mathcal{R}(\eta, \alpha, \alpha, \eta) = g(\mathcal{D}_\alpha \eta, \mathcal{D}_\alpha \eta) + g(\mathcal{D}_\alpha \mathcal{D}_\eta \eta, \alpha), \tag{30}
$$

for any $\alpha \in \Omega^1(\tilde{M})$.

Proof. The symmetry of (29) implies that, η is 2-Killing 1-form if and only if $(\mathcal{L}_\eta \mathcal{L}_\eta g)(\alpha, \alpha) = 0$, for any $\alpha \in \Omega^1(\tilde{M})$. Therefore, we have

$$
g(\mathcal{D}_\eta \mathcal{D}_\alpha \eta, \alpha) + g(\mathcal{D}_{[\alpha,\eta]_\Pi}\eta, \alpha) + g(\mathcal{D}_\alpha \eta, \mathcal{D}_\alpha \eta) = 0, \tag{31}
$$

for any $\alpha \in \Omega^1(\tilde{M})$. The curvature tensor $\mathcal{R}$, is given by

$$
\begin{aligned}
\mathcal{R}(\eta, \alpha, \alpha, \eta) &= \mathcal{R}(\alpha, \eta, \eta, \alpha) \\
&= g(\mathcal{R}(\alpha, \eta)\eta, \alpha) \\
&= g(\mathcal{D}_\alpha \mathcal{D}_\eta \eta, \alpha) - g(\mathcal{D}_\eta \mathcal{D}_\alpha \eta, \alpha) - g(\mathcal{D}_{[\alpha,\eta]_\Pi}\eta, \alpha).
\end{aligned} \tag{32}
$$

After using (31) in (32), provides the result (30). □

There is another characterization for a 2-Killing 1-form η on Riemannian Poisson manifold $(\tilde{M}, g, \Pi)$

$$2\mathcal{R}(\eta,\alpha,\beta,\eta) = 2g(\mathcal{D}_\alpha\eta, \mathcal{D}_\beta\eta) + g(\mathcal{D}_\alpha\mathcal{D}_\eta\eta, \beta) + g(\mathcal{D}_\beta\mathcal{D}_\eta\eta, \alpha), \tag{33}$$

for any $\alpha, \beta \in \Omega^1(\tilde{M})$.

In the following two theorems, we will provide Bochner-type results for 2-Killing 1-form on compact Riemannian Poisson manifold and compact Riemannian Poisson warped product space.

Theorem 4.3. *Let η is a 2-Killing 1-form on n-dimensional compact Riemannian Poisson manifold $(\tilde{M}, g, \Pi)$ with vanishing $\mathcal{D}_\eta\eta$. If $Ric(\eta,\eta) \leq 0$, then η is parallel.*

Proof. Assume that $\{dx_1, ..., dx_n\}$ is a local g-coframe on an open subset U of $\tilde{M}$, then from Proposition 4.2, we obtain

$$\sum_{i=1}^n \mathcal{R}(\eta, dx_i, dx_i, \eta) = \sum_{i=1}^n g(\mathcal{D}_{dx_i}\eta, \mathcal{D}_{dx_i}\eta) + \sum_{i=1}^n g(\mathcal{D}_{dx_i}\mathcal{D}_\eta\eta, dx_i).$$

As $\mathcal{D}_\eta\eta$ vanishes and $\mathcal{R}$ is a curvature tensor therefore the last equation implies that

$$Ric(\eta,\eta) = |\mathcal{D}\eta|^2 \leq 0.$$

This follows the result. □

Theorem 4.4. *Let $(M = \tilde{M}_1 \times_f \tilde{M}_2, g^f, \Pi)$ be a compact Riemannian Poisson warped product space and f is a Casimir function on B also let 1-form $\eta = \eta_1^h + \eta_2^v \in \Omega^1(\tilde{M})$. Then*
(1). *$\eta = \eta_1^h + \eta_2^v$ is parallel if the 1-form η_i is a 2-Killing 1-form, $Ric_i(\eta_i, \eta_i) \leq 0$ and $\mathcal{D}^i_{\eta_i}\eta_i$ vanishes, $i = 1, 2$.*
(2). *$\eta = \eta_1^h$ is parallel if the 1-form η_1 is a 2-Killing 1-form, $Ric_1(\eta_1, \eta_1) \leq 0$ and $\mathcal{D}^1_{\eta_1}\eta_1$ vanishes.*
(3). *$\eta = \eta_2^v$ is parallel if the 1-form η_2 is a 2-Killing 1-form, $Ric_2(\eta_2, \eta_2) \leq 0$ and $\mathcal{D}^2_{\eta_2}\eta_2$ vanishes.*

Proof. Proof is similar to the Theorem 3.11. □

In the following two propositions, we will find the expression for 2-Killing 1-form.

Proposition 4.5. *Let $(\tilde{M} = \tilde{M}_1 \times_f \tilde{M}_2, g^f)$ be a contravariant warped product space and $\mathcal{D}$ is the contravariant Levi-Civita connection associated with pair (g^f, Π) on $\tilde{M}$. Then for any 1-forms $\eta \in \Omega^1(\tilde{M})$, we have*

$$\begin{aligned}
(\mathcal{L}_\eta\mathcal{L}_\eta g^f)(\alpha,\beta) &= \Big[(\mathcal{L}^1_{\eta_1}\mathcal{L}^1_{\eta_1}g_1)(\alpha_1,\beta_1)\Big]^h + \frac{1}{(f^h)^2}\Big[(\mathcal{L}^2_{\eta_2}\mathcal{L}^2_{\eta_2}g_2)(\alpha_2,\beta_2)\Big]^v \\
&+ 2\Big(\mathcal{D}^1_{\eta_1}\big(\frac{g_1(J_1df,\eta_1)}{f^3}\big) + \frac{2g_1(J_1df,\eta_1)^2}{f^4}\Big)^h g_2(\alpha_2,\beta_2)^v \\
&+ 2\Big(\frac{\mathcal{D}^1_{\eta_1}(f)g_1(J_1df,\beta_1)}{f^4} + \frac{g_1(J_1df,\beta_1)g_1(J_1df,\eta_1)}{f^4}\Big)^h g_2(\alpha_2,\eta_2)^v \\
&+ 2\Big(\frac{\mathcal{D}^1_{\eta_1}(f)g_1(J_1df,\alpha_1)}{f^4} + \frac{g_1(J_1df,\alpha_1)g_1(J_1df,\eta_1)}{f^4}\Big)^h g_2(\beta_2,\eta_2)^v \\
&+ 4\Big(\frac{g_1(J_1df,\eta_1)}{f^3}\Big)^h\Big((\mathcal{L}^2_{\eta_2}g_2)(\alpha_2,\beta_2)\Big)^v + 2\Big(\frac{g_1(J_1df,\alpha_1)}{f^3}\Big)^h g_2(\eta_2,\mathcal{D}^2_{\beta_2}\eta_2)^v \\
&+ 2\Big(\frac{g_1(J_1df,\beta_1)}{f^3}\Big)^h g_2(\eta_2,\mathcal{D}^2_{\alpha_2}\eta_2)^v + 4\Big(\frac{g_1(J_1df,\alpha_1)g_1(J_1df,\beta_1)}{f^4}\Big)^h(\|\eta_2\|_2^2)^v,
\end{aligned}$$

for any 1-forms $\alpha, \beta \in \Omega^1(\tilde{M})$.

Proof. See Appendix. □

Proposition 4.6. *Let $(\tilde{M} = \tilde{M}_1 \times_f \tilde{M}_2, g^f, \Pi)$ be a Riemannian Poisson warped product space and f is a Casimir function on $\tilde{M}_1$. Then for any 1-forms $\eta \in \Omega^1(\tilde{M})$, we have*

$$(\mathcal{L}_\eta \mathcal{L}_\eta g^f)(\alpha, \beta) = \Big[(\mathcal{L}^1_{\eta_1}\mathcal{L}^1_{\eta_1} g_1)(\alpha_1, \beta_1)\Big]^h + \frac{1}{(f^h)^2}\Big[(\mathcal{L}^2_{\eta_2}\mathcal{L}^2_{\eta_2} g_2)(\alpha_2, \beta_2)\Big]^v,$$

for any $\alpha, \beta \in \Omega^1(\tilde{M})$.

Proof. Using the property of Casimir function f in Proposition 4.5, provides this result. □

In the following theorem, we will provide necessary and sufficient conditions for 2-Killing 1-form on Riemannian Poisson warped product space.

Theorem 4.7. *Let $(\tilde{M} = \tilde{M}_1 \times_f \tilde{M}_2, g^f, \Pi)$ be a Riemannian Poisson warped product space and f is a Casimir function on $\tilde{M}_1$. Then 1-form $\eta \in \Omega^1(\tilde{M})$ is 2-Killing 1-form if and only if the following conditions holds:*
***(1)**. η_1 is a 2-Killing 1-form on $\tilde{M}_1$.*
***(2)**. η_2 is a 2-Killing 1-form on $\tilde{M}_2$.*

Proof. The if" part is obvious. For the "only if part", let $\eta \in \Omega^1(\tilde{M})$ is 2-Killing 1-form. Putting $\eta = \eta_1^h$ and $\eta = \eta_2^v$ in Proposition 4.6 provide (1) and (2) respectively. □

Now, we will provide a theorem for 2-Killing 1-form. From ([30],eqn. 2.5), Christoffel symbols Γ^{ij}_k defined as

$$\mathcal{D}_{dx_i} dx_j = \Gamma^{ij}_k dx_k. \tag{34}$$

Theorem 4.8. *Let $(\mathbb{R}^2, g, \Pi)$ be a Riemannian Poisson manifold (where g is the cometric of the Riemannian metric $\tilde{g} = (dx^1)^2 + (dx^2)^2$, $\Pi = \Pi^{12}\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}$) and $\eta = \eta_1 dx^1 + \eta_2 dx^2 \in \Omega^1(\mathbb{R}^2)$. Then η is 2- Killing form if and only if*

$$2\mathcal{R}(\eta, dx^1, dx^2, \eta) = -\Big(2(T_1T_3 + T_2T_4) + \frac{\partial(T_5\Pi^{12})}{\partial x^1} + \frac{\partial(T_6\Pi^{12})}{\partial x^2}\Big),$$

where

$$T_1 = \Pi^{12}\frac{\partial \eta_1}{\partial x^2} + \eta_2\frac{\partial \Pi^{12}}{\partial x^1}, \quad T_2 = \Pi^{12}\frac{\partial \eta_2}{\partial x^2} - \eta_1\frac{\partial \Pi^{12}}{\partial x^1},$$
$$T_3 = \Pi^{12}\frac{\partial \eta_1}{\partial x^1} - \eta_2\frac{\partial \Pi^{12}}{\partial x^2}, \quad T_4 = \Pi^{12}\frac{\partial \eta_2}{\partial x^1} + \eta_1\frac{\partial \Pi^{12}}{\partial x^2},$$
$$T_5 = \eta_1\Pi^{12}\frac{\partial \eta_1}{\partial x^2} - \eta_2\Pi^{12}\frac{\partial \eta_1}{\partial x^1} + \eta_1\eta_2\frac{\partial \Pi^{12}}{\partial x^1} + \eta_2^2\frac{\partial \Pi^{12}}{\partial x^2},$$
$$T_6 = \eta_2\Pi^{12}\frac{\partial \eta_2}{\partial x^1} - \eta_1\Pi^{12}\frac{\partial \eta_2}{\partial x^2} + \eta_1\eta_2\frac{\partial \Pi^{12}}{\partial x^2} + \eta_1^2\frac{\partial \Pi^{12}}{\partial x^1}.$$

Proof. Since $\{dx^1, dx^2\}$ is orthonormal coframe field on $\mathbb{R}^2$ therefore (33), implies that

$$2\mathcal{R}(\eta, dx^1, dx^2, \eta) = 2g(\mathcal{D}_{dx^1}\eta, \mathcal{D}_{dx^2}\eta) + g(\mathcal{D}_{dx^1}\mathcal{D}_\eta\eta, dx^2) + g(\mathcal{D}_{dx^2}\mathcal{D}_\eta\eta, dx^1). \tag{35}$$

The local components of $\tilde{g}$ are given by

$$\begin{cases} \tilde{g}_{11} = \tilde{g}(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^1}) = 1, \\ \tilde{g}_{22} = \tilde{g}(\frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^2}) = 1, \\ \tilde{g}_{12} = \tilde{g}(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}) = 0. \end{cases} \tag{36}$$

As g is the cometric of the metric $\tilde{g}$ then its local components are given by

$$\begin{cases} g^{11} = g(dx^1, dx^1) = 1, \\ g^{22} = g(dx^2, dx^2) = 1, \\ g^{12} = g(dx^1, dx^2) = 0. \end{cases} \tag{37}$$

Now, from ([39],eqn. 6.2), Christoffel symbols Γ_k^{ij} (where $i, j, k \in \{1, 2\}$) defined as

$$\Gamma_k^{ij} = \frac{1}{2}\sum_l \sum_m g_{mk}\Big(\Pi^{il}\frac{\partial g^{jm}}{\partial x_l} + \Pi^{jl}\frac{\partial g^{im}}{\partial x_l} - \Pi^{ml}\frac{\partial g^{ij}}{\partial x_l} - g^{li}\frac{\partial \Pi^{jm}}{\partial x_l} - g^{lj}\frac{\partial \Pi^{im}}{\partial x_l}\Big) + \frac{1}{2}\frac{\partial \Pi^{ij}}{\partial x_k}. \tag{38}$$

Therefore from (37) and (3.8), we have

$$\begin{aligned} &\Gamma_1^{11} = 0, \quad \Gamma_1^{12} = \frac{\partial \Pi^{12}}{\partial x^1}, \quad \Gamma_1^{21} = 0, \quad \Gamma_1^{22} = \frac{\partial \Pi^{12}}{\partial x^2}, \\ &\Gamma_2^{11} = -\frac{\partial \Pi^{12}}{\partial x^1}, \quad \Gamma_2^{12} = 0, \quad \Gamma_2^{21} = -\frac{\partial \Pi^{12}}{\partial x^2}, \quad \Gamma_2^{22} = 0. \end{aligned} \tag{39}$$

Hence (34) and (39), conclude that

$$\begin{aligned} &\mathcal{D}_{dx^1}dx^1 = -\frac{\partial \Pi^{12}}{\partial x^1}dx^2, \quad \mathcal{D}_{dx^1}dx^2 = \frac{\partial \Pi^{12}}{\partial x^1}dx^1, \\ &\mathcal{D}_{dx^2}dx^1 = -\frac{\partial \Pi^{12}}{\partial x^2}dx^2, \quad \mathcal{D}_{dx^2}dx^2 = \frac{\partial \Pi^{12}}{\partial x^2}dx^1. \end{aligned} \tag{40}$$

By the properties of contravariant Levi-Civita connection $\mathcal{D}$ and equation (40), we have

$$\mathcal{D}_{dx^1}\eta = T_1 dx^1 + T_2 dx^2, \tag{41}$$
$$\mathcal{D}_{dx^2}\eta = -T_3 dx^1 - T_4 dx^2, \tag{42}$$
$$\mathcal{D}_{\eta}\eta = T_5 dx^1 - T_6 dx^2. \tag{43}$$

Equations (41), (42) and (37), provides

$$g(\mathcal{D}_{dx^1}\eta, \mathcal{D}_{dx^2}\eta) = -T_1T_3 - T_2T_4. \tag{44}$$

Equations (43) and (37), provides

$$g(\mathcal{D}_{dx^1}\mathcal{D}_\eta\eta, dx^2) = -T_5\frac{\partial \Pi^{12}}{\partial x^1} - \Pi^{12}\frac{\partial T_6}{\partial x^2}, \tag{45}$$
$$g(\mathcal{D}_{dx^1}\mathcal{D}_\eta\eta, dx^2) = -T_6\frac{\partial \Pi^{12}}{\partial x^2} - \Pi^{12}\frac{\partial T_5}{\partial x^1}. \tag{46}$$

Using equations (44), (45) and (46) in (35), proves this result. □

Appendix. Proof of Proposition 4.5

Equation (29), is given by

$$\begin{aligned} (\mathcal{L}_\eta\mathcal{L}_\eta g)(\alpha, \beta) &= g(\mathcal{D}_\eta\mathcal{D}_\alpha\eta, \beta) + g(\mathcal{D}_\eta\mathcal{D}_\beta\eta, \alpha) \\ &\quad - g(\mathcal{D}_{[\eta,\alpha]_\Pi}\eta, \beta) - g(\mathcal{D}_{[\eta,\beta]_\Pi}\eta, \alpha) + 2g(\mathcal{D}_\alpha\eta, \mathcal{D}_\beta\eta). \end{aligned} \tag{47}$$

Using (5) and Proposition 2.2 in the first term P_1 of (47), we have

$$\begin{aligned} P_1 &= g(\mathcal{D}_\eta\mathcal{D}_\alpha\eta, \beta) \\ &= g(\mathcal{D}_\eta\mathcal{D}_\alpha\eta, \beta_1^h) + g(\mathcal{D}_\eta\mathcal{D}_\alpha\eta, \beta_2^v). \end{aligned}$$

Assume that $S_1 = \mathcal{D}_\eta \mathcal{D}_\alpha \eta$, therefore

$$\begin{aligned} S_1 &= \mathcal{D}_{\eta_1{}^h}\mathcal{D}_\alpha\eta + \mathcal{D}_{\eta_2{}^v}\mathcal{D}_\alpha\eta \\ &= (\mathcal{D}^1_{\eta_1}\mathcal{D}^1_{\alpha_1}\eta_1)^h + (\mathcal{D}^2_{\eta_2}\mathcal{D}^2_{\alpha_2}\eta_2)^v - \Big(\frac{\mathcal{D}^1_{\eta_1}J_1df}{f^3}\Big)^h g_2(\alpha_2,\eta_2)^v + \Big(\frac{g_1(J_1df,\alpha_1)}{f}\Big)^h(\mathcal{D}^2_{\eta_2}\eta_2)^v \\ &+ \Big(\frac{g_1(J_1df,\eta_1)}{f}\Big)^h(\mathcal{D}^2_{\alpha_2}\eta_2 + \mathcal{D}^2_{\eta_2}\alpha_2)^v + \Big[\Big(\frac{3(\mathcal{D}^1_{\eta_1}f) - g_1(J_1df,\eta_1)}{f^4}\Big)^h g_2(\alpha_2,\eta_2)^v \\ &+ \Big(\frac{g_1(J_1df,\alpha_1)}{f^4}\Big)^h(\|\eta_2\|_2^2)^v - \frac{1}{(f^h)^3}g_2(\mathcal{D}^2_{\alpha_2}\eta_2,\eta_2)^v - \frac{1}{(f^h)^3}\Big(\mathcal{D}^2_{\eta_2}g_2(\alpha_2,\eta_2)\Big)^v\Big](J_1df)^h \\ &+ \Big[\frac{g_1(J_1df,\eta_1)^2}{f^2} - \frac{(\mathcal{D}^1_{\eta_1}f)g_1(J_1df,\eta_1)}{f^2} + \frac{\mathcal{D}^1_{\eta_1}g_1(J_1df,\eta_1)}{f}\Big]^h\alpha_2^v + \Big[\Big(\frac{\mathcal{D}^1_{\eta_1}g_1(J_1df,\alpha_1)}{f}\Big)^h \\ &+ \Big(\frac{g_1(J_1df,\mathcal{D}^1_{\alpha_1}\eta_1)}{f}\Big)^h + \Big(\frac{g_1(J_1df,\alpha_1)g_1(J_1df,\eta_1)}{f^2}\Big)^h - \Big(\frac{(\mathcal{D}^1_{\eta_1}f)g_1(J_1df,\alpha_1)}{f^2}\Big)^h \\ &- \Big(\frac{\|J_1df\|_1^2}{f^4}\Big)^h g_2(\alpha_2,\eta_2)^v\Big]\eta_2^v. \end{aligned}$$

Using S_1 in P_1, provides

$$\begin{aligned} P_1 &= g_1(\mathcal{D}^1_{\eta_1}\mathcal{D}^1_{\alpha_1}\eta_1,\beta_1)^h + \frac{1}{(f^h)^2}g_2(\mathcal{D}^2_{\eta_2}\mathcal{D}^2_{\alpha_2}\eta_2,\beta_2)^v - \Big(\frac{g_1(\mathcal{D}^1_{\eta_1}J_1df,\beta_1)}{f^3}\Big)^h g_2(\alpha_2,\eta_2)^v \\ &+ \Big(\frac{g_1(J_1df,\eta_1)}{f^3}\Big)^h g_2(\mathcal{D}^2_{\alpha_2}\eta_2 + \mathcal{D}^2_{\eta_2}\alpha_2,\beta_2)^v + \Big(\frac{g_1(J_1df,\alpha_1)}{f^3}\Big)^h g_2(\mathcal{D}^2_{\eta_2}\eta_2,\beta_2)^v \\ &+ \Big[\Big(\frac{g_1(J_1df,\alpha_1)}{f^4}\Big)^h(\|\eta_2\|_2^2)^v - \Big(\frac{g_1(J_1df,\eta_1)}{f^4}\Big)^h g_2(\eta_2,\alpha_2)^v + 3\Big(\frac{\mathcal{D}^1_{\eta_1}f}{f^4}\Big)^h g_2(\alpha_2,\eta_2)^v \\ &- \frac{1}{(f^h)^3}g_2(\mathcal{D}^2_{\alpha_2}\eta_2,\eta_2)^v - \frac{1}{(f^h)^3}\Big(\mathcal{D}^2_{\eta_2}g_2(\alpha_2,\eta_2)\Big)^v\Big]g_1(J_1df,\beta_1)^h + \Big[\frac{g_1(J_1df,\eta_1)^2}{f^4} \\ &- \frac{(\mathcal{D}^1_{\eta_1}f)g_1(J_1df,\eta_1)}{f^4} + \frac{\mathcal{D}^1_{\eta_1}g_1(J_1df,\eta_1)}{f^3}\Big]^h g_2(\alpha_2,\beta_2)^v + \Big[\Big(\frac{\mathcal{D}^1_{\eta_1}g_1(J_1df,\alpha_1)}{f^3}\Big)^h \\ &+ \Big(\frac{g_1(J_1df,\mathcal{D}^1_{\alpha_1}\eta_1)}{f^3}\Big)^h + \Big(\frac{g_1(J_1df,\alpha_1)g_1(J_1df,\eta_1)}{f^4}\Big)^h - \Big(\frac{(\mathcal{D}^1_{\eta_1}f)g_1(J_1df,\alpha_1)}{f^4}\Big)^h \\ &- \Big(\frac{\|J_1df\|_1^2}{f^6}\Big)^h g_2(\alpha_2,\eta_2)^v\Big]g_2(\eta_2,\beta_2)^v. \end{aligned}$$

After exchanging α and β in the last equation provides the second term P_2 of (47), is given by

$$\begin{aligned} P_2 &= g(\mathcal{D}_\eta\mathcal{D}_\alpha\eta,\beta) \\ &= g_1(\mathcal{D}^1_{\eta_1}\mathcal{D}^1_{\beta_1}\eta_1,\alpha_1)^h + \frac{1}{(f^h)^2}g_2(\mathcal{D}^2_{\eta_2}\mathcal{D}^2_{\beta_2}\eta_2,\alpha_2)^v - \Big(\frac{g_1(\mathcal{D}^1_{\eta_1}J_1df,\alpha_1)}{f^3}\Big)^h g_2(\beta_2,\eta_2)^v \\ &+ \Big(\frac{g_1(J_1df,\eta_1)}{f^3}\Big)^h g_2(\mathcal{D}^2_{\beta_2}\eta_2 + \mathcal{D}^2_{\eta_2}\alpha_2,\alpha_2)^v + \Big(\frac{g_1(J_1df,\beta_1)}{f^3}\Big)^h g_2(\mathcal{D}^2_{\eta_2}\eta_2,\alpha_2)^v \\ &+ \Big[\Big(\frac{g_1(J_1df,\beta_1)}{f^4}\Big)^h(\|\eta_2\|_2^2)^v - \Big(\frac{g_1(J_1df,\eta_1)}{f^4}\Big)^h g_2(\eta_2,\beta_2)^v + 3\Big(\frac{\mathcal{D}^1_{\eta_1}f}{f^4}\Big)^h g_2(\beta_2,\eta_2)^v \end{aligned}$$

$$- \frac{1}{(f^h)^3} g_2(\mathcal{D}^2_{\beta_2}\eta_2,\eta_2)^v - \frac{1}{(f^h)^3}\Big(\mathcal{D}^2_{\eta_2} g_2(\beta_2,\eta_2)\Big)^v\Big] g_1(J_1df,\alpha_1)^h + \Big[\frac{g_1(J_1df,\eta_1)^2}{f^4}$$
$$- \frac{(\mathcal{D}^1_{\eta_1} f) g_1(J_1df,\eta_1)}{f^4} + \frac{\mathcal{D}^1_{\eta_1} g_1(J_1df,\eta_1)}{f^3}\Big]^h g_2(\beta_2,\alpha_2)^v + \Big[\Big(\frac{\mathcal{D}^1_{\eta_1} g_1(J_1df,\beta_1)}{f^3}\Big)^h$$
$$+ \Big(\frac{g_1(J_1df,\mathcal{D}^1_{\beta_1}\eta_1)}{f^3}\Big)^h + \Big(\frac{g_1(J_1df,\beta_1) g_1(J_1df,\eta_1)}{f^4}\Big)^h - \Big(\frac{(\mathcal{D}^1_{\eta_1} f) g_1(J_1df,\beta_1)}{f^4}\Big)^h$$
$$\Big(\frac{\|J_1df\|_1^2}{f^6}\Big)^h g_2(\beta_2,\eta_2)^v\Big] g_2(\eta_2,\alpha_2)^v.$$

Again using (5) and Proposition 2.2 in the third term P_3 of (47), we have

$$P_3 = g(\mathcal{D}_{[\eta,\alpha]_\Pi}\eta,\beta)$$
$$= g(\mathcal{D}_{[\eta,\alpha]_\Pi}\eta,\beta_1^h) + g(\mathcal{D}_{[\eta,\alpha]_\Pi}\eta,\beta_2^v).$$

Assume that $S_2 = \mathcal{D}_{[\eta,\alpha]_\Pi}\eta$, therefore

$$S_2 = \mathcal{D}_{[\eta,\alpha]_\Pi}\eta_1^h + \mathcal{D}_{[\eta,\alpha]_\Pi}\eta_2^v$$
$$= \mathcal{D}_{[\eta_1,\alpha_1]^h_{\Pi_1}}\eta_1^h + \mathcal{D}_{[\eta_1,\alpha_1]^h_{\Pi_1}}\eta_2^v + \mathcal{D}_{[\eta_2,\alpha_2]^v_{\Pi_2}}\eta_1^h + \mathcal{D}_{[\eta_2,\alpha_2]^v_{\Pi_2}}\eta_2^v$$
$$= (\mathcal{D}^1_{[\eta_1,\alpha_1]_{\Pi_1}}\eta_1)^h + (\mathcal{D}^2_{[\eta_2,\alpha_2]_{\Pi_2}}\eta_2)^v + \Big(\frac{g_1(J_1df,\eta_1)}{f}\Big)^h [\eta_2,\alpha_2]^v_{\Pi_2}$$
$$+ \Big(\frac{g_1(J_1df,[\eta_1,\alpha_1]_{\Pi_1})}{f}\Big)^h \eta_2^v - \Big(\frac{J_1df}{f^3}\Big)^h g_2(\eta_2,[\eta_2,\alpha_2]_{\Pi_2})^v.$$

Using S_2 in P_3, provides

$$P_3 = g_1(\mathcal{D}^1_{[\eta_1,\alpha_1]_{\Pi_1}}\eta_1,\beta_1)^h + \frac{1}{(f^h)^2} g_2(\mathcal{D}^2_{[\eta_2,\alpha_2]_{\Pi_2}}\eta_2,\beta_2)^v$$
$$+ \Big(\frac{g_1(J_1df,\eta_1)}{f^3}\Big)^h g_2([\eta_2,\alpha_2]_{\Pi_2},\beta_2)^v + \Big(\frac{g_1(J_1df,[\eta_1,\alpha_1]_{\Pi_1})}{f^3}\Big)^h g_2(\eta_2,\beta_2)^v - \Big(\frac{g_1(J_1df,\beta_1)}{f^3}\Big)^h g_2(\eta_2,[\eta_2,\alpha_2]_{\Pi_2})^v.$$

After exchanging α and β in the last equation provides the fourth term P_4 of (47), is given by

$$P_4 = g_1(\mathcal{D}^1_{[\eta_1,\beta_1]_{\Pi_1}}\eta_1,\alpha_1)^h + \frac{1}{(f^h)^2} g_2(\mathcal{D}^2_{[\eta_2,\beta_2]_{\Pi_2}}\eta_2,\alpha_2)^v$$
$$+ \Big(\frac{g_1(J_1df,\eta_1)}{f^3}\Big)^h g_2([\eta_2,\beta_2]_{\Pi_2},\alpha_2)^v + \Big(\frac{g_1(J_1df,[\eta_1,\beta_1]_{\Pi_1})}{f^3}\Big)^h g_2(\eta_2,\alpha_2)^v - \Big(\frac{g_1(J_1df,\alpha_1)}{f^3}\Big)^h g_2(\eta_2,[\eta_2,\beta_2]_{\Pi_2})^v.$$

Same as above manipulations the fifth term P_5 of (47), is given by

$$P_5 = g_1(\mathcal{D}^1_{\alpha_1}\eta_1,\mathcal{D}^1_{\beta_1}\eta_1)^h + \frac{1}{(f^h)^2} g_2(\mathcal{D}^2_{\alpha_2}\eta_2,\mathcal{D}^2_{\beta_2}\eta_2)^v + \Big(\frac{g_1(J_1df,\alpha_1)}{f^3}\Big)^h g_2(\mathcal{D}^2_{\beta_2}\eta_2,\eta_2)^v$$
$$+ \Big(\frac{g_1(J_1df,\beta_1)}{f^3}\Big)^h g_2(\mathcal{D}^2_{\alpha_2}\eta_2,\eta_2)^v + \Big(\frac{\|J_1df\|_1^2}{f^6}\Big)^h g_2(\alpha_2,\eta_2)^v g_2(\beta_2,\eta_2)^v$$
$$+ \Big(\frac{g_1(J_1df,\alpha_1) g_1(J_1df,\beta_1)}{f^4}\Big)^h (\|\eta_2\|_2^2)^v + \Big(\frac{g_1(J_1df,\eta_1)^2}{f^4}\Big)^h g_2(\alpha_2,\beta_2)^v$$
$$+ \Big[\frac{g_1(J_1df,\alpha_1) g_1(J_1df,\eta_1)}{f^4} - \frac{g_1(J_1df,\mathcal{D}^1_{\alpha_1}\eta_1)}{f^3}\Big]^h g_2(\beta_2,\eta_2)^v$$
$$+ \Big[\frac{g_1(J_1df,\beta_1) g_1(J_1df,\eta_1)}{f^4} - \frac{g_1(J_1df,\mathcal{D}^1_{\beta_1}\eta_1)}{f^3}\Big]^h g_2(\alpha_2,\eta_2)^v$$

$$+\Big(\frac{g_1(J_1df,\eta_1)}{f^3}\Big)^h\Big(g_2(\mathcal{D}^2_{\alpha_2}\eta_2,\beta_2)+g_2(\mathcal{D}^2_{\beta_2}\eta_2,\alpha_2)\Big)^v.$$

Using terms P_1, P_2, P_3, P_4 and P_5 in (47) and after some manipulations provide the result.

Acknowledgement

The authors wish to express their sincere thanks and gratitude to the referees for his valuable suggestions toward the improvement of the paper.

References

[1] R. Bishop and B. O'Neill, *Manifolds of negative curvature*, Trans. Am. Math. Soc. 145(1969), 1-49.
[2] D. S. Kim and Y. H. Kim, *On compact Einstein warped product spaces with nonpositive scalar curvature* , Proceedings of the American Mathematical Socity, vol.131, No.8 (2003), 2573-2576.
[3] D. S. Kim, *Compact Einstein warped product spaces*, Trends in Mathematics Information Center for Mathematical Sciences, 5(2), (2002), 15.
[4] S. Kim, *Warped products and Einstein metrics*, Journal of Physics A: Mathematical and General, 39 (20) (2006), 1-15.
[5] M. Rimoldi, *A remark on Einstein warped products*, Pacific Journal Of Mathematics, 252 (1) (2011).
[6] A. S. Diallo, *Compact Einstein warped product manifolds*, Afrika Matematika, 25 (2) (2014), 267-270.
[7] Q. Qu and Y. Wang, *Multiply warped products with a quarter-Symmetric connection*, Journal of Mathematical Analysis and Applications, vol. 431, Issue 2, 15 November 2015, 955-987.
[8] F. Dobarro and B. Unal, *Curvature of multiply warped product*, Journal of Geometry and Physics, 55(2005), 75-106.
[9] S. Pahan, B. Pal and A. Bhattacharyya, *On Einstein warped products with a quarter-symmetric connection*, International Journal of Geometric Methods in Modern Physics, vol. 14, No.4 (2017) 1750050 (15 pages).
[10] S. Bochner, *Vector fields and Ricci curvature*, Bull. Amer. Math. Soc., 52 (09) (1946), 776-797.
[11] S. Bochner, *Curvature and Betti numbers*, Ann. of Math., 49 (1948), 379-390.
[12] S. Bochner, *Vector fields on complex and real manifolds*, Ann. of Math., 52 (1950), 642-649.
[13] K. Yano, *On harmonic and Killing vector fields*, Ann. of Math., 55 (1952), 38-45.
[14] K. Yano, *Harmonic and Killing vector fields on compact orientable Riemannian spaces with boundary*, Ann. of Math., 69 (1959), 588-597.
[15] S. Yorozu, *Killing vector fields on complete Riemannian manifolds*, Proc. Amer. Math. Soc., 84 (1982), 115-120.
[16] S. Yorozu, *Killing vector fields on non-compact Riemannian manifolds with boundary*, Kodai Math. J., 5 (1982), 426-433.
[17] S. Deshmukh, *Conformal vector fields and eigenvectors of Laplacian operator*, Math. Phys. Geom., 15 (2012), 63-172.
[18] S. Deshmukh and F.R. A1-Solamy, *Conformal vector fields on a Riemannian manifold*, Balkan J. Geom. Appl., 19(2) (2014), 86-93.
[19] S. Deshmukh, *Geometry of conformal vector fields*, Arab J Math Sci, 23 (2017), 44-73.
[20] K. L. Duggal and R. Sharma, *Conformal Killing vector fields on spacetime solutions of Einstein's equations and initial data*, Nonlinear Anal., 63 (2005), e447-e454.
[21] W. Kuhnel and H. Rademacher, *Conformal vector fields on pseudo-Riemannian spaces*, J. Geom. Appl. 7 (1997), 237–250.
[22] H. K. E1-Sayied, S. Shenawy and N. Syied *Conformal vector fields on doubly warped product manifolds and applications*, Advances in Mathematical Physics, vol 2016, Article ID 6508309, 11 pages, http://dx.doi.org/10.1155/2016/6508309.
[23] T. Opera, *2-Killing vector fields on Riemannian manifolds*, Balkan J. Geom. Appl., 13 (1) (2008), 87-92.
[24] S. Shenawy and B. Ünal, *2-Killing vector fields on warped product manifolds*, International Journal of Mathematics, 26 (9) (2015), 1550065, 17 pages.
[25] Z. Erjavec, *Generalizations of Killing vector fields in Sol Space*, Faculty of Sciences and Mathematics, 33:15 (2019), 4803–4810.
[26] S. D. Poisson, *Sur la variation des constantes arbitraires dans les questions de mécanique*, J. Ecole Poytechnique, 8(15) (1809), 266-344.
[27] S. Lie, *Theorie der Transformationsgruppen*, Zweiter Abschnitt, unter Mitwirkung von Prof. Dr. Friedrich Engel (Teubner, Leipzig, 1890).
[28] I. Vaisman, *Lectures on the geometry of Poisson manifolds*, Progress in Mathematics, vol.118, Birkhauser Verlag, Basel, 1994.
[29] J. P. Dufour, N. T. Zung, *Poisson structures and their normal forms*, Progress in Mathematics, vol.242, Birkh$\breve{a}$user Verlag, Basel, 2005.
[30] R. L. Fernandes, *Connections in Poisson geometry I: Holonomy and invariants*, J. Differential Geometry, 54(2000), 303-365.
[31] M. Boucetta, *Compatibilités des structures pseudo-riemanniennes et des structures de Poisson*, C. R. Acad. Sci. Paris, 333(2001), 763-768.
[32] M. Boucetta, *Poisson manifolds with compatible pseudo-metric and pseudo-Riemannian Lie algebras,*, Differential Geometry and its Applications, 20(2004), 279-291.
[33] Z. Saassai, *A Laplace operator for Poisson manifolds*, Differential Geometry and its Applications, 68 (2020), pages 101576.
[34] R. Nasri, M. Djaa, *On the geometry of the product Riemannian manifold with the Poisson structure*, International Electronic Journal of Geometry, 3 (2010), 1-14.
[35] Y. Ait Amrane, R. Nasri, A. Zeglaoui, *Warped Poisson brackets on warped products*, Journal of Geometric Mechanics, 6 (3) (2014), 279-296.
[36] D. Djebbouri, S. Ouakkas, *Product of statistical manifolds with doubly warped product*, Gen. Math. Notes, 31(2), (2015), 16-28.
[37] B. O'Neill, *Semi-Riemannian Geometry. With Applications to Relativity*, Pure and Applied Mathematics, 103, Academic Press, Inc., New York, 1983.
[38] H. H. Wu, *The Bochner technique in differential geometry*, in: Mathematical Reports, vol.3, Part 2, 1988, pp. 289-538
[39] B. Pal and P. Kumar, *Einstein Poisson warped product space*, Class. Quantum Grav., 38 (2021), 065004 (29pp).