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Modified Bernstein-Kantorovich Operators Reproducing Affine Functions

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Abstract. In the present paper, we introduce a new variant of Bernstein-Kantorovich operators which reproduce affine functions. The approximation rate of the new operators for continuous functions and Voronovskaja's asymptotic estimate are obtained.

1. Introduction

Let $C[0,1]$ be the class of all continuous functions on $[0,1]$. For any given $f(x) \in C[0,1]$, the well-known Bernstein operator is defined by

$$B_n(f,x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x),\ x \in [0,1],$$

where $p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k},\ k = 0, 1, \cdots, n.$

There are lots of different generalizations of Bernstein operators. Among them, the Kantorovich type variant and Durrmeyer type variant are most well known, which are defined by

$$K_n(f,x) := \sum_{k=0}^{n} (n+1) p_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt,$$

$$D_n(f,x) := \sum_{k=0}^{n} (n+1) p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t)dt.$$

$K_n(f,x)$ and $D_n(f,x)$ are called Bernstein-Kantorovich operators and Bernstein-Durrmeyer operators respectively. Both $K_n(f,x)$ and $D_n(f,x)$ are positive operators, but neither $K_n(f,x)$ nor $D_n(f,x)$ can reproduce the affine functions, which is different from classical Bernstein operators. Chen [3], Goodman and Sharma [10] firstly introduced the genuine Bernstein-Durrmeyer defined as follows:

$$D_n^*(f,x) := f(0)p_{n,0}(x) + f(1)p_{n,n}(x) + (n-1)\sum_{k=1}^{n-1} p_{n,k}(x) \int_0^1 p_{n-2,k-1}(t) f(t)dt,$$

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The operators $D_n^*(f,x)$ are limits of the Bernstein-Durrmeyer operators with Jacobi weights. Comparing with the usual Bernstein-Durrmeyer operators, one of the advantages of $D_n^*(f,x)$ is that $D_n^*(f,x)$ reproduce the affine functions. Lots of authors have done many excellent works on both the direct and converse results,Voronovskaja's asymptotic estimate of the approximation by $D_n^*(f,x)$ and its generalizations (see [1], [6]-[11], [16]-[19]).

Analogue constructions of various families of approximation operators such as Szász-Mirakjan operators, Szász-Mirakjan Beta type operators, Srivastava-Gupta operators, Bleimann-Butzer-Hann operators, and so on, were given by many authors ([12], [20]-[25]). The approximation properties of these operators are well investigated.

Recently, Bustamante ([2]) introduced a very interesting modification of the Szász-Mirakjan-Kantorovich operators which reproduce the affine functions. The most important idea in the modification is to replace the integral $\int_{I_{nk}} f(t)dt$ with $\int_{I_{nk}} f(a_k t)dt$ in the definition of the usual Szász-Mirakjan-Kantorovich operators, where $I_{nk} = [k/n, (k+1)/n]$ and $a_k = 2k/(2k+1)$ is a scaling factor. The new modified operators have many excellent approximation properties.

Motivated by the constructions of the genuine Bernstein-Durrmeyer operators and the ideas of Bustamante, we introduce a new variant of the classical Bernstein-Kantorovich operators which reproduce the affine functions. Our new modified Bernstein-Kantorovich operators are define as follows:

$$K_n^*(f,x) := f(0)p_{n,0}(x) + f(1)p_{n,n}(x) + \sum_{k=1}^{n-1}(n+1)p_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(a_{nk}t)dt,$$

where the scaling factor $a_{nk} := \frac{n+1}{n}\cdot\frac{2k}{2k+1}, k = 0, 1, \cdots; n = 1, 2, \cdots$.

Obviously, $K_n^*(f,x)$ is positive linear operator. We will show that $K_n^*(f,x)$ reproduce the affine functions (see Lemma 2). For the approximation rate of $K_n^*(f,x)$ for $f \in C[0,1]$, we have the following Ditzian-Totik type estimates which includes both the uniform estimates and pointwise estimates.

Theorem 1.1. *Let $0 \le \lambda \le 1$ be a fixed number. For any $f \in C[0,1]$, there is a positive constant only depending on λ such that*

$$\left|K_n^*(f,x) - f(x)\right| \le C\omega_{\varphi^\lambda}^2\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right),$$

where $\delta_n(x) = \varphi(x) + \frac{1}{\sqrt{n}}$, $\varphi(x) = \sqrt{x(1-x)}$, and

$$\omega_{\varphi^\lambda}^2(f,t) := \sup_{0<h\le t}\ \sup_{x\pm h\varphi^\lambda(x)\in[0,1]}\left|\Delta_{h\varphi^\lambda}^2 f(x)\right|.$$

We also have the following Voronovskaja's asymptotic estimate of $K_n^*(f,x)$:

Theorem 1.2. *Let $f \in C[0,1]$. If f'' exists at a point $x \in [0,1]$, then*

$$\lim_{n\to\infty} n\left(K_n^*(f,x) - f(x)\right) = \varphi^2(x)f''(x)$$

Throughout the paper, C denotes either a positive absolute constant or a positive constant may depend on some parameters but not on f, x and n. Their values may be deferent in different situations.

2. Auxiliary Lemmas

We need the following some auxiliary lemmas.

Lemma 2.1. ([26]) *It holds that*

$$\sum_{k=0}^{n} \frac{k}{n} p_{nk}(x) = x, \tag{1}$$

$$\sum_{k=0}^{n} \frac{k^2}{n^2} p_{nk}(x) = \frac{n-1}{n}x^2 + \frac{1}{n}x, \tag{2}$$

$$\sum_{k=0}^{n} \frac{k^3}{n^3} p_{nk}(x) = \frac{(n-1)(n-2)}{n^2}x^3 + \frac{3(n-1)}{n^2}x + \frac{1}{n^2}x, \tag{3}$$

$$\sum_{k=0}^{n} \frac{k^4}{n^4} p_{nk}(x) = \frac{(n-1)(n-2)(n-3)}{n^3}x^4 + \frac{6(n-1)(n-2)}{n^3}x^3 + \frac{7(n-1)}{n^3}x^2 + \frac{1}{n^3}x. \tag{4}$$

Lemma 2.2. *It holds that*
(i). $K_n^*(1, x) = 1$;
(ii). $K_n^*(t, x) = x$;
(iii). $K_n^*(t^2, x) = x^2 + \frac{1}{n}(x - x^2) + \frac{1}{12n^2} B_n\left(g_n(t), x\right) - \frac{1}{12n^2+12n+3}x^n$;
(iv).

$$K_n^*(t^3, x) = \frac{(n-1)(n-2)}{n^2}x^3 + \frac{3(n-1)}{n^2}x^2 + \frac{5}{4n^2}x - \frac{1}{4n^2}x^n - \frac{1}{4n^3}B_n\left(\widetilde{g}_n(t), x\right) + \frac{1}{4n^3}\left(1 - \frac{3n+1}{(2n+1)^2}\right)x^n;$$

(v).

$$K_n^*(t^4, x) = \frac{(n-1)(n-2)(n-3)}{n^3}x^4 + \frac{6(n-1)(n-2)}{n^3}x^3 + \frac{15(n-1)}{2n^3}x^2 + \frac{1}{n^3}x$$
$$+x^n\left(\frac{1}{2n^3} - \frac{1}{2n^2} - \frac{1}{2n^4}\frac{\frac{62}{5}n^4 + 16n^3 + 7n^2 + n}{(2n+1)^4}\right) + \frac{1}{2n^4}B_n\left(\overline{g}_n(t), x\right).$$

where

$$g_n(x) := \left(\frac{2nx}{2nx+1}\right)^2;$$
$$\widetilde{g}_n(x) := \frac{4n^2x^2 + nx}{(2nx+1)^2};$$
$$\overline{g}_n(x) := \frac{\frac{62}{5}n^4x^4 + 16n^3x^3 + 7n^2x^2 + nx}{(2nx+1)^4}.$$

Proof. (i) is obvious. By (1), we have

$$\begin{aligned}
K_n^*(t, x) &= p_{n,n}(x) + \sum_{k=1}^{n-1}(n+1)p_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \frac{n+1}{n} \cdot \frac{2k}{2k+1} t\,dt \\
&= p_{n,n}(x) + \sum_{k=1}^{n-1}(n+1)p_{n,k}(x) \cdot \frac{n+1}{n} \cdot \frac{2k}{2k+1} \cdot \frac{1}{2}\left(\frac{(k+1)^2}{(n+1)^2} - \frac{k^2}{(n+1)^2}\right) \\
&= p_{n,n}(x) + \sum_{k=1}^{n-1}(n+1)p_{n,k}(x) \cdot \frac{n+1}{n} \cdot \frac{2k}{2k+1} \cdot \frac{1}{2} \cdot \frac{2k+1}{(n+1)^2} \\
&= p_{n,n}(x) + \sum_{k=1}^{n-1} \frac{k}{n} p_{nk}(x) \\
&= x.
\end{aligned}$$

By (1) and (2), we have

$$
\begin{aligned}
K_n^*(t^2,x) &= p_{n,n}(x)+\sum_{k=1}^{n-1}(n+1)p_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\left(\frac{n+1}{n}\right)^2\cdot\left(\frac{2k}{2k+1}\right)^2 t^2dt\\
&= p_{n,n}(x)+\sum_{k=1}^{n-1}\frac{1}{3n^2}\cdot\left(\frac{2k}{2k+1}\right)^2\cdot(3k^2+3k+1)p_{n,k}(x)\\
&= p_{n,n}(x)+\sum_{k=1}^{n-1}\frac{k^2}{n^2}p_{nk}(x)+\sum_{k=1}^{n-1}\left(\frac{2k}{2k+1}\right)^2\left(\frac{k^2}{n^2}+\frac{k}{n^2}+\frac{1}{3n^2}-\frac{k^2}{n^2}\left(\frac{2k+1}{2k}\right)^2\right)p_{nk}(x)\\
&= \sum_{k=0}^{n}\frac{k^2}{n^2}p_{nk}(x)+\sum_{k=1}^{n-1}\left(\frac{2k}{2k+1}\right)^2\left(\frac{1}{3n^2}-\frac{1}{4n^2}\right)p_{n,k}(x)\\
&= \sum_{k=0}^{n}\frac{k^2}{n^2}p_{nk}(x)+\frac{1}{12n^2}\sum_{k=1}^{n-1}\left(\frac{2n\cdot\frac{k}{n}}{2n\cdot\frac{k}{n}+1}\right)^2 p_{n,k}(x)\\
&= x^2+\frac{x-x^2}{n}+\frac{1}{12n^2}B_n\left(g_n(t),x\right)-\frac{1}{12n^2+12n+3}x^n.
\end{aligned}
$$

By (1)-(3), we have

$$
\begin{aligned}
K_n^*(t^3,x) &= p_{n,n}(x)+\sum_{k=1}^{n-1}(n+1)p_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\left(\frac{n+1}{n}\right)^3\cdot\left(\frac{2k}{2k+1}\right)^3 t^3dt\\
&= p_{n,n}(x)+\sum_{k=1}^{n-1}(n+1)p_{n,k}(x)\left(\frac{n+1}{n}\right)^3\cdot\left(\frac{2k}{2k+1}\right)^3\cdot\frac{1}{4}\left(\frac{(k+1)^4}{(n+1)^4}-\frac{k^4}{(n+1)^4}\right)\\
&= p_{n,n}(x)+\sum_{k=1}^{n-1}p_{n,k}(x)\left(\frac{2k}{2k+1}\right)^3\cdot\frac{1}{4}\cdot(2k+1)(2k^2+2k+1)\cdot\frac{1}{n^3}\\
&= p_{n,n}(x)+\sum_{k=1}^{n-1}p_{n,k}(x)\cdot\frac{k^3}{n^3}\cdot\frac{2(2k^2+2k+1)}{(2k+1)^2}\\
&= p_{n,n}(x)+\sum_{k=1}^{n-1}\frac{k^3}{n^3}p_{n,k}(x)+\sum_{k=1}^{n-1}p_{n,k}(x)\cdot\frac{k^3}{n^3}\cdot\frac{1}{(2k+1)^2}\\
&= p_{n,n}(x)+\sum_{k=1}^{n-1}\frac{k^3}{n^3}p_{n,k}(x)+\frac{1}{4n^2}\sum_{k=1}^{n-1}\frac{k}{n}\cdot\frac{(2k+1)^2-4k-1}{(2k+1)^2}p_{n,k}(x)\\
&= p_{n,n}(x)+\sum_{k=1}^{n-1}\frac{k^3}{n^3}p_{n,k}(x)+\frac{1}{4n^2}\sum_{k=1}^{n-1}\frac{k}{n}p_{n,k}(x)-\frac{1}{4n^3}\sum_{k=1}^{n-1}\frac{4k^2+k}{(2k+1)^2}p_{n,k}(x)\\
&= \frac{(n-1)(n-2)}{n^2}x^3+\frac{3(n-1)}{n^2}x+\frac{1}{n^2}x+\frac{1}{4n^2}\left(x-p_{n,n}(x)\right)-\frac{1}{4n^3}\sum_{k=1}^{n-1}\frac{4k^2+k}{(2k+1)^2}p_{n,k}(x)\\
&= \frac{(n-1)(n-2)}{n^2}x^3+\frac{3(n-1)}{n^2}x^2+\frac{5}{4n^2}x-\frac{1}{4n^2}x^n-\frac{1}{4n^3}B_n\left(\widetilde{g}_n(t),x\right)+\frac{1}{4n^3}\left(1-\frac{3n+1}{(2n+1)^2}\right)x^n.
\end{aligned}
$$

By (1)-(4), we have

$$
K_n^*(t^4,x) = p_{n,n}(x)+\sum_{k=1}^{n-1}(n+1)p_{n,k}(x)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\left(\frac{n+1}{n}\right)^4\cdot\left(\frac{2k}{2k+1}\right)^4 t^4dt
$$

$$
\begin{aligned}
&= p_{n,n}(x)+\sum_{k=1}^{n-1}(n+1)p_{n,k}(x)\left(\frac{n+1}{n}\right)^4\cdot\left(\frac{2k}{2k+1}\right)^4\cdot\frac{1}{5}\left(\frac{(k+1)^5}{(n+1)^5}-\frac{k^5}{(n+1)^5}\right)\\
&= p_{n,n}(x)+\sum_{k=1}^{n-1}\frac{16}{5}\cdot\frac{k^4}{n^4}\left(5k^4+10k^3+10k^2+5k+1\right)\frac{1}{(2k+1)^4}p_{n,k}(x)\\
&= p_{n,n}(x)+\sum_{k=1}^{n-1}\frac{k^4}{n^4}p_{n,k}(x)+\sum_{k=1}^{n-1}\frac{k^4}{n^4}\cdot\frac{8k^2+8k+\frac{11}{5}}{(2k+1)^4}p_{n,k}(x)\\
&= p_{n,n}(x)+\sum_{k=1}^{n-1}\frac{k^4}{n^4}p_{n,k}(x)+\frac{1}{2n^2}\sum_{k=1}^{n-1}\frac{k^2}{n^2}\cdot\frac{2k^2(8k^2+8k+\frac{11}{5})}{(2k+1)^4}p_{n,k}(x)\\
&= p_{n,n}(x)+\sum_{k=1}^{n-1}\frac{k^4}{n^4}p_{n,k}(x)+\frac{1}{2n^2}\sum_{k=1}^{n-1}\frac{k^2}{n^2}p_{n,k}(x)\\
&\quad -\frac{1}{2n^2}\sum_{k=1}^{n-1}\frac{k^2}{n^2}\cdot\frac{16k^3+24k^2-\frac{22}{5}k^2+8k+1}{(2k+1)^4}p_{n,k}(x)\\
&= p_{n,n}(x)++\sum_{k=1}^{n-1}\frac{k^4}{n^4}p_{n,k}(x)+\frac{1}{2n^2}\sum_{k=1}^{n-1}\frac{k^2}{n^2}p_{n,k}(x)-\frac{1}{2n^3}\sum_{k=1}^{n-1}\frac{k}{n}p_{n,k}(x)\\
&\quad +\frac{1}{2n^4}\sum_{k=1}^{n-1}\frac{\frac{62}{5}k^4+16k^3+7k^2+k}{(2k+1)^4}p_{n,k}(x)\\
&= \frac{(n-1)(n-2)(n-3)}{n^3}x^4+\frac{6(n-1)(n-2)}{n^3}x^3+\frac{7(n-1)}{n^3}x^2+\frac{1}{n^3}x\\
&+ \frac{1}{2n^2}\left(\frac{n-1}{n}x^2+\frac{1}{n}x-p_{n,n}(x)\right)-\frac{1}{2n^3}\left(x-p_{n,n}(x)\right)\\
&+ \frac{1}{2n^4}B_n\left(\overline{g}_n(t),x\right)-\frac{1}{2n^4}\cdot\frac{\frac{62}{5}n^4+16n^3+7n^2+n}{(2n+1)^4}p_{n,n}(x)\\
&= \frac{(n-1)(n-2)(n-3)}{n^3}x^4+\frac{6(n-1)(n-2)}{n^3}x^3+\frac{15(n-1)}{2n^3}x^2+\frac{1}{n^3}x\\
&+ x^n\left(\frac{1}{2n^3}-\frac{1}{2n^2}-\frac{1}{2n^4}\frac{\frac{62}{5}n^4+16n^3+7n^2+n}{(2n+1)^4}\right)+\frac{1}{2n^4}B_n\left(\overline{g}_n(t),x\right).
\end{aligned}
$$

□

Lemma 2.3. *It holds that*

$$K_n^*((t-x)^2,x)=\frac{x-x^2}{n}+\frac{1}{12n^2}B_n\left(g_n(t),x\right)-\frac{1}{12n^2+12n+3}x^n, \tag{5}$$

$$
\begin{aligned}
K_n^*((t-x)^4,x) &= \frac{3n-6}{n^3}x^4+\frac{12-6n}{n^3}x^3+\frac{5n-15}{2n^3}x^2+\frac{1}{n^3}x\\
&+ \left(\frac{1-n}{2n^3}+\frac{x}{n^2}-\frac{\frac{62}{5}n^4+16n^3+7n^2+n}{2(2n^2+n)^4}-\frac{(4n+1)x}{(2n^2+n)^2}-\frac{2x^2}{(2n+1)^2}\right)x^n\\
&+ \frac{1}{2n^4}B_n\left(\overline{g}_n(t),x\right)+\frac{x}{n^3}B_n\left(\widetilde{g}_n(t),x\right)+\frac{x^2}{2n^2}B_n\left(g_n(t),x\right).
\end{aligned} \tag{6}
$$

Proof. By using (i)-(iii) of Lemma 2, we have

$$\begin{aligned} K_n^*((t-x)^2,x) &= K_n^*(t^2,x)-2xK_n^*(t,x)+x^2 \\ &= x^2+\frac{x-x^2}{n}+\frac{1}{12n^2}B_n\left(g_n(t),x\right)-\frac{1}{12n^2+12n+3}x^n-x^2 \\ &= \frac{x-x^2}{n}+\frac{1}{12n^2}B_n\left(g_n(t),x\right)-\frac{1}{12n^2+12n+3}x^n, \end{aligned}$$

which proves (5).

By using (i)-(iii) of Lemma 2, we we deduce that

$$\begin{aligned} K_n^*((t-x)^4,x) &= K_n^*(t^4,x)-4xK_n^*(t^3,x)+6x^2K_n^*(t^2,x)-4x^3K_n^*(t,x)+x^4 \\ &= \frac{3n-6}{n^3}x^4+\frac{12-6n}{n^3}x^3+\frac{5n-15}{2n^3}x^2+\frac{1}{n^3}x \\ &\quad+\left(\frac{1-n}{2n^3}+\frac{x}{n^2}-\frac{\frac{62}{5}n^4+16n^3+7n^2+n}{2(2n^2+n)^4}-\frac{(4n+1)x}{(2n^2+n)^2}-\frac{2x^2}{(2n+1)^2}\right)x^n \\ &\quad+\frac{1}{2n^4}B_n\left(\overline{g}_n(t),x\right)+\frac{x}{n^3}B_n\left(\widetilde{g}_n(t),x\right)+\frac{x^2}{2n^2}B_n\left(g_n(t),x\right) \\ &\quad-4x\left(\frac{(n-1)(n-2)}{n^2}x^3+\frac{3(n-1)}{n^2}x^2+\frac{5}{4n^2}x-\frac{1}{4n^2}x^n-\frac{1}{4n^3}B_n\left(\widetilde{g}_n(t),x\right)\right. \\ &\quad\left.+\frac{1}{4n^3}\left(1-\frac{3n+1}{(2n+1)^2}\right)x^n\right)+6x\left(\frac{x-x^2}{n}+\frac{1}{12n^2}B_n\left(g_n(t),x\right)\right. \\ &\quad\left.-\frac{4n^2}{48n^4+48n^3+12n^2}x^n\right)-4x^3+x^4 \\ &= \left(\frac{(n-1)(n-2)(n-3)}{n^3}-\frac{4(n-1)(n-2)}{n^2}-\frac{6}{n}+3\right)x^4 \\ &\quad+\left(\frac{6(n-1)(n-2)}{n^3}-\frac{12(n-1)}{n^2}+\frac{6}{n}\right)x^3+\left(\frac{7(n-1)}{n^3}-\frac{5}{n^2}+\frac{n-1}{2n^3}\right)x^2 \\ &\quad+\frac{1}{n^3}x+\left(\frac{1}{2n^3}-\frac{1}{2n^2}+\frac{x}{n^2}-\frac{\frac{62}{5}n^4+16n^3+7n^2+n}{2(2n^2+n)^4}-\frac{x}{n^3}\left(1-\frac{3n+1}{(2n+1)^2}\right)\right. \\ &\quad\left.-\frac{2x^2}{(2n+1)^2}\right)x^n+\frac{1}{2n^4}B_n\left(\overline{g}_n(t),x\right)+\frac{x}{n^3}B_n\left(\widetilde{g}_n(t),x\right)+\frac{x^2}{2n^2}B_n\left(g_n(t),x\right) \\ &= \frac{3n-6}{n^3}x^4+\frac{12-6n}{n^3}x^3+\frac{5n-15}{2n^3}x^2+\frac{1}{n^3}x \\ &\quad+\left(\frac{1-n}{2n^3}+\frac{x}{n^2}-\frac{\frac{62}{5}n^4+16n^3+7n^2+n}{2(2n^2+n)^4}-\frac{(4n+1)x}{(2n^2+n)^2}-\frac{2x^2}{(2n+1)^2}\right)x^n \\ &\quad+\frac{1}{2n^4}B_n\left(\overline{g}_n(t),x\right)+\frac{x}{n^3}B_n\left(\widetilde{g}_n(t),x\right)+\frac{x^2}{2n^2}B_n\left(g_n(t),x\right). \end{aligned}$$

□

3. Proof of results

3.1. *Proof of Theorem 1*

It is obvious that

$$\|K_n^*(f)\| \le \|f\|, \tag{7}$$

where $\|f\|$ is the uniform norm of f in $C_[0,1]$.

Define

$$K^*_{\varphi^\lambda}(f,t^2) := \inf_{g\in D^2_\lambda}\left\{\|f-g\| + t^2\|\varphi^{2\lambda}g''\| + t^{\frac{2}{1-\frac{\lambda}{2}}}\|g''\|\right\}.$$

where $D^2_\lambda := \left\{f\in C_{[0,1]}, f'\in A.C._{loc}, \|\varphi^{2\lambda}f''\| < +\infty, \|f''\| < \infty\right\}$. It is well known that (see [4]) $K^*_{\varphi^\lambda}(f,t^2) \sim \omega^2_{\varphi^\lambda}(f,t)$. Therefore, for any fixed n, λ and x, we may choose a $g_{n,x,\lambda}(t)\in D^2_\lambda$ such that

$$\|f-g\| \leq C\omega^2_{\varphi^\lambda}\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right), \tag{8}$$

$$\frac{\delta_n^{2(1-\lambda)}(x)}{n}\|\varphi^{2\lambda}g''\| \leq C\omega^2_{\varphi^\lambda}\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right). \tag{9}$$

$$\left(\frac{\delta_n^{2(1-\lambda)}(x)}{n}\right)^{\frac{1}{1-\frac{\lambda}{2}}}\|g''\| \leq C\omega^2_{\varphi^\lambda}\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right). \tag{10}$$

By (7) and (8), we have

$$\begin{aligned}
|K^*_n(f,x)-f(x)| &\leq |K^*_n(f-g,x)| + |f(x)-g(x)| + |K^*_n(g,x)-g(x)| \\
&\leq 2\|f-g\| + |K^*_n(g,x)-g(x)| \\
&\leq C\omega^2_{\varphi^\lambda}\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right) + |K^*_n(g,x)-g(x)|.
\end{aligned} \tag{11}$$

By using Taylor's expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du,$$

Lemma 3, and the following inequality (see [5]):

$$\frac{|t-u|}{\delta_n^{2\lambda}(u)} \leq \frac{|t-x|}{\delta_n^{2\lambda}(x)}, \text{ for any } u \text{ between } x \text{ and } t,$$

we have

$$\begin{aligned}
|K^*_n(g,x)-g(x)| &= \left|K^*_n\left(\int_x^t (t-u)g''(u)du, x\right)\right| \\
&\leq C\|\delta_n^{2\lambda}g''\|K^*_n\left(\frac{(t-x)^2}{\delta_n^{2\lambda}(x)}, x\right) \\
&\leq C\frac{1}{\delta_n^{2\lambda}(x)}\|\delta_n^{2\lambda}g''\|\left(\frac{x-x^2}{n} + \frac{1}{12n^2}B_n\left(g_n(t),x\right) - \frac{1}{12n^2+12n+3}x^n\right) \\
&\leq C\frac{1}{\delta_n^{2\lambda}(x)}\|\delta_n^{2\lambda}g''\|\left(\frac{x-x^2}{n} + \frac{1}{n^2}\right) \\
&\leq C\frac{\|g''\delta_n^{2\lambda}\|}{\delta_n^{2\lambda}(x)}\frac{1}{n}\left(\varphi(x) + \frac{1}{\sqrt{n}}\right)^2 \\
&\leq C\frac{\delta_n^{2(1-\lambda)}(x)}{n}\left\|\left(\varphi^{2\lambda} + \left(\frac{1}{\sqrt{n}}\right)^{2\lambda}\right)g''\right\| \\
&\leq C\left(\frac{\delta_n^{2(1-\lambda)}(x)}{n}\|\varphi^{2\lambda}g''\| + \frac{\delta_n^{2(1-\lambda)}(x)}{n}\left(\frac{1}{\sqrt{n}}\right)^{2\lambda}\|g''\|\right)
\end{aligned}$$

$$\begin{aligned}
&= C\left(\frac{\delta_n^{2(1-\lambda)}(x)}{n}\|\varphi^{2\lambda}g''\| + \left(\frac{\delta_n^{2(1-\lambda)}(x)}{n}\right)^{\frac{2}{2-\lambda}}\left(\frac{\delta_n^{2(1-\lambda)}(x)}{n}\right)^{\frac{-\lambda}{2-\lambda}}\left(\frac{1}{\sqrt{n}}\right)^{2\lambda}\|g''\|\right)\\
&= C\left(\frac{\delta_n^{2(1-\lambda)}(x)}{n}\|\varphi^{2\lambda}g''\| + \left(\frac{\delta_n^{2(1-\lambda)}(x)}{n}\right)^{\frac{2}{2-\lambda}}\left(\frac{1}{n\cdot\delta_n^2(x)}\right)^{\frac{\lambda(1-\lambda)}{2-\lambda}}\|g''\|\right)\\
&\leq C\left(\frac{\delta_n^{2(1-\lambda)}(x)}{n}\|\varphi^{2\lambda}g''\| + \left(\frac{\delta_n^{2(1-\lambda)}(x)}{n}\right)^{\frac{2}{2-\lambda}}\left(\frac{1}{n\cdot\left(\varphi^2(x)+\frac{1}{n}\right)}\right)^{\frac{\lambda(1-\lambda)}{2-\lambda}}\|g''\|\right)\\
&= C\left(\frac{\delta_n^{2(1-\lambda)}(x)}{n}\|\varphi^{2\lambda}g''\| + \left(\frac{\delta_n^{2(1-\lambda)}(x)}{n}\right)^{\frac{2}{2-\lambda}}\left(\frac{1}{n\varphi^2(x)+1}\right)^{\frac{\lambda(1-\lambda)}{2-\lambda}}\|g''\|\right)\\
&\leq C\left(\frac{\delta_n^{2(1-\lambda)}(x)}{n}\|\varphi^{2\lambda}g''\| + \left(\frac{\delta_n^{2(1-\lambda)}(x)}{n}\right)^{\frac{1}{1-\frac{\lambda}{2}}}\|g''\|\right)\\
&\leq C\omega_{\varphi^\lambda}^2\left(f, \frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\right), \qquad (12)
\end{aligned}$$

where in the last inequality, (9) and (10) are applied.

By combining (11) and (12), we obtain Theorem 1.

3.2. *Proof of Theorem 2*

Using Taylor's expansion of f:

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + \epsilon(t,x)(t-x)^2,$$

where $\epsilon(t,x) \to 0$ as $t \to x$. By using linearity of the operators $K_n^*(f,x)$, we have

$$K_n^*(f,x) = f(x) + f'(x)K_n^*(t-x,x) + \frac{1}{2}f''(x)K_n^*((t-x)^2,x) + K_n^*(\epsilon(t,x)(t-x)^2,x)$$

By (5), we have

$$\begin{aligned}
K_n^*(f,x) - f(x) &= \frac{1}{2}\left(\frac{x-x^2}{n} + \frac{1}{12n^2}B_n\left(g_n(t),x\right) - \frac{1}{12n^2+12n+3}x^n\right)f''(x)\\
&+ K_n^*\left(\epsilon(t,x)(t-x)^2,x\right). \qquad (13)
\end{aligned}$$

By using Cauchy-Schwarz's inequality, we have

$$n|K_n^*\left(\epsilon(t,x)(t-x)^2,x\right)| \leq \left(K_n^*(\epsilon^2(t,x),x)\right)^{1/2}\left(n^2K_n^*(t-x)^4,x\right)^{1/2}$$

It follows from (6) that

$$K_n^*\left((t-x)^4,x\right) = O\left(\frac{1}{n^2}\right).$$

Therefore, we have

$$\lim_{n\to\infty} n|K_n^*\left(\epsilon(t,x)(t-x)^2,x\right)| = 0 \qquad (14)$$

By (13) and (14), we have

$$\lim_{n\to\infty} n\left(K_n^*(f,x) - f(x)\right) = \frac{\varphi^2(x)}{2}f''(x).$$

we complete the proof of Theorem 2.

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