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The Competition-Independence Game with Prevention

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Abstract. The competition-independence game, as introduced by Phillips and Slater, is played on a graph by two players, Diminisher and Sweller, who are taking turns in choosing a vertex that is not in the closed neighborhood of any of the previously chosen vertices. The goal of Diminisher is to minimize the (maximal independent) set of chosen vertices at the end of the game, while Sweller wants just the opposite. Assuming that both players are playing optimally according to their goals, two graph invariants arise depending on who starts the game. In this paper, we introduce a variation of the game in which players are allowed at any stage in the game to use an alternative move called prevention. That is, a player can decide that in his/her move he/she will mark (not choose!) a previously unplayed vertex x by which x is prevented to be chosen during the rest of the game; in particular, x is not in the final set of chosen vertices. Given a graph G, and assuming that both players play optimally according to their goals, $\widetilde{I}_{\rm d}(G)$ (resp. $\widetilde{I}_{\rm s}(G)$) denotes the size of the set of chosen vertices in the competition-independence game with prevention if Diminisher (resp. Sweller) moves first. By using the Partition Strategy we prove that for any positive integer n, $\widetilde{I}_{\rm d}(P_n) = \left\lfloor \frac{2n+3}{6} \right\rfloor$ and $\widetilde{I}_{\rm s}(P_n) = \left\lfloor \frac{2n+4}{6} \right\rfloor$. While it is not hard to establish the general bounds, $1 \le \widetilde{I}_{\rm d}(G) \le \left\lfloor \frac{n}{2} \right\rfloor$ and $1 \le \widetilde{I}_{\rm s}(G) \le \left\lceil \frac{n}{2} \right\rceil$, we characterize the classes of (connected) graphs G that attain each of the four bounds. Finally, a close connection of the new game with a version of the coloring game called the packing coloring game is established for graphs with diameter 2, and several open problems are posed.

1. Introduction

The *competition-independence game* is played on a graph G by two players, Diminisher and Sweller, who are taking turns in constructing a maximal independent set of G. On each turn a player chooses a vertex that has not been chosen earlier and is not adjacent to any of the vertices already chosen until there is no such vertex. Upon completion of the game, the resulting set of chosen vertices is a maximal independent set of G. The goal of Diminisher is to make the final set as small as possible while Sweller wants to maximize it. Given a graph G, the game played on G when both players play optimally according to their goals yields two graph invariants known as the *competition-independence numbers*. Notably, $I_{\rm d}(G)$ denotes the size of the resulting maximal independent set in the competition-independence game if Diminisher moves first, and $I_{\rm s}(G)$ denotes the size of the maximal independent set if Sweller starts the game.

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The competition-independence game was initiated by Philips and Slater [15, 16], and was further developed by Goddard and Henning [11]. The game can also be considered as a variation of the domination game, introduced in [7], which received great amount of attention in the last decade that culminated in the monograph [5] surveying the theory of these games. In the book, the game was called the *independent domination game* in accordance with the terminology of domination-type games, but here we use the name from the original paper. Phillips and Slater [16] determined the competition-independence numbers of path and cycles; in particular, if $n \geq 1$, then $I_d(P_n) = \left\lfloor \frac{3n+4}{7} \right\rfloor$ and $I_s(P_n) = \left\lfloor \frac{3n+5}{7} \right\rfloor$. The competition-independence game turns out to be highly non-trivial even when played on trees. Goddard and Henning [11] proved that $I_d(T) \leq \frac{4}{7}n$ and $I_s(T) \geq \frac{3}{8}n$ holds for any tree T of order $n \geq 2$ and maximum degree at most 3. They conjectured that in general trees T the bound $I_d(T) \leq \frac{3}{4}n(T)$ holds, which is known as the *independence game* 3/4-*conjecture* (see [5]). Henning in [12] proposed a conjecture about the Sweller-start competition independence number on trees stating that $I_s(T) \geq \frac{3}{7}n$ for any tree T of order $n \geq 2$. In addition, Worawannotai and Ruksasakchai compared the competition-independence game with the domination game [18].

The *competition-independence game with prevention* is a variation of the competition-independence game in which players have an additional option in every move, which is to prevent a vertex from being put into the independent set that is built during the game. More precisely, while the game is played with the same rules as the competition-independence game, the difference is that on each turn player's move can be of two types. By saying that a player *plays* a vertex x we mean any of the two types of moves are made on vertex x. The types of moves are *choosing* a vertex and *preventing* a vertex from being chosen. When a player *chooses* a vertex x this means that x is put into the independent set M that is being built; in this case x must not coincide nor may x be adjacent to any vertex that has been chosen earlier (and was thus put in M). When a player *prevents* a vertex y this means that y has not been played earlier, and it should not be played until the rest of the game. In particular, y is not in the independent set M at the end of the game. (Note that in the competition-independence game with prevention the resulting set M need not be a maximal independent set of G.) Again, the goal of Diminisher is to make the final set M as small as possible while Sweller wants to maximize it. Given a graph G let $\widetilde{I}_d(G)$ denote the size of the resulting independent set in the competition-independence game with prevention if Diminisher moves first and both players play optimally, and let $\widetilde{I}_s(G)$ denote the size of the resulting independent set if Sweller moves first and both players play optimally. These numbers are called the *competition-independence numbers with prevention*.

The paper is organized as follows. In the next paragraph, we give some necessary definitions that will be used in later sections. In Section 2 we establish the competition-independence numbers of paths, showing that $\widetilde{I}_d(P_n) = \left\lfloor \frac{2n+3}{6} \right\rfloor$ and $\widetilde{I}_s(P_n) = \left\lfloor \frac{2n+4}{6} \right\rfloor$ for any $n \geq 1$. Already this case is non-trivial, and may serve as a good introduction to proof techniques used in dealing with the new game(s). In particular, we present and apply the proof method, which we call the *Partition Strategy*. We follow in Section 3 with some general bounds on both competition-independence numbers with prevention. The bounds are either in terms of the (lower) independence number or in terms of the order. In Section 4 we characterize the connected graphs G, which attain the upper bound $\widetilde{I}_s(G) \leq \left\lceil \frac{n}{2} \right\rceil$, while in Section 5 the connected graphs G attaining the upper bound $\widetilde{I}_d(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$ are characterized. Interestingly, the latter family consists of ten sporadic graphs, while the former family is infinite, yet each graph in it has diameter at most 3. In Section 6, the graphs whose competition-independence numbers are equal to 1 are characterized, while in Section 7 we establish the announced connection between the competition-independence games with prevention and the packing coloring game. The latter is a version of the standard coloring game in which the rules adhere to the packing coloring condition, where two vertices that are given the same color i must be at distance at least $i+1$ apart. We end the paper with concluding remarks and pose several open problems.

Throughout the paper we consider simple undirected graphs. The graph $K_4 - e$ is also called the *diamond*, while P_n, $n \geq 1$, and C_n, $n \geq 3$, denote a *path* and a *cycle*, respectively. The graph with one vertex is *trivial*. The *neighborhood* of a vertex $v \in V(G)$, denoted by $N_G(v)$, is the set of vertices adjacent to v, and the *degree*, $\deg_G(v)$, of $v \in V(G)$, is the cardinality of its neighborhood. The *maximum degree* of vertices in G is denoted by $\Delta(G)$. The *neighborhood* of a set $S \subseteq V(G)$ is $N_G(S) = \cup_{v \in S} N_G(v)$. The *distance* between two vertices x and y of a connected graph G is the length of a shortest path between them, and is denoted by $d_G(x, y)$.

The index G in the above definitions may be omitted if the graph G is understood from the context. The *diameter* of G is $\mathrm{diam(G)} = \max\{d_G(u,v) : u,v \in V(G)\}$. An *independent set* of G is a set of vertices in G, no two of which are adjacent. The size of a largest independent set, $\alpha(G)$, is the *independence number* of G and the *lower independence number* of G, $i(G)$, is defined as the size of a smallest maximal independent set (that is, an independent set that is not a proper subset of any other independent set) in G. A vertex *dominates* itself and its neighbors. When building an independent set during a competition-independence game with prevention we will often say that a vertex is (not) *dominated* (by vertices from M), meaning that the vertex is (not) in the closed neighborhood of any vertex from the independent set M of chosen vertices. For a positive integer n, we write $\{1,\dots,n\}$ shortly as $[n]$, and $\{0,1,\dots,n\}$ will be shortened to $[n]_0$.

2. Paths and the Partition Strategy

In this section, we establish the competition-independence numbers of paths and cycles. We present a detailed proof in the case of paths, which involves the strategy that has been (implicitly) used in the context of domination games and probably other types of games as well. However, it seems that its usefulness comes forth especially in the context of the new game(s), which involve the possibility of prevention. The *Partition Strategy* essentially means that a player in some useful way partitions the vertex set of a graph prior to the game, and during the game each of his moves is generally made in the same set of the partition as the preceding move of the opponent.

Theorem 2.1. *For a positive integer n, $\widetilde{I}_{\mathrm{d}}(P_n) = \left\lfloor \frac{2n+3}{6} \right\rfloor$ and $\widetilde{I}_{\mathrm{s}}(P_n) = \left\lfloor \frac{2n+4}{6} \right\rfloor$.*

Proof. Let $V(P_n) = \{v_1,\dots,v_n\}$, and $k = \left\lfloor \frac{n}{3} \right\rfloor$. For $i \in [k-1]_0$, call $T_i = \{v_{3i+1}, v_{3i+2}, v_{3i+3}\}$ a *triple*, while v_{3i+1}, v_{3i+2} and v_{3i+3} are the *left*, the *middle*, and the *right* vertex of the triple T_i, respectively. Let $R = V(P_n) - \cup_{i=0}^{k-1} T_i$. Note that $R = \emptyset$ if $n = 3k$, $R = \{v_n\}$ if $n = 3k+1$, and $R = \{v_{n-1}, v_n\}$ if $n = 3k+2$. The proof relies on the Partition Strategy, where $V(G)$ is partitioned into triples T_i and also R when $n \neq 3k$.

Dividing the formulas of the theorem into three cases with respect to n modulo 3, we get the following six statements:

$$\begin{array}{ll} \widetilde{I}_{\mathrm{d}}(P_{3k}) = k, & \widetilde{I}_{\mathrm{s}}(P_{3k}) = k, \\ \widetilde{I}_{\mathrm{d}}(P_{3k+1}) = k, & \widetilde{I}_{\mathrm{s}}(P_{3k+1}) = k+1, \\ \widetilde{I}_{\mathrm{d}}(P_{3k+2}) = k+1, & \widetilde{I}_{\mathrm{s}}(P_{3k+2}) = k+1. \end{array} \tag{1}$$

In the first part of the proof we present Diminsher's strategy that provides upper bounds for the six equalities in (1).

In P_{3k}, Diminisher's strategy is to respond to Sweller's preceding move with the following rule:

- If Sweller chooses left/right vertex of a triple, then Diminisher prevents the right/left vertex of the same triple.

In all other cases (that is, if Diminisher starts the game or if Sweller plays a move in which he does not choose a left/right vertex of a triple), Diminisher plays in any triple in which no move had been made, and chooses the middle vertex of that triple. Clearly, if a middle vertex of a triple is chosen, then no other vertex from the triple can be chosen. However, if a right or left vertex of a triple is chosen, this is done only by Sweller (by Diminisher's strategy), and with the subsequent move Diminisher achieves that no other vertex of the triple will be chosen during the game. This yields that at most one vertex of a triple will be chosen during the game, regardless of who starts the game. We infer $\widetilde{I}_{\mathrm{d}}(P_{3k}) \le k$ and $\widetilde{I}_{\mathrm{s}}(P_{3k}) \le k$. In the Diminisher-start game on P_{3k+1}, the first move of Diminisher is to prevent the only vertex of R. Henceforth essentially the Sweller-start game is played on P_{3k}, which implies $\widetilde{I}_{\mathrm{d}}(P_{3k+1}) \le k$. The proofs of $\widetilde{I}_{\mathrm{s}}(P_{3k+1}) \le k+1$, $\widetilde{I}_{\mathrm{d}}(P_{3k+2}) \le k+1$, and $\widetilde{I}_{\mathrm{s}}(P_{3k+2}) \le k+1$, are similar to the above proof. Note that R plays the role of a ("damaged") triple, in which Diminsher need not respond on Sweller's move, since at most one vertex can be chosen in R. Otherwise, the strategy of Diminisher is the same in P_{3k}.

Next, we focus on Sweller's strategies that provide lower bounds for the six equalities in (1). The most involved case turns out to be the Sweller-start game on P_{3k+1}, and we will first present the proof of $\widetilde{I}_{\rm s}(P_{3k+1}) \geq k+1$ in detail. (Other cases follow similar lines, and will be briefly discussed later.)

Let Sweller-start game be played on P_{3k+1}. Sweller starts the game by choosing v_n. Sweller's strategy is to maintain the following property after each of her moves:

(P) if a right vertex of a triple T_i is chosen, then T_{i+1} also has a vertex chosen.

Clearly, (P) holds after the first move of Sweller (choosing v_n). Sweller never chooses the right vertex of a triple. More precisely, her strategy is to respond to Diminisher's moves by the following rules:

- If Diminisher prevents the left/right vertex of a triple in which no move had been made earlier, then Sweller chooses the middle vertex of that triple.
- If Diminisher prevents the middle vertex of a triple in which no moves had been made earlier, then Sweller chooses the left vertex of that triple (which is possible by property (P)).
- If Diminisher chooses the right vertex of T_i and no move has been made in T_{i+1}, then Sweller chooses the middle vertex of T_{i+1}.

In all other cases (that is, if Diminisher chooses the left or the middle vertex of a triple, or if Diminisher chooses the right vertex of T_i and a vertex of T_{i+1} had been chosen earlier, or if Diminisher prevents a vertex in a triple in which moves had been made earlier), then Sweller plays in any triple in which no move had been made (if there is still such a triple), and chooses the middle vertex of that triple.

Note that the third rule ensures that property (P) is maintained throughout the game. Hence the second rule can always be applied by Sweller. Indeed, when Diminisher prevents the middle vertex of T_i in a triple in which no moves had been made, this implies that the right vertex of T_{i-1} (namely, v_{3i}) had not been chosen (for otherwise, property (P) would not hold). Thus playing v_{3i+1} is a legal move of Sweller.

Now, by the first two rules of Diminsher's strategy, in any triple in which Diminisher prevents a vertex, there will always be a vertex chosen (by Sweller in her subsequent move). It is also clear that in all other triples a vertex is chosen, since as long as there is a triple in which no vertex had been played, the game is not over (as the middle vertex can still be chosen). Together with v_n, which is chosen in the beginning of the game, there are at least $k+1$ vertices chosen, yielding $\widetilde{I}_{\rm s}(P_{3k+1}) \geq k+1$.

In the proof of $\widetilde{I}_{\rm d}(P_{3k}) \geq k$, $\widetilde{I}_{\rm s}(P_{3k}) \geq k$, Sweller's strategy applies the same set of rules. The difference in Sweller-start game in P_{3k} is that Sweller starts by choosing the middle vertex of any triple. The proof of $\widetilde{I}_{\rm s}(P_{3k+2}) \geq k+1$ can be directly translated to the proof of $\widetilde{I}_{\rm s}(P_{3k+1}) \geq k+1$ by Sweller starting the game in v_{n-1}. The proof of $\widetilde{I}_{\rm d}(P_{3k+2}) \geq k+1$ is also very similar to the proof of $\widetilde{I}_{\rm s}(P_{3k+1}) \geq k+1$. Indeed, in this game Diminisher starts the game, and Sweller follows with responses according to the described set of rules. Note that a vertex of R is always chosen, and there are several ways in which this happens. First, if Diminisher plays v_{3k}, which is the right vertex of T_{k-1}, then by the adjusted third rule, Sweller plays v_n (the middle vertex of the "damaged" triple R). If Diminisher prevents one of the two vertices of R, then by the adjusted first and second rule, Sweller chooses the other vertex of R. Finally, it is also possible that Diminisher chooses a vertex of R, or, if there is a move of Diminisher that does not trigger Sweller's response by one of the three rules, then Sweller may choose any (middle) vertex of a triple, in which no moves had been made, and this includes choosing v_n. □

In a similar way as for paths, we can establish exact values of the competition-independence number with prevention for cycles. The proof is omitted to contain reasonable length of the paper.

Theorem 2.2. *For any positive integer n, $\widetilde{I}_{\rm d}(C_n) = \left\lfloor \frac{n+1}{3} \right\rfloor$ and $\widetilde{I}_{\rm s}(C_n) = \left\lfloor \frac{2n+3}{6} \right\rfloor$.*

3. General bounds

In this section, we prove several upper and lower bounds on the competition-independence numbers with prevention. The bounds hold for (almost) all graphs, and all of them are sharp.

Proposition 3.1. *For every graph G,*

$$\left\lfloor \frac{i(G)}{2} \right\rfloor \leq \widetilde{I}_{\rm d}(G) \leq \alpha(G) \qquad \textit{and} \qquad \left\lceil \frac{i(G)}{2} \right\rceil \leq \widetilde{I}_{\rm s}(G) \leq \alpha(G),$$

and the bounds are sharp.

Proof. Let G be an arbitrary graph. First, let us prove the lower bounds. Regardless of who starts the game let S be the set of vertices chosen by Sweller, where we assume that Sweller never prevented a vertex, and let D be the set of vertices played by Diminisher (where some and perhaps all of his moves could be preventing a vertex). At the end of the game every vertex in $V(G) \setminus (D \cup S)$ is adjacent to a chosen vertex, which implies that $D \cup S$ contains a maximal independent set. Thus, $|D \cup S| \geq i(G)$. Noting that at least $|S|$ vertices have been chosen in the game, and $|S| = \left\lfloor \frac{|D \cup S|}{2} \right\rfloor$ if Diminisher starts the game and $|S| = \left\lceil \frac{|D \cup S|}{2} \right\rceil$ if Sweller starts the game, we derive the stated lower bounds. The upper bound $\alpha(G)$ in both formulas is a direct consequence of the fact that chosen vertices form an independent set.

Sharpness of the bounds follows from $\widetilde{I}_{\rm d}(K_{2r,2r}) = r$ and $\widetilde{I}_{\rm s}(K_{2r,2r}) = r$, where $r \geq 1$, noting that $i(K_{2r,2r}) = 2r$. The upper bound is sharp in K_n, where $\widetilde{I}_{\rm d}(K_n) = \widetilde{I}_{\rm s}(K_n) = \alpha(K_n) = 1$ for any $n \geq 2$. □

Proposition 3.2. *For every graph G on $n \geq 2$ vertices,*

$$1 \leq \widetilde{I}_{\rm d}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor \qquad \textit{and} \qquad 1 \leq \widetilde{I}_{\rm s}(G) \leq \left\lceil \frac{n}{2} \right\rceil,$$

and the bounds are sharp.

Proof. The lower bound 1 in the above two formulas is obvious, since G has at least two vertices, and prevention of a vertex in the first move of Diminisher still allows Sweller to choose a vertex. The lower bounds are attained in non-trivial complete graphs.

To prove the upper bounds, we present Diminisher's strategy in the game on G, which is essentially the same regardless of who starts the game. Notably, Diminisher prevents any previously non-played vertex of G in each of his moves. Then, the game started by Diminisher ends with at most $\lfloor \frac{n}{2} \rfloor$ and the game started by Sweller ends with at most $\lceil \frac{n}{2} \rceil$ chosen vertices. The bounds are attained, for instance, in a graph consisting of n isolated vertices. □

In the second formula of Proposition 3.2, the lower bound holds also in the trivial graph K_1. However, $\widetilde{I}_{\rm d}(K_1) = 0$.

In the following sections we will characterize the graphs that achieve the bounds in Proposition 3.2. In the case of upper bounds we will restrict to connected graphs.

4. Connected graphs with $\widetilde{I}_{\rm s}(G) = \lceil \frac{n}{2} \rceil$

We start the investigation with a lemma that immediately follows from the fact that during the game a player may choose only a vertex that had not been dominated by vertices so far chosen nor had been prevented by some player earlier in the game.

Lemma 4.1. *Let the competition-independence game with prevention be played on G. If at a certain point in the game there are at most k vertices that are not adjacent to already chosen vertices nor have they been chosen or prevented, then the rest of the game takes at most k moves.*

Lemma 4.2. *Let G be a connected graph on $n = 2k+1$ vertices, $k \geq 1$. Then* $\widetilde{I}_s(G) \leq \lceil \frac{n}{2} \rceil - 1$.

Proof. If $k = 1$, then $G \cong P_3$ or $G \cong K_3$. In either case, $\widetilde{I}_s(G) = 1 = \lceil \frac{n}{2} \rceil - 1$. Next, let $k \geq 2$. Diminisher's strategy that provides $\widetilde{I}_s(G) \leq \lceil \frac{n}{2} \rceil - 1$ is to prevent in each of his moves any previously non-dominated and non-prevented vertex of G. If Sweller chooses any vertex v of G in the first move, he dominates at least two vertices, v and, say, v', since G is a connected graph. Therefore, after the response of Diminisher, there are at most $2k-2$ vertices left in G that have not been dominated nor prevented. By Lemma 4.1, the rest of the game takes at most $2k-2$ moves, and in half of these moves, Diminisher prevents a vertex. Therefore, together with the first move, Sweller choses at most k vertices throughout the game, hence $\widetilde{I}_s(G) \leq k = \lceil \frac{n}{2} \rceil - 1$. □

By using the same strategy of Diminisher as described above, one can easily prove that the number of chosen vertices in a graph of even order is at most $\frac{n}{2} - 1$ if Sweller makes a move of preventing a vertex. We formally state this as follows.

Lemma 4.3. *Let G be a connected graph on $n = 2k$ vertices, $k \geq 1$. If Sweller prevents any vertex of G in any of her moves in the Sweller-start game played on G, then the game ends with at most $\frac{n}{2} - 1$ chosen vertices.*

Next, we prove that a graph having a (not necessarily induced) 4-cycle will not be in the class of graphs with $\widetilde{I}_s(G) = \lceil \frac{n}{2} \rceil$.

Lemma 4.4. *Let G be a connected graph on $n = 2k$ vertices, $k \geq 2$. If G contains a cycle C_4, then* $\widetilde{I}_s(G) \leq \lceil \frac{n}{2} \rceil - 1$.

Proof. Let G be a connected graph on $n = 2k$ vertices, $k \geq 2$, which contains a cycle C isomorphic to C_4. Diminisher's strategy ensuring that no more than $\lceil \frac{n}{2} \rceil - 1$ vertices are chosen in the game, is as follows (note that using Lemma 4.3, we may assume that Sweller never prevents a vertex):

- If Sweller chooses a vertex outside C, then Diminisher prevents a vertex outside C.
- If Sweller chooses a vertex v of C, then Diminisher prevents the antipodal vertex on C (that is, the vertex in C, which is not adjacent to v in C).

Since there are even number of vertices outside C, it follows, by the first rule of Diminisher's strategy, that outside the cycle at most half of the vertices (that is, $k-2$ vertices) will be chosen during the game. (Note that Diminisher is allowed to prevent also vertices that are dominated by chosen vertices.) By the above strategy, Sweller has the first move on C. By choosing a vertex on the cycle, two of its neighbors on C are dominated and so by the second rule of Diminisher's strategy, no other vertex of C will be chosen during the game. Thus, at most $k-2+1 = k-1$ vertices will be chosen during the game. □

Also, a path on 6 vertices (not necessarily induced) is forbidden in our graphs.

Lemma 4.5. *Let G be a connected graph on $n = 2k$ vertices, $k \geq 3$. If G contains a path on 6 vertices, then* $\widetilde{I}_s(G) \leq \lceil \frac{n}{2} \rceil - 1$.

Proof. Let G be a connected graph on $n = 2k$ vertices, $k \geq 3$, and let $P : a_0a_1a_2a_3a_4a_5$ be a path in G.

First, using Lemma 4.3 we observe that Sweller chooses a vertex in every move. Next, until no vertex of P is chosen, if Sweller chooses a vertex of $V(G) \setminus V(P)$, Diminisher responds by preventing a vertex of $V(G) \setminus V(P)$. Note that such a move of Diminisher is possible, since the number of vertices of $V(G) \setminus V(P)$ is even and Diminisher is allowed to prevent already dominated vertices. Hence, Sweller will have to play first on P. When Sweller plays on P, Diminisher's strategy is to respond to Sweller's move using the following rules:

- If Sweller chooses vertex a_1 (resp. a_4), Diminisher responds by preventing vertex a_3 (resp. a_2). Then he continues to prevent unplayed vertices of $V(G) \setminus \{a_0, a_2\}$ (resp. $V(G) \setminus \{a_3, a_5\}$).

  Note that, by choosing a_1 (the case a_4 is symmetric), Sweller dominates vertices a_0 and a_2, which can then no longer be chosen. Hence, at most 2 vertices are chosen on P and at most half (which is $k-3$) vertices will be chosen on $V(G) \setminus V(P)$ during the game. Altogether, no more than $2 + k - 3 = \frac{n}{2} - 1$ vertices are chosen during the game.

- If Sweller chooses vertex a_2 (resp. a_3), Diminisher responds by preventing vertex a_0 (resp. a_5). Then he continues to prevent unplayed vertices of $V(G) \setminus \{a_1, a_3\}$ (resp. $V(G) \setminus \{a_2, a_4\}$).

  Similarly as in the previous case, by choosing vertex a_2 (resp. a_3), Sweller dominates vertices a_1 and a_3 (resp. a_2 and a_4), which can then no longer be chosen. Therefore, preventing every second of the $2k-2$ vertices of $V(G) \setminus \{a_1, a_3\}$ ($V(G) \setminus \{a_2, a_4\}$) ensures Diminisher that no more than $k - 1 = \frac{n}{2} - 1$ vertices are chosen during the game.

- If Sweller chooses vertex a_0 (resp. a_5), Diminisher responds by preventing vertex a_2 (resp. a_3). Then he continues to prevent unplayed vertices outside of $V(G) \setminus V(P)$, until Sweller again chooses a vertex in P. Then Diminisher prevents eventual non-dominated vertex of P.

  In this way, the game ends with at most $k-1 = \frac{n}{2} - 1$ chosen vertices, since only two vertices of P have been chosen, and every second of the remaining $2k-6$ vertices was prevented.

□

Corollary 4.6. *Let G be a connected graph on $n = 2k$ vertices, $k \geq 2$. If G contains a cycle C_k, where $k \geq 4$, then $\widetilde{I_s}(G) \leq \left\lceil \frac{n}{2} \right\rceil - 1$.*

Proof. Let G be a connected graph with even number of vertices, and suppose that G contains a cycle C_k, where $k \geq 4$. The case $k = 4$ follows by Lemma 4.4. If $k \geq 6$, then G contains a path P_6, and the proof follows by Lemma 4.5. Finally, if G contains a 5-cycle C, then since G is connected and has even number of vertices, there exists a vertex v adjacent to a vertex of C. Now, $V(C) \cup \{v\}$ contains a path on 6 vertices, and by Lemma 4.5 we infer $\widetilde{I_s}(G) \leq \left\lceil \frac{n}{2} \right\rceil - 1$. □

Lemma 4.7. *Let G be a connected graph on $n = 2k$ vertices, $k \geq 2$, in which there are no cycles of length greater than 3. If G contains at least two vertices of degree at least 3, then $\widetilde{I_s}(G) \leq \left\lceil \frac{n}{2} \right\rceil - 1$.*

Proof. Assuming the conditions of the statement of the lemma, suppose that Sweller chooses in her first move a non pendant vertex of G. Then, at least three vertices are dominated after such a move, and Diminisher responds by preventing any non dominated vertex of G. Hence, after the first two moves, at least four vertices can no longer be chosen during the game, and let V' be the set of remaining vertices (that can still be chosen). Diminisher's strategy in the rest of the game is to prevent an unplayed vertex of V'. Since $|V'| \leq 2k - 4$, we infer that at most $k - 2$ vertices are chosen from V' during the game, and together with the first move of Sweller, at most $k - 1 = \frac{n}{2} - 1$ vertices are chosen during the entire game. Hence, we assume in the rest of the proof that the first move of Sweller is to choose a pendant vertex p and thus two vertices are dominated after her first move.

Let u and v be vertices of degree at least 3. Denote by u_1, u_2, u_3 three neighbors of u and by v_1, v_2, v_3 three neighbors of v (note that it is possible that $u_i = v_j$ for some $i \in [3]$, $j \in [3]$). Suppose that p is not adjacent to any vertex in $\{u, u_1, u_2, u_3\}$. Then Diminisher responds by choosing u by which at least four vertices become dominated. Hence, after the first two moves at least six vertices (p, its neighbor and vertices in $N[u]$) can no longer be chosen during the game, and let V' be the set of remaining vertices. Diminisher's strategy in the rest of the game is to prevent an unplayed vertex of V'. Since $|V'| \leq 2k - 6$, we infer that at most $k - 3$ vertices are chosen from V' during the game, and together with the first two moves, at most $k - 1 = \frac{n}{2} - 1$ vertices are chosen during the entire game. Since by symmetry the same conclusion holds if p is not adjacent to v and its three neighbors, we may assume that p is adjacent to at least one vertex in $\{u, u_1, u_2, u_3\}$ and at least one vertex in $\{v, v_1, v_2, v_3\}$. There are two possibilities how this can happen. Without loss of generality (since the indices of vertices u_i can be permuted), these possibilities are $u_1 = v_1$ and $u_1 = v$, and in either case p is adjacent to u_1.

Suppose $u_1 = v_1$, and p is adjacent to u_1. We may assume that $\deg_G(u) = \deg_G(v) = 3$ for otherwise, Diminisher could choose a vertex of degree at least four in her next move, and again there are at least six vertices dominated after the first two moves, so we are in the case settled by the previous paragraph. First, we consider the case when u and v are not adjacent. Note that we may assume that $\{v_2, v_3\} \cap \{u_2, u_3\} = \emptyset$ for otherwise G contains a 4-cycle, a contradiction to the assumption. Now, let $V' = V(G) \setminus (N[u] \cup N[v] \cup \{p\})$. Note that $|V'|$ is even. In his response to the first move of Sweller, Diminisher chooses u by which all vertices in $N[u]$ are dominated. Then, his strategy is as follows. Whenever Sweller chooses a vertex in V', he responds by preventing a vertex in V'. Since $|V'|$ is even, Sweller is forced to eventually play on $N[v]$. When Sweller chooses a vertex in $N[v]$, then at most one vertex in $N[v]$ remains undominated, and Diminisher prevents it (if there is one). By this strategy, at most three vertices will be chosen in the set $N[u] \cup N[v] \cup \{p\}$ during the entire game, and at most half (which is $k-4$) of the vertices of V' will be chosen during the entire game. Thus, altogether at most $k-1$ vertices will be chosen during the entire game. Second, suppose that u and v are adjacent, and we may write $u_2 = v$ and $v_2 = u$. The response of Diminisher to Sweller choosing p is to prevent u_3. Now, in the rest of the game Diminisher's strategy is as follows. If Sweller plays in $V' = V(G) \setminus (N[u] \cup N[v] \cup \{p\})$, then Diminsher responds in V' by preventing a vertex. By this strategy, Sweller will be the first to play in $N[u] \cup N[v]$ if there are still any vertices left that can be chosen. There are three possibilities for the move of Sweller, namely, vertices u, v and v_3. The responses of Diminisher are as follows. If Sweller chooses u or v, Diminisher prevents v_3; if Sweller chooses v_3, Diminisher prevents u. By this strategy, there will be at most two vertices chosen in $N[u] \cup N[v] \cup \{p\}$ and at most half (which is $k-3$) vertices chosen in V' during the entire game. This yields the upper bound $k-1$ for the number of chosen vertices.

Now, suppose that p is adjacent to v. Then we may write $u_1 = v$, $v_1 = u$ and $v_2 = p$. That is, u and v are adjacent, and the first chosen vertex p in the game is adjacent to v. If $\deg(u) \geq 4$, then we can conclude the proof as in the beginning of the third paragraph of this proof. Hence we may assume that $N(u) = \{v, u_2, u_3\}$.

If u and v have no common neighbor, then the response of Diminisher to the first move of Sweller is to prevent v_3. Next, whenever Sweller plays in $V' = V(G) \setminus (N[u] \cup \{p, v_3\})$, Diminisher responds by preventing a vertex in V'. Since $|V'|$ is even, it is Sweller who will be the first to play in $N[u]$, if any vertex in $N[u]$ remains to be played at all. Now, if Sweller chooses u_2, Diminisher prevents u_3 and the other way around. (If Sweller chooses u, Diminisher can prevent any vertex.) By this strategy, Diminisher enforces that at most two vertices are chosen from $\{u, u_2, u_3, v, p, v_3\}$, and since at most half of the vertices are chosen from V', we infer that at most $k-1$ vertices will be chosen during the entire game.

Finally, let u and v have a common neighbor, say, $u_3 = v_3$. Consider the set $W = \{u, u_2, u_3, v, p\}$ of five vertices. Since G is connected and of even order, there exists a vertex w adjacent to a vertex in W. Clearly, w is not adjacent to u or p. If w is adjacent to u_2, then G contains the path wu_2uu_3vp on 6 vertices. Therefore, using Lemma 4.5, we immediately infer that $\widetilde{I}_s(G) \leq \left\lceil \frac{n}{2} \right\rceil - 1$. Finally, suppose that w is adjacent to v or u_3. Then the response of Diminisher to the first move of Sweller is to prevent vertex w. In the rest of the game Diminisher's strategy is as follows. If Sweller plays in $V' = V(G) \setminus (W \cup \{w\})$, then Diminisher responds in V' by preventing a vertex. Since $|V'|$ is even, it is Sweller who will be forced to start to play again in W unless all vertices of W are already dominated. Now, if Sweller chooses u_3, Diminisher responds by preventing u_2 and the other way around. (If Sweller chooses u, Diminisher can prevent any vertex, since then none of the vertices in $W \cup \{w\}$ can be chosen anymore.) By this strategy, Diminisher enforces that at most two vertices are chosen from $W \cup \{w\}$, and since at most half of the vertices are chosen from V', we infer that at most $k-1$ vertices will be chosen during the entire game. This completes the proof. □

Lemma 4.8. *Let G be a connected graph on $n = 2k$ vertices, $k \geq 2$, which contains a vertex u of degree at least 3. If $\widetilde{I}_s(G) = \left\lceil \frac{n}{2} \right\rceil$, then each component of $G - u$ has at most two vertices, and odd number of them have only one vertex.*

Proof. Let G be a connected graph on $n = 2k$ vertices, $k \geq 2$, containing a a vertex u of degree at least 3. Since $\widetilde{I}_s(G) = \left\lceil \frac{n}{2} \right\rceil$, we infer by combining Corollary 4.6 and Lemma 4.7 that u is the only vertex of degree at least 3 in G.

First, suppose that all components of $G - u$ are isomorphic to K_1. Then $G \cong K_{1,m}$. Since n is even, we infer that m is odd, so there are indeed odd number of components of $G - u$, which have only one vertex. Hence, assume from now on, that $G - u$ has at least one component, say C, which has at least two vertices. Let $v_1 \in C$ be the neighbor of u. Since u is the only vertex of degree more than 2 in G, we infer that v_1 has only one additional neighbor (apart from u), say v_2. From the same reason, v_2 has at most one additional neighbor beside v_1, say v_3, and so on. By this reasoning we infer that in C there is a path on $|C|$ vertices with v_1 as an endvertex.

Now, if C has three vertices, then all other components of $G-u$ have at most one vertex. Indeed, suppose that D was a component different from C with at least two vertices, say w_1 and w_2, and let $w_1u \in E(G)$. Then $P : v_3v_2v_1uw_1w_2$ is a path on 6 vertices, which implies by Lemma 4.5 that $\widetilde{I}_s(G) < \lceil \frac{n}{2} \rceil$, a contradiction. A similar argument, using Lemma 4.5, shows that C cannot have more than 3 vertices.

Suppose that C has indeed 3 vertices, and $v_1v_2v_3$ is a path in C. Let $w_1, \ldots, w_t$ be the leaves attached to u, and suppose that competition-independence game with prevention is played on G. Note that t is even. If Sweller starts by choosing a leaf, then Diminsher prevents another leaf, and he responds in the same way, as long as Sweller plays on leaves. Hence, Sweller is forced to be the first to play on a vertex of C. If Sweller chooses v_1, Diminisher prevents v_3, and if Sweller chooses v_3, Diminisher prevents v_1. In this way, at most half of the leaf are chosen and only vertex from $\{u, v_1, v_2, v_3\}$ is chosen, which yields $\widetilde{I}_s(G) < \lceil \frac{n}{2} \rceil$, a contradiction. If Sweller does not start with a leaf, then in a similar way Diminisher can ensure that less than $\lceil \frac{n}{2} \rceil$ are chosen. This implies that none of the components of $G - u$ can have more than two vertices, and the first part of the lemma is proved. To see that there are odd number of components with only one vertex, one only needs to note that other components have two vertices, and the order of G is even. □

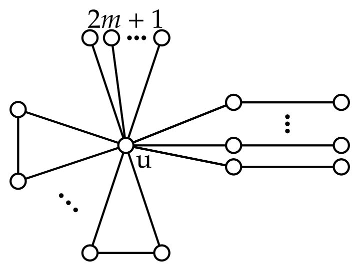

Figure 1: The infinite family of graphs with $\widetilde{I}_s(G) = \lceil \frac{n}{2} \rceil$.

Theorem 4.9. *If G is a connected graph, then $\widetilde{I}_s(G) = \lceil \frac{n}{2} \rceil$ if and only if $G \cong K_1$, $G \cong K_2$, $G \cong P_4$, or G contains a vertex u of degree at least* 3*, each component of $G - u$ has at most two vertices, and odd number of them have only one vertex.*

Proof. First, let G be a graph with $\widetilde{I}_s(G) = \lceil \frac{n}{2} \rceil$. By Lemma 4.2, either G has only one vertex or even number of vertices. Suppose G has no vertex of degree more than 2. By Corollary 4.6, G is a path, and from Theorem 2.1 we infer that $\widetilde{I}_s(G) < \lceil \frac{n}{2} \rceil$, when $n > 4$. Hence G is isomorphic to P_1, P_2 or P_4. Now, let G have a vertex u of degree at least 3. By Lemma 4.8, each component of $G - u$ has at most two vertices, and odd number of them have only one vertex. (See also Fig. 1 depicting the infinite family of graphs with this property.)

For the converse, we first note that $\widetilde{I}_s(K_1) = 1$, $\widetilde{I}_s(K_2) = 1$, and $\widetilde{I}_s(P_4) = 2$. Now, if G contains a vertex u of degree at least 3, with the property that each component of $G - u$ has at most two vertices, and odd number of them have only one vertex, then the strategy of Sweller is as follows. She chooses a leaf as his first move, and follows by responding to Diminsher's moves in the following way. If Diminisher prevents a vertex in a component of $G - u$ with two vertices, then she chooses the other vertex of that component. In all other cases, she chooses a leaf if there is still one non-played leaf, or, otherwise, chooses a vertex in the component of $G - u$ with two vertices. By this strategy, in each component with two vertices one vertex is chosen, and among all $2\ell + 1$ leaves at least $\ell + 1$ leaves are chosen, which gives $\widetilde{I}_s(G) \geq \left\lceil \frac{n}{2} \right\rceil$. By Proposition 3.2, $\widetilde{I}_s(G) \leq \left\lceil \frac{n}{2} \right\rceil$. □

5. Connected graphs with $\widetilde{I}_d(G) = \left\lfloor \frac{n}{2} \right\rfloor$

In this section, we consider Diminisher-start competition-independence game with prevention.

Lemma 5.1. *Let G be a connected graph on n vertices. If Sweller prevents any vertex of G in any of her moves in the Diminisher-start game played on G, then the game ends with at most $\left\lfloor \frac{n}{2} \right\rfloor - 1$ chosen vertices.*

Proof. Let a Diminisher-start game be played on a connected graph G on n vertices. Assume that Sweller prevents a vertex of G in any of her moves. Diminisher's strategy by which in each of his moves he prevents any previously unplayed vertex of G yields that at least $\left\lfloor \frac{n}{2} \right\rfloor + 1$ vertices will be prevented in the game (counting also the vertex prevented by Sweller). Consequently, the game ends with at most $\left\lfloor \frac{n}{2} \right\rfloor - 1$ chosen vertices. □

Lemma 5.2. *Let G be a connected graph on n vertices. If G contains a path on 6 vertices, then $\widetilde{I}_d(G) \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$.*

Proof. Let G be a connected graph on n vertices which contains a (not necessarily induced) path $P : a_0a_1a_2a_3a_4a_5$.

If $V(G) = \{a_0, a_1, a_2, a_3, a_4, a_5\}$, then Diminisher uses the following strategy. In his first move, he chooses a_1, by which a_0 and a_2 can no longer be chosen. By Lemma 5.1, Sweller never prevents a vertex, hence in her response she chooses a vertex with which she dominates at least two vertices that were not dominated before here move. Consequently, at most one vertex remains non-dominated, and Diminisher prevents it in his eventual last move. Thus, at most $2 = \left\lfloor \frac{n}{2} \right\rfloor - 1$ vertices are chosen in the game played on G.

Therefore, assume that $V(G) \neq \{a_0, a_1, a_2, a_3, a_4, a_5\}$, and so there exists at least one vertex in $V(G) \setminus V(P)$. Diminisher's strategy ensuring that at most $\left\lfloor \frac{n}{2} \right\rfloor - 1$ are chosen during the game is as follows:

- If $n = 2k$, then Diminisher prevents in his first move a vertex v such that $G - v$ remains connected.

  Note that $G - v$ is a connected graph that contains odd number of vertices, and the rest of the game translates to a Sweller-start game on $G - v$. Thus, by Lemma 4.2, at most $\left\lceil \frac{2k-1}{2} \right\rceil - 1 = k - 1$ vertices of $G - v$ are chosen during the game. Since in the first move Diminisher only prevented a vertex, we infer that at most $k - 1$ vertices are chosen in the game played on G.

- If $n = 2k + 1$, then Diminisher prevents in his first move a vertex $v \in V(G) \setminus V(P)$ such that $G - v$ is connected.

  It follows that $G - v$ is a connected graph with even number of vertices, which contains a path on 6 vertices. Now Sweller has the first move on $G - v$, and Lemma 4.5 implies that no more than $\left\lceil \frac{2k}{2} \right\rceil - 1 = k - 1$ vertices of $G - v$ are chosen during the game. Together with the first vertex prevented by Diminisher, we conclude that at most $k - 1$ vertices are chosen during the game played on G.

□

Lemma 5.3. *Let G be a connected graph on n vertices. If G contains a cycle C_4, then $\widetilde{I}_d(G) \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$.*

Proof. Let G be a connected graph on n vertices, which contains a cycle C isomorphic to C_4. Diminisher's strategy providing that no more than $\left\lfloor \frac{n}{2} \right\rfloor - 1$ are chosen in the game is as follows:

- If $n = 2k$, then Diminisher prevents in his first move a vertex v such that $G - v$ is connected.

  Note that $G - v$ is a connected graph containing odd number of vertices, and Sweller has the first move in $G - v$. Hence, Lemma 4.2 implies that at most $\left\lceil \frac{2k-1}{2} \right\rceil - 1 = k - 1$ vertices of $G - v$ are chosen in the rest of the game. Considering also the first move of Diminisher (preventing a vertex), we infer that at most $k - 1$ vertices are chosen during the game on G.

- If $n = 2k + 1$, then Diminisher prevents in his first move a vertex v such that $G - v$ is a connected graph containing C (note that, such Diminisher's move is possible since the number of vertices of G is odd, thus there exists at least one vertex outside C).

  Then, $G - v$ is a connected graph with even number of vertices, which contains the cycle C. Now, Sweller starts the game on $G - v$, and by using Lemma 4.4 we infer that at most $\left\lceil \frac{2k}{2} \right\rceil - 1 = k - 1$ vertices of $G - v$ are chosen during the game. Since the first played vertex in the game was prevented by Diminisher, we conclude that at most $k - 1$ vertices are chosen in the game played on G.

□

Corollary 5.4. *Let G be a connected graph on n vertices, and $G \not\cong C_5$. If G contains a cycle C_k, where $k \geq 4$, then $\widetilde{I}_{\rm d}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$.*

Proof. Let G be a connected graph on n vertices, which is not isomorphic to C_5. Assume that G contains a cycle C_k, where $k \geq 4$. If $k = 4$, the result follows by Lemma 5.3. If $k \geq 6$, then G contains a path P_6, thus the proof follows by Lemma 5.2. Finally, suppose that G contains a cycle C_5. Since G is connected and is not isomorphic to C_5 there are only two possibilities. First, if G contains a vertex v adjacent to a vertex of a cycle C_5, then $V(C_5) \cup \{v\}$ contains a path on 6 vertices. Hence, using Lemma 5.2, we infer $\widetilde{I}_{\rm d}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$. Second, let G be isomorphic to a graph obtained from C_5 by adding at least one additional chord. Denote $V(G) = \{a_1, a_2, a_3, a_4, a_5\}$ and suppose without loss of generality that $a_1a_3 \in E(G)$ is a chord in G. To prove that $\widetilde{I}_{\rm d}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 = 1$ we use the following strategy of Diminisher. In the first move, Diminisher prevents vertex a_2, and then responds to Sweller's move by preventing the vertex, which is antipodal on the 4-cycle $a_1a_3a_4a_5a_1$ to the vertex chosen by Sweller. The game ends with only one chosen vertex, which concludes the proof. □

In the following lemma, two graphs on 5 vertices appear, the *bull* (the graph obtained by attaching a single leaf to two vertices of C_3) and graph $\overline{P}$ (the graph obtained from the copy of C_3 and the copy of K_2 by adding an edge between a vertex of C_3 and a vertex of K_2). See Figs. 2h and 2i.

Lemma 5.5. *Let G be a connected graph on n vertices, which contains a cycle C_3 as a proper subgraph and does not contain any cycle of length greater than 3. Then, $\widetilde{I}_{\rm d}(G) = \left\lfloor \frac{n}{2} \right\rfloor$ if and only if G is isomorphic either to the bull or $\overline{P}$.*

Proof. Let G be a connected graph on n vertices which contains a cycle C isomorphic to C_3 and does not contain any cycle of length at least 4. Suppose that $\widetilde{I}_{\rm d}(G) = \left\lfloor \frac{n}{2} \right\rfloor$.

First, assume that G contains a vertex v of degree at least 4. Then, by choosing v in his first move, Diminisher dominates at least 5 vertices, which can then no longer been chosen. In his remaining moves, Diminisher prevents any non-dominated and non-prevented vertex. Thus, among vertices in $V(G) - N[v]$ at most $\left\lceil \frac{n-5}{2} \right\rceil$ vertices are chosen during the game. Together with v, the number of vertices that are chosen during the game of G is $1 + \left\lceil \frac{n-5}{2} \right\rceil \leq \frac{n}{2} - 1$, which is a contradiction. We may thus assume that G does not contain any vertex of degree more than 3.

Next, we prove that $n \leq 6$. Suppose to the contrary that $n \geq 7$. Then, since G is a connected graph, which contains C and has no vertex of degree more than 3, it is straightforward to derive that G contains a path on 6 vertices. Now, using Lemma 5.2 we infer that $\widetilde{I}_{\rm d}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$. Again, this is a contradiction, which implies $n \leq 6$. Moreover, since C is a proper subgraph of G, it follows that $n \geq 4$.

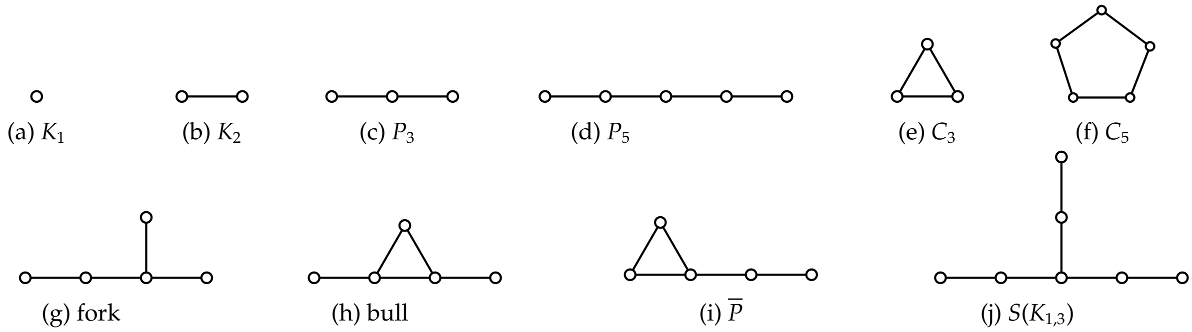

Figure 2: Graphs G with $\widetilde{I}_{\rm d}(G) = \left\lfloor \frac{n}{2} \right\rfloor$

If $n = 4$, G can only be obtained by attaching a single leaf to a vertex of C. Clearly, this is not possible, since by choosing a vertex of degree 3 in the first move, Diminisher dominates all other vertices of G and the game ends with only one chosen vertex.

Assume that $n = 5$. Taking into account all the assumptions (G contains C, does not contain any cycle of length greater than 3 and no vertex of degree at least 4), we infer that G is either a bull or $\overline{P}$.

Finally, let $n = 6$. Lemma 5.2 implies that G does not contain a path on 6 vertices since $\widetilde{I}_{\rm d}(G) = \left\lfloor \frac{n}{2} \right\rfloor$. Thus, G can only be isomorphic to the net graph (the graph obtained by attaching a single leaf to each vertex of C). But in this case, Diminisher has a strategy in the Diminisher-start game on G to ensure at most $\left\lfloor \frac{n}{2} \right\rfloor - 1 = 2$ vertices are chosen during the game. Indeed, if he chooses a vertex of C in his first move, three of its neighbors are dominated and can then no longer be chosen. Clearly, Sweller chooses in her next move another non-dominated vertex, and Diminisher responds by preventing the last non-dominated vertex of G. In this way, the game ends with only 2 chosen vertices, a contradiction.

For the converse, let G be a bull or $\overline{P}$. In either case, Proposition 3.2 implies that $\widetilde{I}_{\rm d}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$. One can easily find strategies of Sweller by which at least $\left\lfloor \frac{n}{2} \right\rfloor = 2$ vertices are chosen for either bull and $\overline{P}$, which concludes the proof. □

The following theorem gives the characterization of the connected graphs G with $\widetilde{I}_{\rm d}(G) = \left\lfloor \frac{n}{2} \right\rfloor$. (The graph $S(K_{1,3})$ is obtained from the star with three leaves by subdividing each of its edges once; see Fig. 2j. The graph *fork* is obtained from the star $K_{1,3}$ by subdividing one of its edges once; see Fig. 2g.)

Theorem 5.6. *Let G be a connected graph on n vertices. Then, $\widetilde{I}_{\rm d}(G) = \left\lfloor \frac{n}{2} \right\rfloor$ if and only if G is either K_1, K_2, P_3, P_5, C_3, C_5, the fork, the bull, $\overline{P}$ or $S(K_{1,3})$ (see Fig. 2).*

Proof. First, consider the sufficient condition, that the graphs G from the list satisfy $\widetilde{I}_{\rm d}(G) = \left\lfloor \frac{n}{2} \right\rfloor$. For some of the graphs in the list, the result follows from previous results. In other cases, the proofs are straightforward. As a demonstration of proof techniques, let us give more details only in the case of the largest graph from the list, which is $S(K_{1,3})$. Sweller's strategy ensuring that at least $\left\lfloor \frac{n}{2} \right\rfloor = 3$ vertices are chosen in the game on $S(K_{1,3})$ is as follows:

- If Diminisher chooses a leaf, Sweller chooses another leaf in her response.
- If Diminisher chooses a neighbor of a leaf, Sweller chooses a non-dominated leaf.
- If Diminisher chooses the center of G, Sweller chooses a leaf.
- If Diminisher prevents a leaf, Sweller chooses its neighbor, if this is possible, otherwise, she chooses a leaf.

- If Diminisher prevents a neighbor of a leaf, Sweller chooses its leaf neighbor.
- If Diminisher prevents the center of G, Sweller chooses any non-dominated and non-prevented vertex of G.

It is straightforward to check that using the described strategy the game ends with at least 3 chosen vertices.

To prove the converse, suppose that G is a connected graph with $\widetilde{I}_{\rm d}(G) = \left\lfloor \frac{n}{2} \right\rfloor$. If $n \le 2$, then we get K_1 and K_2, so assume that $n \ge 3$. We may also assume that $G \not\cong C_3$ and $G \not\cong C_5$. If G contains a cycle, then by Corollary 5.4, the cycle is isomorphic to C_3. Thus, according to Lemma 5.5, G is a bull or $\overline{P}$.

Hence, it remains to consider the case when G is a tree. Now, Lemma 5.2 implies that diam(G) ≤ 4. Since G is connected and $n \ge 3$, we infer that diam(G) ≥ 2.

Let diam(G) = 2. Then, G is a star $K_{1,m}$ with $m \ge 2$. If $m = 2$, then $G \cong P_3$, which is on the list. Otherwise, when $m > 2$, Diminisher has a strategy ensuring that $\widetilde{I}_{\rm d}(G) \le \left\lfloor \frac{n}{2} \right\rfloor - 1$ by playing the center of the star in his first move, which ends the game.

If diam(G) = 3, then let $u, v \in V(G)$ be such that $d(u, v) = 3$. Denote by $P : u = a_0a_1a_2a_3 = v$ a shortest u, v-path in G. If $G \cong P$, then $G \cong P_4$, and using Theorem 2.1 we get a contradiction. Otherwise, G contains a vertex of degree at least 3. However, we observe that G does not contain a vertex of degree at least 4, since by choosing this vertex in the first move of the game, Diminisher can ensure that at most $\left\lfloor \frac{n}{2} \right\rfloor - 1$ vertices will be chosen during the game (similarly, as in the proof of Lemma 5.5). Further, since diam(G) = 3 and G is a tree, a_0 and a_3 are leaves. We infer that G is obtained by attaching another leaf to one or to both non-pendant vertices of P. In the first case, G is a fork. In the second case, choosing a vertex of degree 3 in his first move, Diminisher ensures that at most one new vertex will be chosen in the rest of the game, which gives the contradiction $\widetilde{I}_{\rm d}(G) \le \left\lfloor \frac{n}{2} \right\rfloor - 1$.

Finally, let diam(G) = 4. Let $u, v \in V(G)$ be vertices of G such that $d(u, v) = 4$ and let $P : u = a_0a_1a_2a_3a_4 = v$ be a shortest u, v-path in G. If $G \cong P$, then $G \cong P_5$, which is on the list. Since diam(G) = 4 and G is a tree, a_0 and a_4 are leaves. Moreover, in the same way as earlier we derive that G does not contain a vertex of degree at least 4. Hence, only two leaves can be attached to vertices a_1 and a_3. In addition, if vertex a_2 has neighbors outside P, then a_2 can either be connected a leaf or to one vertex of the graph K_2 or to the center of the path P_3. Clearly, we may assume that G is not isomorphic to $S(K_{1,3})$. Note that each of the graphs constructed in this way contains at least three leaves. If G has an odd number of vertices, then Diminisher's strategy is to prevent a leaf u. Now, the graph $G - u$ has even number of vertices, and the game essentially translates to the Sweller-start game on $G - u$. We notice that $G - u$ is not in the class of extremal graphs from Theorem 4.9. Thus $\widetilde{I}_{\rm d}(G) = \widetilde{I}_{\rm s}(G - u) < \frac{n-1}{2} = \left\lfloor \frac{n}{2} \right\rfloor$. Hence we may assume that G has an even number of vertices. There are six such graphs, and we denote them by G_i, where $i \in [6]$. Graphs G_1 and G_2 have six vertices, and are obtained from P by adding a leaf x; in G_1 vertex x is attached to a_2, while in G_2 vertex x is attached to a_1. In either case, the strategy of Diminisher is to prevent x in his first move. The game translates to Sweller-start game on P_5, which ends with two chosen vertices by Theorem 2.1. Thus $\widetilde{I}_{\rm d}(G_i) = \widetilde{I}_{\rm s}(P_5) = 2 = \frac{6}{2} - 1$ for $i \in [2]$. The next three possibilities for graph G have 8 vertices. Firstly, G_3 is obtained from P by attaching one (additional) leaf on each of a_1, a_2 and a_3. Let the leaf adjacent to a_1 be denoted by x, the leaf adjacent to a_2 by x', and the leaf attached to a_3 by x''. Diminisher starts by preventing x'. Then, if Sweller chooses a leaf (say x) in any of her moves, then Diminisher prevents the other leaf with the same neighbor (that is, u). With this strategy, at most two leaves will be chosen, and we derive $\widetilde{I}_{\rm d}(G_3) \le 3 = \frac{8}{2} - 1$. Secondly, G_4 is obtained from P by attaching one leaf, say x, on a_1 and by connecting a_2 to one vertex of the graph K_2. Now, Diminisher's strategy is to prevent a_2 in his first move. Then, if Sweller chooses a leaf adjacent to a_1, Diminisher prevents the other leaf adjacent to this vertex in his next move. Using the described strategy, at most three vertices will be chosen during the game on G_4 (one from the attached K_2, one from the pair a_3, a_4 and one from the triple a_0, a_1, x). Therefore, $\widetilde{I}_{\rm d}(G_4) \le 3 = \frac{8}{2} - 1$. Thirdly, G_5 is obtained from P by adding a new neighbor x to a_2 and then attaching two leaves, x' and x'', to x. In his first move, Diminisher prevents a_2. Then, if Sweller chooses a leaf adjacent to x (say x'), then in his next move, Diminisher prevents the other leaf adjacent to x (that is x''). Similarly as in the previous case, in this way at most three vertices will be played during the game on G_5 (one from the pair a_0, a_1, one from the pair a_3, a_4 and one from the triple x, x', x''). Hence, $\widetilde{I}_d(G_4) \le 3 = \frac{8}{2} - 1$. Finally, G_6 has 10 vertices and can be described as follows. Let G_5 be obtained from $K_{1,3}$ with the center c (the role of which is taken from a_2) by attaching two leaves to each of the leaf-vertices b_i, $i \in [3]$, of $K_{1,3}$. Let the leaves adjacent to b_i be denoted by d_i^1 and d_i^2, for all $i \in [3]$. The strategy of Diminisher is similar as in the previous case. In his first move, Diminisher prevents c. Then, whenever Sweller chooses a leaf (say d_i^1), in his next move Diminisher prevents the other leaf adjacent to b_i (that is, d_i^2). With this strategy, at most three vertices will be played during the game on G_5 (one from each of the triples b_i, d_i^1, d_i^2), which yields $\widetilde{I}_d(G_5) \le 3 = \frac{10}{2} - 2$. This final case implies the necessary condition for a graph G to enjoy $\widetilde{I}_d(G) = \left\lfloor \frac{n}{2} \right\rfloor$. □

6. Graphs with $\widetilde{I}_d(G) = 1$ or $\widetilde{I}_s(G) = 1$

In this section, we no longer restrict to connected graphs. We start with the characterization of (all) graphs, which attain the value 1 in a Diminisher-start game.

Theorem 6.1. *If G is a graph on $n \ge 2$ vertices, then $\widetilde{I}_d(G) = 1$ if and only if one of the following holds:*

1. $\Delta(G) = n - 1$,
2. *there exists a vertex $u \in V(G)$ such that $\delta(G - u) \ge n - 3$,*
3. $G \cong K_1 + K_1 + K_1$.

Proof. First, let us prove that if G is a graph that satisfies one of the conditions from the list, then $\widetilde{I}_d(G) = 1$. If $\Delta(G) = n - 1$, then also $\gamma(G) = 1$, and Diminisher can complete the game in one move by choosing a vertex with the largest degree. Further, let G be a graph in which there exists a vertex $u \in V(G)$ such that $\delta(G - u) \ge n - 3$. In this case, Diminisher prevents vertex u in the first move. Then, if Sweller chooses a vertex x, then at most one vertex in $G - u$ remains, which is not adjacent to x. If there is such a vertex, Diminisher prevents it, and the game ends with only one chosen vertex. (Otherwise, if all non-prevented vertices in G are adjacent to x, the game is already over.) If the first move of Sweller is not to choose a vertex, then as long as Sweller prevents vertices, Diminisher also prevents vertices. Clearly, it is in Sweller's favor to choose a vertex at least once, but then Diminisher proceeds as described above. Thus, $\widetilde{I}_d(G) = 1$. Lastly, it is clear that $\widetilde{I}_d(K_1 + K_1 + K_1) = 1$.

To prove the reverse direction, suppose that G is a graph on n vertices with $\widetilde{I}_d(G) = 1$. First, consider the case when G is a connected graph. If Diminisher's first move is to choose a vertex $u \in V(G)$, then $\deg(u) = n-1$. Indeed, if there is a vertex in G not adjacent to u, then Sweller can choose it in the next move, which yields $\widetilde{I}_d(G) \ge 2$, a contradiction. In this case, we get the graphs G with $\Delta(G) = n - 1$. Otherwise, suppose that Diminisher's first move is to prevent a vertex $u \in V(G)$. If $\delta(G - u) \ge n - 3$, then we get the class of graphs from the second line of the list. Hence, assume that $\delta(G - u) < n - 3$. Then, Sweller can choose a vertex $v \in V(G - u)$ such that $\deg(v) < n - 3$. There exist at least two vertices $w, z \in V(G - u)$ which are not adjacent to v. Since Diminisher cannot prevent both of them in the next move, Sweller is able to choose a new vertex in her second move. It follows that $\widetilde{I}_d(G) \ge 2$, a contradiction.

Further, assume that G is disconnected. First, observe that if G has at least four connected components, then Sweller has a strategy providing that at least two vertices are chosen during the game by choosing vertex in her first and her second move in a component in which no move had been made before. Next, if G has three components, then we claim that each of them consists of only one vertex. Indeed, if there is a component with at least two vertices, Sweller is able to choose one of them and a vertex of one of the remaining two components, again a contradiction to $\widetilde{I}_d(G) = 1$. Hence, $G \cong K_1 + K_1 + K_1$, which in third line of the list. It remains to consider the case, when G has exactly two components. If none of them is isomorphic to K_1, then Sweller has a strategy ensuring that at least two vertices are chosen in the game. Namely, if Diminisher chooses a vertex in one component, Sweller responds by choosing a vertex in the second component, but if Diminisher prevents a vertex in any of his first two moves, Sweller is able to choose a vertex in the same component in the next move. This yields $\widetilde{I}_{\rm d}(G) \geq 2$, a contradiction. We conclude that one of the two components of G is K_1, and let u be the vertex of K_1. Clearly, Diminisher's optimal first move is to prevent u, for otherwise u will be chosen by Sweller in the next move (and another vertex of $G - u$ can be chosen in Sweller's second move). Now, similarly as above, we derive that $\delta(G - u) \geq n - 3$, for otherwise Sweller can choose a vertex of smallest degree in $G - u$ and enforce at least two chosen vertices in the game. This again yields that G is one of the graphs from the second line of the list. □

Theorem 6.2. *Let G be a graph on n vertices. Then $\widetilde{I}_{\rm s}(G) = 1$ if and only if either $\delta(G) \geq n - 2$ or $G \cong K_1 + K_1$.*

Proof. First, we check that $\delta(G) \geq n - 2$, implies $\widetilde{I}_{\rm s}(G) = 1$. For this purpose we present a strategy of Diminisher to ensure at most one vertex is chosen during the game on G. If Sweller chooses a vertex x in the first move on G, then (since $\delta(G) \geq n - 2$) there exists at most one vertex not adjacent to x, and if it does, Diminisher prevents it in his response. In this way, the game ends with only one chosen vertex. Otherwise, if Sweller prevents a vertex, say v, in the first move, then there exists a vertex which is adjacent to all vertices of G except perhaps v. By choosing this vertex in his response, Diminisher ensures that $\widetilde{I}_{\rm s}(G) = 1$. Second, it is clear that $\widetilde{I}_{\rm s}(K_1 + K_1) = 1$, which ends the proof of the sufficient condition for $\widetilde{I}_{\rm s}(G) = 1$.

For the converse we assume that $\widetilde{I}_{\rm s}(G) = 1$. First, let G be a connected graph. If $\delta(G) < n - 2$, then Sweller can enforce at least two chosen vertices during the game. Namely, in the first move she chooses a vertex with $\deg(v) < n - 2$, and there remain at least two vertices not dominated by v. Since Diminisher can prevent only one of them in next move, Sweller chooses the second one in her second move, thus, $\widetilde{I}_{\rm s}(G) \geq 2$, a contradiction. Thus, $\delta(G) \geq n - 2$, as desired. Finally, if G is a disconnected graph, then it is easy to see that it must have two components or else Sweller can choose two vertices during the game. If one of the components contains at least two vertices, then the first move Sweller chooses a vertex in a component with fewer vertices. In his response, Diminisher can prevent only one vertex, hence there remains an unplayed vertex in the component in which Sweller did not make a move. Sweller chooses a vertex in that component which yield the contradiction $\widetilde{I}_{\rm s}(G) \geq 2$. Thus, we conclude that $G \cong K_1 + K_1$. □

7. Relations with the packing coloring game

Many graph coloring games have been introduced, starting with the game version of graph coloring due to Brams (see Gardner [8]) and Boedlander[2], and followed by a number of variations [1, 3, 9, 13, 14, 17]. Most versions of coloring games are played on a graph G by two players, Alice and Bob, who are taking turns in coloring a (previously uncolored) vertex of a graph. They have opposite goals, Alice aiming to finish the game with as small number of colors as possible, while Bob wants to maximize the number of colors needed in the game to end. They must obey the rule of proper coloring, that is, in each move a vertex chosen by either player has to receive different color than the vertices in its neighborhood that have already received a color (in previous turns). In this paper we propose yet another variation of the coloring game, which is related to a special type of proper coloring – the packing coloring.

Packing coloring was introduced by Goddard, Hedetniemi, Harris and Rall [10] under the name broadcast coloring, though all subsequent papers starting from [6] used the term packing coloring. This type of coloring became quite popular and was recently surveyed in [4], where one can find 68 citations. Given a graph G and a positive integer i, an *i-packing* in G is a subset W of the vertex set of G such that the distance between any two distinct vertices from W is greater than i. Note that a 1-packing is equivalent to an independent set. The *packing chromatic number* of G, denoted by $\chi_\rho(G)$, is the smallest integer k such that the vertex set of G can be partitioned into sets $V_1, \ldots, V_k$, where V_i is an i-packing for each $i \in [k]$. The corresponding mapping $c : V(G) \longrightarrow [k]$ having the property that $c(u) = c(v) = i$ implies $d(u, v) > i$, is a *k-packing coloring*.

The *packing coloring game* is played by two players, Alice and Bob, on a graph G having a fixed number of colors t. They are taking turns in coloring a (previously uncolored) vertex v of a graph by any color i in $[t]$ such that v is at distance greater than i to any vertex of G that received color i in earlier moves. The game ends either when all vertices are colored (in which case Alice wins) or when no legal move is possible. In the latter case Bob wins. In other words, Bob's aim during the game is to reach the situation, where there is a vertex v such that for every color $i \in [t]$ there exists a vertex $u \in V(G)$ that received color i and u is within distance i from v. The minimum number of colors that are needed for Alice to win the packing coloring game on a graph G is the *game packing chromatic number* of G, denoted $\chi_\rho^g(G)$.

A general upper bound on the packing chromatic number in terms of the independence number follows from the fact that a coloring in which vertices of a maximum independent set receive color 1 while all other vertices receive pairwise distinct colors greater than 1 is a packing coloring.

Proposition 7.1. *[10] If G is a graph, then $\chi_\rho(G) \le n(G) - \alpha(G) + 1$, with equality if* $\mathrm{diam}(G) = 2$.

Restricting to diameter 2 graphs and noting that $\chi_\rho^g(G) \ge \chi_\rho(G)$ for any graph G, we immediately get the following bound for the game packing chromatic number for graphs G with diameter 2:

$$\chi_\rho^g(G) \ge n(G) - \alpha(G) + 1.$$

Note that when the packing coloring game is played on diameter 2 graph G two types of moves are possible: either a player gives color 1 to a vertex, which is independent from all other vertices that received color 1, or a player gives a color greater than 1 and distinct from previously chosen colors to a vertex that has not been previously colored. With the latter type of move, the player prevents the selected vertex to be in the (independent) set of vertices colored by color 1. Since Alice wants to minimize the number of colors used in the game, she wants to maximize the number of vertices that are given color 1, while Bob wants just the opposite. Hence, they are essentially playing the competition-independence game with prevention on G, with Alice taking the role of Sweller and Bob that of Diminisher. Since Alice starts the packing coloring game, we derive the following

Corollary 7.2. *If G is a graph with* $\mathrm{diam}(G) = 2$*, then* $\chi_\rho^g(G) = n(G) - \widetilde{I}_{\mathrm{s}}(G) + 1$.

The dual result $\chi_\rho^{gB}(G) = n(G) - \widetilde{I}_{\mathrm{d}}(G) + 1$ for graphs with diameter 2 is obtained in a similar way, where $\chi_\rho^{gB}(G)$ denotes the version of the game packing chromatic number in which Bob starts the game.

8. Concluding remarks

In the special context of Lemma 4.3 we could assume that Sweller never prevents a vertex. However, we could not prove that this holds in general, though it seems plausible.

Question 8.1. *Is it true that an optimal move of Sweller is always to choose a vertex?*

On the other hand, it would be interesting to investigate the graphs in which none of the players needs to use the prevention move. In other words, we ask the following question.

Question 8.2. *In which graphs G, $\widetilde{I}_{\mathrm{d}}(G) = I_{\mathrm{d}}(G)$ or $\widetilde{I}_{\mathrm{s}}(G) = I_{\mathrm{s}}(G)$, respectively?*

It seems reasonable that it is never in Diminisher's favour to let Sweller be the first to play a game. However, we could not prove this in general, therefore we pose the following question.

Question 8.3. *Is there any graph G for which $\widetilde{I}_{\mathrm{d}}(G) > \widetilde{I}_{\mathrm{s}}(G)$ holds?*

If the above question happens to have positive answer, then we would be interested in finding the largest integer k (if it exists) such that for every graph G, $\widetilde{I}_{\mathrm{d}}(G) - \widetilde{I}_{\mathrm{s}}(G) \le k$? Note that if the roles of $\widetilde{I}_{\mathrm{d}}$ and $\widetilde{I}_{\mathrm{s}}$ are reversed, then there is no such integer k. For example take the star $K_{1,n}$, and observe that $\widetilde{I}_{\mathrm{d}}(K_{1,n}) = 1$ while $\widetilde{I}_{\mathrm{s}}(K_{1,n}) = \lceil n/2 \rceil$. If the answer to Question 8.3 is negative, this will be in big contrast with the standard competition-independence numbers for which it was proved in [18] that for any positive integers a and b there exists a connected graph G such that $I_{\mathrm{d}}(G) = a$ and $I_{\mathrm{s}}(G) = b$.

In Sections 4, 5 and 6, we characterized the extremal graphs that attain lower and upper bounds in Proposition 3.2; a similar task with respect to the bounds in Proposition 3.1 triggers the following questions.

Question 8.4. *Which are the graphs G with* $\widetilde{I}_{\rm d}(G) = \alpha(G)$*, and* $\widetilde{I}_{\rm s}(G) = \alpha(G)$*, respectively? Which are the graphs G with* $\widetilde{I}_{\rm d}(G) = \left\lfloor \frac{i(G)}{2} \right\rfloor$*, and* $\widetilde{I}_{\rm s}(G) = \left\lceil \frac{i(G)}{2} \right\rceil$*, respectively?*

The upper bounds $\widetilde{I}_{\rm d}(G) \le \left\lfloor \frac{n}{2} \right\rfloor$ and $\widetilde{I}_{\rm s}(G) \le \left\lceil \frac{n}{2} \right\rceil$ hold in general graphs. It would be interesting to find better bounds in some special classes of graphs. For the standard version of the competition independence games such bounds were studied in trees [11]. Since there are trees T that attain the values $\widetilde{I}_{\rm d}(T) = \left\lfloor \frac{n}{2} \right\rfloor$, and $\widetilde{I}_{\rm s}(T) = \left\lceil \frac{n}{2} \right\rceil$ respectively, the bounds cannot be improved in the class of (all) trees.

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