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Strongly Monotone Solutions of Systems of Nonlinear Differential Equations with Rapidly Varying Coefficients

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Abstract. The two-dimensional systems of first order nonlinear differential equations

$$(\mathrm{S}_1) \quad x' = p(t)y^\alpha, \quad y' = q(t)x^\beta \quad \text{and} \quad (\mathrm{S}_2) \quad x' + p(t)y^\alpha = 0, \quad y' + q(t)x^\beta = 0$$

are analyzed using the theory of rapid variation. This approach allows us to prove that all strongly increasing solutions of system (S_1) (and, respectively, all strongly decreasing solutions of system (S_2)) are rapidly varying functions under the assumption that p and q are rapidly varying. Also, the asymptotic equivalence relations for these solutions are given.

1. Introduction

We consider the two-dimensional first order systems of nonlinear differential equations

$$(\mathrm{S}_1) \qquad x' = p(t)y^\alpha, \quad y' = q(t)x^\beta$$

and

$$(\mathrm{S}_2) \qquad x' + p(t)y^\alpha = 0, \qquad y' + q(t)x^\beta = 0,$$

where α and β are positive constants such that $\alpha\beta < 1$ and p, q are positive, continuous functions on $[a, \infty)$, $a > 0$.

We study positive solutions of (S_i), $i = 1, 2$. By a positive solution of (S_i), $i = 1, 2$ we mean a continuously differentiable couple (x, y) whose components x and y are defined and positive in a neighborhood of infinity and satisfy the system (S_i), $i = 1, 2$ there. Due to the positivity of the coefficients p and q, for system (S_1) both components are then eventually increasing and tend to infinity or to a positive constant, and for system (S_2) both components are then eventually decreasing and tend to zero or to a positive constant. Of the four possible cases for each of the systems, we are interested only in positive solutions of (S_1) and (S_2) such that both components tend to infinity or zero, respectively. Hence, we denote

$$\mathcal{SI} = \{(x, y) \text{ is a positive increasing solution of } (\mathrm{S}_1) : \lim_{t\to\infty} x(t) = \infty, \lim_{t\to\infty} y(t) = \infty\},$$

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which are the so-called *strongly increasing solutions*, and

$$\mathcal{SD} = \{(x, y) \text{ is a positive decreasing solution of } (S_2) : \lim_{t\to\infty} x(t) = 0, \lim_{t\to\infty} y(t) = 0\},$$

which are the so-called *strongly decreasing solutions*.

We notice the connection between the systems (S_i), $i = 1, 2$ and the second order generalized Emden-Fowler differential equation

$$\text{(E)} \qquad \left(p(t)|x'|^{\alpha-1}x'\right)' = q(t)|x|^{\beta-1}x,$$

where α and β are positive constants such that $\alpha > \beta$ and p, q are positive continuous functions on $[a, \infty)$. The equation (E) is called *sublinear*, *half-linear* or *superlinear* depending on if $\alpha > \beta$, $\alpha = \beta$ or $\alpha < \beta$. For the equation (E) we can also defined strongly monotone solutions as follows. A positive solution x of (E) is said to be *strongly increasing* if it satisfies

$$\lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} p(t)|x'(t)|^{\alpha-1}x'(t) = \infty,$$

and *strongly decreasing* if it satisfies

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} p(t)|x'(t)|^{\alpha-1}x'(t) = 0.$$

If x is a strongly increasing solution of (E), putting $y(t) = p(t)x'(t)^\alpha$, we get that (x, y) is a strongly increasing solution of the following system of first order differential equations

$$x' = p(t)^{-1/\alpha}y^{1/\alpha}, \quad y' = q(t)x^\beta. \tag{1.1}$$

Also, if x is a strongly decreasing solution of (E), putting $y(t) = p(t)(-x'(t))^\alpha$, we get that (x, y) is a strongly decreasing solution of the following system of first order differential equations

$$x' + p(t)^{-1/\alpha}y^{1/\alpha} = 0, \quad y' + q(t)x^\beta = 0. \tag{1.2}$$

Conversely, if (x, y) is a strongly increasing solution of (1.1) [resp. strongly decreasing solution of (1.2)], then x is a strongly increasing [resp. strongly decreasing] solution of (E).

The existence of the strongly monotone solutions of (E) has been studied in [2].

Proposition 1.1. *[2, Proposition 1] Sublinear equation* (E) *has strongly increasing solution if and only if*

$$\int_a^\infty \frac{1}{p(t)^{1/\alpha}}\left(\int_a^t q(s)ds\right)^{1/\alpha} dt = \infty \quad \wedge \quad \int_a^\infty q(t)\left(\int_a^t \frac{ds}{p(s)^{1/\alpha}}\right)^\beta dt = \infty.$$

Proposition 1.2. *[2, Proposition 2] Sublinear equation* (E) *has strongly decreasing solution if*

$$\int_a^\infty \frac{1}{p(t)^{1/\alpha}}\left(\int_t^\infty q(s)\, ds\right)^{1/\alpha} dt < \infty \quad \vee \quad \int_a^\infty q(t)\left(\int_t^\infty \frac{ds}{p(s)^{1/\alpha}}\right)^\beta dt < \infty.$$

Based on the mentioned connection between the systems (S_i), $i = 1, 2$ and the equation (E), using Proposition 1.1 and Proposition 1.2, we can derive the conditions (necessary or sufficient) for the existence of strongly increasing solutions of (S_1) and strongly decreasing solutions of (S_2) as follows.

Proposition 1.3. *System* (S_1) *has strongly increasing solution if and only if*

$$\int_a^\infty p(t)\left(\int_a^t q(s)ds\right)^\alpha dt = \infty \quad \wedge \quad \int_a^\infty q(t)\left(\int_a^t p(s)ds\right)^\beta dt = \infty. \tag{1.3}$$

Proposition 1.4. *System* (S_2) *has strongly decreasing solution if*

$$\int_a^\infty p(t)\left(\int_t^\infty q(s)\,ds\right)^\alpha dt < \infty \quad \vee \quad \int_a^\infty q(t)\left(\int_t^\infty p(s)ds\right)^\beta dt < \infty. \tag{1.4}$$

The theory of regularly varying functions has been proved to be very useful in studying the existence and asymptotic behavior of positive solutions of various types of differential equations and systems, see [11–16, 18]. In particular, the existence and precise asymptotic behavior of regularly varying strongly monotone solutions of the systems (S_1) and (S_2) are considered in [8, 9]. This paper represents a continuation of that research in terms of the application of the theory of rapidly varying functions in studying the asymptotic behavior of rapidly varying strongly monotone solutions of these systems. After the pioneer work by Marić [17] dealing with the study of second order linear differential equation in the framework of rapid variation, there are only a few papers related to the application of the theory of rapid variation. In [19–21] half-linear differential equations in the framework of the Karamata theory and the de Haan theory were studied. Also, the existence of regularly and rapidly varying solutions of third order nonlinear differential equations was studied in [10].

Our goal in this paper is to prove that all strongly increasing solutions of (S_1) as well as all strongly decreasing solutions of (S_2) are rapidly varying functions under the assumption that the coefficients of the corresponding system are rapidly varying functions and to give some information about asymptotic behavior of these solutions.

This paper is organized as follows. The basic definitions and properties of the regularly and rapidly varying functions are given in Section 2. Also, the definitions and properties of asymptotic equivalence relations on the class RPV(∞) are presented. Based on those, we introduce new analogous relations on the class RPV($-\infty$) and investigate their properties. In Section 3 we deal with strongly increasing solutions of (S_1). For these solutions, we state and prove the main result as well as some auxiliary lemmas, which help us to prove the main results more elegantly. In Section 4 we turn our attention to the study of strongly decreasing solutions of (S_2). We also state and prove some auxiliary lemmas, and the main result. Section 5 is dedicated to the application of main results to the equation (E). Section 6 presents some illustrative examples.

2. Preliminaries

Let us recall the definitions of regularly and rapidly varying functions.

Definition 2.1. *Let* $f : [a,\infty) \to (0,\infty)$ *be a measurable function.*

(1) f *is regularly varying of index* $\rho \in \mathbb{R}$ *if* $\displaystyle\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho$ *for all* $\lambda > 0$.

(2) f *is rapidly varying of index* ∞ *if* $\displaystyle\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \infty$ *for all* $\lambda > 1$.

(3) f *is rapidly varying of index* $-\infty$ *if* $\displaystyle\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = 0$ *for all* $\lambda > 1$.

The set of all regularly varying functions of index ρ is denoted by RV(ρ), while the set of rapidly varying functions of index ∞ (or $-\infty$) is denoted by RPV(∞) (or RPV($-\infty$)).

The book of Bingaham, Goldie and Teugels [1] is a very good source of information on the theory of regular and rapid variation. Also a more recent contribution to the theory of rapid variation can be found in [6, 7].

Now we present some selected properties of rapidly varying functions.

Proposition 2.2.

(1) If $f, g \in \mathrm{RPV}(\infty)$ and $h \in \mathrm{RV}(\rho)$, $\rho \in \mathbb{R}$, then

(i) $f^p \in \mathrm{RPV}(\infty)$ *for any* $p > 0$.

(ii) $f \cdot h \in \mathrm{RPV}(\infty)$ *and* $f \cdot g \in \mathrm{RPV}(\infty)$.

(2) $f \in \mathrm{RPV}(\infty)$ if and only if $1/f \in \mathrm{RPV}(-\infty)$.

(3) Let $f : [a, \infty) \to (0, \infty)$ be a measurable function, monotone for large t. Then

(i) $f \in \mathrm{RPV}(\infty)$ implies f is increasing for large t and $\lim_{t\to\infty} f(t) = \infty$.

(ii) $f \in \mathrm{RPV}(-\infty)$ implies f is decreasing for large t and $\lim_{t\to\infty} f(t) = 0$.

Proof. The parts (2) and (3) are shown in [22, Proposition 2]. The part (1) of the proposition is easy to prove. □

Next, we consider some useful equivalence relations on the classes $\mathrm{RPV}(\infty)$ and $\mathrm{RPV}(-\infty)$. The following relation is introduced in [1] and further considered in [3, 4].

Definition 2.3. *Let f and g be positive functions in $[a, \infty)$. These two functions are called mutually inversely asymptotic at ∞, denoted by $f(t) \overset{\star}{\sim} g(t)$, $t \to \infty$, if for every $\lambda > 1$ there exists $t_0 = t_0(\lambda)$ such that*

$$f\left(\frac{t}{\lambda}\right) \leq g(t) \leq f(\lambda t), \quad \text{for} \quad \text{all} \quad t \geq t_0.$$

The definition of a stronger relation is given by Elez and Djurčić in [5] as follows.

Definition 2.4. *Let f and g be positive functions in $[a, \infty)$. These two functions are called mutually rapidly equivalent at ∞, denoted by $f(t) \overset{r}{\sim} g(t)$, $t \to \infty$, if*

$$\lim_{t\to\infty} \frac{f(\lambda t)}{g(t)} = \lim_{t\to\infty} \frac{g(\lambda t)}{f(t)} = \infty, \quad \text{for} \quad \text{all} \quad \lambda > 1.$$

In the same paper [5], some very useful properties of relation $\overset{r}{\sim}$ are given.

Proposition 2.5. *Let f and g be positive functions in $[a, \infty)$. Then, the following assertions hold:*

(a) if f and g are measurable functions such that $f(t) \overset{r}{\sim} g(t)$ for $t \to \infty$, then f and g both belong to $\mathrm{RPV}(\infty)$;

(b) the relation $\overset{r}{\sim}$ is an equivalence relation in the class $\mathrm{RPV}(\infty)$.

It is easy to prove the following proposition.

Proposition 2.6. *Let $f, g \in \mathrm{RPV}(\infty)$ and $f(t) \overset{r}{\sim} g(t)$, $t \to \infty$, then*

(a) $f(t)^p \overset{r}{\sim} g(t)^p$, $t \to \infty$ for all $p > 0$,

(b) $h(t) \cdot f(t) \overset{r}{\sim} h(t) \cdot g(t)$, $t \to \infty$ for $h \in \mathrm{RV}(\rho)$, $\rho \in \mathbb{R}$ or $h \in \mathrm{RPV}(\infty)$.

The following proposition, given in [5], represents an analogue of the Karamata's integration theorem for regularly varying functions in the theory of rapidly varying functions.

Proposition 2.7. *Let $f \in \mathrm{RPV}(\infty)$ be a locally bounded function on $[a, \infty)$. Also, let $1/f$ be a locally bounded function on $[a, \infty)$. Then, the following assertions are true:*

(a) $f(t) \overset{r}{\sim} \dfrac{1}{t}\displaystyle\int_a^t f(s)ds, \quad t \to \infty$ *and consequently* $t \cdot f(t) \overset{r}{\sim} \displaystyle\int_a^t f(s)ds, \quad t \to \infty;$

(b) $f(t) \overset{r}{\sim} \dfrac{1}{t\displaystyle\int_t^\infty \frac{ds}{s^2 f(s)}}, \quad t \to \infty;$

(c) $F \in \mathrm{RPV}(\infty)$, *where* $F(t) = \displaystyle\int_a^t f(s)ds, \ t > a;$

(d) $\varphi \in \mathrm{RPV}(\infty)$, *where* $\varphi(t) = \dfrac{1}{\displaystyle\int_t^\infty \frac{ds}{f(s)}}, \ t > a.$

Proposition 2.8. *(i) Let* $f \in \mathrm{RPV}(\infty)$. *Then*

$$\lim_{t\to\infty} F(t,T) = \infty, \quad \text{for every} \quad T \geq a, \quad \text{where} \quad F(t,T) = \int_T^t f(s)ds, \quad t \geq T. \tag{2.1}$$

(ii) Let $g \in \mathrm{RPV}(-\infty)$. *Then*

$$\lim_{t\to\infty} G(t) = 0, \quad \text{where} \quad G(t) = \int_t^\infty g(s)ds, \quad t \geq a. \tag{2.2}$$

Proof.

(i) Fix arbitrary $T \geq a$. Using Proposition 2.7 (c) it follows that $F(t,T)$ is a rapidly varying function of index ∞ and since F is a monotone function, based on Proposition 2.2 (3), we have that (2.1) is satisfied.

(ii) Using Proposition 2.7 (d) and Proposition 2.2 (2) it follows that $G(t)$ is a rapidly varying function of index $-\infty$ and since G is a monotone function, based on Proposition 2.2 (3), we have that (2.2) is satisfied.
□

Now, we introduce two new relations on $\mathrm{RPV}(-\infty)$.

Definition 2.9. *Let f and g be positive functions in $[a,\infty)$. These two functions are called mutually inversely asymptotic at $-\infty$, denoted by* $f(t) \underset{\star}{\sim} g(t), \ t \to \infty$, *if for every* $\lambda > 1$ *there exists* $t_0 = t_0(\lambda)$ *such that*

$$f(\lambda t) \leq g(t) \leq f\left(\frac{t}{\lambda}\right), \quad \text{for all} \quad t \geq t_0.$$

Definition 2.10. *Let f and g be positive functions in $[a,\infty)$. These two functions are called mutually rapidly equivalent at $-\infty$, denoted by* $f(t) \underset{r}{\sim} g(t), \ t \to \infty$, *if*

$$\lim_{t\to\infty} \frac{f(\lambda t)}{g(t)} = \lim_{t\to\infty} \frac{g(\lambda t)}{f(t)} = 0, \quad \text{for all} \quad \lambda > 1.$$

In order to establish a connection between relations $\overset{r}{\sim}$ and $\underset{r}{\sim}$, we give the next proposition.

Proposition 2.11. *Let f and g be positive functions in $[a,\infty)$. Then*

$$f(t) \overset{r}{\sim} g(t), \ t \to \infty \quad \text{if and only if} \quad \frac{1}{f(t)} \underset{r}{\sim} \frac{1}{g(t)}, \ t \to \infty.$$

Proof. The proposition directly follows from the equalities

$$\lim_{t\to\infty}\frac{f(\lambda t)}{g(t)}=\left[\lim_{t\to\infty}\frac{\frac{1}{f(\lambda t)}}{\frac{1}{g(t)}}\right]^{-1}\quad\text{and}\quad\lim_{t\to\infty}\frac{g(\lambda t)}{f(t)}=\left[\lim_{t\to\infty}\frac{\frac{1}{g(\lambda t)}}{\frac{1}{f(t)}}\right]^{-1}$$

□

The next proposition follows from Proposition 2.11, Proposition 2.5 and Proposition 2.2 (2).

Proposition 2.12. *Let f and g be positive functions in $[a,\infty)$. Then, the following assertions hold:*

(a) if f and g are measurable functions such that $f(t)\underset{r}{\sim}g(t)$ for $t\to\infty$, then f and g both belong to $\mathrm{RPV}(-\infty)$*;*

(b) the relation $\underset{r}{\sim}$ is an equivalence relation in the class $\mathrm{RPV}(-\infty)$.

It is easy to prove the following proposition using Proposition 2.11 and Proposition 2.6.

Proposition 2.13. *Let $f,g\in\mathrm{RPV}(-\infty)$ and $f(t)\underset{r}{\sim}g(t)$, $t\to\infty$, then*

(a) $f(t)^p\underset{r}{\sim}g(t)^p$, $t\to\infty$ for all $p>0$,

(b) $f(t)\cdot h(t)\underset{r}{\sim}g(t)\cdot h(t)$, $t\to\infty$ for $h\in\mathrm{RV}(\rho)$, $\rho\in\mathbb{R}$ or $h\in\mathrm{RPV}(-\infty)$.

In the following proposition we give a more convenient form of Proposition 2.7 (b).

Proposition 2.14. *Let $g\in\mathrm{RPV}(-\infty)$ be a locally bounded function on $[a,\infty)$. Then,*

$$t\cdot g(t)\underset{r}{\sim}\int_t^\infty g(s)ds,\quad t\to\infty.$$

Proof. Denote $g(t)=\frac{1}{t^2\,f(t)}$. Since $g\in\mathrm{RPV}(-\infty)$, we conclude that $f\in\mathrm{RPV}(\infty)$. Also, since g is a locally bounded function on $[a,\infty)$, so is $1/f$. Hence, by Proposition 2.7 (b) and Proposition 2.6 (b) we have

$$t\cdot f(t)\overset{r}{\sim}\frac{1}{\displaystyle\int_t^\infty g(s)ds},\ t\to\infty$$

which implies by Proposition 2.11

$$t\cdot g(t)=\frac{1}{t\cdot f(t)}\underset{r}{\sim}\int_t^\infty g(s)ds,\quad t\to\infty.$$

□

3. Strongly increasing solutions of (S_1)

In this section, we deal with strongly increasing solutions of (S_1). Main result of this section is the following theorem which ensures that all positive increasing solutions of (S_1) are strongly increasing and rapidly varying assuming that p,q are rapidly varying functions and gives the asymptotic equivalence relation for these solutions.

Theorem 3.1. *Suppose that p and q are rapidly varying of index ∞. Every positive increasing solution of* (S_1) *is strongly increasing and rapidly varying of index ∞. Moreover, any such solution (x, y) satisfies the asymptotic relation*

$$x(t) \overset{\star}{\sim} X(t), \quad y(t) \overset{\star}{\sim} Y(t), \quad t \to \infty, \tag{3.1}$$

where the functions X and Y are given respectively by

$$X(t) = \left(t^{\alpha+1} p(t)\, q(t)^{\alpha}\right)^{\frac{1}{1-\alpha\beta}}, \tag{3.2}$$

and

$$Y(t) = \left(t^{\beta+1} p(t)^{\beta}\, q(t)\right)^{\frac{1}{1-\alpha\beta}}. \tag{3.3}$$

We first state and prove some auxiliary lemmas, which help us to prove the main result more elegantly. To this end, let us denote by

$$X_{11}(t) = \left(\int_a^t p(s)\, R_1(s) ds\right)^{\frac{1}{\beta+1}}, \quad X_{12}(t) = (P_1(t)\, Q_1(t)^{\alpha})^{\frac{1}{1-\alpha\beta}}, \tag{3.4}$$

$$Y_{11}(t) = \left(\int_a^t q(s)\, R_1(s) ds\right)^{\frac{1}{\alpha+1}}, \quad Y_{12}(t) = \left(P_1(t)^{\beta}\, Q_1(t)\right)^{\frac{1}{1-\alpha\beta}}, \tag{3.5}$$

where

$$P_1(t) = \int_a^t p(s) ds, \quad Q_1(t) = \int_a^t q(s) ds, \quad R_1(t) = \left(\int_a^t p(s)^{\frac{\beta(\alpha+1)}{2\alpha\beta+\alpha+\beta}}\, q(s)^{\frac{\alpha(\beta+1)}{2\alpha\beta+\alpha+\beta}} ds\right)^{\frac{2\alpha\beta+\alpha+\beta}{1-\alpha\beta}}. \tag{3.6}$$

Lemma 3.2. *Suppose that p and q are rapidly varying of index ∞. Then, the following assertions hold:*

(i) $\mathcal{I} = \mathcal{SI}$, where $\mathcal{I}$ denote the set of all positive increasing solutions of (S_1)*;*

(ii) for every $(x, y) \in \mathcal{I}$ there exist positive constants m_i, M_i, $i = 1, 2$ such that

$$m_1\, X_{11}(t) \leq x(t) \leq M_1\, X_{12}(t) \quad \wedge \quad m_2\, Y_{11}(t) \leq y(t) \leq M_2\, Y_{12}(t), \tag{3.7}$$

for large t, where X_{11}, X_{12} and Y_{11}, Y_{12} are given by (3.4) *and* (3.5)*, respectively.*

Proof. Take any $(x, y) \in \mathcal{I}$ defined on $[t_0, \infty)$, $t_0 \geq a$. Integrating both equations in (S_1) on $[t_0, t]$ and taking into account that x and y are increasing, we get

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^t p(s)\, y(s)^{\alpha} ds \geq x(t_0) + y(t_0)^{\alpha} \int_{t_0}^t p(s) ds, \quad t \geq t_0, \\ y(t) &= y(t_0) + \int_{t_0}^t q(s)\, x(s)^{\beta} ds \geq y(t_0) + x(t_0)^{\beta} \int_{t_0}^t q(s) ds, \quad t \geq t_0, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} x(t) &\leq x(t_0) + y(t)^{\alpha} \int_{t_0}^t p(s) ds \leq x(t_0) + y(t)^{\alpha}\, P_1(t), \quad t \geq t_0, \\ y(t) &\leq y(t_0) + x(t)^{\beta} \int_{t_0}^t q(s) ds \leq y(t_0) + x(t)^{\beta}\, Q_1(t), \quad t \geq t_0. \end{aligned} \tag{3.9}$$

Since $p, q \in \mathrm{RPV}(\infty)$, using Proposition 2.8 (i), we conclude that

$$\lim_{t\to\infty} P_1(t) = \infty, \quad \lim_{t\to\infty} Q_1(t) = \infty. \tag{3.10}$$

From (3.8) and (3.10) we have $(x, y) \in \mathcal{SI}$. Hence, $\mathcal{I} = \mathcal{SI}$. Also, from (3.9) and (3.10) we obtain that there exist $K_i > 0$, $i = 1, 2$ and $t_1 \geq t_0$ sufficiently large such that

$$x(t) \leq K_1\, y(t)^\alpha\, P_1(t), \quad t \geq t_1 \tag{3.11}$$

and

$$y(t) \leq K_2\, x(t)^\beta\, Q_1(t), \quad t \geq t_1. \tag{3.12}$$

To find an upper estimate for x, we substitute (3.12) into (3.11) to obtain

$$x(t) \leq K_1\, K_2^\alpha\, x(t)^{\alpha\beta}\, P_1(t)\, Q_1(t)^\alpha, \quad t \geq t_1$$

and similar, to find an upper estimate for y, we substitute (3.11) into (3.12) to obtain

$$y(t) \leq K_1^\beta\, K_2\, y(t)^{\alpha\beta} P_1(t)^\beta\, Q_1(t), \quad t \geq t_1$$

implying that there exist $M_i > 0$, $i = 1, 2$ such that

$$x(t) \leq M_1\, (P_1(t)\, Q_1(t)^\alpha)^{\frac{1}{1-\alpha\beta}} = M_1\, X_{12}(t), \quad t \geq t_1$$

and

$$y(t) \leq M_2\, \left(P_1(t)^\beta\, Q_1(t)\right)^{\frac{1}{1-\alpha\beta}} = M_2\, Y_{12}(t), \quad t \geq t_1.$$

Now we prove the lower estimate for x and y. To this end let $\omega(t) = x(t)^\beta y(t)^\alpha$ and

$$\mu = \frac{\beta(\alpha+1)}{2\alpha\beta+\alpha+\beta}, \quad \nu = \frac{\alpha(\beta+1)}{2\alpha\beta+\alpha+\beta}, \quad \eta = \frac{1-\alpha\beta}{2\alpha\beta+\alpha+\beta}. \tag{3.13}$$

It is easy to verify that $\mu, \nu, \eta > 0$, $\mu + \nu = 1$ and $\dfrac{\beta\nu-\mu}{\beta} = \dfrac{\alpha\mu-\nu}{\alpha} = -\eta$. Applying Young's inequality we get

$$\begin{aligned}\omega'(t) &= \omega(t)\left(\beta\, \frac{p(t)y(t)^\alpha}{x(t)} + \alpha\, \frac{q(t)x(t)^\beta}{y(t)}\right) \geq \omega(t)\, \frac{\beta^\mu\alpha^\nu}{\mu^\mu\nu^\nu} \left(\frac{p(t)y(t)^\alpha}{x(t)}\right)^\mu \left(\frac{q(t)x(t)^\beta}{y(t)}\right)^\nu \\ &= \omega(t)\, \frac{\beta^\mu\alpha^\nu}{\mu^\mu\nu^\nu}\, p(t)^\mu\, q(t)^\nu\, (x(t)^\beta y(t)^\alpha)^{-\eta}\end{aligned}$$

yielding

$$\omega'(t) \geq \frac{\beta^\mu\alpha^\nu}{\mu^\mu\nu^\nu}\, \omega(t)^{1-\eta}\, p(t)^\mu\, q(t)^\nu, \quad t \geq t_1. \tag{3.14}$$

After dividing (3.14) with $\omega(t)^{1-\eta}$ and then integrating the obtained inequality on $[t_1, t]$ we find $k_1 > 0$ such that

$$\omega(t) \geq k_1 \left(\int_{t_1}^t p(s)^\mu\, q(s)^\nu ds\right)^{1/\eta}, \quad t \geq t_1,$$

in the view of (3.13). Using Proposition 2.8 (i) we conclude that $\lim_{t\to\infty} R(t) = \infty$. Therefore, we find $k_2 > 0$ and sufficiently large $t_2 \geq t_1$ such that

$$x(t)^{\beta} y(t)^{\alpha} \geq k_2\, R_1(t), \quad t \geq t_2. \tag{3.15}$$

By substituting $y(t)^{\alpha} = \frac{x'(t)}{p(t)}$ into (3.15) we have

$$x(t)^{\beta} x'(t) \geq k_2\, p(t)\, R_1(t), \quad t \geq t_2. \tag{3.16}$$

Integrating (3.16) from t_2 to t, we find $m_1 > 0$ such that

$$x(t) \geq m_1 \left(\int_a^t p(s) R_1(s) ds \right)^{\frac{1}{\beta+1}} = m_1\, X_{11}(t),$$

for sufficiently large t. Similar arguments lead to the existence of $m_2 > 0$ such that

$$y(t) \geq m_2 \left(\int_a^t q(s) R_1(s) ds \right)^{\frac{1}{\alpha+1}} = m_2\, Y_{11}(t),$$

for sufficiently large t. This completes the proof of Lemma 3.2. □

Next, we show that functions X, X_{11} and X_{12} are in the relation $\overset{r}{\sim}$ under the assumption that p and q are rapidly varying functions of index ∞.

Lemma 3.3. *Suppose that p and q are rapidly varying of index ∞. Then*

$$X_{11}(t) \overset{r}{\sim} X_{12}(t) \overset{r}{\sim} X(t), \quad t \to \infty, \tag{3.17}$$

where the functions X_{11}, X_{12} and X are given by (3.4) *and* (3.2), *respectively.*

Proof. In the view of (3.6), using Proposition 2.7 (a) we have

$$P_1(t) = \int_a^t p(s) ds \overset{r}{\sim} t \cdot p(t), \quad Q_1(t) = \int_a^t q(s) ds \overset{r}{\sim} t \cdot q(t), \quad t \to \infty \tag{3.18}$$

and

$$R_1(t) \overset{r}{\sim} \left(t\, p(t)^{\frac{\beta(\alpha+1)}{2\alpha\beta+\alpha+\beta}} q(t)^{\frac{\alpha(\beta+1)}{2\alpha\beta+\alpha+\beta}} \right)^{\frac{2\alpha\beta+\alpha+\beta}{1-\alpha\beta}} = t^{\frac{2\alpha\beta+\alpha+\beta}{1-\alpha\beta}}\ p(t)^{\frac{\beta(\alpha+1)}{1-\alpha\beta}}\ q(t)^{\frac{\alpha(\beta+1)}{1-\alpha\beta}}, \quad t \to \infty, \tag{3.19}$$

since $\frac{2\alpha\beta+\alpha+\beta}{1-\alpha\beta} > 0$, which implies by Proposition 2.6 (b)

$$p(t)\, R_1(t) \overset{r}{\sim} t^{\frac{2\alpha\beta+\alpha+\beta}{1-\alpha\beta}}\ p(t)^{\frac{\beta+1}{1-\alpha\beta}}\ q(t)^{\frac{\alpha(\beta+1)}{1-\alpha\beta}}, \quad t \to \infty. \tag{3.20}$$

From (3.18) we obtain

$$X_{12}(t) = (P_1(t)\, Q_1(t)^{\alpha})^{\frac{1}{1-\alpha\beta}} \overset{r}{\sim} (t\, p(t)\, t^{\alpha}\, q(t)^{\alpha})^{\frac{1}{1-\alpha\beta}} = X(t), \quad t \to \infty. \tag{3.21}$$

Using (3.20), another application of Proposition 2.7 (a) gives us

$$X_{11}(t) \overset{r}{\sim} \left(t^{\frac{(\alpha+1)(\beta+1)}{1-\alpha\beta}}\ p(t)^{\frac{\beta+1}{1-\alpha\beta}}\ q(t)^{\frac{\alpha(\beta+1)}{1-\alpha\beta}} \right)^{\frac{1}{\beta+1}} = X(t), \quad t \to \infty. \tag{3.22}$$

Since the relation $\overset{r}{\sim}$ is an equivalence relation in RPV(∞), from (3.21) and (3.22) we conclude that (3.17) is satisfied. This completes the proof of Lemma 3.3. □

Similarly, we show that functions Y, Y_{11} and Y_{12} are in the relation $\overset{r}{\sim}$ under the assumption that p and q are rapidly varying functions of index ∞.

Lemma 3.4. *Suppose that p and q are rapidly varying of index ∞. Then*

$$Y_{11}(t) \overset{r}{\sim} Y_{12}(t) \overset{r}{\sim} Y(t), \quad t \to \infty, \tag{3.23}$$

where the functions Y_{11}, Y_{12} and Y are given by (3.5) *and* (3.3), *respectively.*

Proof. From (3.18) we have

$$Y_{12}(t) = \left(P_1(t)^{\beta}\, Q_1(t)\right)^{\frac{1}{1-\alpha\beta}} \overset{r}{\sim} \left(t^{\beta}\, p(t)^{\beta}\, t\, q(t)\right)^{\frac{1}{1-\alpha\beta}} = Y(t), \quad t \to \infty. \tag{3.24}$$

Using (3.19) and Proposition 2.6 (b) we get

$$q(t)\, R_1(t) \overset{r}{\sim} t^{\frac{2\alpha\beta+\alpha+\beta}{1-\alpha\beta}}\, p(t)^{\frac{\beta(\alpha+1)}{1-\alpha\beta}}\, q(t)^{\frac{\alpha+1}{1-\alpha\beta}}, \quad t \to \infty,$$

implying

$$Y_{11}(t) \overset{r}{\sim} \left(t^{\frac{(\alpha+1)(\beta+1)}{1-\alpha\beta}}\, p(t)^{\frac{\beta(\alpha+1)}{1-\alpha\beta}}\, q(t)^{\frac{\alpha+1}{1-\alpha\beta}}\right)^{\frac{1}{\alpha+1}} = Y(t), \quad t \to \infty, \tag{3.25}$$

where we use Proposition 2.7 (a) once again. Since the relation $\overset{r}{\sim}$ is an equivalence relation in RPV(∞), from (3.24) and (3.25) we conclude that (3.23) is satisfied. This completes the proof of Lemma 3.4. □

Proof of Theorem 3.1: Since p and q are rapidly varying of index ∞ we conclude that (3.10) is satisfied implying that both condition in (1.3) are satisfied, so that $\mathcal{SI} \neq \emptyset$ or consequently using Lemma 3.2(i) $\mathcal{I} = \mathcal{SI} \neq \emptyset$.

Take any $(x, y) \in \mathcal{I}$. From Lemma 3.2 (ii) we have that there exist positive constants m_i, M_i, $i = 1, 2$ such that (3.7) holds for large t. For brevity, we will not mention the phrase "for large t" repeatedly.

First, we show that x and y are rapidly varying functions of index ∞. Fix arbitrary $\lambda > 1$. In view of (3.7) we obtain

$$m_1\, X_{11}(\lambda t) \le x(\lambda t) \le M_1\, X_{12}(\lambda t) \quad \wedge \quad m_2\, Y_{11}(\lambda t) \le y(\lambda t) \le M_2\, Y_{12}(\lambda t), \tag{3.26}$$

and

$$\frac{1}{M_1 X_{12}(t)} \le \frac{1}{x(t)} \le \frac{1}{m_1 X_{11}(t)} \quad \wedge \quad \frac{1}{M_2 Y_{12}(t)} \le \frac{1}{y(t)} \le \frac{1}{m_2 Y_{11}(t)}. \tag{3.27}$$

From (3.26) and (3.27) we get

$$\frac{m_1}{M_1}\, \frac{X_{11}(\lambda\, t)}{X_{12}(t)} \le \frac{x(\lambda\, t)}{x(t)} \le \frac{M_1}{m_1}\, \frac{X_{12}(\lambda\, t)}{X_{11}(t)} \quad \wedge \quad \frac{m_2}{M_2}\, \frac{Y_{11}(\lambda\, t)}{Y_{12}(t)} \le \frac{y(\lambda\, t)}{y(t)} \le \frac{M_2}{m_2}\, \frac{Y_{12}(\lambda\, t)}{Y_{11}(t)}. \tag{3.28}$$

Lemma 3.3 and Lemma 3.4 ensures that functions X_{11} and X_{12} are in the relation $\overset{r}{\sim}$, as well as functions Y_{11} and Y_{12}, which by definition means

$$\lim_{t\to\infty} \frac{X_{11}(\lambda\, t)}{X_{12}(t)} = \lim_{t\to\infty} \frac{X_{12}(\lambda\, t)}{X_{11}(t)} = \infty \quad \wedge \quad \lim_{t\to\infty} \frac{Y_{11}(\lambda\, t)}{Y_{12}(t)} = \lim_{t\to\infty} \frac{Y_{12}(\lambda\, t)}{Y_{11}(t)} = \infty. \tag{3.29}$$

Since λ was arbitrary, combining (3.28) and (3.29) gives us

$$\lim_{t\to\infty} \frac{x(\lambda t)}{x(t)} = \infty \quad \wedge \quad \lim_{t\to\infty} \frac{y(\lambda t)}{y(t)} = \infty$$

for all $\lambda > 1$. Thus, $x, y \in \mathrm{RPV}(\infty)$.

It remains to establish the asymptotic relation (3.1). Fix arbitrary $\lambda > 1$. Let m_i and M_i, $i = 1, 2$ be positive numbers that satisfy (3.7). By Lemma 3.3 and Lemma 3.4 we have (3.17) and (3.23), or consequently

$$M_1\, X_{12}(t) \leq X(\lambda\ t) \quad \wedge \quad X\left(\frac{t}{\lambda}\right) \leq m_1\, X_{11}(t),$$

and

$$M_2\, Y_{12}(t) \leq Y(\lambda\ t) \quad \wedge \quad Y\left(\frac{t}{\lambda}\right) \leq m_2\, Y_{11}(t),$$

which in the view of (3.7) implies

$$X\left(\frac{t}{\lambda}\right) \leq x(t) \leq X(\lambda\ t) \quad \wedge \quad Y\left(\frac{t}{\lambda}\right) \leq y(t) \leq Y(\lambda\ t),$$

that is (3.1). This completes the proof of Theorem 3.1. □

4. Strongly decreasing solutions of (S_2)

Now we turn our attention to the study of strongly decreasing solutions of (S_2) . The following theorem is the main result of this section.

Theorem 4.1. *Suppose that p and q are rapidly varying of index $-\infty$. Every strongly decreasing solution of* (S_2) *is rapidly varying of index $-\infty$. Moreover, any such solution (x, y) satisfies the asymptotic relation*

$$x(t) \underset{\star}{\sim} X(t), \quad y(t) \underset{\star}{\sim} Y(t), \quad t \to \infty, \tag{4.1}$$

where the functions X and Y are given by (3.2) *and* (3.3)*, respectively.*

Lemma 4.2. *If $(x, y) \in \mathcal{SD}$, then there exist positive constants l_i, $i = 1, 2$ such that*

$$l_1\, X_{21}(t) \leq x(t) \leq X_{22}(t) \quad \wedge \quad l_2\, Y_{21}(t) \leq y(t) \leq Y_{22}(t), \tag{4.2}$$

for large t, where X_{21}, X_{22} and Y_{21}, Y_{22} are given by

$$X_{21}(t) = \left(\int_t^\infty p(s)\, R_2(s)ds\right)^{\frac{1}{\beta+1}}, \quad X_{22}(t) = (P_2(t)\, Q_2(t)^\alpha)^{\frac{1}{1-\alpha\beta}}, \tag{4.3}$$

$$Y_{21}(t) = \left(\int_t^\infty q(s)\, R_2(s)ds\right)^{\frac{1}{\alpha+1}}, \quad Y_{22}(t) = \left(P_2(t)^\beta\, Q_2(t)\right)^{\frac{1}{1-\alpha\beta}}, \tag{4.4}$$

with

$$P_2(t) = \int_t^\infty p(s)ds, \quad Q_2(t) = \int_t^\infty q(s)ds, \quad R_2(t) = \left(\int_t^\infty p(s)^{\frac{\beta(\alpha+1)}{2\alpha\beta+\alpha+\beta}}\, q(s)^{\frac{\alpha(\beta+1)}{2\alpha\beta+\alpha+\beta}} ds\right)^{\frac{2\alpha\beta+\alpha+\beta}{1-\alpha\beta}}. \tag{4.5}$$

Proof. Let $(x, y) \in \mathcal{SD}$. Using the fact that $x(t) \to 0$ and $y(t) \to 0$ as $t \to \infty$ and that x, y are decreasing, integration of (S_2) on (t, ∞) gives us

$$x(t) = \int_t^\infty p(s)y(s)^\alpha ds \leq y(t)^\alpha \int_t^\infty p(s)ds = y(t)^\alpha\, P_2(t), \tag{4.6}$$

$$y(t) = \int_t^\infty q(s)x(s)^\beta ds \leq x(t)^\beta \int_t^\infty q(s)ds = x(t)^\beta\, Q_2(t). \tag{4.7}$$

By substituting (4.7) into (4.6) we obtain

$$x(t) \le (P_2(t)\, Q_2(t)^{\alpha})^{\frac{1}{1-\alpha\beta}} = X_{22}(t),$$

and by substituting (4.6) into (4.7) we get

$$y(t) \le \left(P_2(t)^{\beta}\, Q_2(t)\right)^{\frac{1}{1-\alpha\beta}} = Y_{22}(t).$$

Thus, the right-hand side of the inequality (4.2) is proved.

Now we prove the left-hand side of (4.2). Setting $\omega(t) = x(t)^{\beta} y(t)^{\alpha}$ and μ, ν, η as in (3.13), application of Young's inequality gives

$$-\omega'(t) = \omega(t)\left(\beta\, \frac{p(t)y(t)^{\alpha}}{x(t)} + \alpha\, \frac{q(t)x(t)^{\beta}}{y(t)}\right) \ge \omega(t)\, \frac{\beta^{\mu}\alpha^{\nu}}{\mu^{\mu}\nu^{\nu}} \left(\frac{p(t)y(t)^{\alpha}}{x(t)}\right)^{\mu} \left(\frac{q(t)x(t)^{\beta}}{y(t)}\right)^{\nu} = \frac{\beta^{\mu}\alpha^{\nu}}{\mu^{\mu}\nu^{\nu}}\, \omega(t)^{1-\eta}\, p(t)^{\mu}\, q(t)^{\nu}.$$

Then, there is $k_1 > 0$ such that

$$-\omega(t)^{\eta-1}\, \omega'(t) \ge k_1\, p(t)^{\mu}\, q(t)^{\nu}. \tag{4.8}$$

Since $\omega(t) \to 0$ as $t \to \infty$ and $\eta > 0$, integrating (4.8) from t to ∞ we have

$$\omega(t) \ge k_2 \left(\int_t^{\infty} p(s)^{\mu}\, q(s)^{\nu} ds\right)^{1/\eta}$$

or

$$x(t)^{\beta} y(t)^{\alpha} \ge k_2 \left(\int_t^{\infty} p(s)^{\mu}\, q(s)^{\nu} ds\right)^{1/\eta} = k_2\, R_2(t), \tag{4.9}$$

for some $k_2 > 0$. Using first equation of (S_2), we substitute $y(t)^{\alpha}$ into (4.9) to get

$$-x(t)^{\beta} x'(t) \ge k_2\, p(t)\, R_2(t). \tag{4.10}$$

Integrating (4.10) from t to ∞, we find $l_1 > 0$ such that

$$x(t) \ge l_1 \left(\int_t^{\infty} p(s) R_2(s) ds\right)^{\frac{1}{\beta+1}} = l_1\, X_{21}(t).$$

Similar arguments lead to the existence of $l_2 > 0$ such that

$$y(t) \ge l_2 \left(\int_t^{\infty} q(s) R_2(s) ds\right)^{\frac{1}{\alpha+1}} = l_2\, Y_{21}(t).$$

This completes the proof of Lemma 4.2. □

Lemma 4.3. *Suppose that p and q are rapidly varying of index* $-\infty$*. Then*

$$X_{21}(t) \underset{r}{\sim} X_{22}(t) \underset{r}{\sim} X(t), \quad t \to \infty, \tag{4.11}$$

where the functions X_{21}, X_{22} *and* X *are given by* (4.3) *and* (3.2)*, respectively.*

Proof. In the view of (4.5), applying Proposition 2.14 we conclude that

$$P_2(t) = \int_t^{\infty} p(s) ds \underset{r}{\sim} t \cdot p(t), \quad Q_2(t) = \int_t^{\infty} q(s) ds \underset{r}{\sim} t \cdot q(t), \quad t \to \infty \tag{4.12}$$

and

$$R_2(t) \underset{r}{\sim} t^{\frac{2\alpha\beta+\alpha+\beta}{1-\alpha\beta}}\, p(t)^{\frac{\beta(\alpha+1)}{1-\alpha\beta}}\, q(t)^{\frac{\alpha(\beta+1)}{1-\alpha\beta}}, \quad t \to \infty. \tag{4.13}$$

From (4.12) we have

$$X_{22}(t) \underset{r}{\sim} X(t), \quad t \to \infty. \tag{4.14}$$

Multiplying (4.13) with $p \in \mathrm{RPV}(-\infty)$ gives us by Proposition 2.13 (b)

$$p(t)\, R_2(t) \underset{r}{\sim} t^{\frac{2\alpha\beta+\alpha+\beta}{1-\alpha\beta}}\, p(t)^{\frac{\beta+1}{1-\alpha\beta}}\, q(t)^{\frac{\alpha(\beta+1)}{1-\alpha\beta}}, \quad t \to \infty,$$

implying

$$X_{21}(t) \underset{r}{\sim} \left(t^{\frac{(\alpha+1)(\beta+1)}{1-\alpha\beta}}\, p(t)^{\frac{\beta+1}{1-\alpha\beta}}\, q(t)^{\frac{\alpha(\beta+1)}{1-\alpha\beta}} \right)^{\frac{1}{\beta+1}} = X(t), \quad t \to \infty, \tag{4.15}$$

where we use Proposition 2.14 once again. Combining (4.14) and (4.15), we conclude that (4.11) is satisfied. This completes the proof of Lemma 4.3. □

Lemma 4.4. *Suppose that p and q are rapidly varying of index $-\infty$. Then*

$$Y_{21}(t) \underset{r}{\sim} Y_{22}(t) \underset{r}{\sim} Y(t), \quad t \to \infty, \tag{4.16}$$

where the functions Y_{21}, Y_{22} and Y are given by (4.4) *and* (3.3), *respectively.*

Proof. Directly using (4.12) we obtain

$$Y_{22}(t) \underset{r}{\sim} Y(t), \quad t \to \infty. \tag{4.17}$$

Using Proposition 2.13 (b), from (4.13) we get

$$q(t)\, R_2(t) \underset{r}{\sim} t^{\frac{2\alpha\beta+\alpha+\beta}{1-\alpha\beta}}\, p(t)^{\frac{\beta(\alpha+1)}{1-\alpha\beta}}\, q(t)^{\frac{\alpha+1}{1-\alpha\beta}}, \quad t \to \infty,$$

which with application of Proposition 2.14 implies

$$Y_{21}(t) \underset{r}{\sim} Y(t), \quad t \to \infty. \tag{4.18}$$

From (4.17) and (4.18) we have that (4.16) is satisfied. This completes the proof of Lemma 4.4. □

Proof of Theorem 4.1: Since p and q are rapidly varying of index $-\infty$ using Proposition 2.8 (ii) we conclude that

$$\lim_{t\to\infty} P_2(t) = 0 \quad \wedge \quad \lim_{t\to\infty} Q_2(t) = 0$$

or respectively

$$\int_a^\infty p(t)dt < \infty \quad \wedge \quad \int_a^\infty q(t)dt < \infty$$

implying that both condition in (1.4) are satisfied, so that $\mathcal{SD} \neq \emptyset$.

Take any $(x, y) \in \mathcal{SD}$. From Lemma 4.2, we have that there exist positive constants l_i, $i = 1, 2$ such that (4.2) holds for large t.

That $x, y \in \mathrm{RPV}(-\infty)$ and satisfy the asymptotic relation (4.1) can be proved in the same way as in the proof of Theorem 3.1, using Lemma 4.3 and Lemma 4.4.

5. Application to the generalized Emden-Fowler equation

This section is dedicated to applying the main results to the generalized Emden-Fowler equation (E). We derive new results, that give the conditions under which all strongly monotone solutions of the equation (E) are rapidly varying functions if p, q are rapidly varying and determine the asymptotic behavior of these solutions in terms of relations $\overset{\star}{\sim}$ or $\underset{\star}{\sim}$.

Indeed, if x is a strongly increasing solution of (E), putting $y(t) = p(t)x'(t)^\alpha$, we get that (x, y) is a strongly increasing solution of the system (1.1). In order to study the equation (E) in the framework of rapid variation we need to require that $p \in \mathrm{RPV}(-\infty)$ and $q \in \mathrm{RPV}(\infty)$, from which follows that $p^{-1/\alpha}, q \in \mathrm{RPV}(\infty)$. We can now apply the Theorem 3.1 to the system (1.1), concluding that every strongly increasing solution of (1.1) is rapidly varying of index ∞. Also, the component x satisfies the asymptotic relation

$$x(t) \overset{\star}{\sim} \left(t^{\frac{1}{\alpha}+1}\, p(t)^{-\frac{1}{\alpha}}\, q(t)^{\frac{1}{\alpha}}\right)^{\frac{1}{1-\frac{\beta}{\alpha}}} = \left(t^{\alpha+1}\, \frac{q(t)}{p(t)}\right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty.$$

Similarly, if x is a strongly decreasing solution of (E), putting $y(t) = p(t)(-x'(t))^\alpha$, we get that (x, y) is a strongly decreasing solution of the system (1.2). We should assume $p \in \mathrm{RPV}(\infty)$ and $q \in \mathrm{RPV}(-\infty)$ whence it follows that $p^{-1/\alpha}, q \in \mathrm{RPV}(-\infty)$. Applying Theorem 4.1 on the system (1.2) for the component x we conclude that

$$x(t) \underset{\star}{\sim} \left(t^{\frac{1}{\alpha}+1}\, p(t)^{-\frac{1}{\alpha}}\, q(t)^{\frac{1}{\alpha}}\right)^{\frac{1}{1-\frac{\beta}{\alpha}}} = \left(t^{\alpha+1}\, \frac{q(t)}{p(t)}\right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty.$$

Therefore, we have the following two theorems.

Theorem 5.1. *Suppose that* $p \in \mathrm{RPV}(-\infty)$, $q \in \mathrm{RPV}(\infty)$. *Every strongly increasing solution of* (E) *is rapidly varying of index* ∞. *Moreover, any such solution* x *satisfies the asymptotic relation*

$$x(t) \overset{\star}{\sim} \left(t^{\alpha+1}\, \frac{q(t)}{p(t)}\right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty.$$

Theorem 5.2. *Suppose that* $p \in \mathrm{RPV}(\infty)$, $q \in \mathrm{RPV}(-\infty)$. *Every strongly decreasing solution of* (E) *is rapidly varying of index* $-\infty$. *Moreover, any such solution* x *satisfies the asymptotic relation*

$$x(t) \underset{\star}{\sim} \left(t^{\alpha+1}\, \frac{q(t)}{p(t)}\right)^{\frac{1}{\alpha-\beta}}, \quad t \to \infty.$$

6. Examples

Now, we present two examples that illustrate main results stated by Theorem 3.1 and Theorem 4.1.

Example 6.1. Consider the system

$$x' = p_1(t)\, y^\alpha, \quad y' = q_1(t)\, x^\beta, \qquad 0 < \alpha < 1, \quad 0 < \beta < 1, \tag{6.1}$$

where

$$p_1(t) = \beta^{-1}\, e^{\frac{1-\beta}{\beta} t} \quad \wedge \quad q_1(t) = \alpha^{-1}\, e^{\frac{1-\alpha}{\alpha} t}.$$

Since $p_1,\ q_1 \in \mathrm{RPV}(\infty)$ applying Theorem 3.1 we conclude that every strongly increasing solution of (6.1) is rapidly varying of index ∞ and any such solution (x, y) satisfies the asymptotic relation

$$x(t) \overset{\star}{\sim} X_1(t), \quad y(t) \overset{\star}{\sim} Y_1(t) \quad t \to \infty, \tag{6.2}$$

where

$$X_1(t) = (t^{\alpha+1}\, p_1(t)\, q_1(t)^{\alpha})^{\frac{1}{1-\alpha\beta}} = (\alpha^{\alpha}\, \beta)^{\frac{1}{\alpha\beta-1}}\ t^{\frac{\alpha+1}{1-\alpha\beta}}\ e^{\frac{t}{\beta}} \quad \wedge \quad Y_1(t) = (t^{\beta+1}\, p_1(t)^{\beta}\, q_1(t))^{\frac{1}{1-\alpha\beta}} = \left(\alpha\, \beta^{\beta}\right)^{\frac{1}{\alpha\beta-1}}\ t^{\frac{\beta+1}{1-\alpha\beta}}\ e^{\frac{t}{\alpha}}.$$

It is easy to check that $(x_1(t), y_1(t)) = (e^{\frac{t}{\beta}}, e^{\frac{t}{\alpha}})$ is such an solution of (6.1), since

$$\lim_{t\to\infty} \frac{x_1(\lambda t)}{X_1(t)} = \lim_{t\to\infty} \frac{X_1(\lambda t)}{x_1(t)} = \infty \quad \wedge \quad \lim_{t\to\infty} \frac{y_1(\lambda t)}{Y_1(t)} = \lim_{t\to\infty} \frac{Y_1(\lambda t)}{y_1(t)} = \infty,$$

implying that (x_1, y_1) satisfies the asymptotic relation (6.2) and $x_1, y_1 \in \mathrm{RPV}(\infty)$.

Example 6.2. Consider the system

$$x' + p_2(t)\, y^{\alpha} = 0, \quad y' + q_2(t)\, x^{\beta} = 0, \qquad 0 < \alpha < 1, \quad 0 < \beta < 1, \tag{6.3}$$

where

$$p_2(t) = \beta^{-1}\, e^{\frac{\beta-1}{\beta} t} \quad \wedge \quad q_2(t) = \alpha^{-1}\, e^{\frac{\alpha-1}{\alpha} t}.$$

Since $p_2, q_2 \in \mathrm{RPV}(-\infty)$ applying Theorem 4.1 we obtain that every strongly decreasing solution of (6.3) is rapidly varying of index $-\infty$ and any such solution (x, y) satisfies the asymptotic relation

$$x(t) \underset{\star}{\sim} X_2(t), \quad y(t) \underset{\star}{\sim} Y_2(t) \quad t \to \infty, \tag{6.4}$$

where

$$X_2(t) = (t^{\alpha+1}\, p_2(t)\, q_2(t)^{\alpha})^{\frac{1}{1-\alpha\beta}} = (\alpha^{\alpha}\, \beta)^{\frac{1}{\alpha\beta-1}}\ t^{\frac{\alpha+1}{1-\alpha\beta}}\ e^{-\frac{t}{\beta}} \quad \wedge \quad Y_2(t) = (t^{\beta+1}\, p_2(t)^{\beta}\, q_2(t))^{\frac{1}{1-\alpha\beta}} = \left(\alpha\, \beta^{\beta}\right)^{\frac{1}{\alpha\beta-1}}\ t^{\frac{\beta+1}{1-\alpha\beta}}\ e^{-\frac{t}{\alpha}}.$$

It is easy to check that $(x_2(t), y_2(t)) = (e^{-\frac{t}{\beta}}, e^{-\frac{t}{\alpha}})$ is such an solution of (6.3), since

$$\lim_{t\to\infty} \frac{x_2(\lambda t)}{X_2(t)} = \lim_{t\to\infty} \frac{X_2(\lambda t)}{x_2(t)} = 0 \quad \wedge \quad \lim_{t\to\infty} \frac{y_2(\lambda t)}{Y_2(t)} = \lim_{t\to\infty} \frac{Y_2(\lambda t)}{y_2(t)} = 0,$$

implying that (x_2, y_2) satisfies the asymptotic relation (6.4) and $x_2, y_2 \in \mathrm{RPV}(-\infty)$.

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