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Matrix Transformations Between Certain New Sequence Spaces over Ultrametric Fields

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Abstract. Throughout the present paper, K denotes a complete, non-trivially valued, ultrametric (or non-archimedean) field. Sequences, infinite series and infinite matrices have their entries in K. The sequence spaces $m^\lambda, c^\lambda, c_0^\lambda$ were introduced in K earlier by the author in [8–10] and some studies were made. The purpose of the present paper is to characterize the matrix classes (c_0^λ, c_0^μ), (c_0^λ, m^μ), (c_0^λ, c^μ) and (c^λ, c_0^μ).

1. Introduction and Preliminaries

Throughout the present paper, K denotes a complete, non-trivially valued, ultrametric (or non-archimedean) field. Entries of sequences, infinite series and infinite matrices are in K. Given a sequence $x = \{x_k\}$ in K and an infinite matrix $A = (a_{nk})$, $a_{nk} \in K$, $n, k = 0, 1, 2, \ldots$, let

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \ n = 0, 1, 2, \ldots,$$

where we suppose that the series on the right converge. $A(x) = \{(Ax)_n\}$ is called the A-transform of the sequence $x = \{x_k\}$.

If X, Y are sequence spaces, we write $A = (a_{nk}) \in (X, Y)$ if $\{(Ax)_n\} \in Y$, whenever $x = \{x_k\} \in X$. In the sequel, m, c, c_0 respectively denote the ultrametric Banach spaces of bounded, convergent and null sequences in K under the ultrametric norm

$$||x|| = \sup_{k \geq 0} |x_k|, \ x = \{x_k\} \in m, c, c_0.$$

Following Kangro [1], the author of the present paper introduced the analogues in ultrametric analysis of the concepts of λ-convergence, λ-boundedness etc. and made a study in [8–10]. We continue the study in the present paper. For a detailed investigation of the above concepts λ-convergence, λ-boundedness etc. in the classical case, a standard reference is [1]. For a study of summability theory and its applications in the classical case, the reader can refer to [2, 3, 6].

To make the paper self-contained, we recall the following definitions [8–10].

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Definition 1.1. *Let $\lambda = \{\lambda_n\}$ be a sequence in K such that*

$$0 < |\lambda_n| \nearrow \infty, \; n \to \infty.$$

A sequence $\{x_n\}$ in K is said to be convergent with speed λ or λ-convergent if $\{x_n\} \in c$ with $\lim_{n\to\infty} x_n = s$ (say) and

$$\lim_{n\to\infty} \lambda_n(x_n - s) \text{ exists.}$$

Let c^λ denote the set of all λ-convergent sequences in K. From the definition, we have,

$$c^\lambda \subset c.$$

In the above context, we note that the sequences

$$e_k = \{0, 0, \dots, 0, 1, 0, \dots\},$$

1 occurring in the kth place only, $k = 0, 1, 2, \dots;$

$$e = \{1, 1, 1, \dots\}$$

and

$$e^\lambda = \left\{\frac{1}{\lambda_0}, \frac{1}{\lambda_1}, \dots\right\}$$

all belong to c^λ.

Definition 1.2. *A sequence $\{x_n\}$ in K is said to be bounded with speed λ or λ-bounded, if $x = \{x_n\} \in c$ with $\lim_{n\to\infty} x_n = s$ and*

$$\{\lambda_n(x_n - s)\} \text{ is bounded.}$$

Let m^λ denote the set of all λ-bounded sequences in K. Note that

$$c^\lambda \subset m^\lambda \subset c.$$

Definition 1.3. *Let c_0^λ denote the set of all sequences $x = \{x_n\}$ in K such that $\{x_n\} \in c$ with $\lim_{n\to\infty} x_n = s$ and*

$$\lim_{n\to\infty} \lambda_n(x_n - s) = 0.$$

Note again that

$$c_0^\lambda \subset c^\lambda \subset m^\lambda \subset c.$$

The following results can be easily proved.

Theorem 1.4 ([5, 7]). *$A = (a_{nk}) \in (c_0, c_0)$ if and only if*

$$\sup_{n,k} |a_{nk}| < \infty; \tag{1}$$

and

$$\lim_{n\to\infty} a_{nk} = 0, \; k = 0, 1, 2, \dots. \tag{2}$$

Theorem 1.5 ([5, 7]). *$A = (a_{nk}) \in (c, c_0)$ if and only if (1), (2) hold and*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 0. \tag{3}$$

Theorem 1.6 ([5, 7]). *$A = (a_{nk}) \in (c_0, c)$ if and only if (1) holds and*

$$\lim_{n\to\infty} a_{nk} = a_k,\ k = 0, 1, 2, \ldots. \tag{4}$$

In such a case,

$$\lim_{n\to\infty} (Ax)_n = \sum_{k=0}^{\infty} a_k x_k. \tag{5}$$

Proof. Leaving out the former part, we prove (5). Let $x = \{x_k\} \in c_0$.

$$\begin{aligned}(Ax)_n &= \sum_{k=0}^{\infty} a_{nk} x_k \\ &= \sum_{k=0}^{\infty} (a_{nk} - a_k) x_k + \sum_{k=0}^{\infty} a_k x_k.\end{aligned}$$

Since $x = \{x_k\} \in c_0$, given $\epsilon > 0$, there exists a positive integer N such that

$$|x_k| < \frac{\epsilon}{H},\ k > N,$$

where $|a_{nk}| \leq H$, $n, k = 0, 1, 2, \ldots$.
Since

$$\lim_{n\to\infty} a_{nk} = a_k,\ k = 0, 1, 2, \ldots, N,$$

there exists a positive integer M such that

$$|a_{nk} - a_k| < \frac{\epsilon}{L},\ k = 0, 1, 2, \ldots, N \text{ and } n > M,$$

where $|x_k| \leq L$, $k = 0, 1, 2, \ldots$.
Thus, for $n > M$, we have,

$$\begin{aligned}\left|\sum_{k=0}^{\infty} (a_{nk} - a_k) x_k\right| &= \left|\sum_{k=0}^{N} (a_{nk} - a_k) x_k + \sum_{k>N} (a_{nk} - a_k) x_k\right| \\ &\leq Max\left[\max_{0\leq k\leq N} |a_{nk} - a_k||x_k|, \max_{k>N} |a_{nk} - a_k||x_k|\right] \\ &\leq Max\left[\frac{\epsilon}{L} L, \frac{\epsilon}{H} H\right] \\ &= \epsilon,\end{aligned}$$

from which it follows that

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} (a_{nk} - a_k) x_k = 0.$$

Consequently

$$\lim_{n\to\infty} (Ax)_n = \sum_{k=0}^{\infty} a_k x_k,$$

completing the proof. □

Theorem 1.7 (Kojima-Schur)(see [4, 5, 7]). *$A = (a_{nk}) \in (c, c)$ if and only if (1), (4) hold and*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = a \ \text{ exists.} \tag{6}$$

In such a case,

$$\lim_{n\to\infty} (Ax)_n = \sum_{k=0}^{\infty} a_k(x_k - s) + sa \tag{7}$$

where $x = \{x_k\} \in c$ with $\lim_{k\to\infty} x_k = s$.

2. Main Results

Let $\mu = \{\mu_n\}$ be a sequence in K such that

$$0 < |\mu_n| \nearrow \infty, \ n \to \infty.$$

We now prove the main results in this section.

Theorem 2.1. *$A = (a_{nk}) \in (c_0^\lambda, c_0^\mu)$ if and only if*

$$A(e), A(e_k) \in c_0^\mu, \ k = 0, 1, 2, \ldots; \tag{8}$$

$$\sup_{n,k} \left| \frac{a_{nk}}{\lambda_k} \right| < \infty; \tag{9}$$

and

$$\sup_{n,k} \left| \frac{\mu_n(a_{n,k} - a_k)}{\lambda_k} \right| < \infty, \tag{10}$$

where $\lim_{n\to\infty} a_{nk} = a_k$, $k = 0, 1, 2, \ldots$.

Proof. Necessity. Let $A \in (c_0^\lambda, c_0^\mu)$. Since $e, e_k \in c_0^\lambda$, $k = 0, 1, 2, \ldots$, it follows that $A(e), A(e_k) \in c_0^\mu$, $k = 0, 1, 2, \ldots$, i.e., (8) holds. Since $A(e_k) \in c_0^\mu$, $\lim_{n\to\infty} a_{nk} = a_k$, $k = 0, 1, 2, \ldots$.

Since $A(e) \in c_0^\mu$, $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = a$.
Next, let $x = \{x_k\} \in c_0^\lambda$ so that $x = \{x_k\} \in c$.
Let $\lim_{k\to\infty} x_k = s$ and

$$\beta_k = \lambda_k(x_k - s).$$

So,

$$\begin{aligned} (Ax)_n &= \sum_{k=0}^{\infty} a_{nk} x_k \\ &= \sum_{k=0}^{\infty} a_{nk} \left(\frac{\beta_k}{\lambda_k} + s \right) \\ &= \sum_{k=0}^{\infty} \frac{a_{nk}}{\lambda_k} \beta_k + s \sum_{k=0}^{\infty} a_{nk}. \end{aligned} \tag{11}$$

Now, $\{(Ax)_n\} \in c$, $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk}$ exists and $\{\beta_k\} \in c_0$.
Using (11), the infinite matrix

$$\left(\frac{a_{nk}}{\lambda_k}\right) \in (c_0, c).$$

In view of Theorem 1.6,

$$\sup_{n,k} \left|\frac{a_{nk}}{\lambda_k}\right| < \infty,$$

i.e., (9)*holds*

and

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{a_{nk}}{\lambda_k}\beta_k = \sum_{k=0}^{\infty} \frac{a_k}{\lambda_k}\beta_k.$$

Taking the limit as $n \to \infty$ in (11), we get,

$$y = \lim_{n\to\infty} (Ax)_n = \sum_{k=0}^{\infty} \frac{a_k}{\lambda_k}\beta_k + sa. \tag{12}$$

Using (11) and (12), we have,

$$(Ax)_n - y = \sum_{k=0}^{\infty} \frac{a_{nk} - a_k}{\lambda_k}\beta_k + s\left(\sum_{k=0}^{\infty} a_{nk} - a\right),$$

and consequently,

$$\mu_n\{(Ax)_n - y\} = \sum_{k=0}^{\infty} \frac{\mu_n(a_{nk} - a_k)}{\lambda_k}\beta_k + s\mu_n\left(\sum_{k=0}^{\infty} a_{nk} - a\right). \tag{13}$$

Since $\{(Ax)_n\} \in c_0^{\mu}$,

$$\lim_{n\to\infty} \mu_n\{(Ax)_n - y\} \text{ exists.}$$

Since $A(e) \in c_0^{\mu}$,

$$\lim_{n\to\infty} \mu_n\left(\sum_{k=0}^{\infty} a_{nk} - a\right) \text{ exists.}$$

Using (13) and the fact that $\{\beta_k\} \in c_0$, the infinite matrix

$$\left(\frac{\mu_n(a_{nk} - a_k)}{\lambda_k}\right) \in (c_0, c_0).$$

In view of Theorem 1.4, we have,

$$\sup_{n,k} \left|\frac{\mu_n(a_{nk} - a_k)}{\lambda_k}\right| < \infty,$$

i.e., (10)*holds*.

Sufficiency. Let the conditions (8), (9) and (10) hold. Let $x = \{x_k\} \in c_0^{\lambda}$. So $x = \{x_k\} \in c$ with $\lim_{k\to\infty} x_k = s$. Because of (8),

$$\lim_{n\to\infty} a_{nk} = a_k,\ k = 0, 1, 2, \ldots;$$

and

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = a.$$

Now, (11) holds. In view of (9) and the fact that

$$\lim_{n\to\infty} \frac{a_{nk}}{\lambda_k} = \frac{a_k}{\lambda_k},\ k = 0, 1, 2, \ldots,$$

using Theorem 1.6, it follows that the infinite matrix

$$\left(\frac{a_{nk}}{\lambda_k}\right) \in (c_0, c).$$

Since $\{\beta_k\} \in c_0$,

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{a_{nk}}{\lambda_k}\beta_k \text{ exists,}$$

i.e., $\lim_{n\to\infty} (Ax)_n$ exists, using (11).

At this stage, we note that (13) also holds and

$$\lim_{n\to\infty} \mu_n(a_{nk} - a_k) = 0,\ k = 0, 1, 2, \ldots.$$

Now, using (10) and Theorem 1.4, the infinite matrix

$$\left(\frac{\mu_n(a_{nk} - a_k)}{\lambda_k}\right) \in (c_0, c_0).$$

Since $\{\beta_k\} \in c_0$, we have,

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{\mu_n(a_{nk} - a_k)}{\lambda_k}\beta_k = 0.$$

Already,

$$\lim_{n\to\infty} \mu_n\left(\sum_{k=0}^{\infty} a_{nk} - a\right) = 0.$$

Using (13), we conclude that

$$\lim_{n\to\infty} \mu_n\{(Ax)_n - y\} = 0,$$

i.e., $\{(Ax)_n\} \in c_0^{\mu}$,

completing the proof of the theorem. □

Using Theorem 1.4 and Theorem 1.6, we can establish the following theorem in a similar fashion.

Theorem 2.2. *$A = (a_{nk}) \in (c_0^\lambda, m^\mu)$ if and only if*

$$A(e), A(e_k) \in m^\mu,\ k = 0, 1, 2, \ldots; \tag{14}$$

and (9), (10) hold.

Next, we prove the following result.

Theorem 2.3. *$A = (a_{nk}) \in (c^\lambda, c_0^\mu)$ if and only if*

$$A(e), A(e^\lambda), A(e_k) \in c_0^\mu,\ k = 0, 1, 2, \ldots; \tag{15}$$

(9) and (10) hold.

Proof. Necessity. Let $A = (a_{nk}) \in (c^\lambda, c_0^\mu)$. Since $e, e^\lambda, e_k \in c^\lambda$, it follows that $A(e), A(e^\lambda), A(e_k) \in c_0^\mu$, $k = 0, 1, 2, \ldots$, i.e., (15) holds. Thus,

$$\lim_{n\to\infty} a_{nk} = a_k,\ k = 0, 1, 2, \ldots;$$

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = a;$$

and

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{a_{nk}}{\lambda_k} = a^\lambda.$$

Let, now, $x = \{x_k\} \in c^\lambda$. So $\lim_{k\to\infty} x_k = s$ (say). Let, as usual,

$$\beta_k = \lambda_k(x_k - s).$$

Then $\{\beta_k\} \in c$. Let $\lim_{k\to\infty} \beta_k = \beta$. Note that (11) holds and $\{(Ax)_n\} \in c$. Hence the infinite matrix

$$\left(\frac{a_{nk}}{\lambda_k}\right) \in (c, c).$$

In view of Theorem 1.7, (9) holds. Also,

$$y = \lim_{n\to\infty} (Ax)_n = \sum_{k=0}^{\infty} \frac{a_k}{\lambda_k}(\beta_k - \beta) + \beta a^\lambda + sa.$$

Consequently,

$$\begin{aligned}
(Ax)_n - y &= \sum_{k=0}^{\infty} \frac{a_{nk}}{\lambda_k}\beta_k + s\sum_{k=0}^{\infty} a_{nk} - \sum_{k=0}^{\infty} \frac{a_k}{\lambda_k}(\beta_k - \beta) - \beta a^\lambda - sa \\
&= \sum_{k=0}^{\infty} \frac{a_{nk}}{\lambda_k}(\beta_k - \beta) + \beta\sum_{k=0}^{\infty} \frac{a_{nk}}{\lambda_k} + s\sum_{k=0}^{\infty} a_{nk} - \sum_{k=0}^{\infty} \frac{a_k}{\lambda_k}(\beta_k - \beta) - \beta a^\lambda - sa \\
&= \sum_{k=0}^{\infty} \frac{a_{nk} - a_k}{\lambda_k}(\beta_k - \beta) + \beta\left(\sum_{k=0}^{\infty} \frac{a_{nk}}{\lambda_k} - a^\lambda\right) + s\left(\sum_{k=0}^{\infty} a_{nk} - a\right)
\end{aligned}$$

and so

$$\mu_n\{(Ax)_n - y\} = \sum_{k=0}^{\infty} \frac{\mu_n(a_{nk} - a_k)}{\lambda_k}(\beta_k - \beta) + \beta\mu_n\left(\sum_{k=0}^{\infty} \frac{a_{nk}}{\lambda_k} - a^{\lambda}\right) + s\mu_n\left(\sum_{k=0}^{\infty} a_{nk} - a\right). \tag{16}$$

We note that since $\{(Ax)_n\} \in c_0^{\mu}$,

$$\lim_{n\to\infty} \mu_n\{(Ax)_n - y\} = 0;$$

Since $A(e^{\lambda}) \in c_0^{\mu}$,

$$\lim_{n\to\infty} \mu_n\left(\sum_{k=0}^{\infty} \frac{a_{nk}}{\lambda_k} - a^{\lambda}\right) = 0;$$

Since $A(e) \in c_0^{\mu}$,

$$\lim_{n\to\infty} \mu_n\left(\sum_{k=0}^{\infty} a_{nk} - a\right) = 0.$$

Thus, using (16), it follows that the infinite matrix

$$\left(\frac{\mu_n(a_{nk} - a_k)}{\lambda_k}\right) \in (c_0, c_0).$$

In view of Theorem 1.4, (10) holds.
Sufficiency. Let (9), (10) and (15) hold. Note that (11) holds. Because of (9) and the fact that

$$\lim_{n\to\infty} \frac{a_{nk}}{\lambda_k} = \frac{a_k}{\lambda_k},\ k = 0, 1, 2, \ldots$$

and

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{a_{nk}}{\lambda_k} = a^{\lambda},$$

we have,

$$\left(\frac{a_{nk}}{\lambda_k}\right) \in (c, c).$$

Since $\{\beta_k\} \in c$,

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{a_{nk}}{\lambda_k}\beta_k \text{ exists.}$$

In view of (11), $\{(Ax)_n\} \in c$. At this juncture, we note that (16) holds. Because of (10) and the fact that

$$\lim_{n\to\infty} \frac{\mu_n(a_{nk} - a_k)}{\lambda_k} = 0,\ k = 0, 1, 2, \ldots,$$

we have,

$$\left(\frac{\mu_n(a_{nk}-a_k)}{\lambda_k}\right) \in (c_0, c_0).$$

Hence

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{\mu_n(a_{nk}-a_k)}{\lambda_k}(\beta_k - \beta) = 0,$$

observing that $\{\beta_k - \beta\} \in c_0$. Now, appealing to (16), we conclude that

$$\lim_{n\to\infty} \mu_n\{(Ax)_n - y\} = 0,$$
$$i.e., \{(Ax)_n\} \in c_0^{\mu},$$

completing the proof of the theorem. □

Using Theorem 1.6, we can establish the following theorem in a similar fashion.

Theorem 2.4. *$A = (a_{nk}) \in (c_0^{\lambda}, c^{\mu})$ if and only if*

$$A(e), A(e_k) \in c^{\mu},\ k = 0, 1, 2, \ldots; \tag{17}$$

(9) and (10) hold.

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