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Lifts of Metallic Structure on a Cross-Section

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Abstract. The purpose of the present work is to study the behavior of the cross-section of the metallic structure in M to the tangent bundle TM.

1. Introduction

The study of tensor fields and connections on a cross-section in the tangent bundle over the manifold M was initiated by Yano [19], Tani [21], Okubo and Houh [18], Houh and Ishihara [20], etc. Fattaev [23] studied the lifts of vector fields to the semitensor bundle of the type (2, 0) in 2008. Recently, Yildirim [22] investigated lifts of vector felds on a cross-section in the semi-tensor bundle of a tensor bundle of type (2,0). Yano and Ishihara [5] studied the cross-section of an almost complex structure F i.e. $F^2 = -I$ in an almost complex manifold M. This paper is to study the behavior of the cross-section of the metallic structure Ψ i.e. $\Psi^2 - \alpha\Psi - \beta I = 0, \alpha$ and β are positive integers, in the differentiable manifold M to the tangent bundle TM, which generalizes the notion of almost complex structure F introduced by Yano and Ishihara [5]. The metallic structure have been studied by numerous investigators [3, 4, 7, 12, 16, 24, 25]

In an n-dimensional differentiable manifold M, $T_p(M)$ is the tangent space at a point p of M i.e. the set of all tangent vectors of M at p. Then the set $TM = \bigcup_{p\in M} T_p(M)$ is the tangent bundle over the manifold M [8, 10, 14, 15].

The following notations will be used throughout the paper: let $\mathfrak{I}^0_0(M), \mathfrak{I}^1_0(M), \mathfrak{I}^0_1(M), \mathfrak{I}^1_1(M)$ be the set of functions, vector fields, 1-forms and tensor fields of type (1,1) in M, respectively. Similarly, let $\mathfrak{I}^0_0(TM), \mathfrak{I}^1_0(TM), \mathfrak{I}^0_1(TM), \mathfrak{I}^1_1(TM)$ be the set of functions, vector fields, 1-forms and tensor fields of type (1,1) in TM, respectively.

If f is a function in M, we write f^C for the function in $T(M)$ defined by

$$f^C = i(df) \tag{1}$$

and call f^C the complete lift of the function f. The complete lift f^C of a function f has the local expression

$$f^C = y^i \partial_i f = \partial f \tag{2}$$

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with respect to the induced coordinates in $T(M)$, where (1) and (2) are partial differential equations.

Suppose that $X \in \Im^1_0(M)$. We define a vector field X^C in $T(M)$ by

$$X^C f^C = (Xf)^C \tag{3}$$

f being an arbitrary function in M and call X^C the complete lift of X in $T(M)$. The complete lift X^C of X with components x^h in M has components

$$X^C : \begin{bmatrix} x^h \\ \partial x^h \end{bmatrix} \tag{4}$$

with respect to the induced coordinates in $T(M)$.

The complete lifts to a unique algebraic isomorphism of the tensor algebra $\Im(M)$ into the tensor algebra $\Im(T(M))$ with respect to constant coefficients by mathematical operators

$$(P \otimes Q)^C = P^C \otimes Q^V + P^V \otimes Q^C, (P + R)^C = P^C + R^C,$$

where P, Q and R being arbitrary elements of $\Im(M)$ and $\Im^s_r(M)$ represents the set of all tensor fields of type (r, s) in M [11, 13].

Metallic structure: Let M be a differentiable manifold of class C^∞. A tensor field Ψ of type (1,1) on M is called the metallic structure if Ψ satisfies the equation

$$\Psi^2 - \alpha\Psi - \beta I = 0, \tag{5}$$

where α, β are positive integers [1, 9].

The complete lift Ψ^C of the metallic structure Ψ has the local expression [5]

$$\Psi^C = \begin{bmatrix} \Psi^h_i & 0 \\ \partial\Psi^h_i & \Psi^h_i \end{bmatrix}. \tag{6}$$

Nijenhuis tensor: The Nijenhuis tensor N_Ψ of Ψ is given by [17]

$$N_\Psi(X, Y) = [\Psi X, \Psi Y] - \Psi[\Psi X, Y] - \Psi[X, \Psi Y] + \Psi^2[X, Y], \ \ \forall X, Y \in \Im^1_0(M). \tag{7}$$

The metallic structure Ψ is said to be integrable if $N_\Psi(X, Y) = 0$.

2. Lifts of metallic structure on a cross-section

Let V be a vector field in an n-dimensional manifold M and TM its tangent bundle. An n-dimensional submanifold $\beta_V(M)$ of TM is called the cross section determined by V, where β_V is a mapping $\beta_V : M \to TM$. If the vector field V has local components $V^h(x)$ in M, then the cross section is locally defined by [5]

$$x^h = x^h, y^h = V^h(x) \tag{8}$$

with respect to the induced coordinates $(x^A) = (x^h, y^h)$ in TM. Let x^h be the local component of a field $X \in \Im^1_0(M)$ and the local components of the vector field BX is

$$BX : (B^A_i X^i) = \begin{bmatrix} x^h \\ x^i \partial_i V^h \end{bmatrix} \tag{9}$$

in TM, where BX is tangent to $\beta_V(M)$ and defined globally along submanifold $\beta_V(M)$. The local component of a vector field DX is

$$DX : (D^A_i X^i) = \begin{bmatrix} 0 \\ x^h \end{bmatrix}, \tag{10}$$

which is tangent to the fibre, since a fibre is locally expressed by x^h =constant, $y^h = y^h, y^h$ are parameters. From (9) and (10), we have

$$[BX, BY] = B[X, Y], \qquad [DX, DY] = 0 \tag{11}$$

for any $X, Y \in \mathfrak{I}_0^1(M)$. By the definitions of complete and vertical lifts and equations (9) and (10), we have along $\beta_V(M)$ the formulas

$$X^C = BX + D(L_V X), \quad X^V = DX \tag{12}$$

for any $X \in \mathfrak{I}_0^1(M)$, where $L_V X$ denotes the Lie derivative of X with respect to V.
The complete lift X^C and vertical lift X^V of a vector field X in M along $\beta_V(M)$ has components of the form

$$X^C : \begin{bmatrix} x^h \\ L_V x^h \end{bmatrix}, X^V : \begin{bmatrix} 0 \\ x^h \end{bmatrix}. \tag{13}$$

The complete lift Ψ^C of an element Ψ of $\mathfrak{I}_1^1(M)$ along $\beta_V(M)$ in M to $T(M)$ has components of the form

$$\Psi^C : \begin{bmatrix} \Psi_i^h & 0 \\ L_V \Psi_i^h & \Psi_i^h \end{bmatrix} \tag{14}$$

and then, we have along the cross section $\beta_V(M)$ the formula

$$\Psi^C(BX) = B(\Psi X) + D(L_V \Psi)X \tag{15}$$

for any $X \in \mathfrak{I}_0^1(M)$. When $\Psi^C(BX)$ is tangent to $\beta_V(M)$, then Ψ^C is said to leave $\beta_V(M)$ invariant. Thus we have

Theorem 2.1 [5] The complete lift Ψ^C of an element Ψ of $\mathfrak{I}_1^1(M)$ leaves the cross section $\beta_V(M)$ invariant iff $L_V \Psi = 0$.

Theorem 2.2 Let Ψ be an almost product structure in M and satisfies the condition $L_V \Psi = 0, V$ is a vector field in M, then $\Psi^{C\#}$ is a metallic structure on the cross section in $T(M)$ determined by V.

Proof. The complete lift Ψ of an element Ψ of $\mathfrak{I}_1^1(M)$ leaves the cross section $\beta_V(M)$ invariant. Let us define an element $\Psi^{C\#} \in \mathfrak{I}_1^1(\beta_V(M))$ by

$$\Psi^{C\#}(BX) = \Psi^C(BX) = \Psi(BX), \forall X \in \mathfrak{I}_1^1(\beta_V(M)). \tag{16}$$

The element $\Psi^{C\#}$ is called the tensor field induced on $\beta_V(M)$ from Ψ^C. Since Ψ is a metallic structure in M and $L_V \Psi = 0$ i.e

$$\Psi^2 - \alpha\Psi - \beta I = 0 \quad \textit{and} \quad L_V \Psi = 0, \tag{17}$$

from (16), we have

$$(\Psi^{C\#})^2 - \alpha\Psi^{C\#} - \beta I = 0. \tag{18}$$

Hence, $\Psi^{C\#}$ is a metallic structure in $\beta_V(M)$.

Let N_Ψ and N_{Ψ^C} be Nijenhuis tensors of $\Psi \in \mathfrak{I}_1^1(M)$ and of the complete lift Ψ^C of Ψ respectively. From page [5, pg. 36], we obtain

$$N_\Psi^C = (N_\Psi)^C$$

which implies from (15),

$$N_{\Psi^C}(BX, BY) = B(N_\Psi(X, Y)) + D((L_V N_\Psi)(X, Y) \tag{19}$$

for any $X, Y \in \mathfrak{I}_0^1(M)$. Thus we have from (19)
Theorem 2.3 Let N_Ψ and N_{Ψ^C} be Nijenhuis tensors of $\Psi \in \mathfrak{I}_1^1(M)$ and of the complete lift Ψ^C of Ψ respectively. Then , in order that $N_{\Psi^C}(BX, BY)$ is tangent to the cross-section $\beta_V(M)$ determined by $V \in \mathfrak{I}_0^1(M)$ for any $X, Y \in \mathfrak{I}_0^1(M)$, it is necessary and sufficient that $L_V N_\Psi = 0$.

Suppose that the complete lift Ψ^C of an element Ψ of $\mathfrak{I}_1^1(M)$ leave the cross section $\beta_V(M)$ invariant. From (16) and (11), we obtain

$$\begin{aligned} N_{\Psi^C}(BX, BY) &= [\Psi^C(BX), \Psi^C(BY)] - \Psi^C[\Psi^C(BX), (BY)] - \Psi^C[(BX), \Psi^C(BY)] \\ &\quad + (\Psi^C)^2[\Psi^C(BX), \Psi^C(BY)] \\ &= [\Psi^{C\#}(BX), \Psi^{C\#}(BY)] - \Psi^{C\#}[\Psi^{C\#}(BX), (BY)] - \Psi^{C\#}[(BX), \Psi^{C\#}(BY)] \\ &\quad + (\Psi^{C\#})^2[\Psi^{C\#}(BX), \Psi^{C\#}(BY)] \end{aligned} \tag{20}$$

i.e.

$$N_{\Psi^C}(BX, BY) = N_{\Psi^{C\#}}(BX, BY) \tag{21}$$

for any $X, Y \in \mathfrak{I}_0^1(M)$. From(19),

$$N_{\Psi^C}(BX, BY) = B(N_\Psi(X, Y)) + D((L_V N_\Psi)(X, Y) \tag{22}$$

for any $X, Y \in \mathfrak{I}_0^1(M)$. As $L_V\Psi = 0$ implies that $L_V N_\Psi = 0$. Thus we have
Theorem 2.4 Let the complete lift Ψ^C of an element Ψ of $\mathfrak{I}_1^1(M)$ leave the cross section $\beta_V(M)$ invariant. Then

$$N_{\Psi^{C\#}} = 0$$

iff

$$N_\Psi = 0.$$

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