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The MP Weak Group Inverse and its Application

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Abstract. In this paper, we introduce a new generalized inverse, called MPWG inverse of a complex square matrix. We investigate characterizations, representations, and properties for this new inverse. Then, by using the core-EP decomposition, we discuss the relationships between MPWG inverse and other generalized inverses. A variant of the successive matrix squaring computational iterative scheme is given for calculating the MPWG inverse. The Cramer rule for the solution of a singular equation $Ax = b$ is also presented. Moreover, the MPWG inverse being used in solving appropriate systems of linear equations is established. Finally, we analyze the MPWG binary relation.

1. Introduction

Throughout this paper, we denote the set of all $m{\times}n$ complex matrices by $\mathbb{C}^{m\times n}$. For $A \in \mathbb{C}^{m\times n}$, the symbols A^*, $rank(A)$, $N(A)$, and $R(A)$ stand for the conjugate transpose, the rank, the null space and the range space of A, respectively. Moreover, I_n will refer to the $n{\times}n$ identity matrix. Let $A{\in}\mathbb{C}^{n\times n}$, the smallest positive integer k for which $rank(A^k) = rank(A^{k+1})$ is called the index of A and is denoted by $Ind(A)$. Then $\mathbb{C}_k^{m\times n}$ represents all $m{\times}n$ complex matrices sets with index k. $P_{E,F}$ represents the projector on the subspace E along the subspace F. For $A \in \mathbb{C}^{n\times n}$, P_A stands for the orthogonal projection onto $R(A)$. The symbol $\mathbb{C}_n^{OP}$ represents the subset of $\mathbb{C}^{n\times n}$ including orthogonal projectors (Hermitian idempotent matrices), i.e., $\mathbb{C}_n^{OP} = \{A|A{\in}\mathbb{C}^{n\times n}, A^2 = A = A^*\}$. $\mathbb{C}_n^{CM}$ represents the subset of all $n{\times}n$ complex matrices sets with index 1.

Next, let's review the definitions of some common generalized inverses.

For $A{\in}\mathbb{C}^{m\times n}$, the Moore-Penrose inverse $A^\dagger$ of A is the unique matrix $X{\in}\mathbb{C}^{n\times m}$ satisfying the following four Penrose equations [1]:

$$AXA = A, \qquad XAX = X, \qquad (AX)^* = AX, \qquad (XA)^* = XA.$$

The Moore-Penrose inverse can be used to represent orthogonal projectors $P_A := AA^\dagger$ onto $R(A)$ and $Q_A := A^\dagger A$ onto $R(A^*)$, respectively. A matrix $X{\in}\mathbb{C}^{n\times m}$ that satisfies the equality $AXA = A$ is called an inner inverse or $\{1\}$-inverse of A, and a matrix $X{\in}\mathbb{C}^{n\times m}$ that satisfies the equality $XAX = X$ is called an outer inverse or $\{2\}$-inverse of A.

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For $A\in\mathbb{C}^{m\times n}$ with $rank(A) = r$, and suppose T be a subspace of $\mathbb{C}^n$ of dimension $s\leq r$, S be a subspace of $\mathbb{C}^m$ of dimension $m - s$.. According to the common terminology, X is a $\{2\}$-inverse of A with prescribed range T and null space S if

$$XAX = X, \qquad R(X) = T, \qquad N(X) = S.$$

If such X exists, X is unique and denoted by $A^{(2)}_{T,S}$.

The Drazin inverse is a kind of outer inverse defined for square matrices. For $A\in\mathbb{C}^{n\times n}$ and $Ind(A) = k$, the Drazin inverse A^D of A is the unique matrix $X\in\mathbb{C}^{n\times n}$ satisfying the following three equations [1]:

$$A^{k+1}X = A^k, \qquad XAX = X, \qquad AX = XA.$$

In particular, if $Ind(A) = 1$, $A^D = A^\#$ is the group inverse of A.

For $A\in\mathbb{C}^{n\times n}$, the core inverse $A^{\circledS}$ of A is the unique matrix $X\in\mathbb{C}^{n\times n}$ satisfying the conditions [2]:

$$AA^{\circledS} = P_A, \qquad R(A^{\circledS})\subseteq R(A).$$

$A^{\circledS}$ exists if and only if $Ind(A) = 1$.

For $A\in\mathbb{C}^{n\times n}$ and $Ind(A) = k$, the core-EP inverse $A^{\textcircled{\dagger}}$ of A is the unique matrix $X\in\mathbb{C}^{n\times n}$ satisfying the following conditions [3]:

$$X = XAX, \qquad R(A^k) = R(X) = R(X^*).$$

Obviously, the core-EP inverse is an outer inverse of A. Recall that, by [4], the core-EP inverse can be expressed as $A^{\textcircled{\dagger}} = A^DA^k(A^k)^\dagger$.

The weak group inverse is proposed by Wang and Chen [5] for square matrices of an arbitrary index as an extension of the group inverse. For $A\in\mathbb{C}^{n\times n}$, the weak group inverse $A^{\textcircled{w}}$ of A is the uniquely determined matrix that satisfying:

$$AX^2 = X, \qquad AX = A^{\textcircled{\dagger}}A.$$

Notice that, by [5], we have $A^{\textcircled{w}} = (A^{\textcircled{\dagger}})^2A$.

The BT-inverse of $A\in\mathbb{C}^{n\times n}$, denoted by $A^\diamond$, which was defined in [6] can be written by $(AP_A)^\dagger$ [6, 7]. The DMP-inverse of $A\in\mathbb{C}^{n\times n}_k$, written by $A^{D,\dagger}$, was defined in [8] as the unique matrix $X\in\mathbb{C}^{n\times n}$ satisfying $XAX = X, XA = A^DA$ and $A^kX = A^kA^\dagger$. Moreover, it was proved that $A^{D,\dagger} = A^DAA^\dagger$. Also, the dual DMP-inverse of A was introduced in [8], namely $A^{\dagger,D} = A^\dagger AA^D$ [8, 9]. The CMP-inverse of $A\in\mathbb{C}^{n\times n}_k$, written by $A^{C,\dagger}$ was defined in [10] as the unique matrix $X\in\mathbb{C}^{n\times n}_k$ satisfying $XAX = X, AX = AA^DAA^\dagger$ and $XA = A^\dagger AA^DA$. Moreover, it was proved that $A^{C,\dagger} = A^\dagger AA^DAA^\dagger$ [10, 11]. The (B,C)-inverse of $A\in\mathbb{C}^{m\times n}$, denoted by $A^{(B,C)}$ [12], is the unique matrix $X\in\mathbb{C}^{n\times m}$ satisfying $XAB = B, CAX = C, N(X) = N(C)$ and $R(X) = R(B)$, where $B, C\in\mathbb{C}^{n\times m}$ [9, 12].

Recently, two new generalized inverses have emerged by combining Moore-Penrose inverse and the weak group inverse, which are the weak core inverse (WCI) $A^{\textcircled{w},\dagger}$ and the dual weak core inverse (d-WCI) $A^{\dagger,\textcircled{w}}$, respectively [13]. Precisely, the weak core inverse of $A\in\mathbb{C}^{n\times n}$ presents a unique solution to the matrix system

$$X = XAX, \qquad AX = CA^\dagger, \qquad XA = A^DC,$$

where C is the weak core part of A with $C = AA^{\textcircled{w}}A$. Notice that $A^{\textcircled{w},\dagger} = A^{\textcircled{w}}AA^\dagger$, $A^{\dagger,\textcircled{w}} = A^\dagger AA^{\textcircled{w}}$.

The main structure of this paper is as follows.

(1) In Section 2, some preliminaries are given.

(2) In Section 3, we introduce the MP weak group inverse(MPWG inverse) and give some representations and characterizations.

(3) In Section 4, we discuss the relationships between the MPWG inverse and other generalized inverses by the core-EP decomposition.

(4) In Section 5, we develop the SMS method for finding the MPWG inverse.

(5) In Section 6, the Cramer rule for the solution of a singular equation $Ax = b$ is generalized.

(6) In Section 7, we give the application of the MPWG inverse in solving linear equations.

(7) Finally, in Section 8, we analyse the MPWG binary relation.

2. Preliminaries

In this article, we will use the core-EP decomposition. First, let's review it.

Wang gave the core-EP decomposition in the document [14]. Let $A\in\mathbb{C}^{n\times n}$ with $\text{Ind}(A)=k$, $rank(A^k)=p$. Then, one has $A=A_1+A_2$, where $A_1\in\mathbb{C}_n^{CM}$, $A_2^k=0$, $A_1^*A_2=A_2A_1=0$.

Further, there exists an unitary matrix $U\in\mathbb{C}^{n\times n}$ such that

$$A=U\begin{pmatrix}T & S\\ 0 & N\end{pmatrix}U^*,\ \ A_1=U\begin{pmatrix}T & S\\ 0 & 0\end{pmatrix}U^*,\ \ A_2=U\begin{pmatrix}0 & 0\\ 0 & N\end{pmatrix}U^*, \tag{2.1}$$

where $T\in\mathbb{C}^{p\times p}$ is nonsingular; $S\in\mathbb{C}^{p\times(n-p)}$; $N\in\mathbb{C}^{(n-p)\times(n-p)}$ is nilpotent of index k, i.e., $N^k=0$.

Lemma 2.1. *[5, 16–18] Let $A\in\mathbb{C}_k^{n\times n}$ be as in (2.1). Then*

(1) $A^\dagger=U\begin{pmatrix}T^*\triangle & -T^*\triangle SN^\dagger\\ (I_{n-p}-N^\dagger N)S^*\triangle & N^\dagger-(I_{n-p}-N^\dagger N)S^*\triangle SN^\dagger\end{pmatrix}U^*$,

(2) $A^D=U\begin{pmatrix}T^{-1} & (T^{k+1})^{-1}\widetilde{T}\\ 0 & 0\end{pmatrix}U^*$,

(3) $A^{\oplus}=U\begin{pmatrix}T^{-1} & 0\\ 0 & 0\end{pmatrix}U^*$,

(4) $A^\diamond=U\begin{pmatrix}T^*\triangle_1 & -T^*\triangle_1 SN^\diamond\\ (P_N-P_{N^\diamond})S^*\triangle_1 & N-(P_N-P_{N^\diamond})S^*\triangle_1 SN^\diamond\end{pmatrix}U^*$,

(5) $A^{D,\dagger}=U\begin{pmatrix}T^{-1} & (T^{k+1})^{-1}\widetilde{T}NN^\dagger\\ 0 & 0\end{pmatrix}U^*$,

(6) $A^{\dagger,D}=U\begin{pmatrix}T^*\triangle & T^*\triangle T^{-k}\widetilde{T}\\ (I_{n-p}-N^\dagger N)S^*\triangle & (I_{n-p}-N^\dagger N)S^*\triangle T^{-k}\widetilde{T}\end{pmatrix}U^*$,

(7) $A^{C,\dagger}=U\begin{pmatrix}T^*\triangle & T^*\triangle T^{-k}\widetilde{T}NN^\dagger\\ (I_{n-p}-N^\dagger N)S^*\triangle & (I_{n-p}-N^\dagger N)S^*\triangle T^{-k}\widetilde{T}NN^\dagger\end{pmatrix}U^*$,

where $\widetilde{T}=\sum_{j=0}^{k-1}T^jSN^{k-1-j}$, $\triangle=[TT^*+S(I_{n-p}-N^\dagger N)S^*]^{-1}$, $\triangle_1=[TT^*+S(P_N-P_{N^\diamond})S^*]^{-1}$.

Lemma 2.2. *[19] Let $A\in\mathbb{C}_k^{n\times n}$ be as in (2.1). Then*

$$rank(A)=rank(A^2)\Leftrightarrow N=0.$$

In which case, we have

$$A^\#=U\begin{pmatrix}T^{-1} & T^{-2}S\\ 0 & 0\end{pmatrix}U^*,\qquad A^{\textcircled{\#}}=U\begin{pmatrix}T^{-1} & 0\\ 0 & 0\end{pmatrix}U^*.$$

Lemma 2.3. *[5, 13, 14, 17] Let $A\in\mathbb{C}_k^{n\times n}$ be as in (2.1). Then*

(1) $AA^\dagger=U\begin{pmatrix}I_p & 0\\ 0 & NN^\dagger\end{pmatrix}U^*$,

(2) $A^\dagger A=U\begin{pmatrix}T^*\triangle T & T^*\triangle S(I-N^\dagger N)\\ (I_{n-p}-N^\dagger N)S^*\triangle T & (I_{n-p}-N^\dagger N)S^*\triangle S(I-N^\dagger N)+N^\dagger N\end{pmatrix}U^*$,

(3) $A^{\textcircled{W}}=(A^{\oplus})^2A=U\begin{pmatrix}T^{-1} & T^{-2}S\\ 0 & 0\end{pmatrix}U^*$,

(4) $A^{\textcircled{w},\dagger} = A^{\textcircled{w}}AA^{\dagger} = U\begin{pmatrix} T^{-1} & T^{-2}SNN^{\dagger} \\ 0 & 0 \end{pmatrix}U^*$,

(5) $A^{\dagger,\textcircled{w}} = A^{\dagger}AA^{\textcircled{w}} = U\begin{pmatrix} T^*\triangle & T^*\triangle T^{-1}S \\ (I_{n-p} - N^{\dagger}N)S^*\triangle & (I_{n-p} - N^{\dagger}N)S^*\triangle T^{-1}S \end{pmatrix}U^*$,

where $\triangle = [TT^* + S(I_{n-p} - N^{\dagger}N)S^*]^{-1}$.

Lemma 2.4. *[5, 13] The following statements concerning $A^{\textcircled{w}}$ are true.*
(1) *$A^{\textcircled{w}}$ is an outer inverse of A,*
(2) $R(A^{\textcircled{w}}) = R(A^k)$,
(3) $A^{\textcircled{w}}A^{k+1} = A^k$,
(5) *$AA^{\textcircled{w}} = A^kB$ for some matrix B,*
(6) *$A^{\textcircled{w}} = A^kZ$ for some matrix Z.*

Lemma 2.5. *[20] Let $A \in \mathbb{C}^{n\times n}$ with $rank(A) = r>0$. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that*

$$A = U\begin{pmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{pmatrix}U^*, \tag{2.2}$$

where $\Sigma = diag(\sigma_1 I_{r_1}, \sigma_2 I_{r_2}, \ldots, \sigma_t I_{r_t})$ is the diagonal matrix of singular values of A, $\sigma_1>\sigma_2>\ldots>\sigma_t>0$, $r_1+r_2+\ldots+r_t = r$, and $K \in C^{r\times r}$, $L \in \mathbb{C}^{r\times(n-r)}$ satisfy $KK^* + LL^* = I_r$.

Lemma 2.6. *[21, 22] Let $A \in C^{n\times n}$ be a matrix written as in* (2.2)*. Then,*
(1) *the Moore–Penrose inverse of A is*

$$A^{\dagger} = U\begin{pmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{pmatrix}U^*,$$

(2) *the core-EP inverse of A is*

$$A^{\textcircled{\dagger}} = U\begin{pmatrix} (\Sigma K)^{\textcircled{\dagger}} & 0 \\ 0 & 0 \end{pmatrix}U^*.$$

3. Definition, characterizations and representations of the MPWG inverse

According to the Moore–Penrose inverse and the weak group inverse of A, we establish a new inverse which is called the MP weak group inverse. Here we give the definition as follows.

Let $A \in \mathbb{C}^{n\times n}$. C is the weak core part of A. We consider the following system of equations:

$$XAX = X, \;\; AX = A^DC, \;\; XA = A^{\dagger}A^{\textcircled{w}}A^2. \tag{3.1}$$

Theorem 3.1. *Let $A \in \mathbb{C}^{n\times n}$ with $Ind(A) = k$. C is the weak core part of A. The system* (3.1) *is consistent and its unique solution is the matrix $X = A^{\dagger}A^DC$.*

Proof. We will check that the matrix $X = A^{\dagger}A^DC$ satisfies the three equations in system (3.1).

By $AA^D = A^DA$ and $CA^DC = C$, we can get

$$\begin{aligned} XAX &= (A^{\dagger}A^DC)A(A^{\dagger}A^DC) = A^{\dagger}A^DCAA^{\dagger}A^D(AA^{\textcircled{w}}A) \\ &= A^{\dagger}A^DCA^DC = A^{\dagger}A^DC = X. \end{aligned}$$

On the other hand,

$$\begin{aligned} AX &= AA^{\dagger}A^DC = AA^{\dagger}A^DAA^{\textcircled{w}}A \\ &= AA^{\dagger}AA^DA^{\textcircled{w}}A = AA^DA^{\textcircled{w}}A = A^DAA^{\textcircled{w}}A = A^DC. \end{aligned}$$

From (6) in Lemma 2.4, $A^{\textcircled{w}} = A^kZ$ for some matrix Z, and because $A^DA^{k+1} = A^k$, then $XA = A^{\dagger}A^DCA = A^{\dagger}A^DAA^{\textcircled{w}}A^2 = A^{\dagger}A^DAA^kZA^2 = A^{\dagger}A^kZA^2 = A^{\dagger}A^{\textcircled{w}}A^2$.

For the uniqueness, we assume that X_1 and X_2 are two solutions of the system (3.1). From $AX_1 = A^DC = AX_2$, $X_1A = A^{\dagger}A^{\textcircled{w}}A^2 = X_2A$, we have $X_1 = (X_1A)X_1 = (X_2A)X_1 = X_2(AX_1) = X_2AX_2 = X_2$. The uniqueness is proved. □

Definition 3.2. *Let $A \in \mathbb{C}^{n\times n}$ with $Ind(A) = k$. C is the weak core part of A. The MP weak group inverse (or, in short, MPWG inverse) of A, denoted as $A^{\dagger,WG}$, is defined to be the solution of the system* (3.1).

Theorem 3.3. *Let $A \in C^{n\times n}$ with $Ind(A) = k$. Then*

$$A^{\dagger,WG} = A^{\dagger}A^{\textcircled{w}}A. \tag{3.2}$$

Proof. From (6) in Lemma 2.4, we have
$A^{\dagger,WG} = A^{\dagger}A^{D}C = A^{\dagger}A^{D}AA^{\textcircled{w}}A = A^{\dagger}A^{D}AA^{k}ZA = A^{\dagger}A^{k}ZA = A^{\dagger}A^{\textcircled{w}}A.$ □

Example 3.4. *Let*

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check that Ind(A)=2. It can be obtained by calculation that the Moore-Penrose inverse, the Drazin inverse and the WG inverse are

$$A^{\dagger} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad A^{D} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A^{\textcircled{w}} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

the BT inverse, the core EP inverse, the DMP inverse and the CMP inverse are

$$A^{\diamond} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A^{\textcircled{\dagger}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A^{D,\dagger} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A^{C,\dagger} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then we can get

$$A^{\textcircled{w},\dagger} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A^{\dagger,\textcircled{w}} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A^{\dagger,WG} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Through this example, we can see that $A^{\dagger,WG}$ is different from other common generalized inverses.

Theorem 3.5. *Let $A \in \mathbb{C}^{n\times n}$ with $Ind(A) = k$. C is the weak core part of A. Then the following statements are equivalent:*
(1)$X = A^{\dagger,WG} = A^{\dagger}A^{\textcircled{w}}A$,
(2)$X = XA^{D}C, XA^{k} = A^{\dagger}A^{k}$,
(3)$A^{\dagger}AX = X, AX = A^{\textcircled{w}}A$,
(4)$XA^{k} = A^{\dagger}A^{k}, XA^{\textcircled{w}}A = X$,
(5)$X = XA^{D}C, R(X) = R(A^{\dagger}A^{k})$ *and AX is idempotent,*
(6)$AX = A^{D}C, R(X) \subseteq R(A^{*})$.

Proof. That (1) implies all other items (2) − (6) can be checked directly.

(2)⇒(1). Since $A^{\textcircled{W}} = A^kZ$, it follows that

$$\begin{aligned} X &= XA^DC = XA^DAA^{\textcircled{W}}A = XA^DAA^{\textcircled{W}}A \\ &= XA^kZA = A^\dagger A^kZA = A^\dagger A^{\textcircled{W}}A. \end{aligned}$$

(3)⇒(1). It is obvious that $X = A^\dagger AX = A^\dagger A^{\textcircled{W}}A$.

(4)⇒(1). From (6) in Lemma 2.4, and $XA^k = A^\dagger A^k$ we have

$$X = XA^{\textcircled{W}}A = XA^kZA = A^\dagger A^kZA = A^\dagger A^{\textcircled{W}}A.$$

(5)⇒(2). Since AX is idempotent, it follows that $AX-(AX)^2 = (A-AXA)X = 0$, so $R(A^\dagger A^k) = R(X)\subseteq N(A-AXA)$. We can get that $(A - AXA)A^\dagger A^k = 0$, i.e., $A^k = AXA^k$. Multiplying the last equality by $A^\dagger$ from the left side, we get $A^\dagger A^k = A^\dagger AXA^k$.

Furthermore, from $(I - A^\dagger A)A^\dagger A^k = 0$, we have $R(X) = R(A^\dagger A^k)\subseteq N(I - A^\dagger A)$. Then, $(I - A^\dagger A)X = 0$, i.e., $X = A^\dagger AX$. Hence $XA^k = A^\dagger AXA^k = A^\dagger A^k$.

Finally, since $R(I - A^\dagger A)\subseteq N((A^k)^*A^2)) = N(X)$, we have that $X = A^\dagger AX$. Hence $XA^k = A^\dagger A^k$.

(6)⇒(1). Let $X = A^{\dagger,WG}$, clearly, from (3.1) we obtain $AX = A^DC$. On the other hand, from $A^\dagger AA^{\dagger,WG} = A^\dagger AA^\dagger A^{\textcircled{W}}A = A^{\dagger,WG}$, we can get $R(X)\subseteq R(A^\dagger A) = R(A^*)$.

In order to show that system (6) has a unique solution, assume that both X_1 and X_2 satisfy (6), that is $AX_1 = A^DC = AX_2$, $R(X_1)\subseteq R(A^*)$ and $R(X_2)\subseteq R(A^*)$, so we can get $R(X_1 - X_2)\subseteq R(A^*)$. Since $A(X_1 - X_2) = 0$, we obtain $R(X_1 - X_2)\subseteq N(A) = R(A^*)^\perp$. Therefore, $R(X_1 - X_2)\subseteq (R(A^*)^\perp)\cap R(A^*) = 0$. Thus, $X_1 = X_2$. □

According to the decompositions of $A^\dagger$ and $A^{\textcircled{W}}$, we can easily get the following two inferences.

Corollary 3.6. *Let $A\in\mathbb{C}^{n\times n}$ be a matrix written as in* (2.1). *Then*

$$A^{\dagger,WG} = A^\dagger A^{\textcircled{W}}A = U\begin{pmatrix} T^*\triangle & T^*\triangle(T^{-1}S + T^{-2}SN) \\ (I_{n-p} - Q_N)S^*\triangle & (I_{n-p} - Q_N)S^*\triangle(T^{-1}S + T^{-2}SN) \end{pmatrix}U^*, \tag{3.3}$$

where $\triangle = [TT^* + S(I_{n-p} - N^\dagger N)S^*]^{-1}$.

Remark 3.7. *Using the core-EP decomposition , we can get that*

$$AA^{\textcircled{W}}A^\dagger = U\begin{pmatrix} T^{-1} & 0 \\ 0 & 0 \end{pmatrix}U^* = A^{\textcircled{\dagger}}.$$

The expression $AA^{\textcircled{W}}A^\dagger$ can not be considered as a new generalized inverse of A.

Corollary 3.8. *Let $A\in\mathbb{C}^{n\times n}$ be a matrix written as in* (2.2). *Then*

$$A^{\dagger,WG} = A^\dagger A^{\textcircled{W}}A = A^\dagger(A^{\textcircled{\dagger}})^2A^2 = U\begin{pmatrix} K^*\Sigma^{-1}(\Sigma K)^{\textcircled{W}}\Sigma K & K^*\Sigma^{-1}(\Sigma K)^{\textcircled{W}}\Sigma L \\ L^*\Sigma^{-1}(\Sigma K)^{\textcircled{W}}\Sigma K & L^*\Sigma^{-1}(\Sigma K)^{\textcircled{W}}\Sigma L \end{pmatrix}U^*.$$

Theorem 3.9. *Let $A\in\mathbb{C}^{n\times n}$ with $Ind(A) = k$. Then*

(1) $A^{\dagger,WG} = A^\dagger(AA^{\textcircled{\dagger}}A)^\# A$,

(2) $A^{\dagger,WG} = A^\dagger(A^{\textcircled{\dagger}})^2A^2 = A^\dagger(A^2)^{\textcircled{\dagger}}A^2$,

(3) $A^{\dagger,WG} = A^\dagger A^k(A^{k+2})^{\textcircled{\#}}A^2$,

(4) $A^{\dagger,WG} = A^\dagger(A^2P_{A^k})^\dagger A^2$.

Proof. From Theorem 3.8 and Theorem 3.9 in reference [5], we have $A^{\textcircled{W}} = (AA^{\textcircled{\dagger}}A)^\# = (A^{\textcircled{\dagger}})^2A = (A^2)^{\textcircled{\dagger}}A = A^k(A^{k+2})^{\textcircled{\#}}A = (A^2P_{A^k})^\dagger A$, so (1) − (4) are established. □

In the following result, we give a new representation of the MPWG inverse as an outer inverse with prescribed range and null space.

Theorem 3.10. *Let $A\in\mathbb{C}^{n\times n}$ with $Ind(A)=k$. Then*

$$A^{\dagger,WG}=A^{(2)}_{R(A^{\dagger}A^{k}),N((A^{k})^{*}A^{2})}.$$

Proof. By the definition, since $A^{\dagger,WG}$ satisfies the equation $XAX=X$, $A^{\dagger,WG}$ is an outer inverse of A. From $A^{\dagger,WG}=A^{\dagger}A^{\textcircled{w}}A$ and $AA^{\dagger,WG}=A^{\textcircled{w}}A$, we have

$$N(A^{\textcircled{w}}A)\subseteq N(A^{\dagger}A^{\textcircled{w}}A)=N(A^{\dagger,WG})\subseteq N(AA^{\dagger,WG})=N(A^{\textcircled{w}}A).$$

On the other hand,

$$N(A^{\textcircled{w}}A)\subseteq N(AA^{\textcircled{w}}A)=N(A^{\textcircled{\#}}A^{2})\subseteq N((A^{\textcircled{\#}})^{2}A^{2})=N(A^{\textcircled{w}}A).$$

Therefore,

$$N(A^{\dagger,WG})=N(A^{\textcircled{w}}A)=N(A^{\textcircled{\#}}A^{2}).$$

Then, $x\in N(A^{\dagger,WG})$ if and only if $A^{2}x\in N(A^{\textcircled{\#}})=N((A^{k})^{*})$. Therefore, $x\in N(A^{\dagger,WG})$ if and only if $x\in N((A^{k})^{*}A^{2})$. And we can get

$$R(A^{\dagger}A^{k})=R(A^{\dagger,WG}A^{k})\subseteq R(A^{\dagger,WG})=R(A^{\dagger}A^{\textcircled{w}}A)=R(A^{\dagger}A^{k}ZA)\subseteq R(A^{\dagger}A^{k}).$$

Therefore, $A^{\dagger,WG}=A^{(2)}_{R(A^{\dagger}A^{k}),N((A^{k})^{*}A^{2})}$. □

Theorem 3.11. *Let $A\in\mathbb{C}^{n\times n}$ with $Ind(A)=k$. Then*
(1) *$AA^{\dagger,WG}$ is a projector onto the column space of A^{k} along the null space of $(A^{k})^{*}A^{2}$.*
(2) *$A^{\dagger,WG}A$ is a projector onto the column space of $A^{\dagger}A^{k}$ along the null space of $(A^{k})^{*}A^{3}$.*

Proof. Since, by definition, $A^{\dagger,WG}$ is an outer inverse of A, we obtain that $AA^{\dagger,WG}$ and $A^{\dagger,WG}A$ are idempotents and $N(AA^{\dagger,WG})=N(A^{\dagger,WG})$ and $R(A^{\dagger,WG}A)=R(A^{\dagger,WG})$.

(1) It is obviously that $R(A^{\textcircled{w}})=R(A^{\textcircled{w}}AA^{\textcircled{w}})\subseteq R(A^{\textcircled{w}}A)\subseteq R(A^{\textcircled{w}})$. Therefore, $R(AA^{\dagger,WG})=R(A^{\textcircled{w}}A)=R(A^{\textcircled{w}})=R(A^{k})$. On the other hand, $N(AA^{\dagger,WG})=N(A^{\dagger,WG})=N((A^{k})^{*}A^{2})$.

(2) First, we are going to prove that $N(A^{\dagger,WG}A)=N((A^{k})^{*}A^{3})$ holds. In fact, $x\in N(A^{\dagger,WG}A)$ if and only if $A^{3}x\in N(A^{\textcircled{\#}})=N((A^{k})^{*})$. Therefore, $x\in N(A^{\dagger,WG}A)$ if and only if $x\in N((A^{k})^{*}A^{3})$. Besides, $R(A^{\dagger,WG}A)=R(A^{\dagger,WG})=R(A^{\dagger}A^{k})$. □

Theorem 3.12. *Let $A\in\mathbb{C}^{n\times n}$ with $Ind(A)=k$. Then $A^{\dagger,WG}$ is a $(A^{\dagger}A^{k},A^{\textcircled{w}}A)$-inverse of A.*

Proof. From Lemma 2.4 and Theorem 3.5 we can get

$$A^{\dagger,WG}AA^{\dagger}A^{k}=A^{\dagger}A^{\textcircled{w}}A^{2}A^{\dagger}A^{k}=A^{\dagger}A^{\textcircled{w}}A^{k+1}=A^{\dagger}A^{k},$$

and

$$A^{\textcircled{w}}AAA^{\dagger,WG}=A^{\textcircled{w}}A^{2}A^{\dagger}A^{\textcircled{w}}A=A^{\textcircled{w}}AA^{\textcircled{w}}A=A^{\textcircled{w}}A.$$

On the other hand, from Theorem 3.10 we have

$$R(A^{\dagger,WG})=R(A^{\dagger}A^{k}),N(A^{\dagger,WG})=N(A^{\textcircled{w}}A).$$

□

Corollary 3.13. *Let $A\in\mathbb{C}^{n\times n}$ with $Ind(A)=k$. For $l\geq k$,*

$$A^{\dagger,WG}=A^{\dagger}A^{l}(A^{l+2})^{\dagger}A^{2}. \tag{3.4}$$

Proof. According to [15, Theorem 2.1], it follows $A^{\textcircled{w}}=A^{l}(A^{l+2})^{\dagger}A$. By the corresponding Theorem 3.3, we get the equality (3.4). □

Corollary 3.14. *Let $A \in \mathbb{C}^{n\times n}$ with $Ind(A) = k$. Then*

$$A^{\dagger,WG} = A^{\dagger}A^{k}((A^{k})^{*}A^{k+2})^{\dagger}(A^{k})^{*}A^{2}.$$

Proof. Using $A^{\dagger,WG} = A^{(2)}_{R(A^{\dagger}A^{k}),N((A^{k})^{*}A^{2})}$, we obtain on the basis of the Urquhart formula [23, 24],

$$A^{\dagger,WG} = A^{\dagger}A^{k}((A^{k})^{*}A^{2}AA^{\dagger}A^{k})^{\dagger}(A^{k})^{*}A^{2} = A^{\dagger}A^{k}((A^{k})^{*}A^{k+2})^{\dagger}(A^{k})^{*}A^{2}.$$

□

Theorem 3.15. *Let $A \in \mathbb{C}^{n\times n}$ be arbitrary square matrix. For $P = I - AA^{\dagger,WG}$ and $Q = I - A^{\dagger,WG}A$, the matrix expressions $A + P$ and $A - P$ are nonsingular. Furthermore,*

$$A^{\dagger,WG} = (I - Q)(A \pm P)^{-1}(I - P).$$

Proof. Let A be represented as in (2.1), Then

$$I_n - P = AA^{\dagger,WG} = U\begin{pmatrix} I_p & T^{-1}S + T^{-2}SN \\ 0 & 0 \end{pmatrix}U^{*},$$

$$I_n - Q = A^{\dagger,WG}A = U\begin{pmatrix} T^{*}\triangle T & T^{*}\triangle(S + T^{-1}SN + T^{-2}SN^{2}) \\ (I_{n-p} - N^{\dagger}N)S^{*}\triangle T & (I_{n-p} - N^{\dagger}N)S^{*}\triangle(S + T^{-1}SN + T^{-2}SN^{2}) \end{pmatrix}U^{*}.$$

Since

$$P = U\begin{pmatrix} 0 & -(T^{-1}S + T^{-2}SN) \\ 0 & I_{n-p} \end{pmatrix}U^{*},$$

we have

$$A \pm P = U\begin{pmatrix} T & S \mp (T^{-1}S + T^{-2}SN) \\ 0 & N \pm I_{n-p} \end{pmatrix}U^{*}.$$

Notice that T and $N \pm I$ are invertible, we deduce that $A + P$ and $A - P$ are invertible and

$$(A \pm P)^{-1} = U\begin{pmatrix} T^{-1} & -T^{-1}(S \mp (T^{-1}S + T^{-2}SN))(N \pm I_{n-p})^{-1} \\ 0 & (N \pm I_{n-p})^{-1}I_{n-p} \end{pmatrix}U^{*}.$$

Therefore,

$$\begin{aligned} &(I - Q)(A \pm P)^{-1}(I - P) \\ &= U\begin{pmatrix} T^{*}\triangle & T^{*}\triangle(T^{-1}S + T^{-2}SN) \\ (I_{n-p} - Q_N)S^{*}\triangle & (I_{n-p} - Q_N)S^{*}\triangle(T^{-1}S + T^{-2}SN) \end{pmatrix}U^{*} \\ &= A^{\dagger,WG}. \end{aligned}$$

□

Example 3.16. *Consider matrix $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ in Example 3.4. Since Ind(A)=2, one can verify*

$$A^{\dagger,WG} = A^{\dagger}A^{\textcircled{w}}A = A^{\dagger}A^{2}(A^{4})^{\dagger}A^{2} = A^{\dagger}A^{2}((A^{2})^{*}A^{4})^{\dagger}(A^{2})^{*}A^{2} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Further calculation gives

$$P = I - AA^{\dagger,WG} = I - \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$Q = I - A^{\dagger,WG}A = I - \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & -1 \\ -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since

$$A + P = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

it follows that

$$(A + P)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} (I - Q)(A + P)^{-1}(I - P) &= \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} = A^{\dagger,WG}. \end{aligned}$$

Similarly, $A^{\dagger,WG} = (I - Q)(A - P)^{-1}(I - P)$ *can also be checked.*

4. Relationship with other generalized inverses

In this chapter, we discuss the equivalence between the MPWG inverse and other known generalized inverses by using the core-EP decomposition. And in this section, we remember

$$A^{\dagger,WG} = U \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} U^*,$$

where $G_1 = T^*\triangle$, $G_2 = T^*\triangle(T^{-1}S + T^{-2}SN)$, $G_3 = (I_{n-p} - N^\dagger N)S^*\triangle$, $G_4 = (I_{n-p} - N^\dagger N)S^*\triangle(T^{-1}S + T^{-2}SN)$, $\triangle = [TT^* + S(I_{n-p} - N^\dagger N)S^*]^{-1}$.

Theorem 4.1. *Let* $A \in \mathbb{C}^{n\times n}$ *be a matrix with* $Ind(A) = k$ *written as in* (2.1). *Then*

(1) $A^{\dagger,WG} = A \Leftrightarrow T^2 = I_p$, $S = 0$ *and* $N = 0$.

(2) $A^{\dagger,WG} = A^* \Leftrightarrow TT^* = I_p$, $S = 0$ *and* $N = 0$.

(3) $A^{\dagger,WG} = P_A \Leftrightarrow A \in \mathbb{C}_n^{OP}$.

(4) $A^{\dagger,WG} = Q_A \Leftrightarrow T = I_p$ *and* $N = 0$.

Proof. (1)

$$A^{\dagger,WG} = A \Leftrightarrow \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} = \begin{pmatrix} T & S \\ 0 & N \end{pmatrix}$$
$$\Leftrightarrow T^*\triangle = T,\ S = SN^\dagger N,\ T(T^{-1}S + T^{-2}SN) = S \text{ and } 0 = N$$
$$\Leftrightarrow T^2 = I_p,\ S = 0 \text{ and } N = 0.$$

(2)

$$A^{\dagger,WG} = A^* \Leftrightarrow \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} = \begin{pmatrix} T^* & 0 \\ S^* & N^* \end{pmatrix}$$
$$\Leftrightarrow T^*\triangle = T^*,\ T^*(T^{-1}S + T^{-2}SN) = 0,$$
$$(I_{n-p} - N^\dagger N)S^*\triangle = S^* \text{ and } S^*(T^{-1}S + T^{-2}SN) = N^*$$
$$\Leftrightarrow \triangle = I,\ T^{-1}S + T^{-2}SN = 0,\ SN^\dagger N = 0,\ N^* = 0$$
$$\Leftrightarrow TT^* = I_p,\ S = 0 \text{ and } N = 0.$$

(3)

$$A^{\dagger,WG} = P_A \Leftrightarrow A^{\dagger,WG} = AA^\dagger$$
$$\Leftrightarrow \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} = \begin{pmatrix} I_p & 0 \\ 0 & NN^\dagger \end{pmatrix}$$
$$\Leftrightarrow T^*\triangle = I_p,\ T^{-1}S + T^{-2}SN = 0,\ (I_{n-p} - N^\dagger N)S^*\triangle = 0 \text{ and } 0 = NN^\dagger$$
$$\Leftrightarrow T = I_p,\ S = 0 \text{ and } N = 0.$$

Then from [19], we know that it is equivalent to $A \in C_n^{OP}$.

(4)

$$A^{\dagger,WG} = Q_A \Leftrightarrow A^{\dagger,WG} = A^\dagger A$$
$$\Leftrightarrow \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} =$$
$$\begin{pmatrix} T^*\triangle T & T^*\triangle S - T^*\triangle SN^\dagger N \\ (I_{n-p} - N^\dagger N)S^*\triangle T & N^\dagger N + (I_{n-p} - N^\dagger N)S^*\triangle S(I_{n-p} - N^\dagger N) \end{pmatrix}$$
$$\Leftrightarrow T = I_p \text{ and } N = 0.$$

□

Theorem 4.2. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix with $Ind(A) = k$ written as in (2.1). Then*
(1) $A^{\dagger,WG} = A^{D,\dagger} \Leftrightarrow S = SN^\dagger N$, $(TS + SN)(I - NN^\dagger) = 0$ *and* $SN^3 = 0$.
(2) $A^{\dagger,WG} = A^{\dagger,D} \Leftrightarrow SN^2 = 0$.
(3) $A^{\dagger,WG} = A^{C,\dagger} \Leftrightarrow SN^3 = 0$ *and* $(TS + SN)(I - NN^\dagger) = 0$.
(4) $A^{\dagger,WG} = A^{Ⓦ,\dagger} \Leftrightarrow S = SN^\dagger N$ *and* $S + T^{-1}SN - SNN^\dagger = 0$.
(5) $A^{\dagger,WG} = A^{\dagger,Ⓦ} \Leftrightarrow SN = 0$.
(6) $A^{\dagger,WG}A = A^\dagger A \Leftrightarrow N = 0$.

Proof. (1)

$$A^{\dagger,WG} = A^{D,\dagger} \Leftrightarrow \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} = \begin{pmatrix} T^{-1} & (T^{k+1})^{-1}\widetilde{T}NN^\dagger \\ 0 & 0 \end{pmatrix}$$
$$\Leftrightarrow T^*\triangle = T^{-1},\ S = SN^\dagger N \text{ and } T^{-2}S + T^{-3}SN = (T^{k+1})^{-1}\widetilde{T}NN^\dagger$$
$$\Leftrightarrow S = SN^\dagger N,\ (TS + SN)(I - NN^\dagger) = 0 \text{ and } SN^3 = 0.$$

(2)

$$A^{\dagger,WG} = A^{\dagger,D} \Leftrightarrow \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} = \begin{pmatrix} T^*\triangle & T^*\triangle T^{-k}\widetilde{T} \\ (I_{n-p} - N^\dagger N)S^*\triangle & (I_{n-p} - N^\dagger N)S^*\triangle T^{-k}\widetilde{T} \end{pmatrix}$$
$$\Leftrightarrow T^{-1}S + T^{-2}SN = T^{-k}\widetilde{T}$$
$$\Leftrightarrow SN^2 = 0.$$

(3)

$$A^{\dagger,WG} = A^{C,\dagger} \Leftrightarrow \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} = \begin{pmatrix} T^*\triangle & T^*\triangle T^{-k}\widetilde{T}NN^\dagger \\ (I_{n-p} - N^\dagger N)S^*\triangle & (I_{n-p} - N^\dagger N)S^*\triangle T^{-k}\widetilde{T}NN^\dagger \end{pmatrix}$$
$$\Leftrightarrow T^{-1}S + T^{-2}SN = T^{-k}\widetilde{T}NN^\dagger$$
$$\Leftrightarrow SN^3 = 0 \text{ and } (TS + SN)(I - NN^\dagger) = 0.$$

(4)

$$A^{\dagger,WG} = A^{\textcircled{W},\dagger} \Leftrightarrow \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} = \begin{pmatrix} T^{-1} & T^{-2}SNN^\dagger \\ 0 & 0 \end{pmatrix}$$
$$\Leftrightarrow S = SN^\dagger N \text{ and } T^{-2}S + T^{-3}SN = T^{-2}SNN^\dagger$$
$$\Leftrightarrow S = SN^\dagger N \text{ and } S + T^{-1}SN - SNN^\dagger = 0.$$

(5)

$$A^{\dagger,WG} = A^{\dagger,\textcircled{W}} \Leftrightarrow \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} = \begin{pmatrix} T^*\triangle & T^*\triangle T^{-1}S \\ (I_{n-p} - N^\dagger N)S^*\triangle & (I_{n-p} - N^\dagger N)S^*\triangle T^{-1}S \end{pmatrix}$$
$$\Leftrightarrow T^{-1}S + T^{-2}SN = T^{-1}S$$
$$\Leftrightarrow SN = 0.$$

(6)

$$A^{\dagger,WG}A = A^\dagger A \Leftrightarrow \begin{pmatrix} T^*\triangle T & T^*\triangle(S + T^{-1}SN + T^{-2}SN^2) \\ (I_{n-p} - N^\dagger N)S^*\triangle T & (I_{n-p} - N^\dagger N)S^*\triangle(S + T^{-1}SN + T^{-2}SN^2) \end{pmatrix} =$$
$$\begin{pmatrix} T^*\triangle T & T^*\triangle S(I - N^\dagger N) \\ (I_{n-p} - N^\dagger N)S^*\triangle T & (I_{n-p} - N^\dagger N)S^*\triangle S(I - N^\dagger N) + N^\dagger N \end{pmatrix}$$
$$\Leftrightarrow S + T^{-1}SN + T^{-2}SN^2 = S(I - N^\dagger N) \text{ and}$$
$$(I_{n-p} - N^\dagger N)S^*\triangle(S + T^{-1}SN + T^{-2}SN^2) = (I_{n-p} - N^\dagger N)S^*\triangle(I - N^\dagger N) + N^\dagger N$$
$$\Leftrightarrow T^{-1}SN + T^{-2}SN^2 = -SN^\dagger N, N^\dagger N = 0$$
$$\Leftrightarrow N = 0.$$

□

Remark 4.3. *When A is an EP matrix, we have*

$$A^{\dagger,WG} = A^\dagger = A^\# = A^{\textcircled{\#}} = A^{\textcircled{W}} = A^{\textcircled{\dagger}} = A^\diamond.$$

Theorem 4.4. *Let $A \in \mathbb{C}^{n\times n}$ with $Ind(A) = k$. Then the following statements are equivalent:*
(1) $A^{\dagger,WG} = A^D$.
(2) $AA^{\dagger,WG} = A^{\dagger,WG}A$.

Proof. Simple computations show that

$$\begin{aligned} A^{\dagger,WG} = A^D \Leftrightarrow & \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} = \begin{pmatrix} T^{-1} & (T^{k+1})^{-1}\widetilde{T} \\ 0 & 0 \end{pmatrix} \\ \Leftrightarrow & T^*\triangle = T^{-1},\ T^*\triangle(T^{-1}S + T^{-2}SN) = (T^{k+1})^{-1}\widetilde{T} \text{ and } S = SN^\dagger N \\ \Leftrightarrow & SN^2 = 0 \text{ and } S = SN^\dagger N, \end{aligned}$$

and

$$\begin{aligned} AA^{\dagger,WG} = A^{\dagger,WG}A \Leftrightarrow & \begin{pmatrix} I_p & T^{-1}S + T^{-2}SN \\ 0 & 0 \end{pmatrix} = \\ & \begin{pmatrix} T^*\triangle T & T^*\triangle(S + T^{-1}SN + T^{-2}SN^2) \\ (I_{n-p} - N^\dagger N)S^*\triangle T & (I_{n-p} - N^\dagger N)S^*\triangle(S + T^{-1}SN + T^{-2}SN^2) \end{pmatrix} \\ \Leftrightarrow & S = SN^\dagger N \text{ and } T^{-1}S + T^{-2}SN = T^{-1}(S + T^{-1}SN + T^{-2}SN^2) \\ \Leftrightarrow & SN^2 = 0, \text{ and } S = SN^\dagger N. \end{aligned}$$

Therefore, (1) and (2) are equivalent. □

Theorem 4.5. *Let $A\in\mathbb{C}^{n\times n}$ with $Ind(A) = k$. Then the following statements are equivalent:*
(1) $A\in\mathbb{C}_n^{EP}$.
(2) $AA^{\dagger,WG} = AA^\dagger$.
(3) $AA^{\dagger,WG} = A^\dagger A$.
(4) $A^{\dagger,WG}A = AA^\dagger$.

Proof. We already know that $A\in\mathbb{C}_n^{EP}$ is equivalent to $S = 0$ and $N = 0$.

(2)

$$\begin{aligned} AA^{\dagger,WG} = AA^\dagger \Leftrightarrow & \begin{pmatrix} I_p & T^{-1}S + T^{-2}SN \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_p & 0 \\ 0 & NN^\dagger \end{pmatrix} \\ \Leftrightarrow & T^{-1}S + T^{-2}SN = 0, NN^\dagger = 0 \\ \Leftrightarrow & S = 0 \text{ and } N = 0. \end{aligned}$$

(3)

$$\begin{aligned} AA^{\dagger,WG} = A^\dagger A \Leftrightarrow & \begin{pmatrix} I_p & T^{-1}S + T^{-2}SN \\ 0 & 0 \end{pmatrix} = \\ & \begin{pmatrix} T^*\triangle T & T^*\triangle S(I_{n-p} - N^\dagger N) \\ (I_{n-p} - N^\dagger N)S^*\triangle T & (I_{n-p} - N^\dagger N)S^*\triangle S(I_{n-p} - N^\dagger N) + N^\dagger N \end{pmatrix} \\ \Leftrightarrow & S = SN^\dagger N,\ N^\dagger N = 0 \text{ and } T^{-1}S + T^{-2}SN = T^{-1}S(I - N^\dagger N) \\ \Leftrightarrow & S = 0 \text{ and } N = 0. \end{aligned}$$

(4)

$$\begin{aligned} A^{\dagger,WG}A = AA^\dagger \Leftrightarrow & \begin{pmatrix} T^*\triangle T & T^*\triangle(S + T^{-1}SN + T^{-2}SN^2) \\ (I_{n-p} - N^\dagger N)S^*\triangle T & (I_{n-p} - N^\dagger N)S^*\triangle(S + T^{-1}SN + T^{-2}SN^2) \end{pmatrix} = \\ & \begin{pmatrix} I_p & 0 \\ 0 & NN^\dagger \end{pmatrix} \\ \Leftrightarrow & S = SN^\dagger N,\ NN^\dagger = 0 \text{ and } T^{-1}(S + T^{-1}SN + T^{-2}SN^2) = 0 \\ \Leftrightarrow & S = 0 \text{ and } N = 0. \end{aligned}$$

Therefore, the above conditions are equivalent. □

5. Successive matrix squaring algorithm for the MPWG inverse

In this section, we give successive matrix squaring algorithms for computing the MPWG inverse. The development of the SMS iterations start from the transformations.

Since

$$(A^{k+2})^{\dagger}A^2(AA^{\dagger,WG}) = (A^{k+2})^{\dagger}A^3A^{\dagger}A^k(A^{k+2})^{\dagger}A^2 = (A^{k+2})^{\dagger}A^{k+2}(A^{k+2})^{\dagger}A^2 = (A^{k+2})^{\dagger}A^2,$$

we have

$$A^{\dagger,WG} = A^{\dagger,WG} - \beta((A^{k+2})^{\dagger}A^2AA^{\dagger,WG} - (A^{k+2})^{\dagger}A^2) = (I - \beta(A^{k+2})^{\dagger}A^3)A^{\dagger,WG} + \beta(A^{k+2})^{\dagger}A^2.$$

Observe the following matrices

$$P = I - \beta(A^{k+2})^{\dagger}A^3,\ Q = \beta(A^{k+2})^{\dagger}A^2,\ \beta > 0.$$

It is obvious that $A^{\dagger,WG}$ is the unique solution of $X = PX + Q$. Then an iterative procedure for computing the MPWG inverse $A^{\dagger,WG}$ can be defined as follows

$$X_1 = Q,\ \ X_{m+1} = PX_m + Q. \tag{5.1}$$

This algorithm can be implemented in parallel by considering the block matrix

$$T = \begin{pmatrix} P & Q \\ 0 & I \end{pmatrix} \text{ and } T^m = \begin{pmatrix} P^m & \sum_{i=0}^{m-1} P^iQ \\ 0 & I \end{pmatrix}.$$

The top right block of T^m is X_m, the mth approximation to $A^{\dagger,WG}$. The matrix power T^m can be computed by the successive squaring, i.e.

$$T_0 = T,\ T_{i+1} = T_i^2,\ i = 0, 1, ..., \jmath,$$

where the integer $\jmath$ is such that $2^{\jmath} \geq m$.

The following theorem gives the sufficient condition for the convergence of the iterative process (5.1).

Theorem 5.1. *Let $A \in \mathbb{C}^{n\times n}$ with $Ind(A) = k$ and $rank(A^k) = r$.Then the approximation*

$$X_{2^m} = \sum_{i=0}^{2^m-1}(I - \beta(A^{k+2})^{\dagger}A^3)^i\beta(A^{k+2})^{\dagger}A^2, \tag{5.2}$$

defined by the iterative process (5.1) *converges to the MPWG inverse $A^{\dagger,WG}$ if the spectral radius $\rho(I - X_1A) \leq 1$. Moreover, the following error estimation holds:*

$$\|A^{\dagger,WG} - X_{2^m}\| \leq \|(I - X_1A)^{2^m}\|.$$

As a result,

$$\lim_{m\to\infty} \sup \sqrt[2^m]{\|A^{\dagger,WG} - X_{2^m}\|} \leq \rho(I - X_1A).$$

Proof. We know that

$$A^{\dagger,WG}AA^{\dagger,WG} = A^{\dagger,WG}, X_{2^m}AA^{\dagger,WG} = X_{2^m}.$$

By the mathematical induction, we can get

$$I - X_{2^m}A = (I - X_1A)^{2^m}.$$

Therefore,

$$\begin{aligned}\|A^{\dagger,WG} - X_{2^m}\| &= \|A^{\dagger,WG} - X_{2^m}AA^{\dagger,WG}\| \\ &= \|(I - X_{2^m}A)A^{\dagger,WG}\| \\ &\leq \|A^{\dagger,WG}\|\|I - X_{2^m}A\| \\ &= \|A^{\dagger,WG}\|\|(I - X_1A)^{2^m}\|,\end{aligned}$$

and

$$\begin{aligned}\lim_{m\to\infty}\sup \sqrt[2^m]{\|A^{\dagger,WG} - X_{2^m}\|} &\leq \lim_{m\to\infty}\sup \sqrt[2^m]{\|A^{\dagger,WG}\|\|(I - X_1A)^{2^m}\|}\\ &= \rho(I - X_1A).\end{aligned}$$

In the last equality, we use the fact that $\lim_{m\to\infty} \|B^n\|^{\frac{1}{n}} = \rho(B)$ for any square matrix B.

If β is a real parameter such that , where $\max_{1\leq i\leq s} \left|1 - \beta\lambda_i\right| < 1, \lambda_i (i = 1, 2, ..., s)$ are the nonzero eigenvalues of $(A^{k+2})^\dagger A^2$. Then

$$\rho(I - X_1A) = \rho(I - \beta(A^{k+2})^\dagger A^3) \leq 1.$$

It completes the proof. □

Example 5.2. *Consider the following matrix [10]:*

$$A = \begin{pmatrix} 2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}, Ind(A) = 2.$$

Let

$$P = I - \beta(A^4)^\dagger A^3,\ Q = \beta(A^4)^\dagger A^2,\ \beta = 0.8.$$

The eigenvalues λ_i of QA are included in the set $\{0, 0, 0.4\}$. The nonzero eigenvalues λ_i satisfy

$$\max_i \left|1 - \lambda_i\right| = 1 - 0.4 = 0.6 < 1.$$

Then we obtain the satisfactory approximation for $A^{\dagger,WG}$ after the 6th iteration of the successive matrix squaring algorithm.

$$(T^2)^6 \approx \begin{pmatrix} 0.0000 & 0 & 0 & 0.5000 & 0 & 0\\ 0 & 1.0000 & 0 & 0 & 0 & 0\\ 0 & 0 & 1.0000 & 0 & 0 & 0\\ 0 & 0 & 0 & 1.0000 & 0 & 0\\ 0 & 0 & 0 & 0 & 1.0000 & 0\\ 0 & 0 & 0 & 0 & 0 & 1.0000\end{pmatrix}.$$

The upper right corner of $(T^2)^6$ is an approximation of the MPWG inverse

$$A^{\dagger,WG} = \begin{pmatrix} 0.5000 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

6. The Cramer rule for the solution of a singular equation Ax = b

We study the relationship between the MPWG inverse $A^{\dagger,WG}$ and an invertible bordered matrix.

Theorem 6.1. *Let $A\in\mathbb{C}^{n\times n}$ with $Ind(A) = k$. Let $U\in C^{n\times r}$ and $V^*\in C^{n\times r}$ having full column rank such that*

$$R(A^\dagger A^k) = N(V),\ N((A^k)^* A^2) = R(U).$$

Then the bordered matrix

$$X = \begin{pmatrix} A & U\\ V & 0\end{pmatrix}$$

is nonsingular and

$$X^{-1} = \begin{pmatrix} A^{\dagger,WG} & (I - A^{\dagger,WG}A)V^\dagger\\ U^\dagger(I - AA^{\dagger,WG}) & -U^\dagger(A - AA^{\dagger,WG}A)V^\dagger\end{pmatrix}. \tag{6.1}$$

Proof. Since $R(A^{\dagger,WG}) = R(A^\dagger A^k) = N(V)$, we obtain $VA^{\dagger,WG} = 0$. By

$$R(I - AA^{\dagger,WG}) = N(AA^{\dagger,WG}) = N(A^{\dagger,WG}) = N((A^k)^*A^2) = R(U) = R(UU^\dagger),$$

we can obtain

$$UU^\dagger(I - AA^{\dagger,WG}) = (I - AA^{\dagger,WG}).$$

Let

$$Y = \begin{pmatrix} A^{\dagger,WG} & (I - A^{\dagger,WG}A)V^\dagger \\ U^\dagger(I - AA^{\dagger,WG}) & -U^\dagger(A - AA^{\dagger,WG}A)V^\dagger \end{pmatrix},$$

we have

$$\begin{aligned} XY &= \begin{pmatrix} AA^{\dagger,WG} + UU^\dagger(I - AA^{\dagger,WG}) & A(I - A^{\dagger,WG}A)V^\dagger - UU^\dagger(A - AA^{\dagger,WG}A)V^\dagger \\ VA^{\dagger,WG} & V(I - A^{\dagger,WG}A)V^\dagger \end{pmatrix} \\ &= \begin{pmatrix} AA^{\dagger,WG} + (I - AA^{\dagger,WG}) & A(I - A^{\dagger,WG}A)V^\dagger - UU^\dagger(I - AA^{\dagger,WG})AV^\dagger \\ VA^{\dagger,WG} & VV^\dagger - VA^{\dagger,WG}AV^\dagger \end{pmatrix} \\ &= \begin{pmatrix} I_n & A(I - A^{\dagger,WG}A)V^\dagger - (I - AA^{\dagger,WG})AV^\dagger \\ 0 & VV^\dagger - 0 \end{pmatrix} \\ &= \begin{pmatrix} I_n & 0 \\ 0 & I_r \end{pmatrix} \\ &= I_{n+r}. \end{aligned}$$

In an analogous way, it is possible to verify that $YX = I$. Thus, X is nonsingular and $X^{-1} = Y$. □

Using the relationship between the MPWG inverse and a nonsingular bordered matrix, we give the Cramer rule for solving a singular linear equation $Ax = B$. $A(ij \to b_j)$ denotes the matrix obtained by replacing ith column of A with b_j, where b_j is the jth column of B.

Theorem 6.2. *Let $A, B \in \mathbb{C}^{n\times n}$ and $Ind(A) = k$. If $R(B) \subseteq R(A^k)$, then the restricted matrix equation*

$$AX = B, R(X) \subseteq R(A^\dagger A^k) \tag{6.2}$$

has unique solution $X = A^{\dagger,WG}B$.

Proof. If $R(B) \subseteq R(A^k)$, then $AA^{\dagger,WG}B = P_{R(A^k)}B = B$. Clearly, $X = A^{\dagger,WG}B$ is a solution of (6.2). $X = A^{\dagger,WG}B$ also satisfies the restricted condition because $R(X) \subseteq R(A^{\dagger,WG}) = R(A^\dagger A^k)$. Finally, we show the uniqueness of X. If X_1 also satisfies (6.2), we can get $R(X_1) \subseteq R(A^\dagger A^k)$, then

$$X = A^{\dagger,WG}B = A^{\dagger,WG}AX_1 = P_{R(A^\dagger A^k)}X_1 = X_1.$$

□

Theorem 6.3. *Let $A, B \in \mathbb{C}^{n\times n}$ and $Ind(A) = k$. Let $U \in C^{n\times r}$ and $V^* \in C^{n\times r}$ having full column rank such that $R(A^\dagger A^k) = N(V), N((A^k)^*A^2) = R(U)$. If $R(B) \subseteq R(A^k)$, then the unique solution $X = A^{\dagger,WG}B$ of the singular linear equation* (6.2) *is given by*

$$x_{ij} = \frac{det\begin{pmatrix} A(i \to b_j) & U \\ V(i \to 0) & 0 \end{pmatrix}}{det\begin{pmatrix} A & U \\ V & 0 \end{pmatrix}}, i = 1, 2, ...n, j = 1, 2, ...n. \tag{6.3}$$

Proof. Since $X = A^{\dagger,WG}B \in R(A^{\dagger}A^{k}) = N(V)$ and $B \in R(A^{k}) = AR(A^{\dagger}A^{k})$, we have

$$VX = 0, \quad (I - AA^{\dagger,WG})B = 0. \tag{6.4}$$

It follows from (6.4) that the solution of $AX = B$ satisfies

$$\begin{pmatrix} A & U \\ V & 0 \end{pmatrix}\begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} B \\ 0 \end{pmatrix}. \tag{6.5}$$

By Theorem 6.1, the coefficient matrix of (6.5) is nonsingular. Using (6.1) and (6.4), we obtain

$$\begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} A^{\dagger,WG} & (I - A^{\dagger,WG}A)V^{\dagger} \\ U^{\dagger}(I - AA^{\dagger,WG}) & -U^{\dagger}(A - AA^{\dagger,WG}A)V^{\dagger} \end{pmatrix}\begin{pmatrix} B \\ 0 \end{pmatrix} = \begin{pmatrix} A^{\dagger,WG}B \\ 0 \end{pmatrix}.$$

Thus $X = A^{\dagger,WG}B$ and (6.3) follows from the classical Cramer rule [1, Chapter 3]. □

7. Applications

We need to apply the MPWG inverse to solve the appropriate linear equations.

Theorem 7.1. *Let $A \in \mathbb{C}^{n\times n}$ and $Ind(A) = k$, the equation*

$$(A^{k+2})^{*}A^{3}x = (A^{k+2})^{*}A^{2}b, \ \ b \in C^{n}, \tag{7.1}$$

is consistent and its general solution is

$$x = A^{\dagger,WG}b + (I - A^{\dagger,WG}A)y, \tag{7.2}$$

for arbitrary $y \in C^{n}$.

Proof. Suppose that x has the form (7.2). Applying $A^{\dagger,WG} = A^{\dagger}A^{k}(A^{k+2})^{\dagger}A^{2}$, we have

$$\begin{aligned}(A^{k+2})^{*}A^{3}A^{\dagger,WG} &= (A^{k+2})^{*}A^{3}A^{\dagger}A^{k}(A^{k+2})^{\dagger}A^{2} = (A^{k+2})^{*}A^{k+2}(A^{k+2})^{\dagger}A^{2} \\ &= (A^{k+2})^{*}A^{2}.\end{aligned}$$

Therefore $(A^{k+2})^{*}A^{3}A^{\dagger,WG}b = (A^{k+2})^{*}A^{2}b$, which implies that (7.1) holds for x.

For a solution x to (7.1), we obtain

$$A^{\dagger,WG}b = A^{\dagger}A^{k}(A^{k+2})^{\dagger}A^{2}b = A^{\dagger}A^{k}(A^{k+2})^{\dagger}((A^{k+2})^{\dagger})^{*}(A^{k+2})^{*}A^{2}b = A^{\dagger,WG}Ax.$$

Now, we get

$$x = A^{\dagger,WG}b + x - A^{\dagger,WG}Ax = A^{\dagger,WG}b + (I - A^{\dagger,WG}A)x.$$

i.e., x possesses the form (7.2). □

8. The MPWG binary relation

In this section, we first give the definition of the MPWG relation: $A \overset{\dagger,WG}{\leq} B$ if and only if $A^{\dagger,WG}A = A^{\dagger,WG}B$ and $AA^{\dagger,WG} = BA^{\dagger,WG}$, where A and B are square matrices of the same size.

Naturally, we will consider whether this binary relationship can become a partial order. The answer to this question is No. A binary relation is called a partial order if it is reflexive, transitive, and anti-symmetric on a non-empty set. Next, we give a concrete example to prove that this relationship is not satisfied antisymmetry.

Example 8.1. *Consider the matrices*

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since

$$A^{\dagger,WG} = B^{\dagger,WG} = \begin{pmatrix} 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 \end{pmatrix},$$

$A^{\dagger,WG}A = A^{\dagger,WG}B, AA^{\dagger,WG} = BA^{\dagger,WG}$ *and* $B^{\dagger,WG}B = B^{\dagger,WG}A, BB^{\dagger,WG} = AB^{\dagger,WG}$. *Clearly,* $A \overset{\dagger,WG}{\leq} B$ *and* $B \overset{\dagger,WG}{\leq} A$ *hold, but* $A \neq B$. *The MPWG relation can not become a partial order.*

9. Conclusion

In this paper, the definition, representations and characterizations of the MPWG inverse are given. The equivalence conditions between various famous generalized inverses and the MPWG inverse are proved. For Cramer rule and SMS iterative algorithm, we also give relevant theorems. Moreover, the MPWG inverse can be applied to solving equations. We believe that the research on MPWG inverse will be popularized in the near future.

Some perspectives for further researches can be described as follows:

1. Our further goal is to study more properties and characteristics of MPWG inverse.
2. In addition, we can further extend MPWG inverse to tensors.

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