



Geometry of Contact Skew CR-Warped Product Submanifolds of Sasakian Manifolds

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Abstract. In this paper, we study warped products of contact skew-CR submanifolds, called contact skew CR-warped products. We establish a lower bound relationship between the squared norm of the second fundamental form and the warping function. The equality case of the inequality is investigated and some special cases of derived inequality are given. Furthermore, we provide non-trivial examples of such submanifolds.

1. Introduction

The concept of skew CR-submanifolds of almost Hermitian manifolds was given by G. S. Ronsse [21] to unify and generalize the concepts of holomorphic, totally real, CR, slant, semi-slant and pseudo-slant (hemi-slant in the sense of B. Sahin [22]) submanifolds by exploiting the behavior of the bounded symmetric linear operator. Later, this idea is extended to the contact geometry by Tripathi in [24] with the name of almost semi-invariant submanifolds as a generalized class of invariant, anti-invariant, slant, contact CR, bi-slant submanifolds of contact metric manifolds.

On the other hand, the warped products of skew CR-submanifolds of Kaehler manifolds were studied by B. Sahin in [23] as a generalization of CR-warped products introduced by B.-Y. Chen in his seminal work [9–12] and of warped product hemi-slant submanifolds, studied by B. Sahin in [22]. Later on, the contact version of skew CR-warped products of cosymplectic manifolds appeared in [17]. Recently, we studied warped product skew CR-submanifolds of Kenmotsu manifolds in [20]. For up-to-date survey on warped product manifolds and warped product submanifolds we refer to B.-Y. Chen's books [13, 15] and his survey article [14].

In this paper, we study the contact skew CR-warped product submanifolds by considering the base manifold is the Riemannian product of invariant and proper slant submanifolds of a Sasakian manifold and the fiber of warped product is an anti-invariant submanifold.

2020 *Mathematics Subject Classification.* 53B25; 53B20; 53C15; 53C25; 53C42; 53D10

Keywords. warped products; slant; semi-slant submanifolds; pseudo-slant submanifolds; contact skew CR-submanifolds; Sasakian manifolds

Received: 06 December 2021; Accepted: 19 February 2022

Communicated by Mica S. Stanković

This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia under grant no. (KEP-PhD-88-130-38). The authors, therefore, acknowledge with thanks DSR technical and financial support.

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The paper is organized as follows: In Section 2, we give some basic formulas and definitions for almost contact metric manifolds and their submanifolds. In Section 3, we recall the definition of skew CR-submanifolds and provide two non-trivial examples. In this section, we also find some useful relations for contact skew CR-warped products those are essential to derive our main result. In Section 4, we derive a lower bound relation for the squared norm of the second fundamental form in terms of components of the gradient of warping function along both factors of a base manifold. The equality case is also considered. In Section 5, we give some special cases of our derived inequality. In Section 6, we give two non-trivial examples of skew CR-warped products in Euclidean spaces.

2. Preliminaries

A $(2m + 1)$ -dimensional differentiable manifold \tilde{M} is called an *almost contact manifold* if there is an almost contact structure (φ, ξ, η) consisting of a $(1, 1)$ tensor field φ , a vector field ξ and a 1-form η satisfying [3]

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0, \tag{1}$$

where $I : T\tilde{M} \rightarrow T\tilde{M}$ is the identity mapping. From the definition it follows that the $(1, 1)$ -tensor field φ has constant rank $2m$ (cf. [3]). An almost contact manifold $(\tilde{M}, \varphi, \eta, \xi)$ is said to be *normal* when the tensor field $N_\varphi = [\varphi, \varphi] + 2d\eta \otimes \xi$ vanishes identically, where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ . It is known that any almost contact manifold $(\tilde{M}, \varphi, \eta, \xi)$ admits a Riemannian metric \tilde{g} such that

$$\tilde{g}(\varphi X, \varphi Y) = \tilde{g}(X, Y) - \eta(X)\eta(Y) \tag{2}$$

for any $X, Y \in \Gamma(T\tilde{M})$, where the $\Gamma(T\tilde{M})$ is the Lie algebra of vector fields on \tilde{M} . This metric \tilde{g} is called a *compatible metric* and the manifold \tilde{M} together with the structure $(\varphi, \xi, \eta, \tilde{g})$ is called an *almost contact metric manifold*. As an immediate consequence of (2), one has $\eta(X) = \tilde{g}(X, \xi)$, $\eta(\xi) = 1$ and $\tilde{g}(\varphi X, Y) = -\tilde{g}(X, \varphi Y)$. Hence the fundamental 2-form Φ of \tilde{M} is defined $\Phi(X, Y) = \tilde{g}(X, \varphi Y)$ and the manifold is said to be *contact metric manifold* if $\Phi = d\eta$. If ξ is a Killing vector field with respect to \tilde{g} , the contact metric structure is called a *K-contact structure*. A normal contact metric manifold is said to be a *Sasakian manifold*. An almost contact metric manifold is Sasakian if and only if

$$(\tilde{\nabla}_X \varphi)Y = \tilde{g}(X, Y)\xi - \eta(Y)X \tag{3}$$

for all $X, Y \in \Gamma(T\tilde{M})$, where $\tilde{\nabla}$ is the Levi-Civita connection of \tilde{g} . From the formula (3), it follows that $\tilde{\nabla}_X \xi = -\varphi X$. A Sasakian manifold is always a *K-contact manifold* and the converse is true in the dimension three.

Let M be a submanifold of a Riemannian manifold \tilde{M} equipped with a Riemannian metric \tilde{g} . We use the same symbol g for both the metrics \tilde{g} of \tilde{M} and the induced metric g on the submanifold M . Let $\Gamma(TM)$ the Lie algebra of vector fields on M and $\Gamma(T^\perp M)$, the set of all vector fields normal to M . If we denote by ∇ , the Levi-Civita connection of M , then the Gauss and Weingarten formulas are respectively given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{4}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N, \tag{5}$$

for any vector field $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where ∇^\perp is the normal connection in the normal bundle, σ is the second fundamental form and A_N is the shape operator (corresponding to the normal vector field N) for the immersion of M into \tilde{M} . They are related by $g(\sigma(X, Y), N) = g(A_N X, Y)$.

A submanifold M is said to be totally geodesic if $\sigma = 0$ and totally umbilical if $\sigma(X, Y) = g(X, Y)H$, $\forall X, Y \in \Gamma(TM)$, where $H = \frac{1}{n} \sum_{i=1}^n \sigma(e_i, e_i)$ is the mean curvature vector of M . For any $x \in M$ and $\{e_1, \dots, e_n, \dots, e_{2m+1}\}$ is an orthonormal frame of $T_x \tilde{M}$ such that e_1, \dots, e_n are tangent to M at x . Then, we set

$$\|\sigma\|^2 = \sum_{i,j=1}^n g(\sigma(e_i, e_j), \sigma(e_i, e_j)), \quad \sigma_{ij}^r = g(\sigma(e_i, e_j), e_r), \quad i, j \in \{1, \dots, n\}, \quad r \in \{n + 1, \dots, 2m + 1\}. \tag{6}$$

According to the behaviour of the tangent bundle of a submanifold under the action of the almost contact structure tensor φ of the ambient manifold, there are two well-known classes of submanifolds, namely, φ -invariant submanifolds and φ -anti-invariant submanifolds. In the first case the tangent space of the submanifold remains invariant under the action of the almost contact structure tensor φ whereas in the second case it is mapped into the normal space.

Later, A. Bejancu [1] generalized the concept of invariant and anti-invariant submanifolds in to a semi-invariant submanifold (also known as contact CR-submanifold [18], [33]). A submanifold M tangent to the structure vector field ξ of an almost contact metric manifold \tilde{M} is called a *contact CR-submanifold* if there exists a pair of orthogonal distributions $\mathfrak{D} : x \rightarrow \mathfrak{D}_x$ and $\mathfrak{D}^\perp : x \rightarrow \mathfrak{D}_x^\perp, \forall x \in M$ such that $TM = \mathfrak{D} \oplus \mathfrak{D}^\perp \oplus \langle \xi \rangle$, where $\langle \xi \rangle$ is the 1-dimensional distribution spanned by the structure vector field ξ with \mathfrak{D} is invariant, i.e., $\varphi\mathfrak{D} = \mathfrak{D}$ and \mathfrak{D}^\perp is anti-invariant, i.e., $\varphi\mathfrak{D}^\perp \subseteq T^\perp M$. Obviously, invariant and anti-invariant submanifolds are contact CR-submanifolds with $\mathfrak{D}^\perp = \{0\}$ and $\mathfrak{D} = \{0\}$, respectively.

Slant submanifolds in complex geometry were defined and studied by B.-Y. Chen [7, 8]. In [19], A. Lotta introduced the contact version of slant submanifolds. Let M be a submanifold of an almost contact metric manifold \tilde{M} . Let \mathfrak{D} be a differentiable distribution on M . For any non-zero vector $X \in \mathfrak{D}_x$, the angle $\theta_{\mathfrak{D}}(X)$ between φX and \mathfrak{D}_x is a slant angle of X with respect to the distribution \mathfrak{D} . If the slant angle $\theta_{\mathfrak{D}}(X)$ is constant, i.e., it is independent of the choice $x \in M$ and $X \in \mathfrak{D}_x$, then \mathfrak{D} is called a θ -slant distribution and $\theta_{\mathfrak{D}}(X) = \theta_{\mathfrak{D}}$ is called the slant angle of the distribution \mathfrak{D} . A submanifold M tangent to ξ is said to be *slant* if for any $x \in M$ and any $X \in T_x M$, linearly independent to ξ , the angle between φX and $T_x M$ is a constant $\theta \in [0, \pi/2]$, called the *slant angle* of M in \tilde{M} . Invariant and anti-invariant submanifolds are θ -slant submanifolds with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. A slant submanifold which is neither invariant nor anti-invariant is called *proper slant*. For more details, we refer to [5, 8].

For any vector field $X \in \Gamma(TM)$, we have

$$\varphi X = TX + FX, \quad (7)$$

where TX and FX are the tangential and normal components of φX , respectively. For a slant submanifold of almost contact metric manifolds we have the following useful result.

Theorem 2.1. [5] *Let M be a submanifold of an almost contact metric manifold \tilde{M} , such that $\xi \in \Gamma(TM)$. Then M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$T^2 = \lambda(-I + \eta \otimes \xi). \quad (8)$$

Furthermore, if θ is slant angle, then $\lambda = \cos^2 \theta$.

Following relations are straightforward consequence of (8)

$$g(TX, TY) = \cos^2 \theta [g(X, Y) - \eta(X)\eta(Y)], \quad (9)$$

$$g(FX, FY) = \sin^2 \theta [g(X, Y) - \eta(X)\eta(Y)] \quad (10)$$

for any $X, Y \in \Gamma(TM)$.

Beside these classes of submanifolds of almost contact metric manifolds there are some other submanifolds. J.L. Caberizo et al. defined and studied semi-slant submanifolds of Sasakian manifolds in [4]. A submanifold M of an almost contact metric manifold \tilde{M} is said to be a *semi-slant submanifold* if there exists a pair of orthogonal distributions \mathfrak{D} and \mathfrak{D}^θ on M such that \mathfrak{D} is φ -invariant and \mathfrak{D}^θ is proper slant with slant angle θ with $TM = \mathfrak{D} \oplus \mathfrak{D}^\theta \oplus \langle \xi \rangle$.

Pseudo-slant submanifolds were defined by Carriazo in [6] under the name of *anti-slant submanifolds* as a particular class of bi-slant submanifolds. Later, he called these classes of submanifolds as *pseudo-slant submanifolds*. A submanifold M of an almost contact metric manifold \tilde{M} is said to be a *pseudo-slant submanifold* if there exists a pair of orthogonal distributions \mathfrak{D}^\perp and \mathfrak{D}^θ on M such that $TM = \mathfrak{D}^\perp \oplus \mathfrak{D}^\theta \oplus \langle \xi \rangle$ with \mathfrak{D}^\perp is anti-invariant, that is, $\varphi(\mathfrak{D}^\perp) \subseteq T^\perp M$ and \mathfrak{D}^θ is a proper slant distribution with angle θ .

3. Contact skew CR-warped product submanifolds

Skew CR-submanifolds introduced by Ronsse [21] for almost Hermitian manifolds. Later, for contact metric manifolds, Tripathi [24] studied contact skew CR-submanifolds under the name almost semi-invariant submanifolds by exploiting the behaviour of a natural bounded symmetric linear operator $T^2 = Q$ on the submanifold. From (2) and (7), it is easy to see that $g(TX, Y) = -g(X, TY)$, for any $X, Y \in \Gamma(TM)$, which implies that $g(QX, Y) = g(X, QY)$, i.e., Q is a symmetric operator, therefore its eigenvalues are real and diagonalizable. Moreover, its eigenvalues are bounded by -1 and 0 .

Since $\xi \in \Gamma(TM)$, then we have $TM = \langle \xi \rangle \oplus \langle \xi \rangle^\perp$ where $\langle \xi \rangle$ is the distribution spanned by ξ and $\langle \xi \rangle^\perp$ is the orthogonal complementary distribution of $\langle \xi \rangle$ in M . For any $x \in M$, we may write

$$\mathfrak{D}_x^\lambda = \ker(Q + \lambda^2(x)I)_x,$$

where I is the identity transformation and $\lambda(x) \in [0, 1]$ such that $-\lambda^2(x)$ is an eigenvalue of $Q(x)$. We note that $\mathfrak{D}_x^1 = \ker F$ and $\mathfrak{D}_x^0 = \ker T$. \mathfrak{D}_x^1 is the maximal φ -invariant subspace of T_xM and \mathfrak{D}_x^0 is the maximal φ -anti-invariant subspace of T_xM . From now on, we denote the distributions \mathfrak{D}^1 and \mathfrak{D}^0 by \mathfrak{D} and \mathfrak{D}^\perp , respectively. Since Q_x is symmetric and diagonalizable, for some integer k if $-\lambda_1^2(x), \dots, -\lambda_k^2(x)$ are the eigenvalues of Q at $x \in M$, then $\langle \xi \rangle_x^\perp$ can be decomposed as direct sum of mutually orthogonal eigenspaces, i.e.

$$\langle \xi \rangle_x^\perp = \mathfrak{D}_x^{\lambda_1} \oplus \mathfrak{D}_x^{\lambda_2} \dots \oplus \mathfrak{D}_x^{\lambda_k}.$$

Each $\mathfrak{D}_x^{\lambda_i}, 1 \leq i \leq k$, is a T -invariant subspace of T_xM . Moreover if $\lambda_i \neq 0$, then $\mathfrak{D}_x^{\lambda_i}$ is even dimensional. We say that a submanifold M of an almost contact metric manifold \tilde{M} is a generic submanifold if there exists an integer k and functions $\lambda_i, 1 \leq i \leq k$ defined on M with values in $(0, 1)$ such that

- (1) Each $-\lambda_i^2(x), 1 \leq i \leq k$ is a distinct eigenvalue of Q with

$$T_xM = \mathfrak{D}_x \oplus \mathfrak{D}_x^\perp \oplus \mathfrak{D}_x^{\lambda_1} \oplus \dots \oplus \mathfrak{D}_x^{\lambda_k} \oplus \langle \xi \rangle_x$$

for any $x \in M$.

- (2) The dimensions of $\mathfrak{D}_x, \mathfrak{D}_x^\perp$ and $\mathfrak{D}_x^{\lambda_i}, 1 \leq i \leq k$ are independent on $x \in M$.

Moreover, if each λ_i is constant on M , then M is called a skew CR-submanifold. Thus, we observe that CR-submanifolds are a particular class of skew CR-submanifolds with $k = 0, \mathfrak{D} \neq \{0\}$ and $\mathfrak{D}^\perp \neq \{0\}$. And slant submanifolds are also a particular class of skew CR-submanifolds with $k = 1, \mathfrak{D} = \{0\}, \mathfrak{D}^\perp = \{0\}$ and λ_1 is constant. Moreover, if $\mathfrak{D}^\perp = \{0\}, \mathfrak{D} \neq 0$ and $k = 1$, then M is a semi-slant submanifold. Furthermore, if $\mathfrak{D} = \{0\}, \mathfrak{D}^\perp \neq \{0\}$ and $k = 1$, then M is a pseudo-slant (or hemi-slant) submanifold.

A submanifold M of an almost contact metric manifold \tilde{M} is said to be a contact skew CR-submanifold of order 1 if M is a skew CR-submanifold such that $k = 1$ and λ_1 is constant. In this case, the tangent and normal bundles of M are decomposed by

$$TM = \mathfrak{D} \oplus \mathfrak{D}^\perp \oplus \mathfrak{D}^\theta \oplus \langle \xi \rangle, \quad T^\perp M = \varphi \mathfrak{D}^\perp \oplus F \mathfrak{D}^\theta \oplus \mu$$

where μ is a φ -invariant normal subbundle of $T^\perp M$.

We provide the following examples of contact skew CR-submanifolds of order 1 in the Euclidean spaces.

Example 3.1. Consider the Euclidean 11-space \mathbb{R}^{11} with cartesian coordinates $(x_1, \dots, x_5, y_1, \dots, y_5, t)$ and the standard Euclidean metric \langle, \rangle . Define the almost contact structure on \mathbb{R}^{11} as follows:

$$\varphi \left(\frac{\partial}{\partial x_i} \right) = - \left(\frac{\partial}{\partial y_i} \right), \quad \varphi \left(\frac{\partial}{\partial y_j} \right) = \left(\frac{\partial}{\partial x_j} \right), \quad \varphi \left(\frac{\partial}{\partial t} \right) = 0; \quad 1 \leq i, j \leq 5.$$

Then, it is easy to see that $(\mathbb{R}^{11}, \varphi, \xi, \eta, \langle, \rangle)$ is an almost contact metric manifold with $\xi = \frac{\partial}{\partial t}$ and $\eta = dt$. Let M be a submanifold of \mathbb{R}^{11} defined by the immersion $\psi : M \rightarrow \mathbb{R}^{11}$ as follows:

$$\psi(u, v, w, r, s, t) = (u + v, \cosh w, kr, \cos r, \cos s, u - v, \sinh w, s, \sin r, \sin s, t)$$

for any non-zero constant k . Then the tangent space of M is spanned by the following vectors:

$$X_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}, X_2 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_1}, X_3 = \sinh w \frac{\partial}{\partial x_2} + \cosh w \frac{\partial}{\partial y_2},$$

$$X_4 = k \frac{\partial}{\partial x_3} - \sin r \frac{\partial}{\partial x_4} + \cos r \frac{\partial}{\partial y_4}, X_5 = -\sin s \frac{\partial}{\partial x_5} + \frac{\partial}{\partial y_3} + \cos s \frac{\partial}{\partial y_5}, X_6 = \frac{\partial}{\partial t}.$$

Hence, we find that φX_3 is orthogonal to TM , thus $\mathfrak{D}^\perp = \text{Span}\{X_3\}$ is an anti-invariant distribution and $\mathfrak{D} = \text{Span}\{X_1, X_2\}$ is an invariant distribution; while $\mathfrak{D}^\theta = \text{Span}\{X_4, X_5\}$ is a slant distribution with slant angle $\theta = \cos^{-1}\left(\frac{k}{\sqrt{2(1+k^2)}}\right)$. Hence, the tangent space is decomposed $TM = \mathfrak{D} \oplus \mathfrak{D}^\perp \oplus \mathfrak{D}^\theta \oplus \langle \xi \rangle$, i.e., M is a contact skew CR-submanifold of order 1.

Example 3.2. Let M be a submanifold \mathbb{R}^9 given by

$$x_1 = u, y_1 = -v, x_2 = r, y_2 = s, x_3 = s \cos \theta, y_3 = s \sin \theta, x_4 = \cos w, y_4 = -\sin w, t = t.$$

It is easy to find that the local frame of TM is spanned by

$$X_1 = \frac{\partial}{\partial x_1}, X_2 = -\frac{\partial}{\partial y_1}, X_3 = \frac{\partial}{\partial x_2}, X_4 = \cos \theta \frac{\partial}{\partial x_3} + \frac{\partial}{\partial y_2} + \sin \theta \frac{\partial}{\partial y_3},$$

$$X_5 = -\sin w \frac{\partial}{\partial x_4} - \cos w \frac{\partial}{\partial y_4}, X_6 = \frac{\partial}{\partial t}.$$

Then, using the almost contact structure of \mathbb{R}^9 defined in Example 3.1, we find that φX_5 is orthogonal to TM , thus $\mathfrak{D}^\perp = \text{Span}\{X_5\}$ is an anti-invariant distribution and $\mathfrak{D} = \text{Span}\{X_1, X_2\}$ is an invariant distribution; while $\mathfrak{D}^\theta = \text{Span}\{X_3, X_4\}$ is a slant distribution with slant angle $\theta = 45^\circ$. Hence, M is a contact skew CR-submanifold of order 1.

Let (B, g_B) and (F, g_F) be two Riemannian manifolds and f be a positive smooth function on B . Consider the product manifold $B \times F$ with canonical projections $\pi_1 : B \times F \rightarrow B$ and $\pi_2 : B \times F \rightarrow F$. Then the manifold $M = B \times_f F$ is said to be *warped product* if it is equipped with the following warped metric

$$g(X, Y) = g_B(\pi_1^*(X), \pi_1^*(Y)) + (f \circ \pi_1)^2 g_F(\pi_2^*(X), \pi_2^*(Y)) \tag{11}$$

for all $X, Y \in \Gamma(TM)$ and $'*$ stands for derivation maps. The function f is called *the warping function* and a warped product manifold M is said to be *trivial* or simply a Riemannian product of B and F if f is constant.

Proposition 3.3. [2] For $X, Y \in \Gamma(TB)$ and $Z, W \in \Gamma(TF)$, we obtain for the warped product manifold $M = B \times_f F$ that

- (i) $\nabla_X Y \in \Gamma(TB)$,
- (ii) $\nabla_X Z = \nabla_Z X = X(\ln f)Z$,
- (iii) $\nabla_Z W = \nabla'_Z W - \frac{g(Z, W)}{f} \vec{\nabla} f$,

where ∇ and ∇' denote the Levi-Civita connections on M and F , respectively and $\vec{\nabla} f$ is the gradient of f defined by $g(\vec{\nabla} f, X) = X(f)$.

Remark 3.4. It is also important to note that for a warped product $M = B \times_f F$; B is totally geodesic and F is totally umbilical in M [2, 9].

In this section, we study warped products of contact skew CR-submanifolds of order 1 of a Sasakian manifold \tilde{M} which we define as: A warped product submanifolds of the form $M = B \times_f M_\perp$ is called a *contact skew CR-warped product submanifold* if $B = M_T \times M_\theta$ is the product of M_T and M_θ , called semi-slant product, where M_T , M_\perp and M_θ are invariant, anti-invariant and proper slant submanifolds of \tilde{M} , respectively. Throughout this paper, we assume the structure vector field ξ tangent to the submanifold. For this reason, on a contact skew CR-warped product $M = B \times_f M_\perp$, two case arise either ξ is tangent to M_\perp or ξ is tangent to B . When, $\xi \in \Gamma(TM_\perp)$, then we have the following non-existence result.

Theorem 3.5. Let $M = B \times_f M_\perp$ be a contact skew CR-warped product submanifold with $B = M_T \times M_\theta$ of a Sasakian manifold \tilde{M} such that ξ is tangent to M_\perp . Then M is simply a Riemannian product submanifold of \tilde{M} .

Proof. For any $U_1 + U_2 = U \in \Gamma(TB)$, where $U_1 \in \Gamma(TM_T)$ and $U_2 \in \Gamma(TM_\theta)$, we have

$$\tilde{\nabla}_U \xi = -\phi U = -\phi U_1 - T U_2 - F U_2.$$

Using (4) and equating the tangential components, we derive

$$\nabla_U \xi = -\phi U_1 - T U_2.$$

Then using Proposition 3.3 (ii), we get

$$U(\ln f)\xi = -\phi U_1 - T U_2.$$

Taking the inner product with ξ in the above relation, we find that $U(\ln f) = 0$, i.e., f is constant, which proves the theorem completely. \square

From now, for the simplicity we denote the tangent spaces of M_T , M_\perp and M_θ by the same symbols \mathfrak{D} , \mathfrak{D}^\perp and \mathfrak{D}^θ , respectively.

Now, if we consider $\xi \in \Gamma(TB)$, then there are two possibilities that either ξ is tangent to M_T or tangent to M_θ . For this, we have the following useful results.

Lemma 3.6. Let $M = B \times_f M_\perp$ be a contact skew CR-warped product submanifold of order 1 of a Sasakian manifold \tilde{M} such that ξ is tangent to B and $B = M_T \times M_\theta$, where M_T and M_θ are invariant and proper slant submanifolds of \tilde{M} , respectively. Then, we have

- (i) $\xi(\ln f) = 0$,
- (ii) $g(\sigma(X, Y), \phi Z) = 0$,
- (iii) $g(\sigma(X, V), \phi Z) = -g(\sigma(X, Z), FV) = 0$,

for any $X, Y \in \Gamma(\mathfrak{D})$, $V \in \Gamma(\mathfrak{D}^\theta)$ and $Z \in \Gamma(\mathfrak{D}^\perp)$.

Proof. For any $Z \in \Gamma(\mathfrak{D}^\perp)$, we have $\tilde{\nabla}_Z \xi = -\phi Z$, by using (4), we find that $\nabla_Z \xi = 0$, $\sigma(Z, \xi) = -\phi Z$. Using Proposition 3.3, we get the first part of the lemma. For the second part, we have

$$g(\sigma(X, Y), \phi Z) = g(\tilde{\nabla}_X Y, \phi Z) = -g(\tilde{\nabla}_X \phi Y, Z) + g((\tilde{\nabla}_X \phi)Y, Z).$$

for any $X, Y \in \Gamma(\mathfrak{D})$ and $Z \in \Gamma(\mathfrak{D}^\perp)$. Using (3) and the orthogonality of vector fields, we derive

$$g(\sigma(X, Y), \phi Z) = g(\tilde{\nabla}_X Z, \phi Y) = g(\nabla_X Z, \phi Y).$$

Again, using Proposition 3.3, we find that $g(\sigma(X, Y), \phi Z) = X(\ln f)g(Z, \phi Y) = 0$, which is (ii). Similarly, for any $X \in \Gamma(\mathfrak{D})$, $V \in \Gamma(\mathfrak{D}^\theta)$ and $Z \in \Gamma(\mathfrak{D}^\perp)$, we have

$$g(\sigma(X, V), \phi Z) = g(\tilde{\nabla}_X V, \phi Z) = -g(\tilde{\nabla}_X \phi V, Z) + g((\tilde{\nabla}_X \phi)V, Z).$$

Again, from (3), (7) and the orthogonality of vector fields, we obtain

$$g(\sigma(X, V), \varphi Z) = -g(\tilde{\nabla}_X TV, Z) + g(\tilde{\nabla}_X FV, Z) = g(\nabla_X Z, TV) - g(A_{FV}X, Z).$$

Then from Proposition 3.3, we get $g(\sigma(X, V), \varphi Z) = X(\ln f)g(Z, TV) - g(\sigma(X, Z), FV)$. Hence, by the orthogonality of vector fields, the second term vanishes identically which gives the first equality of (iii). On the other hand, for any $X \in \Gamma(\mathfrak{D})$, $V \in \Gamma(\mathfrak{D}^\theta)$ and $Z \in \Gamma(\mathfrak{D}^\perp)$, we have

$$g(\sigma(X, V), \varphi Z) = g(\tilde{\nabla}_V X, \varphi Z) = -g(\tilde{\nabla}_V \varphi X, Z) + g((\tilde{\nabla}_V \varphi)X, Z).$$

Again, using the structure equation of Sasakian manifold, the orthogonality of vector fields and Proposition 3.3, we get $g(\sigma(X, V), \varphi Z) = 0$, which is the second equality. Hence, the proof is complete. \square

Lemma 3.7. *Let $M = B \times_f M_\perp$ be a contact skew CR-warped product submanifold of order 1 of a Sasakian manifold \tilde{M} such that ξ is tangent to B . Then*

$$g(\sigma(U, V), \varphi Z) = g(\sigma(U, Z), FV) \tag{12}$$

for any $U, V \in \Gamma(\mathfrak{D}^\theta)$ and $Z \in \Gamma(\mathfrak{D}^\perp)$.

Proof. For any $U, V \in \Gamma(\mathfrak{D}^\theta)$ and $Z \in \Gamma(\mathfrak{D}^\perp)$, we have

$$g(\sigma(U, V), \varphi Z) = g(\tilde{\nabla}_U V, \varphi Z) = -g(\tilde{\nabla}_U \varphi V, Z) + g((\tilde{\nabla}_U \varphi)V, Z).$$

Using (3), (7) and the orthogonality of vector fields, we find

$$g(\sigma(U, V), \varphi Z) = -g(\tilde{\nabla}_U TV, Z) - g(\tilde{\nabla}_U FV, Z) = g(\nabla_U Z, TV) + g(A_{FV}U, Z).$$

By Proposition 3.3 and the orthogonality of vector field, we obtain $g(\sigma(U, V), \varphi Z) = g(\sigma(U, Z), FV)$, which proves the lemma completely. \square

Lemma 3.8. *Let $M = B \times_f M_\perp$ be a contact skew CR-warped product submanifold of order 1 of a Sasakian manifold \tilde{M} such that ξ is tangent to B . Then, we have*

$$g(\sigma(\varphi X, Z), \varphi W) = X(\ln f)g(Z, W) \tag{13}$$

for any $X \in \Gamma(\mathfrak{D})$ and $Z, W \in \Gamma(\mathfrak{D}^\perp)$.

Proof. For any $X \in \Gamma(\mathfrak{D})$ and $Z, W \in \Gamma(\mathfrak{D}^\perp)$, we have

$$g(\sigma(X, Z), \varphi W) = g(\tilde{\nabla}_Z X, \varphi W) = -g(\tilde{\nabla}_Z \varphi X, W) + g((\tilde{\nabla}_Z \varphi)X, W).$$

Using Proposition 3.3, structure equation (3) and the orthogonality of vector fields, we find

$$g(\sigma(X, Z), \varphi W) = -\varphi X(\ln f)g(Z, W) - \eta(X)g(Z, W). \tag{14}$$

Interchanging X by φX and using (1), we find (13), which completes the proof. \square

A warped product $M = B \times_f F$ is said to be *mixed totally geodesic* if $\sigma(X, Z) = 0$, for any $X \in \Gamma(TB)$ and $Z \in \Gamma(TF)$. From Lemma 3.8, we have the following consequence for a mixed totally geodesic warped product.

Theorem 3.9. *Let $M = B \times_f M_\perp$ be a $\mathfrak{D} - \mathfrak{D}^\perp$ mixed totally geodesic contact skew CR-warped product submanifold of order 1 of a Sasakian manifold \tilde{M} such that ξ is tangent to B . Then M is simply a Riemannian product manifold.*

Proof. The proof of this theorem follows from (13) and the mixed totally geodesic condition. \square

Lemma 3.10. Let $M = B \times_f M_\perp$ be a contact skew CR-warped product submanifold of order 1 of a Sasakian manifold \tilde{M} such that ξ is tangent to B . Then

- (i) $g(\sigma(Z, W), FV) - g(\sigma(Z, V), \varphi W) = (TV(\ln f) + \eta(V))g(Z, W)$,
- (ii) $g(\sigma(Z, W), FTV) - g(\sigma(Z, TV), \varphi W) = -\cos^2 \theta V(\ln f)g(Z, W)$

for any $Z, W \in \Gamma(\mathfrak{D}^\perp)$ and $V \in \Gamma(\mathfrak{D}^\theta)$.

Proof. For any $V \in \Gamma(\mathfrak{D}^\theta)$ and $Z, W \in \Gamma(\mathfrak{D}^\perp)$, we have

$$g(\sigma(Z, V), \varphi W) = g(\tilde{\nabla}_Z V, \varphi W) = -g(\tilde{\nabla}_Z \varphi V, W) + g((\tilde{\nabla}_Z \varphi)V, W).$$

Using (3) and (7), we derive

$$g(\sigma(Z, V), \varphi W) = -g(\tilde{\nabla}_Z TV, W) - g(\tilde{\nabla}_Z FV, W) - \eta(V)g(Z, W),$$

which on using Proposition 3.3 (ii) implies that

$$g(\sigma(Z, W), FV) - g(\sigma(Z, V), \varphi W) = (TV(\ln f) + \eta(V))g(Z, W),$$

which is (i). Interchanging V by TV in (i) and using Theorem 2.1, we find (ii), which ends the proof. \square

4. Inequality for $\|\sigma\|^2$

Let $M = B \times_f M_\perp$ be a n -dimensional contact skew CR-warped product submanifold of a $(2m + 1)$ -dimensional Sasakian manifold \tilde{M} with $B = M_T \times M_\theta$ and ξ is tangent to B . If $\dim M_T = m_1$, $\dim M_\perp = m_2$ and $\dim M_\theta = m_3$, then, clearly we have $n = m_1 + m_2 + m_3$. We denote the tangent bundle of M_T , M_\perp and M_θ by \mathfrak{D} , \mathfrak{D}^\perp and \mathfrak{D}^θ , respectively. Since, $\xi \in \Gamma(TB)$, then we have two cases: either $\xi \in \Gamma(\mathfrak{D})$ or $\xi \in \Gamma(\mathfrak{D}^\theta)$. If we consider $\xi \in \Gamma(\mathfrak{D})$ then we set the orthonormal frame fields of M as follows: $\mathfrak{D} = \text{Span}\{e_1, \dots, e_p, e_{p+1} = \varphi e_1, \dots, e_{2p} = \varphi e_p, e_{m_1} = e_{2p+1} = \xi\}$, $\mathfrak{D}^\perp = \text{Span}\{e_{m_1+1} = \tilde{e}_1, \dots, e_{m_1+m_2} = \tilde{e}_{m_2}\}$ and $\mathfrak{D}^\theta = \text{Span}\{e_{m_1+m_2+1} = e_1^*, \dots, e_{m_1+m_2+s} = e_s^*, e_{m_1+m_2+s+1} = e_{s+1}^* = \sec \theta \text{Te}_1^*, \dots, e_n = e_{m_3}^* = \sec \theta \text{Te}_s^*\}$. Then, the normal subbundles of $T^\perp M$ are spanned by $\varphi \mathfrak{D}^\perp = \text{Span}\{e_{n+1} = \tilde{e}_1 = \varphi \tilde{e}_1, \dots, e_{n+m_2} = \tilde{e}_{m_2} = \varphi \tilde{e}_{m_2}\}$, $F\mathfrak{D}^\theta = \text{Span}\{e_{n+m_2+1} = \tilde{e}_{m_2+1} = \csc \theta \text{Fe}_1^*, \dots, e_{n+m_2+s} = \tilde{e}_{m_2+s} = \csc \theta \text{Fe}_s^*, e_{n+m_2+s+1} = \tilde{e}_{m_2+s+1} = \csc \theta \sec \theta \text{FTe}_1^*, \dots, e_{n+m_2+m_3} = \tilde{e}_{m_2+m_3} = \csc \theta \sec \theta \text{FTe}_s^*\}$ and $\mu = \text{Span}\{e_{n+m_2+m_3+1} = \tilde{e}_{m_2+m_3+1}, \dots, e_{2m+1} = \tilde{e}_{2(m-m_2-m_3)-m_1+1}\}$.

Now, using the above orthonormal frame fields and some results of previous sections, we derive the following main result of this paper.

Theorem 4.1. Let $M = B \times_f M_\perp$ be a $\mathfrak{D}^\perp - \mathfrak{D}^\theta$ mixed totally geodesic contact skew CR-warped product submanifold of order 1 of a Sasakian manifold \tilde{M} . Then we have:

- (i) If ξ is tangent to M_T , then

$$\|\sigma\|^2 \geq 2m_2 (\|\nabla^T(\ln f)\|^2 + 1) + m_2 \cot^2 \theta \|\nabla^\theta(\ln f)\|^2.$$

- (ii) If ξ is tangent to M_θ , then

$$\|\sigma\|^2 \geq 2m_2 \|\nabla^T(\ln f)\|^2 + m_2 \cot^2 \theta \|\nabla^\theta(\ln f)\|^2,$$

where $m_2 = \dim M_\perp$ and $\nabla^T(\ln f)$ and $\nabla^\theta(\ln f)$ are the gradient components along M_T and M_θ , respectively.

- (iii) If the equality sign holds in above inequalities, then B is totally geodesic and M_\perp is a totally umbilical in \tilde{M} .

Proof. From the definition of the second fundamental from σ , we have

$$\|\sigma\|^2 = \sum_{i,j=1}^n g(\sigma(e_i, e_j), \sigma(e_i, e_j)) = \sum_{r=n+1}^{2m+1} \sum_{i,j=1}^n g(\sigma(e_i, e_j), e_r).$$

According to the constructed frame filed, the above relation takes the form

$$\|\sigma\|^2 = \sum_{r=n+1}^{n+m_2} \sum_{i,j=1}^n g(\sigma(e_i, e_j), e_r)^2 + \sum_{r=n+m_2+1}^{n+m_2+m_3} \sum_{i,j=1}^n g(\sigma(e_i, e_j), e_r)^2 + \sum_{r=n+m_2+m_3+1}^{2m+1} \sum_{i,j=1}^n g(\sigma(e_i, e_j), e_r)^2. \tag{15}$$

Leaving the last μ - components in (15). Then, we can spilt the above relation for the orthogonal spaces as follows

$$\begin{aligned} \|\sigma\|^2 &\geq \sum_{r=1}^{m_2} \sum_{i,j=1}^{m_1} g(\sigma(e_i, e_j), \bar{e}_r)^2 + 2 \sum_{r=1}^{m_2} \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} g(\sigma(e_i, \bar{e}_j), \bar{e}_r)^2 + \sum_{r=1}^{m_2} \sum_{i,j=1}^{m_2} g(\sigma(\bar{e}_i, \bar{e}_j), \bar{e}_r)^2 \\ &+ 2 \sum_{r=1}^{m_2} \sum_{i=1}^{m_2} \sum_{j=1}^{m_3} g(\sigma(\bar{e}_i, e_j^*), \bar{e}_r)^2 + \sum_{r=1}^{m_2} \sum_{i,j=1}^{m_3} g(\sigma(e_i^*, e_j^*), \bar{e}_r)^2 + 2 \sum_{r=1}^{m_2} \sum_{i=1}^{m_1} \sum_{j=1}^{m_3} g(\sigma(e_i, e_j^*), \bar{e}_r)^2 \\ &+ \sum_{r=m_2+1}^{m_2+m_3} \sum_{i,j=1}^{m_1} g(\sigma(e_i, e_j), \bar{e}_r)^2 + 2 \sum_{r=m_2+1}^{m_2+m_3} \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} g(\sigma(e_i, \bar{e}_j), \bar{e}_r)^2 + \sum_{r=m_2+1}^{m_2+m_3} \sum_{i,j=1}^{m_2} g(\sigma(\bar{e}_i, \bar{e}_j), \bar{e}_r)^2 \\ &+ 2 \sum_{r=m_2+1}^{m_2+m_3} \sum_{i=1}^{m_2} \sum_{j=1}^{m_3} g(\sigma(\bar{e}_i, e_j^*), \bar{e}_r)^2 + \sum_{r=m_2+1}^{m_2+m_3} \sum_{i,j=1}^{m_3} g(\sigma(e_i^*, e_j^*), \bar{e}_r)^2 + 2 \sum_{r=m_2+1}^{m_2+m_3} \sum_{i=1}^{m_1} \sum_{j=1}^{m_3} g(\sigma(e_i, e_j^*), \bar{e}_r)^2. \end{aligned} \tag{16}$$

We have no relation for warped product for the third, seventh, eleventh and twelfth terms, so leaving these terms. Then, using Lemma 3.6 (ii) and Lemma 3.7 with the hypothesis of theorem, we derive

$$\begin{aligned} \|\sigma\|^2 &\geq 2 \sum_{r=1}^{m_2} \sum_{i=1}^p \sum_{j=1}^{m_2} g(\sigma(e_i, \bar{e}_j), \varphi \bar{e}_r)^2 + 2 \sum_{r=1}^{m_2} \sum_{i=1}^p \sum_{j=1}^{m_2} g(\sigma(\varphi e_i, \bar{e}_j), \varphi \bar{e}_r)^2 + 2 \sum_{r=1}^{m_2} \sum_{j=1}^{m_2} g(\sigma(e_{2p+1}, \bar{e}_j), \varphi \bar{e}_r)^2 \\ &+ \sum_{r=1}^s \sum_{i,j=1}^{m_2} g(\sigma(\bar{e}_i, \bar{e}_j), \csc \theta F e_r^*)^2 + \sum_{r=1}^s \sum_{i,j=1}^{m_2} g(\sigma(\bar{e}_i, \bar{e}_j), \csc \theta \sec \theta F T e_r^*)^2. \end{aligned} \tag{17}$$

Since, for a submanifold M of a Sasakian manifold $\sigma(U, \xi) = -\varphi U$, for any $U \in \Gamma(TM)$, using this fact in the third term of (17). Also, using Lemma 3.8 and Lemma 3.10 with the $\mathfrak{D}^\perp - \mathfrak{D}^\theta$ mixed totally geodesic condition, we derive

$$\begin{aligned} \|\sigma\|^2 &\geq 2 \sum_{j,r=1}^{m_2} \sum_{i=1}^p (\varphi e_i(\ln f) + \eta(e_i))^2 g(\bar{e}_j, \bar{e}_r)^2 + 2 \sum_{j,r=1}^{m_2} \sum_{i=1}^p (e_i(\ln f))^2 g(\bar{e}_j, \bar{e}_r)^2 + 2 \sum_{j,r=1}^{m_2} g(\sigma(\varphi \bar{e}_j, \varphi \bar{e}_r))^2 \\ &+ \csc^2 \theta \sum_{r=1}^s \sum_{i,j=1}^{m_2} (T e_r^*(\ln f) + \eta(e_r^*))^2 g(\bar{e}_i, \bar{e}_j)^2 + \cot^2 \theta \sum_{r=1}^s \sum_{i,j=1}^{m_2} (e_r^*(\ln f))^2 g(\bar{e}_i, \bar{e}_j)^2. \end{aligned} \tag{18}$$

Now, we consider both cases: (i) When $\xi \in \Gamma(\mathfrak{D})$, then we have

$$\begin{aligned} \|\sigma\|^2 &\geq 2m_2 \sum_{i=1}^{2p+1} (e_i(\ln f))^2 - 2m_2 (e_{2p+1}(\ln f))^2 + 2m_2 + m_2 \csc^2 \theta \sum_{r=1}^{m_3} (T e_r^*(\ln f))^2 \\ &+ m_2 \cot^2 \theta \sum_{r=1}^s (e_r^*(\ln f))^2 - m_2 \csc^2 \theta \sum_{r=s+1}^{m_3} (T e_r^*(\ln f))^2. \end{aligned}$$

Now, using gradient definition and Lemma 3.6 (i), we find

$$\begin{aligned} \|\sigma\|^2 &\geq 2m_2 \left(\|\nabla^T(\ln f)\|^2 + 1 \right) + m_2 \csc^2 \theta \|\nabla^\theta(\ln f)\|^2 + m_2 \cot^2 \theta \sum_{r=1}^s (e_r^*(\ln f))^2 \\ &\quad - m_2 \csc^2 \theta \sec^2 \theta \sum_{r=1}^s g(Te_r^*, \nabla^\theta(\ln f))^2 \\ &= 2m_2 \left(\|\nabla^T(\ln f)\|^2 + 1 \right) + m_2 \csc^2 \theta \|\nabla^\theta(\ln f)\|^2, \end{aligned}$$

which is inequality (i). If $\xi \in \Gamma(\mathfrak{D}^\theta)$, then from (17), we obtain

$$\begin{aligned} \|\sigma\|^2 &\geq 2m_2 \|\nabla^T(\ln f)\|^2 + m_2 \csc^2 \theta \sum_{r=1}^{m_3} g(e_r^*, \nabla^\theta(\ln f))^2 + m_2 \csc^2 \theta + m_2 \cot^2 \theta \sum_{r=1}^s (e_r^*(\ln f))^2 \\ &\quad - m_2 \csc^2 \theta \sum_{r=1}^s g(e_{r+s}^*, \nabla^\theta(\ln f))^2 - m_2 \csc^2 \theta \\ &= 2m_2 \|\nabla^T(\ln f)\|^2 + m_2 \csc^2 \theta \|\nabla^\theta(\ln f)\|^2 + m_2 \cot^2 \theta \sum_{r=1}^s (e_r^*(\ln f))^2 \\ &\quad - m_2 \csc^2 \theta \sec^2 \theta \sum_{r=1}^s g(Te_r^*, \nabla^\theta(\ln f))^2 \\ &= 2m_2 \|\nabla^T(\ln f)\|^2 + m_2 \cot^2 \theta \|\nabla^\theta(\ln f)\|^2, \end{aligned}$$

which is inequality (ii). For the equality case, From the leaving and vanishing terms in (15) and (16), we obtain

$$\sigma(\mathfrak{D}, \mathfrak{D}) = 0, \quad \sigma(\mathfrak{D}^\perp, \mathfrak{D}^\theta) = 0, \quad \sigma(\mathfrak{D}^\theta, \mathfrak{D}^\theta) = 0, \quad \sigma(\mathfrak{D}, \mathfrak{D}^\theta) = 0. \tag{19}$$

Then, from (19) with the Remark 3.4, we conclude that B is totally geodesic in \tilde{M} . Also, we find

$$\sigma(\mathfrak{D}, \mathfrak{D}^\perp) \subseteq \varphi \mathfrak{D}^\perp, \quad \sigma(\mathfrak{D}^\perp, \mathfrak{D}^\perp) \subseteq F\mathfrak{D}^\theta. \tag{20}$$

Thus, by Remark 3.4 with (19) and (20), we deduce that M_\perp is totally umbilical in \tilde{M} . Hence, the theorem is proved completely. \square

5. Special cases of Theorem 4.1

There are two special cases of Theorem 4.1:

1. If $\mathfrak{D}^\theta = \{0\}$ i.e., $\dim M_\theta = 0$ in a contact skew CR-warped product, then it reduces to contact CR-warped products of the form $M = M_T \times_f M_\perp$ studied in [18]. In this case, the statement of Theorem 4.1 will be: *Let $M = M_T \times_f M_\perp$ be a contact CR-warped product submanifold of a Sasakian manifold \tilde{M} such that ξ is tangent to M_T , where M_T and M_\perp are invariant and anti-invariant submanifolds of \tilde{M} with their real dimensions m_1, m_2 , respectively. Then we have:*

(i) *The squared norm of the second fundamental form σ satisfies*

$$\|\sigma\|^2 \geq 2m_2 \left(\|\nabla^T(\ln f)\|^2 + 1 \right).$$

where $\nabla^T(\ln f)$ is the gradient of $\ln f$ along M_T .

(ii) If the equality sign holds in above inequality, then M_T is totally geodesic and M_\perp is a totally umbilical in \tilde{M} .

Which is the main result of [18].

2. On the other hand, if $\mathfrak{D} = \{0\}$ in a contact skew CR-warped product, then it will change into a pseudo-slant warped product of the form $M = M_\theta \times_f M_\perp$ studied in [30]. In this case, Theorem 4.2 of [30] is a particular case of Theorem 4.1 as follows:

Corollary 5.1. (Theorem 4.2 of [30]) Let $M = M_\theta \times_f M_\perp$ be a mixed totally geodesic warped product submanifold of a Sasakian manifold \tilde{M} such that $\xi \in \Gamma(\mathfrak{D}^\theta)$, where M_θ is a proper slant submanifold and M_\perp is an m_2 -dimensional anti-invariant submanifold of \tilde{M} . Then we have:

(i) The squared norm of the second fundamental form of M satisfies

$$\|\sigma\|^2 \geq m_2 \cot^2 \theta \|\nabla^\theta(\ln f)\|^2$$

where $\nabla^\theta \ln f$ is the gradient of $\ln f$ along M_θ .

(ii) If the equality sign in (i) holds identically, then M_θ is totally geodesic in \tilde{M} and M_\perp is a totally umbilical submanifold of \tilde{M} .

6. Examples

We construct the following non-trivial examples of Riemannian products and contact skew CR-warped products in Euclidean spaces.

Example 6.1. Let M be a submanifold of Euclidean 9-space \mathbb{R}^9 with the cartesian coordinates $(x_1, \dots, x_4, y_1, \dots, y_4, t)$ and the almost contact structure defined in Example 3.1. If M is given by the equations

$$x_1 = u_1, y_1 = v_1, x_2 = u_2, y_2 = v_2, x_3 = \sin v_2, y_3 = \cos v_2, x_4 = \cos w^2, y_4 = \sin w^2, t = t,$$

then, the tangent space TM is spanned by X_1, X_2, X_3, X_4, X_5 and X_6 , where

$$X_1 = \frac{\partial}{\partial x_1}, X_2 = \frac{\partial}{\partial y_1}, X_3 = \frac{\partial}{\partial x_2}, X_4 = \cos v_2 \frac{\partial}{\partial x_3} + \frac{\partial}{\partial y_2} - \sin v_2 \frac{\partial}{\partial y_3},$$

$$X_5 = -2w \sin w^2 \frac{\partial}{\partial x_4} + 2w \cos w^2 \frac{\partial}{\partial y_4}, X_6 = \frac{\partial}{\partial t}.$$

Then, we find that $\mathfrak{D} = \text{Span}\{X_1, X_2\}$ is an invariant distribution and $\mathfrak{D}^\perp = \text{Span}\{X_5\}$ is an anti-invariant distribution. Moreover, $\mathfrak{D}^\theta = \text{Span}\{X_3, X_4\}$ is a slant distribution with slant angle $\theta = 45^\circ$. Hence, M is a skew CR-submanifold of \mathbb{R}^9 . Clearly, each distribution is integrable. If M_T, M_θ and M_\perp integral manifolds of $\mathfrak{D}, \mathfrak{D}^\theta$ and \mathfrak{D}^\perp , respectively, then M is a Riemannian product submanifold of $B = M_T \times M_\theta$ and M_\perp in \mathbb{R}^9 .

Example 6.2. Consider the Euclidean space \mathbb{R}^{13} with the cartesian coordinates $(x_1, \dots, x_6, y_1, \dots, y_6, z)$ and the almost contact structure

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial y_i}, \quad \varphi\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j}, \quad \varphi\left(\frac{\partial}{\partial z}\right) = 0, \quad 1 \leq i, j \leq 6.$$

It is clear that \mathbb{R}^{13} is an almost contact metric manifold with respect to the given structure and standard Euclidean metric tensor of \mathbb{R}^{13} . Let M be a submanifold of \mathbb{R}^{13} defined by the immersion $\psi : \mathbb{R}^7 \rightarrow \mathbb{R}^{13}$ as follows

$$\psi(u, v, w, r, s, t, z) = (u \cos(w+r), u \sin(w+r), v \cos(w-r), v \sin(w-r), k(u+v), s+t, z)$$

$$v \cos(w+r), v \sin(w+r), u \cos(w-r), u \sin(w-r), -k(u-v), -s+t, z)$$

for non-zero vectors and a scalar $k \neq 0$. Let the tangent space of M is spanned by the following vectors

$$\begin{aligned} X_1 &= \cos(w+r) \frac{\partial}{\partial x_1} + \sin(w+r) \frac{\partial}{\partial x_2} + k \frac{\partial}{\partial x_5} + \cos(w-r) \frac{\partial}{\partial y_3} + \sin(w-r) \frac{\partial}{\partial y_4} - k \frac{\partial}{\partial y_5}, \\ X_2 &= \cos(w-r) \frac{\partial}{\partial x_3} + \sin(w-r) \frac{\partial}{\partial x_4} + k \frac{\partial}{\partial x_5} + \cos(w+r) \frac{\partial}{\partial y_1} + \sin(w+r) \frac{\partial}{\partial y_2} + k \frac{\partial}{\partial y_5}, \\ X_3 &= -u \sin(w+r) \frac{\partial}{\partial x_1} + u \cos(w+r) \frac{\partial}{\partial x_2} - v \sin(w-r) \frac{\partial}{\partial x_3} + v \cos(w-r) \frac{\partial}{\partial x_4} \\ &\quad - v \sin(w+r) \frac{\partial}{\partial y_1} + v \cos(w+r) \frac{\partial}{\partial y_2} - u \sin(w-r) \frac{\partial}{\partial y_3} + u \cos(w-r) \frac{\partial}{\partial y_4}, \\ X_4 &= -u \sin(w+r) \frac{\partial}{\partial x_1} + u \cos(w+r) \frac{\partial}{\partial x_2} + v \sin(w-r) \frac{\partial}{\partial x_3} - v \cos(w-r) \frac{\partial}{\partial x_4} \\ &\quad - v \sin(w+r) \frac{\partial}{\partial y_1} + v \cos(w+r) \frac{\partial}{\partial y_2} + u \sin(w-r) \frac{\partial}{\partial y_3} - u \cos(w-r) \frac{\partial}{\partial y_4}, \\ X_5 &= \frac{\partial}{\partial x_6} - \frac{\partial}{\partial y_6}, \quad X_6 = \frac{\partial}{\partial x_6} + \frac{\partial}{\partial y_6}, \quad X_7 = \frac{\partial}{\partial z}. \end{aligned}$$

Then, the distribution $\mathfrak{D}^\perp = \text{Span}\{X_3, X_4\}$ is an anti-invariant distribution. It is easy to see that $\mathfrak{D} = \text{Span}\{X_5, X_6\}$ is an invariant distribution and $\mathfrak{D}^\theta = \text{Span}\{X_1, X_2\}$ is a slant distribution with slant angle $\theta = \cos^{-1}\left(\frac{k}{1+k^2}\right)$. Hence, M is a proper skew CR-submanifold of order 1 of \mathbb{R}^{13} such that $\xi = \frac{\partial}{\partial z}$ is tangent to M . It is easy to observe that each distribution is integrable. If we denote the integral manifolds of \mathfrak{D} , \mathfrak{D}^θ and \mathfrak{D}^\perp by M_T, M_θ and M_\perp , respectively, then the induced metric tensor g of M is given by

$$\begin{aligned} g &= 2(1+k^2)(du^2 + dv^2) + 2(ds^2 + dt^2) + dz^2 + 2(u^2 + v^2)(dw^2 + dr^2) \\ &= g_B + 2(u^2 + v^2)g_{M_\perp}. \end{aligned}$$

Hence, M is a skew CR-warped product submanifold of \mathbb{R}^{13} with the warping function $f = \sqrt{2(u^2 + v^2)}$ and the warped product metric g such that $(B, g_1) = (M_T \times M_\theta, g_1)$ with product metric $g_1 = 2(1+k^2)(du^2 + dv^2) + 2(ds^2 + dt^2) + dz^2$.

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