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Comments on the Paper "Global Optimal Approximate Solutions of Best Proximity Points"

Moosa Gabeleh[a], Pradip Ramesh Patle[b]

[a]*Department of Mathematics, Ayatollah Boroujerdi University, Boroujerd, Iran*
[b]*Department of Mathematics, ASET, Amity University Madhya Pradesh Gwalior (India) 474020*

Abstract. In a recent paper "Filomat, 35:5 (2021), 1555–1564" the authors proved some best proximity point theorems for classes of contractive pair of mappings as well as cyclic ψ-contractions under some sufficient conditions. In this article, we present some counterexamples of the existence results of contractive pair of mappings and present the corrected version of these results. We also show that the best proximity point theorem for cyclic ψ-contractions is a straightforward consequence of Boyd-Wong fixed point theorem by dropping an additional assumption.

1. Introduction and Preliminaries

Suppose (Ω, d) is a metric space. A mapping $T : A \subseteq \Omega \to \Omega$ is called *contraction* if

$$d(Tx, Ty) \leq kd(x, y),$$

for some $k \in (0, 1)$ and for all $x, y \in A$. We also say that T is *nonexpansive* provided that

$$d(Tx, Ty) \leq d(x, y),$$

for all $x, y \in A$.
The *Banach contraction principle* states that if A is a nonempty and closed subset of a complete metric space (Ω, d) and $T : A \to A$ is a contraction mapping, then T has a unique fixed point in A and for any $x_0 \in A$ if we define $x_n = T^n x_0$ for all $n \in \mathbb{N}$, then the sequence $\{x_n\}$ converges to the fixed point of T.
On the other hand, if Ω is a reflexive Banach space having normal structure, then every nonexpansive self-mapping defined on a nonempty, bounded, closed and convex subset of Ω has a fixed point. This fact is well-known as *Kirk's fixed point theorem* ([7]).

Let (Δ, Λ) be a pair of nonempty subsets of a metric space (Ω, d) and $\Gamma : \Delta \to \Lambda$ be a non-self mapping. A point $p \in \Delta$ is said to be a *best proximity point* for the mapping Γ provided that

$$d(p, \Gamma p) = \text{dist}(\Delta, \Lambda) := \inf\{d(x, y) : (x, y) \in \Delta \times \Lambda\}.$$

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Communicated by Vladimir Rakočević
Email addresses: Gabeleh@abru.ac.ir, gab.moo@gmail.com (Moosa Gabeleh), pradip.patle12@gmail.com, prpatle@gwa.amity.edu (Pradip Ramesh Patle)

We refer to [4–6] for some existence results of best proximity points. The set of all best proximity points of the non-self mapping Γ is denoted by $P_\Gamma(\Delta, \Lambda)$. The *proximal pair* of (Δ, Λ) is the pair (Δ^0, Λ^0) given by

$$\Delta^0 = \{\delta \in \Delta : d(\delta, \lambda) = \text{dist}(\Delta, \Lambda) \text{ for some } \lambda \in \Lambda\},$$

$$\Lambda^0 = \{\lambda \in \Lambda : d(\delta, \lambda) = \text{dist}(\Delta, \Lambda) \text{ for some } \delta \in \Delta\}.$$

Just recently in [3], the authors studied the existence of best proximity points for various classes of non-self mappings under some sufficient conditions. Before stating these existence results, we recall some basic notions of [3].

Definition 1.1. ([3]) *Let Δ and Λ be nonempty subsets of a metric space (Ω, d) and let $\Gamma : \Delta \to \Lambda$ and $\Upsilon : \Lambda \to \Delta$ be two non-self mappings. The pair (Γ, Υ) is said to be a contractive pair if*

$$d(\Gamma\delta, \Upsilon\lambda) \leq \alpha d(\delta, \lambda) + (1 - \alpha)\text{dist}(\Delta, \Lambda),$$

for some $\alpha \in (0, 1)$ and for all $\delta \in \Delta$ and $\lambda \in \Lambda$.

Definition 1.2. ([3]) *Let Δ and Λ be nonempty subsets of a metric space (Ω, d). A mapping $\Gamma : \Delta \cup \Lambda \to \Delta \cup \Lambda$ is called a cyclic ψ-contraction if it satisfies*

(i) $\Gamma(\Delta) \subseteq \Lambda,\ \Gamma(\Lambda) \subseteq \Delta$,
(ii) $d(\Gamma\delta, \Gamma\lambda) \leq \psi\Big(d(\delta, \lambda)\Big), \quad \forall \delta, \lambda \in \Delta \cup \Lambda$,

where $\psi : [0, \infty) \to [0, \infty)$ an upper semicontinuous from the right on $(0, \infty) - \{\text{dist}(\Delta, \Lambda)\}$ and satisfies $0 \leq \psi(t) < t$ for all $t > 0$ with $t \neq \text{dist}(\Delta, \Lambda)$.

Remark 1.3. *If in Definition 1.2 we have $\Delta = \Lambda$, then $\Gamma : \Delta \to \Delta$ is said to be a contraction self-mapping in the sense of Boyd-Wong ([1]). Boyd-Wong fixed point theorem states that if Δ is a closed subset of a complete metric space (Ω, d), then the Boyd-Wong contraction mapping $\Gamma : \Delta \to \Delta$ has a unique fixed point and for any element $\delta_0 \in \Delta$ the Picard's iteration sequence $\delta_n := \Gamma^n\delta_0$ converges to the fixed point of Γ.*

In what follows we present the main existence results of best proximity points for nonself-mappings.

Theorem 1.4. (Theorem 2.3 of [3]) *Let Δ and Λ be nonempty and closed subsets of a metric space (Ω, d) and (Γ, Υ) be a contractive pair. If both non-self mappings Γ and Υ are contraction, then there exists $(\delta, \lambda) \in \Delta \times \Lambda$ such that $d(\delta, \Gamma\delta) = d(\lambda, \Upsilon\lambda) = \text{dist}(\Delta, \Lambda)$.*

Theorem 1.5. (Theorem 2.7 of [3]) *Let Δ and Λ be nonempty subsets of a normed linear space Ω such that Δ^0 and Λ^0 are nonempty and convex. Also assume that $\Gamma : \Delta \to \Lambda$ and $\Upsilon : \Lambda \to \Delta$ are such that $\|\Gamma\delta - \Upsilon\lambda\| \leq \|\delta - \lambda\|$ for all $(\delta, \lambda) \in \Delta \times \Lambda$. If both non-self mappings Γ and Υ are nonexpansive, then there exists $(\delta, \lambda) \in \Delta \times \Lambda$ such that $\|\delta - \Gamma\delta\| = \|\lambda - \Upsilon\lambda\| = \text{dist}(\Delta, \Lambda)$.*

Theorem 1.6. (Theorem 2.9 of [3]) *Let Δ and Λ be nonempty subsets of a normed linear space Ω such that Δ is compact. Assume that $\Gamma : \Delta \to \Lambda$ and $\Upsilon : \Lambda \to \Delta$ are such that*

$$\|\Gamma\delta - \Upsilon\lambda\| < \|\delta - \lambda\|,$$

for all $(\delta, \lambda) \in (\Delta \times \Lambda) - (\Delta^0 \times \Lambda^0)$. If Γ is upper semicontinuous, then $P_\Gamma(\Delta, \Lambda)$ is a nonempty compact set.

Theorem 1.7. (Theorem 3.2 of [3]) *Let Δ and Λ be nonempty and closed subsets of a complete metric space (Ω, d) such that*

$$\text{diam}(\Delta) < \text{dist}(\Delta, \Lambda). \tag{1}$$

Suppose that $\Gamma : \Delta \cup \Lambda \to \Delta \cup \Lambda$ is a cyclic ψ-contraction. Then $P_\Gamma(\Delta, \Lambda) \neq \emptyset$. Further, if $\delta_0 \in \Delta$, and $\delta_{n+1} = \Gamma\delta_n$ for all $n \in \mathbb{N} \cup \{0\}$, then the sequence $\{\delta_{2n}\}$ converges to the best proximity point of Γ.

Our main result will show that Theorem 1.4, Theorem 1.5 and Theorem 1.6 are incorrect. Moreover, Theorem 1.7 is a particular cases of Boyd-Wong fixed point theorem without the condition (1).

2. Main results

The following simple example is a counterexample of Theorem 1.4.

Example 2.1. Consider $\Omega = [-1,1] - \{0\}$ with the usual metric. Let $\Delta = [-1,0)$ and $\Lambda = (0,1]$ and define $\Gamma : \Delta \to \Lambda$ and $\Upsilon : \Lambda \to \Delta$ with $\Gamma(\delta) = -\frac{\delta}{2}$ and $\Upsilon(\lambda) = -\frac{\lambda}{2}$. Clearly, Δ and Λ are closed with $\mathrm{dist}(\Delta, \Lambda) = 0$. Moreover, (Γ, Υ) is a contractive pair and both non-self mappings Γ and Υ are contraction, but Γ and Υ have no best proximity point which is a fixed point in this case.

It is remarkable to note that even if in Theorem 1.4 the completeness assumption of the metric space (Ω, d) is considered, the conclusion is again invalid. The next example illustrates this fact.

Example 2.2. Consider the Banach space ℓ_1 with the canonical basis $\{e_n\}$. Given $k \in (0, \frac{1}{2}]$, let

$$\Delta = \{(k + k^{2n})e_{2n} : n \in \mathbb{N}\}, \quad \Lambda = \{(k + k^{2m-1})e_{2m-1} : m \in \mathbb{N}\}.$$

Then Δ and Λ are closed. Also, for any $m, n \in \mathbb{N}$ we have

$$\|(k + k^{2n})e_{2n} - (k + k^{2m-1})e_{2m-1}\|_1 = 2k + k^{2n} + k^{2m-1} \xrightarrow{n,m\to\infty} 2k,$$

which implies that $\mathrm{dist}(\Delta, \Lambda) = 2k$. Define the non-self mappings $\Gamma : \Delta \to \Lambda$ and $\Upsilon : \Lambda \to \Delta$ as

$$\Gamma\Big((k + k^{2n})e_{2n}\Big) = (\frac{k}{2} + k^{2n+1})e_{2n+1}, \quad \Upsilon\Big((k + k^{2m-1})e_{2m-1}\Big) = (\frac{k}{2} + k^{2m})e_{2m}, \ \forall m, n \in \mathbb{N}.$$

For $\alpha := \frac{1}{2}$, we obtain the following observations about the mappings Γ and Υ.
♣ Γ is a contraction with the contractive constant α. Indeed, if $x, y \in \Delta$, then $x = (k+k^{2n})e_{2n}$ and $y = (k+k^{2m})e_{2m}$ for seme $n, m \in \mathbb{N}$. If $n = m$, the result follows. So, if $n \neq m$, then

$$\begin{aligned}
\|\Gamma x - \Gamma y\|_1 &= \|\Gamma\Big((k + k^{2n})e_{2n}\Big) - \Gamma\Big((k + k^{2m})e_{2m}\Big)\|_1 \\
&= \|(\frac{k}{2} + k^{2n+1})e_{2n+1} - (\frac{k}{2} + k^{2m+1}e_{2m+1})\|_1 \\
&= k + k^{2n+1} + k^{2m+1} = k + k\Big(k^{2n} + k^{2m}\Big) \\
&\leq k + \frac{1}{2}\Big(k^{2n} + k^{2m}\Big) = \frac{1}{2}\|(k + k^{2n})e_{2n} - (k + k^{2m})e_{2m}\|_1 \\
&= \alpha\|x - y\|_1.
\end{aligned}$$

♣ Υ is a contraction with the contractive constant α. In fact if $x, y \in \Lambda$, then $x = (k + k^{2n-1})e_{2n-1}$ and $y = (k + k^{2m-1})e_{2m-1}$ for seme $n, m \in \mathbb{N}$. If $n = m$, there is nothing to prove. Let $n \neq m$. Therefore,

$$\begin{aligned}
\|\Upsilon x - \Upsilon y\|_1 &= \|\Upsilon\Big((k + k^{2n-1})e_{2n-1}\Big) - \Upsilon\Big((k + k^{2m-1})e_{2m-1}\Big)\|_1 \\
&= \|(\frac{k}{2} + k^{2n})e_{2n} - (\frac{k}{2} + k^{2m}e_{2m})\|_1 \\
&= k + k^{2n} + k^{2m} = k + k\Big(k^{2n-1} + k^{2m-1}\Big) \\
&\leq k + \frac{1}{2}\Big(k^{2n-1} + k^{2m-1}\Big) = \frac{1}{2}\|(k + k^{2n-1})e_{2n-1} - (k + k^{2m-1})e_{2m-1}\|_1 \\
&= \alpha\|x - y\|_1.
\end{aligned}$$

♣ (Γ, Υ) is a contractive pair. To see this, let $x \in \Delta$ and $y \in \Lambda$. Then $x = (k + k^{2n})e_{2n}$ and $y = (k + k^{2m-1})e_{2m-1}$

for seme $n, m \in \mathbb{N}$. Thus

$$\begin{aligned}\|\Gamma x - \Upsilon y\|_1 &= \|\Gamma\Big((k+k^{2n})e_{2n}\Big) - \Upsilon\Big((k+k^{2m-1})e_{2m-1}\Big)\|_1\\ &= \|(\frac{k}{2}+k^{2n+1})e_{2n+1} - (\frac{k}{2}+k^{2m}e_{2m})\|_1\\ &= k + k^{2n+1} + k^{2m} = k + k\Big(k^{2n}+k^{2m-1}\Big)\\ &\leq 2k + \frac{1}{2}\Big(k^{2n}+k^{2m-1}\Big)\\ &= k + \frac{1}{2}\Big(k^{2n}+k^{2m-1}\Big) + k\\ &= \alpha\|x-y\|_1 + (1-\alpha)\mathrm{dist}(\Delta,\Lambda).\end{aligned}$$

Hence, all of the assumptions of Theorem 1.4 hold in the Banach space ℓ_1 but Δ^0 and Λ^0 are empty and so, there is no any best proximity point for Γ and Υ which implies that Theorem 1.4 is incorrect even in complete metric spaces.

Here, we present a revised version of Theorem 1.4 which is a direct consequence of the Banach contraction principle.

Theorem 2.1. *Let Δ and Λ be nonempty subsets of a complete metric space (Ω, d) and (Γ, Υ) be a contractive pair. If Δ^0 is nonempty, closed and both non-self mappings Γ and Υ are contraction, then Γ and Υ have a best proximity point.*

Proof. Let $\delta \in \Delta^0$. Then there exists an element $\lambda \in \Lambda$ such that $d(\delta, \lambda) = \mathrm{dist}(\Delta, \Lambda)$. In view of the fact that (Γ, Υ) be a contractive pair, there exists $\alpha \in (0, 1)$ such that

$$d(\Gamma\delta, \Upsilon\lambda) \leq \alpha d(\delta, \lambda) + (1-\alpha)\mathrm{dist}(\Delta, \Lambda) \leq d(\delta, \lambda) = \mathrm{dist}(\Delta, \Lambda),$$

which ensures that $\Gamma\delta \in \Lambda^0$, that is, $\Gamma(\Delta^0) \subseteq \Lambda^0$. Similarly, $\Upsilon(\Lambda^0) \subseteq \Delta^0$. Now consider the self-mapping $\Upsilon\Gamma : \Delta^0 \to \Delta^0$. Now if β and γ are contractive constants of the mappings Γ and Υ respectively, then for any $x, y \in \Delta^0$ we have

$$d\Big(\Upsilon\Gamma(x), \Upsilon\Gamma(y)\Big) \leq \gamma d\Big(\Gamma(x), \Gamma(y)\Big) \leq \gamma\beta d(x, y),$$

which implies that $\Upsilon\Gamma$ is a contraction self-mapping with the contractive constant $\gamma\beta \in (0, 1)$. Because of the fact that Δ^0 is closed and by the Banach contraction principle, we obtain $\Upsilon\Gamma$ has a unique fixed point, called $p \in \Delta^0$. We now have

$$d(\Gamma p, p) = d(\Gamma p, \Upsilon\Gamma p) \leq \alpha d(p, \Gamma p) + (1-\alpha)\mathrm{dist}(\Delta, \Lambda),$$

which concludes that $d(p, \Gamma p) = \mathrm{dist}(\Delta, \Lambda)$. Besides, $\Gamma p \in \Lambda^0$ and

$$d(\Gamma p, \Upsilon\Gamma p) = \mathrm{dist}(\Delta, \Lambda),$$

that is, Γp is a best proximity point of Υ and we are finished. □

In the next example, we show that Theorem 1.5 is incorrect.

Example 2.3. Consider the Banach space $\mathbb{R}^2$ with the norm $\|.\|_1$. Assume that

$$\Delta = \mathbb{R} \times \{0\}, \quad \Lambda = \mathbb{R} \times \{1\}.$$

Obviously,

$$\mathrm{dist}(\Delta, \Lambda) = 1, \quad \Delta^0 = \Delta,\ \Lambda^0 = \Lambda.$$

Now define the non-self mappings $\Gamma : \Delta \to \Lambda$ and $\Upsilon : \Lambda \to \Delta$ with

$$\Gamma(x, 0) = (x + 1, 1), \quad \Lambda(y, 1) = (y + 1, 0), \ \forall x, y \in \mathbb{R}.$$

We have the following observations:
♠ Γ is nonexpansive. Indeed, if $(x, 0), (y, 0) \in \Delta$, then

$$\|\Gamma(x, 0) - \Gamma(y, 0)\|_1 = \|(x + 1, 1) - (y + 1, 1)\|_1 = |x - y| = \|(x, 0) - (y, 0)\|_1.$$

Similarly, Υ is nonexpansive.
♠ For any $(x, 0) \in \Delta$ and $(y, 1) \in \Lambda$ we have

$$\|\Gamma(x, 0) - \Upsilon(y, 1)\|_1 = \|(x + 1, 1) - (y + 1, 0)\|_1 = |x - y| + 1 = \|(x, 0) - (y, 1)\|_1.$$

Hence, all of the conditions of Theorem 1.5 satisfy. Besides, for all $(x, 0) \in \Delta$ we have

$$\|(x, 0) - \Gamma(x, 0)\|_1 = \|(x, 0) - (x + 1, 1)\|_1 = 2 > 1 = \mathrm{dist}(\Delta, \Lambda).$$

Moreover, for any $(y, 1) \in \Lambda$ we have

$$\|(y, 1) - \Upsilon(y, 1)\|_1 = \|(y, 1) - (y + 1, 0)\|_1 = 2 > 1 = \mathrm{dist}(\Delta, \Lambda).$$

Therefore, the mappings Γ and Υ have no any best proximity point.

Remark 2.2. *In order to present the corrected version of Theorem 1.5, we purpose the following way:*
• *If the pair (Δ, Λ) is a nonempty, bounded, closed and convex pair of subsets of a uniformly convex Banach space Ω, then the result follows from Corollary 2.1 of [2]. In this case, we do not need the nonexpansiveness of the non-self mappings Γ and Υ.*

In what follows we show that Theorem 1.6 is not true.
At first step, we note that if $\Delta^0 = \emptyset$, then the existence of best proximity points cannot be guaranteed in Theorem 1.6. So we claim that under the assumptions on the considered pair (Δ, Λ) in Theorem 1.6, the set Δ^0 may be empty.

Example 2.4. Consider the Banach space ℓ_1 with the canonical basis $\{e_n\}$. Given $k \in (0, 1)$, let

$$\Delta = \{e_2\}, \quad \Lambda = \{(1 + k^{2m-1})e_{2m-1} : m \in \mathbb{N}\}.$$

Then Δ is compact. Also, for any $m \in \mathbb{N}$ we have

$$\mathrm{dist}(\Delta, \Lambda) \le \|e_2 - (1 + k^{2m-1})e_{2m-1}\|_1 = 2 + k^{2m-1} \longrightarrow^{m\to\infty} 2,$$

which implies that $\mathrm{dist}(\Delta, \Lambda) = 2$. Obviously, $\Delta^0 = \Lambda^0 = \emptyset$.

It is worth noticing that under the conditions of Theorem 1.6, if moreover, $\Delta^0 \neq \emptyset$, then the result may not obtain. Let us illustrate this fact with the following example.

Example 2.5. Consider $X = \mathbb{R}$ with the usual metric. Let $\Delta = \{1, 3, 5\}$ and $\Lambda = \{2, 4, 6\}$. Clearly, $\Delta^0 = \Delta$ and $\Lambda^0 = \Lambda$ and that both Δ and Λ are compact. Now define $\Gamma : \Delta \to \Lambda$ and $\Upsilon : \Lambda \to \Delta$ as follows:

$$\Gamma(1) = 4, \ \Gamma(3) = 6, \ \Gamma(5) = 2,$$

$$\Upsilon(2) = 5, \ \Upsilon(4) = 1, \ \Upsilon(6) = 3.$$

Then Γ is continuous. Since $(\Delta \times \Lambda) - (\Delta^0 \times \Lambda^0) = \emptyset$, so the condition

$$\|\Gamma\delta - \Upsilon\lambda\| < \|\delta - \lambda\|,$$

for all $(\delta, \lambda) \in (\Delta \times \Lambda) - (\Delta^0 \times \Lambda^0)$ is **meaningless**. Note that Γ and Υ does not have any best proximity point.

Remark 2.3. *We refer to Theorem 4 of [8] for an analogous result of Theorem 1.6 in the setting of metric spaces where the pair* (Δ, Λ) *satisfies a geometric property, called property UC.*

Finally, we improve Theorem 1.6 in the aspects of both statement and proof. Indeed, we show that Theorem 1.7 is a particular case of Boyd-Wong fixed point theorem.

Theorem 2.4. *Let* Δ *and* Λ *be nonempty and closed subsets of a complete metric space* (Ω, d). *Suppose that* $\Gamma : \Delta \cup \Lambda \to \Delta \cup \Lambda$ *is a cyclic* ψ*-contraction. Then* $P_\Gamma(\Delta, \Lambda) \neq \emptyset$. *Further, if* $\delta_0 \in \Delta$, *and* $\delta_{n+1} = \Gamma\delta_n$ *for all* $n \in \mathbb{N} \cup \{0\}$, *then the sequence* $\{\delta_{2n}\}$ *converges to the best proximity point of* Γ.

Proof. Since Γ is cyclic, we conclude that Γ^2 is a self-mapping on Δ. Besides, from the condition (*ii*) of Definition 1.2, for all $x, y \in \Delta$ we have

$$\begin{aligned} d(\Gamma^2 x, \Gamma^2 y) &\leq \psi\Big(d(\Gamma x, \Gamma y)\Big) \\ &\leq \psi^2\Big(d(x, y)\Big) \leq \psi\Big((d(x, y)\Big). \end{aligned}$$

This implies that Γ^2 is a contraction self-mapping in the sense of Boyd-Wong on the closed subset Δ of the complete metric space (Ω, d). Hence it has a unique fixed point, called $p \in \Delta$. Thus $\Gamma^2 p = p$. Also, for any $\delta_0 \in \Delta$ the sequence $(\Gamma^2)^n\delta_0 = \Gamma^{2n}\delta_0$ converges to the point p. Now set $r := d(p, \Gamma p)$. If $r > \mathrm{dist}(\Delta, \Lambda)$, then by the assumption on the control function ψ we have $\psi(r) < r$. Besides,

$$\begin{aligned} r &= d(p, \Gamma p) = d(\Gamma^2 p, \Gamma p) \\ &\leq \psi\Big(d(p, \Gamma p)\Big) \\ &= \psi(r), \end{aligned}$$

which is a contradiction and so we must have $d(p, \Gamma p) = \mathrm{dist}(\Delta, \Lambda)$, that is, $p \in P_\Gamma(\Delta, \Lambda)$ and this completes the proof of theorem. □

Remark 2.5. *It is worth noticing that we have dropped the condition of* $\mathrm{diam}(\Delta) < \mathrm{dist}(\Delta, \Lambda)$ *in Theorem 2.4 where as it was considered in Theorem 1.7.*

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