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Algorithmic and Analytical Approach of Solutions of a System of Generalized Multi-Valued Nonlinear Variational Inclusions

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Abstract. The main contributions of this paper is twofold. First, our primary concern is to suggest a new iterative algorithm using the P-η-proximal-point mapping technique and Nadler's technique for finding the approximate solutions of a system of generalized multi-valued nonlinear variational-like inclusions. Under some appropriate conditions imposed on the parameters and mappings involved in the system of generalized multi-valued nonlinear variational-like inclusions, the strong convergence of the sequences generated by our proposed iterative algorithm to a solution of the aforesaid system is proved. Second, the $H(.,.)$-η-cocoercive mapping considered in [R. Ahmad, M. Dilshad, M. Akram, Resolvent operator technique for solving a system of generalized variational-like inclusions in Banach sapces, Filomat 26(5)(2012) 897–908] is investigated and analyzed, and the fact that under the assumptions imposed on $H(.,.)$-η-cocoercive mapping, every $H(.,.)$-η-cocoercive mapping is P-η-accretive and is not a new one is pointed out. At the same time, some important comments on $H(.,.)$-η-cocoercive mapping and the results given in the above-mentioned paper are stated.

1. Introduction

The study of variational inequalities has a long history and interest in these types of inequalities is caused by their wide applications in solving a large variety of problems arising in many diverse fields of pure and applied science, such as mechanics, economics, engineering science, physics, elasticity, game theory, optimization and control, and so forth. For this reason, the theory of variational inequalities has always been an important subject as it evolved through the last decades, and the mathematical literature dedicated to this is growing rapidly. In the course of the past few decades, because of their extraordinary utility and broad applicability in many branches of sciences, variational inequalities have received a lot of attention and many interesting generalizations of them are appeared in the literature. For a detailed description of these generalizations along with relevant commentaries, the reader is referred to [4–7, 9, 10, 14, 20] and the references therein. Without doubt, among the generalizations, variational inclusions are the most important and well known ones, and in the last two decades the study of various types of variational inclusion problems and related optimization problems has become a rapidly growing area of research, see, for example, [1, 3, 8, 11, 12, 15–19, 24, 26–28, 32, 33, 35–37, 39] and the references contained therein. With the

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purpose of constructing iterative algorithms for solving various kinds of variational inequality problems and other related optimization problems in the setting of different spaces, in the past several decades, many interesting methods are designed and planned. Among the methods existing in the literature, the proximal-point mapping method (resolvent operator technique) as a useful and significant generalization of projection method is of interest and importance. For references in this regard and some detailed information, we refer the interested reader to [1, 3, 15–19, 24, 26, 27, 29, 33, 34, 36, 37, 39] and the references given therein.

In the last two decades, the notions of monotone, maximal monotone, accretive and m-accretive operators, which the beginning of the study and formulating of them comes back to the sixties, have been developed and generalized in different contexts. In 2001, Huang and Fang [24] succeeded to introduce the concept of maximal η-monotone operator as a generalization of maximal monotone operator. The same authors [25] introduced the notion of generalized m-accretive (also referred to as m-η-accretive or η-m-accretive [12]) mapping as a generalization of maximal η-monotone operators and m-accretive mappings. Subsequently, another successfully efforts in this direction led to the emergence of several other extensions of maximal monotone operators and m-accretive mappings which for example one can refer to H-monotone operators [16], H-accretive mappings [15] and (H, η)-monotone operators [19]. With the goal of defining and the introduction of a wider class of accretive mappings as a unifying framework for the generalized monotone and generalized accretive operators existing in the literature, the efforts in this direction have been continued and Kazmi and Khan [27], and Peng and Zhu [33] were the first, independently, to introduce and study the notion of P-η-accretive mapping in a Banach space setting. They defined the P-η-proximal-point mapping associated with a P-η-accretive mapping and gave some properties concerning it. The systems of variational inclusions involving P-η-accretive mappings are considered in [27, 33] and the existence of a unique solution for the above-mentioned systems of variational inclusions is proved under some suitable conditions. By using the P-η-proximal-point mapping technique, they proposed Mann-type iterative algorithms for finding the approximate solution of the aforesaid systems of variational inclusions. In the meanwhile, they studied the convergence analysis of the sequences generated by the Mann-type iterative algorithms proposed in [27, 33].

Recently, Ahmad et al. [3] introduced and studied another class of generalized accretive mappings, the so-called $H(.,.)$-η-cocoercive mappings as a generalization of P-η-accretive and $H(.,.)$-accretive mappings. They used the resolvent operator associated with an $H(.,.)$-η-cocoercive operator to suggest an iterative algorithm for solving a system of generalized variational-like inclusions in q-uniformly smooth Banach spaces. Moreover, they proved the strong convergence of the sequences generated by the proposed iterative algorithm to a solution of the above mentioned system.

The paper is structured as follows. Section 2 provides the basic definitions and preliminaries concerning P-η-accretive mappings. In Sect. 3, a new system of generalized multi-valued nonlinear variational inclusions (in short, SGMNVI) is considered and its equivalence with a fixed point problem is proved under some appropriate conditions. The obtained equivalence and Nadler's technique are employed to construct a new iterative algorithm for finding the approximate solution of the SGMNVI. We study the convergence analysis of the sequences generated by our proposed iterative algorithm under some imposed conditions on the parameters and mappings involved in the SGMNVI. In the final section, the notion of $H(.,.)$-η-cocoercive operator introduced and studied by Ahmad et al. [3] is investigated and analyzed. The fact that contrary to the claim of the authors in [3], under the conditions imposed on it, every $H(.,.)$-η-cocoercive operator is actually a P-η-accretive mapping and is not a new one is pointed out. At the same time, we give some important comments on $H(.,.)$-η-cocoercive operators and with the help of them we discuss the results appeared in [1].

2. Notation, basic definitions and fundamental properties

In what follows, unless otherwise stated, we always let X be a real Banach space with a norm $\|.\|$, d be the metric induced by the norm $\|.\|$, X^* be the topological dual space of X, $\langle .,. \rangle$ be the dual pair between X and X^*, and 2^X (resp. $CB(X)$) denote the family of all the nonempty (resp. nonempty closed and bounded) subsets of X. Further, let $D(.,.)$ be the Hausdorff metric of $CB(X)$ defined by

$$D(A,B)=\max\{\sup_{x\in A}\inf_{y\in B}\|x-y\|,\sup_{y\in B}\inf_{x\in A}\|x-y\|\},\quad \forall A,B\in CB(X).$$

For a given multi-valued mapping $M: X\to 2^X$,

(i) the set Range(M) defined by

$$\text{Range}(M)=\{y\in X:\exists x\in X:(x,y)\in M\}=\bigcup_{x\in X}M(x)$$

is called the range of M;

(ii) the set Graph(M) defined by

$$\text{Graph}(M)=\{(x,u)\in X\times X: u\in M(x)\},$$

is called the graph of M.

For a Banach space X, the unit sphere of X, denoted by S_X, is the set of all elements of X having norm 1. Recall that a Banach space X is strictly convex if for each x and y in S_X such that $x\neq y$ and each λ in $(0,1)$, $\|\lambda x+(1-\lambda)y\|<1$, i.e., S_X is strictly convex. As a consequence of this definition, the condition that for x and y in S_X such that $x\neq y$, $2-\|x+y\|>0$ is equivalent to X being strictly convex and provides us a characterization of strict convexity. X is said to be smooth if for every $x\in S_X$ there exits a unique x^* in X^* such that $\|x^*\|=\langle x^*,x\rangle=1$. It is well known that X is smooth if X^* is strictly convex, and that X is strictly convex if X^* is smooth. A Banach space X is uniformly convex if for each ε in $(0,2]$, $2\delta_X(\varepsilon)=\inf\{2-\|x+y\|: x,y\in S_X,\|x-y\|\geq\varepsilon\}$ is positive. It is said to be uniformly smooth whenever given $\varepsilon>0$ there exists $\delta>0$ such that for all $x\in S_X$ and $y\in X$ with $\|y\|\leq\delta$, then $\|x+y\|+\|x-y\|<2+\varepsilon\|y\|$.

The functions $\delta_X:[0,2]\to[0,1]$ and $\rho_X:\mathbb{R}^+\to\mathbb{R}^+$ given by

$$\delta_X(\varepsilon):=\inf\{1-\frac{1}{2}\|x+y\|: x,y\in S_X,\|x-y\|\geq 2\varepsilon\}$$

and

$$\rho_X(\tau):=\sup\{\frac{\|x+\tau y\|+\|x-\tau y\|}{2}-1: x,y\in S_X\}$$

are respectively called the modulus of convexity and smoothness of X. In the light of the definitions of the functions δ_X and ρ_X, a Banach space X is

(i) uniformly convex if and only if δ_X is strictly positive for every $\varepsilon\in(0,2]$;

(ii) uniformly smooth if and only if $\lim\limits_{\tau\to 0}\frac{\rho_X(\tau)}{\tau}=0$.

It is worthwhile to stress that in the definitions of $\delta_X(\varepsilon)$ and $\rho_X(\tau)$, one can as well take the infimum and supremum over all vectors $x,y\in X$ with $\|x\|,\|y\|\leq 1$.

A Banach space X is uniformly convex (resp. uniformly smooth) if and only if X^* is uniformly smooth (resp. uniformly convex). The spaces l^p, L^p and W^p_m, $1<p<\infty$, $m\in\mathbb{N}$, are uniformly convex as well as uniformly smooth, see [13, 22, 30]. In the meanwhile, the modulus of convexity and smoothness of a Hilbert space and the spaces l^p, L^p and W^p_m, $1<p<\infty$, $m\in\mathbb{N}$, can be found in [13, 22, 30].

For a real constant $q>1$, a mapping $J_q: X\to 2^{X^*}$ satisfying the condition

$$J_q(x)=\{x^*\in X^*:\langle x,x^*\rangle=\|x\|^q,\|x^*\|=\|x\|^{q-1}\},\quad \forall x\in X,$$

is called the *generalized duality mapping* of X. In particular, J_2 is the usual normalized duality mapping. It is known that, in general, $J_q(x)=\|x\|^{q-2}J_2(x)$, for all $x\neq 0$ and J_q is single-valued if X^* is strictly convex. If X is a Hilbert space, then J_2 becomes the identity mapping on X.

A Banach space X is uniformly convex (resp., uniformly smooth) if and only if the dual X^* is uniformly smooth (resp., uniformly convex). Note that J_q is single-valued if X is uniformly smooth.

For a real constant $q > 1$, X is called *q-uniformly smooth* if there exists a constant $C > 0$ such that $\rho_X(\tau) \leq C\tau^q$, for all $\tau \in \mathbb{R}^+$. It is well known that (see e.g. [38]) L_q (or l_q) is q-uniformly smooth for $1 \leq q \leq 2$ and is 2-uniformly smooth if $q > 2$.

In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [38] proved the following result.

Lemma 2.1. *Let X be a real uniformly smooth Banach space. For a real constant $q > 1$, X is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in X$,*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

We also recall the following concepts and some known results which shall be used in the sequel.

Definition 2.2. *Let X be a real q-uniformly smooth Banach space and let $T : X \to X$ and $\eta : X \times X \to X$ be the mappings. Then T is said to be*

(i) *η-accretive if*

$$\langle T(x) - T(y), J_q(\eta(x, y))\rangle \geq 0, \quad \forall x, y \in X;$$

(ii) *strictly η-accretive if T is η-accretive and equality holds if and only if $x = y$;*

(iii) *r-strongly η-accretive if there exists a constant $r > 0$ such that*

$$\langle T(x) - T(y), J_q(\eta(x, y))\rangle \geq r\|x - y\|^q, \quad \forall x, y \in X;$$

(iv) *η-cocoercive with constant k if there exists a constant $k > 0$ such that*

$$\langle T(x) - T(y), J_q(\eta(x, y))\rangle \geq k\|T(x) - T(y)\|^q, \quad \forall x, y \in X;$$

(v) *γ-relaxed η-cocoercive (as referred to as η-relaxed cocoercive with constant γ, see, for example [3, Definition 2.2(ii)]) if there exists a constant $\gamma > 0$ such that*

$$\langle T(x) - T(y), J_q(\eta(x, y))\rangle \geq -\gamma\|T(x) - T(y)\|^q, \quad \forall x, y \in X;$$

(vi) *α-expansive if there exists a constant $\alpha > 0$ such that*

$$\|T(x) - T(y)\| \geq \alpha\|T(x) - T(y)\|, \quad \forall x, y \in X;$$

(vii) *β-lipschitz continuous if there exists a constant $\beta > 0$ such that*

$$\|T(x) - T(y)\| \leq \beta\|x - y\|, \quad \forall x, y \in X.$$

Definition 2.3. [15, Definition 1.2] *Let X be a real q-uniformly smooth Banach space, $P : X \to X$ be a single-valued mapping and $M : X \to 2^X$ be a multi-valued mapping. M is said to be*

(i) *accretive if*

$$\langle u - v, J_q(x - y)\rangle \geq 0, \quad \forall (x, u), (y, v) \in \text{Graph}(M);$$

(ii) *m-accretive if M is accretive and $(I + \lambda M)(X) = X$ holds for all $\lambda > 0$, where I is the identity mapping on X;*

(iii) *P-accretive if M is accretive and $(P + \lambda M)(X) = X$ holds for every $\lambda > 0$.*

Chidume et al. [12] defined a class of η-accretive mappings the so-called m-η-accretive (also referred to as generalized m-accretive [25]) mappings as a generalization of the class of m-accretive mappings as follows.

Definition 2.4. [12] *Let X be a real q-uniformly smooth Banach space, $\eta : X \times X \to X$ be a vector-valued mapping. The multi-valued mapping $M : X \to 2^X$ is said to be*

(i) *η-accretive if*

$$\langle u - v, J_q(\eta(x, y))\rangle \geq 0, \quad \forall (x, u), (y, v) \in \text{Graph}(M);$$

(ii) *m-η-accretive if M is η-accretive and $(I + \lambda M)(X) = X$ holds for all $\lambda > 0$, where I is the identity mapping on X.*

We note that M is an m-η-accretive mapping if and only if M is η-accretive and there is no other η-accretive mapping whose graph contains strictly Graph(M). The m-η-accretivity is to be understood in terms of inclusion of graphs. If $M : X \to 2^X$ is an m-η-accretive mapping, then adding anything to its graph so as to obtain the graph of a new multi-valued mapping, destroys the η-accretivity. If fact, the extended mapping is no longer η-accretive. In other words, for every pair $(x, u) \in X \times X \setminus \text{Graph}(M)$ there exists $(y, v) \in \text{Graph}(M)$ such that $\langle u - v, J_q(\eta(x, y))\rangle < 0$. Taking into account of the above-mentioned arguments, a necessary and sufficient condition for a multi-valued mapping $M : X \to 2^X$ to be m-η-accretive is that the property

$$\langle u - v, J_q(\eta(x, y))\rangle \geq 0, \quad \forall (y, v) \in \text{Graph}(M)$$

is equivalent to $u \in M(x)$. The above characterization of m-η-accretive mappings provides a useful and manageable way for recognizing that an element u belongs to $M(x)$.

Kazmi and Khan [27] and subsequently Peng and Zhu [33] introduced and studied another class of generalized accretive operators the so-called P-η-accretive (also referred to as (H, η)-accretive) mappings as an extension of m-η-accretive mappings as follows.

Definition 2.5. [27, 33] *Let X be a real q-uniformly smooth Banach space, $P : X \to X$ and $\eta : X \times X \to X$ be two single-valued mappings and $M : X \to 2^X$ be a multi-valued mapping. M is said to be P-η-accretive if M is η-accretive and $(P + \lambda M)(X) = X$ holds for every constant $\lambda > 0$.*

The following example illustrates that for given mappings $\eta : X \times X \to X$ and $P : X \to X$, a P-η-accretive mapping may be neither P-accretive nor m-η-accretive.

Example 2.6. Let $m, n \in \mathbb{N}$ be arbitrary but fixed and let $M_{m\times n}(\mathbb{F})$ be the space of all $m \times n$ matrices with real or complex entries. Then

$$M_{m\times n}(\mathbb{F}) = \{A = \begin{pmatrix} a_{ij} \end{pmatrix} | a_{ij} \in \mathbb{F}, i = 1, 2, \dots, m; j = 1, 2, \dots, n; \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}\}$$

is a 2-uniformly smooth Banach space with respect to the Hilbert-Schmidt norm

$$\|A\| = \Big(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\Big)^{\frac{1}{2}}, \quad \forall A \in M_{m\times n}(\mathbb{F})$$

induced by the Hilbert-Schmidt inner product

$$\langle A, B\rangle = tr(A^*B) = \sum_{i=1}^{m}\sum_{j=1}^{n} \bar{a}_{ij}b_{ij}, \quad \forall A, B \in M_{m\times n}(\mathbb{F}),$$

where tr denotes the trace, that is, the sum of the diagonal entries, A^* denotes the Hermitian conjugate (or adjoint) of the matrix A, that is, $A^* = \overline{A^t}$, the complex conjugate of the transpose A, the bar denotes complex conjugation and superscript denotes the transpose of the entries. For $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$, let E_{ij} be the $m \times n$ such that (i, j)-entry equals to one and all other entries equal to zero. Then the set $\{E_{ij} : i = 1, 2, \dots, m; j = 1, 2, \dots, n\}$ is called the set matrix-units and form a basis of $M_{m\times n}(\mathbb{F})$. Any matrix $A = \begin{pmatrix} a_{ij} \end{pmatrix} \in M_{m\times n}(\mathbb{F})$ can be written as $A = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}E_{ij}$. If $m = n$, then $\{E_{ij} : i, j = 1, 2, \dots, n\}$ is the set of matrix units of the space $M_{n\times n}(\mathbb{F}) = M_n(\mathbb{F})$, that is, the space of all $n \times n$ real or complex matrices, and for any $A = \begin{pmatrix} a_{ij} \end{pmatrix} \in M_{m\times n}(\mathbb{F})$, we have $A = \sum_{i,j=1}^{n} a_{ij}E_{ij}$. Furthermore, $I_n = \sum_{i=1}^{n} E_{ii}$, where for each $k \in \{1, 2, \dots, n\}$,

$E_{kk} = \begin{pmatrix} e_{ij} \end{pmatrix}$ is an $n \times n$ matrix with the entry $e_{kk} = 1$ and 0's everywhere else, is a representation of the identity matrix I_n in $M_n(\mathbb{F})$. Indeed, $I_n = \begin{pmatrix} \delta_{ij} \end{pmatrix}$ and

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

is the Kronecker delta. Let us denote by $D_n(\mathbb{R})$ the space of all diagonal $n \times n$ matrices with real entries, that is, the (i, j)-entry is an arbitrary real number if $i = j$, and is zero if $i \neq j$. Then

$$D_n(\mathbb{R}) = \{A = \begin{pmatrix} a_{ij} \end{pmatrix} | a_{ij} \in \mathbb{R}, a_{ij} = 0 \text{ if } i \neq j; i, j = 1, 2, \ldots, n\}$$

is a subspace of $M_{n\times n}(\mathbb{R}) = M_n(\mathbb{R})$ with respect to the operations of addition and scalar multiplication defined on $M_n(\mathbb{R})$, and the Hilbert-Schmidt inner product on $D_n(\mathbb{R})$, and the Hilbert-Schmidt norm induced by it become as $\langle A, B\rangle = tr(A^*B) = tr(AB)$ and $\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{tr(AA)} = \Big(\sum\limits_{i=1}^{n} a_{ii}^2\Big)^{\frac{1}{2}}$, respectively. Let the mappings $M : D_n(\mathbb{R}) \to 2^{D_n(\mathbb{R})}$, $\eta : D_n(\mathbb{R}) \times D_n(\mathbb{R}) \to D_n(\mathbb{R})$ and $P : D_n(\mathbb{R}) \to D_n(\mathbb{R})$ be defined, respectively, by

$$M(A) = \begin{cases} \{E_{ii} - E_{kk} : i = 1, 2, \ldots, n; i \neq k\}, & A = E_{kk}, \\ -A + E_{kk}, & A \neq E_{kk}, \end{cases}$$

$$\eta(A, B) = \begin{cases} C, & A, B \neq E_{kk}, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

and $P(A) = \beta A + \gamma E_{kk}$, for all $A = \begin{pmatrix} a_{ij} \end{pmatrix}, B = \begin{pmatrix} b_{ij} \end{pmatrix} \in D_n(\mathbb{R})$, where $C = \begin{pmatrix} c_{ij} \end{pmatrix}$ is an $n \times n$ matrix with the entries

$$c_{ij} = \begin{cases} \alpha_i e^{l_i(a_{ii}+b_{ii})}(b_{ii}^{q_i} - a_{ii}^{q_i}), & i = j, \\ 0, & i \neq j, \end{cases}$$

where for $i = 1, 2, \ldots, n$, $\alpha_i, l_i (i = 1, 2, \ldots, n), \beta, \gamma \in \mathbb{R}$ are arbitrary constants such that $\beta < 0 < \alpha_i$ for each $i \in \{1, 2, \ldots, n\}$, q_i $(i = 1, 2, \ldots, n)$ are arbitrary but fixed odd natural numbers, $\mathbf{0}$ is the zero vector (the zero matrix) of the space $D_n(\mathbb{R})$, and $k \in \{1, 2, \ldots, n\}$ is an arbitrary but fixed natural number. Then for any $A = \begin{pmatrix} a_{ij} \end{pmatrix}, B = \begin{pmatrix} b_{ij} \end{pmatrix} \in D_n(\mathbb{R})$, $A \neq B \neq E_{kk}$, we have

$$\langle M(A) - M(B), J_2(A - B)\rangle = \langle B - A, A - B\rangle = -\|A - B\|^2 = -\sum_{i=1}^{n}(a_{ii} - b_{ii})^2 < 0,$$

which means that M is not accretive and so it is not a P-accretive mapping.

For any given $A = \begin{pmatrix} a_{ij} \end{pmatrix}, B = \begin{pmatrix} b_{ij} \end{pmatrix} \in D_n(\mathbb{R})$, $A \neq B \neq E_{kk}$, we obtain

$$\begin{aligned} \langle M(A) - M(B), J_2(\eta(A, B))\rangle &= \langle M(A) - M(B), \eta(A, B)\rangle \\ &= tr(\begin{pmatrix} b_{ij} - a_{ij} \end{pmatrix}\begin{pmatrix} c_{ij} \end{pmatrix}) \\ &= \sum_{i=1}^{n} \alpha_i (b_{ii} - a_{ii})^2 e^{l_i(a_{ii}+b_{ii})} \sum_{s=1}^{q_i} b_{ii}^{q_i-s} a_{ii}^{s-1}. \end{aligned}$$

Since for each $i \in \{1, 2, \ldots, n\}$, q_i is an odd natural number, it follows that $\sum\limits_{s=1}^{q_i} b_{ii}^{q_i-s} a_{ii}^{s-1} \geq 0$ for each $i \in \{1, 2, \ldots, n\}$. Thus, the preceding relation implies that

$$\langle M(A) - M(B), J_2(\eta(A, B))\rangle \geq 0, \quad \forall A, B \in D_n(\mathbb{R}), A \neq B \neq E_{kk}.$$

For each of the cases when $A \neq B = E_{kk}$, $B \neq A = E_{kk}$ and $A = B = E_{kk}$, thanks to the fact that $\eta(A, B) = \mathbf{0}$, we infer that

$$\langle u - v, J_2(\eta(A, B))\rangle = 0, \quad \forall u \in M(A), v \in M(B).$$

Therefore, M is an η-accretive mapping. Taking into account that for any $E_{kk} \neq A \in D_n(\mathbb{R})$,

$$\|(I + M)(A)\|^2 = \|A - A + E_{kk}\|^2 = \|E_{kk}\|^2 = \langle E_{kk}, E_{kk}\rangle = tr(E_{kk}E_{kk}) = \sum_{i=1}^{n} e_{ii}^2 = e_{kk}^2 = 1 > 0$$

and $(I + M)(E_{kk}) = \{E_{ii} : i = 1, 2, \dots, n; i \neq k\}$, where I is the identity mapping on $X = D_n(\mathbb{R})$, we deduce that $\mathbf{0} \notin (I + M)(D_n(\mathbb{R}))$. This fact ensures that $I + M$ is not surjective, and so M is not an m-η-accretive mapping. For any given constant $\lambda > 0$ and $A \in D_n(\mathbb{R})$, by taking $Q = \frac{1}{\beta-\lambda}A + \frac{\gamma+\lambda}{\lambda-\beta}E_{kk}$ ($\lambda \neq \beta$ because $\beta < 0$), it follows that

$$\begin{aligned}(P + \lambda M)(Q) = (P + \lambda M)(\frac{1}{\beta - \lambda}A + \frac{\gamma + \lambda}{\lambda - \beta}E_{kk}) &= \frac{\beta}{\beta - \lambda}A + \frac{\beta(\gamma + \lambda)}{\lambda - \beta}E_{kk} + \gamma E_{kk} \\ &- \frac{\lambda}{\beta - \lambda}A - \frac{\lambda(\gamma + \lambda)}{\lambda - \beta}E_{kk} + \lambda E_{kk} = A.\end{aligned}$$

Thereby, the mapping $P + \lambda M$ is surjective for any real constant $\lambda > 0$ and so M is a P-η-accretive mapping.

The following example shows that for given mappings $P : X \to X$ and $\eta : X \times X \to X$, an m-η-accretive mapping need not be P-η-accretive.

Example 2.7. Suppose that the space $D_n(\mathbb{R})$ is the same as in Example 2.6 and let the mappings $P, M : D_n(\mathbb{R}) \to D_n(\mathbb{R})$ and $\eta : D_n(\mathbb{R}) \times D_n(\mathbb{R}) \to D_n(\mathbb{R})$ be defined, respectively, by $P(A) = P(\begin{pmatrix} a_{ij} \end{pmatrix}) = \begin{pmatrix} a'_{ij} \end{pmatrix}$, $M(A) = M(\begin{pmatrix} a_{ij} \end{pmatrix}) = \begin{pmatrix} a''_{ij} \end{pmatrix}$ and $\eta(A, B) = \eta(\begin{pmatrix} a_{ij} \end{pmatrix}, \begin{pmatrix} b_{ij} \end{pmatrix}) = \begin{pmatrix} c_{ij} \end{pmatrix}$ for all $A = \begin{pmatrix} a_{ij} \end{pmatrix}, B = \begin{pmatrix} b_{ij} \end{pmatrix} \in D_n(\mathbb{R})$, where for each $i, j \in \{1, 2, \dots, n\}$,

$$a'_{ij} = \begin{cases} a_{ii}^2, & i = j, \\ 0, & i \neq j, \end{cases}$$

$$a''_{ij} = \begin{cases} \alpha_i a_{ii}, & i = j, \\ 0, & i \neq j, \end{cases}$$

and

$$c_{ij} = \begin{cases} \beta_i e^{k_i(a_{ii}+b_{ii})}(a_{ii}^{q_i} - b_{ii}^{q_i}), & i = j, \\ 0, & i \neq j, \end{cases}$$

$k_i \in \mathbb{R}$ and $\alpha_i, \beta_i > 0$ are arbitrary but fixed, and q_i are arbitrary but fixed odd natural numbers. Then, for any $A = \begin{pmatrix} a_{ij} \end{pmatrix}, B = \begin{pmatrix} b_{ij} \end{pmatrix} \in D_n(\mathbb{R})$, we get

$$\begin{aligned}\langle M(A) - M(B), J_2(\eta(A, B))\rangle &= \langle M(A) - M(B), \eta(A, B)\rangle \\ &= tr\Big(\begin{pmatrix} a''_{ij} - b''_{ij} \end{pmatrix}\begin{pmatrix} c_{ij} \end{pmatrix}\Big) \\ &= tr\Big(\begin{pmatrix} \widetilde{a}_{ij} \end{pmatrix}\Big) \\ &= \sum_{i=1}^{n} \alpha_i\beta_i(a_{ii} - b_{ii})^2 e^{k_i(a_{ii}+b_{ii})} \sum_{l=1}^{q_i} a_{ii}^{q_i-l} b_{ii}^{l-1},\end{aligned} \tag{1}$$

where for each $i, j \in \{1, 2, \dots, n\}$,

$$\widetilde{a}_{ij} = \begin{cases} \alpha_i\beta_i(a_{ii} - b_{ii})e^{k_i(a_{ii}+b_{ii})}(a_{ii}^{q_i} - b_{ii}^{q_i}), & i = j, \\ 0, & i \neq j. \end{cases}$$

Since for each $i \in \{1,2,\ldots,n\}$, q_i is an odd natural number, it can be easily observed that $\sum_{l=1}^{q_i} a_{ii}^{q_i-l} b_{ii}^{l-1} \geq 0$, for each $i \in \{1,2,\ldots,n\}$. Consequently, from (1) it follows that M is an η-accretive mapping.

Let for each $i \in \{1,2,\ldots,n\}$, the mapping $\widehat{f_i} : \mathbb{R} \to \mathbb{R}$ be defined by $\widehat{f_i}(x) = x^2 + \alpha_i x$, for all $x \in \mathbb{R}$. Then, for any $A = \begin{pmatrix} a_{ij} \end{pmatrix} \in D_n(\mathbb{R})$, we obtain $(P+M)(A) = (P+M)(\begin{pmatrix} a_{ij} \end{pmatrix}) = \begin{pmatrix} \widehat{a_{ij}} \end{pmatrix}$, where for each $i,j \in \{1,2,\ldots,n\}$,

$$\widehat{a_{ij}} = \begin{cases} a_{ii}^2 + \alpha_i a_{ii}, & i = j, \\ 0, & i \neq j, \end{cases} = \begin{cases} \widehat{f_i}(a_{ii}), & i = j, \\ 0, & i \neq j. \end{cases}$$

In virtue of the fact that for each $i \in \{1,2,\ldots,n\}$, $\widehat{f_i}(x) = x^2 + \alpha_i x = (x + \frac{\alpha_i}{2})^2 - \frac{\alpha_i^2}{4} \geq -\frac{\alpha_i^2}{4}$, it follows that for each $i \in \{1,2,\ldots,n\}$, $\widehat{f_i}(\mathbb{R}) = [-\frac{\alpha_i^2}{4}, +\infty) \neq \mathbb{R}$. This fact implies that $(P+M)(D_n(\mathbb{R})) \neq D_n(\mathbb{R})$, that is, $P+M$ is not surjective, and so M is not P-η-accretive. Now, let $\lambda > 0$ be an arbitrary constant and let for each $i \in \{1,2,\ldots,n\}$, the mapping $\widehat{g_i} : \mathbb{R} \to \mathbb{R}$ be defined by $\widehat{g_i}(x) = (1 + \lambda\alpha_i)x$, for all $x \in \mathbb{R}$. Then, for any $A = \begin{pmatrix} a_{ij} \end{pmatrix} \in D_n(\mathbb{R})$, it yields $(I + \lambda M)(A) = (I + \lambda M)(\begin{pmatrix} a_{ij} \end{pmatrix}) = \begin{pmatrix} a_{ij}^\dagger \end{pmatrix}$, where for each $i,j \in \{1,2,\ldots,n\}$,

$$a_{ij}^\dagger = \begin{cases} (1 + \lambda\alpha_i)a_{ii}, & i = j, \\ 0, & i \neq j, \end{cases} = \begin{cases} \widehat{g_i}(a_{ii}), & i = j, \\ 0, & i \neq j, \end{cases}$$

where I is the identity mapping on $D_n(\mathbb{R})$. Since $\widehat{g_i}(\mathbb{R}) = \mathbb{R}$ for each $i \in \{1,2,\ldots,n\}$, it follows that $(I + \lambda M)(D_n(\mathbb{R})) = D_n(\mathbb{R})$, that is, $I + \lambda M$ is surjective. Taking into account the arbitrariness in the choice of $\lambda > 0$, we conclude that M is an m-accretive mapping.

Example 2.8. Let the space $D_n(\mathbb{R})$ be the same as in Example 2.6 and assume that the mappings $P_1, P_2, M : D_n(\mathbb{R}) \to D_n(\mathbb{R})$ and $\eta : D_n(\mathbb{R}) \times D_n(\mathbb{R}) \to D_n(\mathbb{R})$ are defined, respectively, by $P_1(A) = P_1(\begin{pmatrix} a_{ij} \end{pmatrix}) = \begin{pmatrix} a'_{ij} \end{pmatrix}$, $P_2(A) = P_2(\begin{pmatrix} a_{ij} \end{pmatrix}) = \begin{pmatrix} a''_{ij} \end{pmatrix}$, $M(A) = M(\begin{pmatrix} a_{ij} \end{pmatrix}) = \begin{pmatrix} a'''_{ij} \end{pmatrix}$, and $\eta(A,B) = \eta(\begin{pmatrix} a_{ij} \end{pmatrix}, \begin{pmatrix} b_{ij} \end{pmatrix}) = \begin{pmatrix} c_{ij} \end{pmatrix}$, for all $A = \begin{pmatrix} a_{ij} \end{pmatrix}, B = \begin{pmatrix} b_{ij} \end{pmatrix} \in D_n(\mathbb{R})$, where for each $i,j \in \{1,2,\ldots,n\}$,

$$a'_{ij} = \begin{cases} \frac{2a_{ii}^2 - 1}{a_{ii}^2 + 1} - \varrho\sqrt[k]{a_{ii}}, & i = j, \\ 0, & i \neq j, \end{cases}$$

$$a''_{ij} = \begin{cases} 3a_{ii} + 2 + |a_{ii} - 2|, & i = j, \\ 0, & i \neq j, \end{cases}$$

$$a'''_{ij} = \begin{cases} \varrho\sqrt[k]{a_{ii}}, & i = j, \\ 0, & i \neq j, \end{cases}$$

and

$$c_{ij} = \begin{cases} \gamma\theta^{\sigma a_{ii} b_{ii}}(a_{ii} - b_{ii}), & i = j, \\ 0, & i \neq j, \end{cases}$$

where $\gamma, \varrho, \theta > 0$ and $\sigma \in \mathbb{R}$ are arbitrary constants, and k is an arbitrary but fixed odd natural number. In view of the fact that $(D_n(\mathbb{R}), \|.\|)$ is a finite dimensional normed space, we infer that it is a 2-uniformly smooth Banach space. Then, for any $A = \begin{pmatrix} a_{ij} \end{pmatrix}, B = \begin{pmatrix} b_{ij} \end{pmatrix} \in D_n(\mathbb{R})$, it yields

$$\begin{aligned} \langle M(A) - M(B), J_2(\eta(A,B)) \rangle &= \langle M(A) - M(B), \eta(A,B) \rangle \\ &= tr\Big(\begin{pmatrix} a'''_{ij} - b'''_{ij} \end{pmatrix}\begin{pmatrix} c_{ij} \end{pmatrix}\Big) \\ &= \gamma\varrho\sum_{i=1}^{n}(\sqrt[k]{a_{ii}} - \sqrt[k]{b_{ii}})\theta^{\sigma a_{ii} b_{ii}}(a_{ii} - b_{ii}). \end{aligned}$$

For any $i \in \{1,2,\ldots,n\}$,

(i) if $a_{ii} = b_{ii} = 0$, then $(\sqrt[k]{a_{ii}} - \sqrt[k]{b_{ii}})(a_{ii} - b_{ii}) = 0$;

(ii) if $a_{ii} \neq 0$ and $b_{ii} = 0$, then $(\sqrt[k]{a_{ii}} - \sqrt[k]{b_{ii}})(a_{ii} - b_{ii}) = a_{ii}\sqrt[k]{a_{ii}} = \sqrt[k]{a_{ii}^{k+1}}$;

(iii) if $a_{ii} = 0$ and $b_{ii} \neq 0$, then $(\sqrt[k]{a_{ii}} - \sqrt[k]{b_{ii}})(a_{ii} - b_{ii}) = b_{ii}\sqrt[k]{b_{ii}} = \sqrt[k]{b_{ii}^{k+1}}$;

(iv) if $a_{ii}, b_{ii} \neq 0$, then $\sqrt[k]{a_{ii}} - \sqrt[k]{b_{ii}} = \frac{a_{ii}-b_{ii}}{\sum\limits_{t=1}^{k}\sqrt[k]{a_{ii}^{k-t}b_{ii}^{t-1}}}$.

Since k is an odd natural number, it follows that $\sqrt[k]{a_{ii}^{k+1}}, \sqrt[k]{b_{ii}^{k+1}} > 0$ and $\sum\limits_{t=1}^{k}\sqrt[k]{a_{ii}^{k-t}b_{ii}^{t-1}} > 0$. These facts guarantee that $(\sqrt[k]{a_{ii}} - \sqrt[k]{b_{ii}})(a_{ii} - b_{ii}) > 0$ and $\sum\limits_{i=1}^{n}(\sqrt[k]{a_{ii}} - \sqrt[k]{b_{ii}})(a_{ii} - b_{ii}) = \sum\limits_{i=1}^{n}\frac{(a_{ii}-b_{ii})^2}{\sum\limits_{t=1}^{k}\sqrt[k]{a_{ii}^{k-t}b_{ii}^{t-1}}} > 0$. Taking into account that $\gamma, \varrho > 0$, in the light of the above-mentioned discussions, we deduce that for all $A = \begin{pmatrix} a_{ij} \end{pmatrix}, B = \begin{pmatrix} b_{ij} \end{pmatrix} \in D_n(\mathbb{R})$,

$$\langle M(A) - M(B), J_2(\eta(A,B))\rangle = \gamma\varrho\sum_{i=1}^{n}(\sqrt[k]{a_{ii}} - \sqrt[k]{b_{ii}})\theta^{\sigma a_{ii}b_{ii}}(a_{ii} - b_{ii}) = \gamma\varrho\sum_{i=1}^{n}\frac{\theta^{\sigma a_{ii}b_{ii}}(a_{ii} - b_{ii})^2}{\sum\limits_{t=1}^{k}\sqrt[k]{a_{ii}^{k-t}b_{ii}^{t-1}}} \geq 0,$$

i.e., M is an accretive mapping. Assume that the function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) := \frac{2x^2-1}{x^2+1}$ for all $x \in \mathbb{R}$. Then, for any $A = \begin{pmatrix} a_{ij} \end{pmatrix} \in D_n(\mathbb{R})$, we get

$$(P_1 + M)(A) = (P_1 + M)(\begin{pmatrix} a_{ij} \end{pmatrix}) = \begin{pmatrix} a'_{ij} + a'''_{ij} \end{pmatrix} = \begin{pmatrix} \widetilde{a}_{ij} \end{pmatrix},$$

where for each $i, j \in \{1, 2, \ldots, n\}$,

$$\widetilde{a}_{ij} = \begin{cases} \frac{2a_{ii}^2-1}{a_{ii}^2+1}, & i = j, \\ 0, & i \neq j, \end{cases} = \begin{cases} f(a_{ii}), & i = j, \\ 0, & i \neq j. \end{cases}$$

In virtue of the fact that $f(\mathbb{R}) = [-1, 2)$, we conclude that $(P_1 + M)(D_n(\mathbb{R})) \neq D_n(\mathbb{R})$, which means that the mapping $P_1 + M$ is not surjective, and so M is not a P_1-η-accretive mapping. Now, let the real constant λ be chosen arbitrarily but fixed and suppose that the function $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) := 3x+2+|x-2|+\lambda\varrho\sqrt[k]{x}$, for all $x \in \mathbb{R}$. Then, for any $A = \begin{pmatrix} a_{ij} \end{pmatrix} \in D_n(\mathbb{R})$, we obtain

$$(P_2 + \lambda M)(A) = (P_2 + \lambda M)(\begin{pmatrix} a_{ij} \end{pmatrix}) = \begin{pmatrix} a''_{ij} + \lambda a'''_{ij} \end{pmatrix} = \begin{pmatrix} \widehat{a}_{ij} \end{pmatrix},$$

where for each $i, j \in \{1, 2, \ldots, n\}$,

$$\widehat{a}_{ij} = \begin{cases} 3a_{ii} + 2 + |a_{ii} - 2| + \lambda\varrho\sqrt[k]{a_{ii}}, & i = j, \\ 0, & i \neq j, \end{cases} = \begin{cases} g(a_{ii}), & i = j, \\ 0, & i \neq j. \end{cases}$$

Relying on the fact that $g(\mathbb{R}) = \mathbb{R}$, it follows that $(P_2 + \lambda M)(D_n(\mathbb{R})) = D_n(\mathbb{R})$, that is, $P_2 + \lambda M$ is a surjective mapping. Since the positive real constant λ was arbitrary, we deduce that M is a P_2-η-accretive mapping.

In accordance with Example 2.6, for given mappings $P : X \to X$ and $\eta : X \times X \to X$, a P-η-accretive mapping need not be m-η-accretive. The following proposition states conditions under which for given mappings $P : X \to X$ and $\eta : X \times X \to X$, every P-η-accretive mapping is m-η-accretive.

Proposition 2.9. [27, Theorem 3.1] *Let X be a real q-uniformly smooth Banach space, $\eta : X \times X \to X$ be a vector-valued mapping, $P : X \to X$ be a strictly η-accretive mapping, and $M : X \to 2^X$ be a P-η-accretive mapping, and let $x, u \in X$ be two given points. If $\langle u - v, J_q(\eta(x, y))\rangle \geq 0$ holds, for all $(y, v) \in$ Graph(M), then $u \in M(x)$, that is, M is an m-η-accretive mapping.*

Regarding to Example 2.7, for given mappings $P : X \to X$ and $\eta : X \times X \to X$, an m-η-accretive mapping may not be P-η-accretive. In the next result, the sufficient conditions for guaranteeing that for given mappings $P : X \to X$ and $\eta : X \times X \to X$, an m-η-accretive mapping to be P-η-accretive are provided. Before proceeding to it, we need to recall the following concepts.

Definition 2.10. *Let X be a real q-uniformly smooth Banach space. A single-valued mapping $P : X \to X$ is said to be coercive if*

$$\lim_{\|x\|\to+\infty} \frac{\langle P(x), J_q(x)\rangle}{\|x\|} = +\infty.$$

Definition 2.11. *Let X be a real q-uniformly smooth Banach space and $P : X \to X$ be a single-valued mapping. P is said to be bounded, if $P(A)$ is a bounded subset of X, for every bounded subset A of X. We say that P is a hemi-continuous mapping if for any $x, y, z \in X$, the function $t \longmapsto \langle P(x + ty), J_q(z)\rangle$ is continuous at 0^+.*

Proposition 2.12. *Let X be a real q-uniformly smooth Banach space, $\eta : X \times X \to X$ be a vector-valued mapping, and $P : X \to X$ be a bounded, coercive, hemi-continuous and η-accretive mapping. If $M : X \to 2^X$ is an m-η-accretive mapping, then M is P-η-accretive.*

Proof. Taking into consideration the fact that P is bounded, coercive, hemi-continuous and η-accretive, invoking Theorem 3.1 of Guo [21, P.401], we conclude that $P + \lambda M$ is surjective for every $\lambda > 0$, i.e., $(P + \lambda M)(X) = X$ holds for every $\lambda > 0$. Accordingly, M is a P-η-accretive mapping. This completes the proof. □

Lemma 2.13. [33, Theorem 3.1(b)] *Let X be a real q-uniformly smooth Banach space, $\eta : X \times X \to X$ be a vector-valued mapping, $P : X \to X$ be a strictly η-accretive mapping, and $M : X \to 2^X$ be a P-η-accretive mapping. Then, the mapping $(P + \lambda M)^{-1}$ is single-valued for every real constant $\lambda > 0$.*

Based on Lemma 2.13, one can define the P-η-resolvent operator $R^{P,\eta}_{M,\lambda}$ associated with a P-η-accretive mapping M and an arbitrary real constant $\lambda > 0$ as follows.

Definition 2.14. *[27, 33] Let X be a real q-uniformly smooth Banach space, $\eta : X \times X \to X$ be a vector-valued mapping, $P : X \to X$ be a strictly η-accretive mapping, $M : X \to 2^X$ be a P-η-accretive mapping, and $\lambda > 0$ be an arbitrary real constant. The resolvent operator $R^{P,\eta}_{M,\lambda} : X \to X$ associated with P, η, M and λ is defined by*

$$R^{P,\eta}_{M,\lambda}(u) = (P + \lambda M)^{-1}(u), \quad \forall u \in X.$$

Definition 2.15. *A vector-valued mapping $\eta : X \times X \to X$ is said to be τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that $\|\eta(x, y)\| \leq \tau\|x - y\|$, for all $u, v \in X$.*

Under some suitable conditions imposed on the mappings and parameter, the authors [33] proved the Lipschitz continuity of the resolvent operator $R^{P,\eta}_{M,\lambda}$ associated with a P-η-accretive mapping M and an arbitrary real constant $\lambda > 0$ and compute an estimate of its Lipschitz constant as follows.

Lemma 2.16. [33, Lemma 2.4] *Let X be a real q-uniformly smooth Banach space, $\eta : X \times X \to X$ be τ-Lipschitz continuous, $P : X \to X$ be an r-strongly η-accretive mapping, $M : X \to 2^X$ be a P-η-accretive mapping, and $\lambda > 0$ be an arbitrary real constant. Then, the P-η-proximal mapping $R^{P,\eta}_{M,\lambda} : X \to X$ is Lipschitz continuous with constant $\frac{\tau^{q-1}}{r}$, i.e.,*

$$\|R^{P,\eta}_{M,\lambda}(u) - R^{P,\eta}_{M,\lambda}(v)\| \leq \frac{\tau^{q-1}}{r}\|u - v\|, \quad \forall u, v \in X.$$

3. Formulation of the Problem, Iterative Algorithms and Convergence Results

Let for each $i \in \{1,2\}$, X_i be a real q_i-uniformly smooth Banach space with dual space X_i^* and norm $\|.\|_i$, and $\langle .,. \rangle_i$ be the dual pair between X_i and X_i^*. Assume that for $i = 1,2$, $f_i, p_i : X_i \to X_i$, $S_i : X_1 \times X_2 \to X_i$ and $Q_i : X_j \times X_i \to X_i$ ($j \in \{1,2\}\backslash\{i\}$) are the mappings. Further, let for $i = 1,2$, $F_i : X_i \to CB(X_i)$, $M_i : X_i \to 2^{X_i}$ and $T_i : X_j \to CB(X_j)$ ($j \in \{1,2\}\backslash\{i\}$) be the multi-valued mappings. We consider the following system of generalized multi-valued nonlinear variational inclusions (SGMNVI): find $(x,y) \in X_1 \times X_2$, $u \in F_1(x)$, $v \in F_2(y)$, $w \in T_1(y)$ and $t \in T_2(x)$ such that

$$\begin{cases} 0 \in S_1(p_1(x), v) + Q_1(w,t) + M_1(f_1(x)), \\ 0 \in S_2(u, p_2(y)) + Q_2(t,w) + M_2(f_2(y)). \end{cases} \tag{2}$$

If $q_i = q$ for $i = 1,2$, $S_1 = S$, $S_2 = T$, $M_1 = M$, $M_2 = N$, $F_1 = E$, $F_2 = F$, $Q_1 = Q_2 \equiv 0$, $f_1 = f$, $f_2 = g$, $p_1 = p$ and $p_2 = d$, then the SGMNVI (2) collapses to the following generalized multi-valued nonlinear variational inclusions system: find $(x,y) \in X_1 \times X_2$, $u \in E(x)$, $v \in F(y)$ such that

$$\begin{cases} 0 \in S(p(x), v) + M(f(x)), \\ 0 \in T(u, d(y)) + N(g(y)). \end{cases} \tag{3}$$

A special case of the system (3) where the underlying spaces are Hilbert spaces and the multi-valued mappings M and N are A-monotone operators is considered in [28]. It should be remarked that for suitable and appropriate choices of the mappings $S_i, Q_i, F_i, T_i, M_i, f_i, p_i$ and the spaces X_i ($i = 1,2$), the SGMNVI (2) reduces to various classes of variational inclusions and variational inequalities, see for example, [17–19, 23, 28, 32, 33, 36, 37, 39] and the references therein.

In order to construct an iterative algorithm for approximating the solution of the SGMNVI (2), we require the lemma mentioned below, in which the equivalence between the SGMNVI (2) and a fixed point problem is stated.

Lemma 3.1. *Let $X_i, F_i, S_i, T_i, Q_i, M_i, f_i, p_i$ ($i = 1,2$) be the same as in the* SGMNVI (2)*. Suppose further that for each $i \in \{1,2\}$, $\eta_i : X_i \times X_i \to X_i$ is a vector-valued mapping, $P_i : X_i \to X_i$ is a strictly η_i-accretive mapping, and M_i is a P_i-η_i-accretive mapping. Then $(x,y) \in X_1 \times X_2$, $(u,v) \in F_1(x) \times F_2(y)$ and $(w,t) \in T_1(y) \times T_2(x)$ are the solution of the* SGMNVI (2)*, if and only if*

$$\begin{cases} f_1(x) = R^{P_1,\eta_1}_{M_1,\lambda}[P_1(f_1(x)) - \lambda(S_1(p_1(x), v) + Q_1(w,t))], \\ f_2(y) = R^{P_2,\eta_2}_{M_2,\rho}[P_2(f_2(y)) - \rho(S_2(u, p_2(y)) + Q_2(t,w))], \end{cases} \tag{4}$$

where $\lambda, \rho > 0$ are two constants.

Proof. The conclusions follow directly from Definition 2.14 and some simple arguments. □

As an immediate consequence of the above result, we obtain the following conclusion.

Lemma 3.2. *Suppose that X_i ($i = 1,2$), $S, T, E, F, M, N, f, g, p, d$ are the same as in the system* (3)*. Further, let for each $i \in \{1,2\}$, $\eta_i : X_i \to X_i$ be a vector-valued mapping, $P_i : X_i \to X_i$ be a strictly η_i-accretive mapping, M be a P_1-η_1-accretive mapping and N be a P_2-η_2-accretive mapping. Then $(x,y) \in X_1 \times X_2$ and $(u,v) \in F(x) \times F(y)$ are the solution of the system* (3) *if and only if*

$$\begin{cases} f(x) = R^{P_1,\eta_1}_{M,\lambda}[P_1(f(x)) - \lambda S_1(p(x), v)], \\ g(y) = R^{P_2,\eta_2}_{N,\rho}[P_2(g(y)) - \rho T(u, d(y))], \end{cases}$$

where $\lambda, \rho > 0$ are two constants.

Lemma 3.3. [31] *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a multi-valued mapping. Then, for any $\varepsilon > 0$ and for any given $x, y \in X$, $u \in T(x)$, there exists $v \in T(y)$ such that*

$$d(u,v) \leq (1+\varepsilon) D(T(x), T(y)),$$

where $D(.,.)$ is the Hausdorff metric on $CB(X)$.

The fixed point formulation (4) and Nadler's technique [31] enable us to construct the following iterative algorithm for approximating the solution of the SGMNVI (2).

Algorithm 3.4. *Let $X_i, F_i, S_i, T_i, Q_i, f_i, p_i$ $(i = 1, 2)$ be the same as in the* SGMNVI (2). *Suppose that for each $i \in \{1, 2\}$, $\eta_i : X_i \times X_i \to X_i$ is a vector-valued mapping, $P_i : X_i \to X_i$ is a strictly η_i-accretive mapping and $M_i : X_i \to 2^{X_i}$ is a P_i-η_i-accretive mapping. For any given $(x_0, y_0) \in X_1 \times X_2$, $(u_0, v_0) \in F_1(x_0) \times F_2(y_0)$ and $(w_0, t_0) \in T_1(y_0) \times T_2(x_0)$, define the iterative sequences $\{(x_n, y_n)\}_{n=0}^{\infty}$, $\{(w_n, t_n)\}_{n=0}^{\infty} \subseteq \bigcup\limits_{n=0}^{\infty} F_1(x_n) \times F_2(y_n)$ and $\{(w_n, t_n)\}_{n=0}^{\infty} \subseteq \bigcup\limits_{n=0}^{\infty} T_1(y_n) \times T_2(x_n)$ in $X_1 \times X_2$ in the following way:*

$$\begin{cases} x_{n+1} = (1-\alpha_1)x_n + \alpha_1\{x_n - f_1(x_n) + R^{P_1,\eta_1}_{M_1,\lambda}[P_1(f_1(x_n)) \\ \qquad -\lambda(S_1(p_1(x_n), v_n) + Q_1(w_n, t_n))]\} + \alpha_1 e_n + r_n, \\ y_{n+1} = (1-\alpha_2)y_n + \alpha_2\{y_n - f_2(y_n) + R^{P_2,\eta_2}_{M_2,\rho}[P_2(f_2(y_n)) \\ \qquad -\rho(S_2(u_n, p_2(y_n)) + Q_2(t_n, w_n))]\} + \alpha_2 l_n + k_n, \\ u_n \in F_1(x_n); \|u_{n+1} - u_n\|_1 \le (1 + (1+n)^{-1})D_1(F_1(x_{n+1}), F_1(x_n)), \\ v_n \in F_2(x_n); \|v_{n+1} - v_n\|_2 \le (1 + (1+n)^{-1})D_2(F_2(y_{n+1}), F_2(y_n)), \\ w_n \in T_1(y_n); \|w_{n+1} - w_n\|_2 \le (1 + (1+n)^{-1})D_2(T_1(y_{n+1}), T_1(y_n)), \\ t_n \in T_2(x_n); \|t_{n+1} - t_n\|_1 \le (1 + (1+n)^{-1})D_1(T_2(x_{n+1}), T_2(x_n)), \end{cases} \tag{5}$$

where $n = 0, 1, 2, \ldots$; $\lambda, \rho > 0$ are constants, $\alpha_1, \alpha_2 \in (0, 1]$ are two parameters such that $\alpha_1 + \alpha_2 \in (0, 1]$ and $\{(e_n, l_n)\}_{n=0}^{\infty}$ and $\{(r_n, k_n)\}_{n=0}^{\infty}$ are two sequences in $X_1 \times X_2$ to take into account a possible inexact computation of the resolvent operator point satisfying the following conditions:

$$\begin{cases} \lim\limits_{n\to\infty} \|e_n\|_1 = \lim\limits_{n\to\infty} \|r_n\|_1 = \lim\limits_{n\to\infty} \|l_n\|_2 = \lim\limits_{n\to\infty} \|k_n\|_2 = 0, \\ \sum\limits_{n=0}^{\infty} \|e_{n+1} - e_n\|_1 < \infty, \ \sum\limits_{n=0}^{\infty} \|r_{n+1} - r_n\|_1 < \infty, \\ \sum\limits_{n=0}^{\infty} \|l_{n+1} - l_n\|_2 < \infty, \ \sum\limits_{n=0}^{\infty} \|k_{n+1} - k_n\|_2 < \infty. \end{cases} \tag{6}$$

If $q_i = q$ for $i = 1, 2$, $S_1 = S$, $S_2 = T$, $M_1 = M$, $M_2 = N$, $F_1 = E$, $F_2 = F$, $Q_1 = Q_2 \equiv 0$, $f_1 = f$, $f_2 = g$, $p_1 = p$, $p_2 = d$, and $e_n = r_n = l_n = k_n = 0$, then Algorithm 3.4 collapses to the following algorithm.

Algorithm 3.5. *Suppose that X_i $(i = 1, 2)$, S, T, E, F, f, g, p, d are the same as in the system* (3). *Let for each $i \in \{1, 2\}$, $\eta_i : X_1 \times X_2 \to X_i$ be a vector-valued mapping, $P_i : X_i \to X_i$ be a strictly η_i-accretive mapping, $M : X_1 \to 2^{X_1}$ be a P_1-η_1-accretive mapping and $N : X_2 \to 2^{X_2}$ be a P_2-η_2-accretive mapping. For any given $(x_0, y_0) \in X_1 \times X_2$, $u_0 \in E(x_0)$ and $v_0 \in F(y_0)$, define the iterative sequences $\{(x_n, y_n)\}_{n=0}^{\infty}$ in $X_1 \times X_2$, $\{u_n\}_{n=0}^{\infty}$ in X_1 and $\{v_n\}_{n=0}^{\infty}$ in X_2 in the following way:*

$$\begin{cases} x_{n+1} = (1-\alpha_1)x_n + \alpha_1\{x_n - f(x_n) + R^{P_1,\eta_1}_{M,\lambda}[P_1(f(x_n)) - \lambda S(p(x_n), v_n)]\}, \\ y_{n+1} = (1-\alpha_2)y_n + \alpha_2\{y_n - g(y_n) + R^{P_2,\eta_2}_{N,\rho}[P_2(g(y_n)) - \rho T(u_n, d(y_n))]\}, \\ u_n \in E(x_n); \|u_{n+1} - u_n\|_1 \le (1 + (1+n)^{-1})D_1(E(x_{n+1}), E(x_n)), \\ v_n \in F(y_n); \|v_{n+1} - v_n\|_2 \le (1 + (1+n)^{-1})D_2(F(y_{n+1}), F(y_n)), \end{cases}$$

where $n = 0, 1, 2, \ldots$; $\lambda, \rho > 0$ are two constants, and $\alpha_1, \alpha_2 \in (0, 1]$ are two parameters the same as in Algorithm 3.4.

We are now in a position to give the main result of this section concerning the strong convergence of the sequences generated by our suggested iterative algorithm to a solution of the SGMNVI (2). For this purpose, we need to recall the following definitions.

Definition 3.6. *A multi-valued mapping $T : X \to CB(X)$ is said to be D-Lipschitz continuous with constant δ, if there exists a constant $\delta > 0$ such that*

$$D(T(x), T(y)) \le \delta\|x - y\|, \quad \forall x, y \in X.$$

Definition 3.7. *Let X be a real q-uniformly smooth Banach space. A mapping $f : X \to X$ is said to be*

(i) *(γ, μ)-relaxed cocoercive if there exist two constants $\gamma, \mu > 0$ such that*

$$\langle f(x) - f(y), J_q(x-y)\rangle \geq -\gamma\|f(x) - f(y)\|^q + \mu\|x-y\|^q, \quad \forall x, y \in X;$$

(ii) *δ-strongly accretive if there exists a constant $\delta > 0$ such that*

$$\langle f(x) - f(y), J_q(x-y)\rangle \geq \delta\|x-y\|^q, \quad \forall x, y \in X.$$

Definition 3.8. *Let X be a real q-uniformly smooth Banach space. Further, let $p : X \to X$, $S : X \times X \to X$ and $\eta : X \times X \to X$ be the mappings. S is said to be*

(i) *(ξ, π)-relaxed η-cocoercive with respect to p in the first argument if there exist two constants $\xi, \pi > 0$ such that for all $x, y, u \in X$,*

$$\langle S(p(x), u) - S(p(y), u), J_q(\eta(x, y))\rangle \geq -\xi\|S(p(x), u) - S(p(y), u)\|^q + \pi\|x-y\|^q;$$

(ii) *(ς, ϱ)-relaxed η-cocoercive with respect to p in the second argument if there exist two constants $\varsigma, \varrho > 0$ such that for all $x, y, u \in X$,*

$$\langle S(u, p(x)) - S(u, p(y)), J_q(\eta(x, y))\rangle \geq -\varsigma\|S(u, p(x)) - S(u, p(y))\|^q + \varrho\|x-y\|^q;$$

(iii) *k-strongly η-accretive with respect to p in the first argument if there exists a constant $k > 0$ such that*

$$\langle S(p(x), u) - S(p(y), u), J_q(\eta(x, y))\rangle \geq k\|x-y\|^q, \quad \forall x, y, u \in X;$$

(iv) *γ-strongly η-accretive with respect to p in the second argument if there exists a constant $\gamma > 0$ such that*

$$\langle S(u, p(x)) - S(u, p(y)), J_q(\eta(x, y))\rangle \geq \gamma\|x-y\|^q, \quad \forall x, y, u \in X;$$

(v) *ϑ-Lipschitz continuous with respect to p in the first argument if there exists a constant $\vartheta > 0$ such that*

$$\|S(p(x), u) - S(p(y), u)\| \leq \vartheta\|x-y\|, \quad \forall x, y, u \in X;$$

(vi) *δ-Lipschitz continuous with respect to p in the second argument if there exists a constant $\delta > 0$ such that*

$$\|S(u, p(x)) - S(u, p(y))\| \leq \delta\|x-y\|, \quad \forall x, y, u \in X.$$

Definition 3.9. *Let X be a real q-uniformly smooth Banach space. A mapping $Q : X \times X \to X$ is said to be (θ, μ)-mixed Lipschitz continuous in the first and second arguments if there exist two constants $\theta, \mu > 0$ such that*

$$\|Q(x, y) - Q(x', y')\| \leq \theta\|x - x'\| + \mu\|y - y'\|, \quad \forall x, x', y, y' \in X.$$

Theorem 3.10. *Let for each $i \in \{1, 2\}$, X_i be a q_i-uniformly smooth Banach space with $q_i > 1$, $\eta_i : X_i \times X_i \to X_i$ be a τ_i-Lipschitz continuous mapping, $P_i : X_i \to X_i$ be a θ_i-strongly η_i-accretive and ϱ_i-Lipschitz continuous mapping, and $M_i : X_i \to 2^{X_i}$ be a P_i-η_i-accretive mapping. Suppose that for each $i \in \{1, 2\}$, $f_i : X_i \to X_i$ is a (ξ_i, δ_i)-relaxed cocoercive and λ_{f_i}-Lipschitz continuous mapping, and $Q_i : X_j \times X_i \to X_i$ for $j \in \{1, 2\}\backslash\{i\}$ is $(\lambda_{Q_i}, \lambda'_{Q_i})$-mixed Lipschitz continuous in the first and second arguments. Let $S_1 : X_1 \times X_2 \to X_1$ be $(\gamma_{S_1}, \delta_{S_1})$-relaxed η_1-cocoercive and ϖ_1-Lipschitz continuous with respect to p_1 in the first argument and ϖ_2-Lipschitz continuous with respect to p_1 in the second argument, and $S_2 : X_1 \times X_2 \to X_2$ be $(\gamma_{S_2}, \delta_{S_2})$-relaxed η_2-cocoercive and π_2-Lipschitz continuous with respect to p_2 in the second argument and π_1-Lipschitz continuous with respect to p_2 in the first argument. Let for each $i \in \{1, 2\}$, the mapping $F_i : X_i \to CB(X_i)$ be D_i-Lipschitz continuous with constant $\lambda_{D_{F_i}}$ and for each $i \in \{1, 2\}$ and $j \in \{1, 2\}\backslash\{i\}$, $T_i : X_j \to CB(X_j)$ be D_j-Lipschitz continuous with constant $\lambda_{D_{T_i}}$. If there exist two constants $\lambda, \rho > 0$ such that*

$$\begin{cases} 1 - \alpha_1 + \alpha_1 \sqrt[q_1]{1 - q_1\delta_1 + (q_1\xi_1 + c_{q_1})\lambda_{f_1}^{q_1}} + \frac{\alpha_1\tau_1^{q-1}}{\theta_1}(\mu_1 + \lambda\lambda_{Q_1}\lambda_{D_{T_2}}) \\ + \frac{\alpha_2\tau_2^{q-1}\rho(\pi_1\lambda_{D_{F_1}} + \lambda_{Q_2}\lambda_{D_{T_2}})}{\theta_2} < 1, \\ 1 - \alpha_2 + \alpha_2 \sqrt[q_2]{1 - q_2\delta_2 + (q_2\xi_2 + c_{q_2})\lambda_{f_2}^{q_2}} + \frac{\alpha_2\tau_2^{q-1}}{\theta_2}(\mu_2 + \rho\lambda_{Q_2}\lambda_{D_{T_1}}) \\ + \frac{\alpha_1\tau_1^{q-1}\lambda(\varpi_2\lambda_{D_{F_2}} + \lambda_{Q_1}\lambda_{D_{T_1}})}{\theta_1} < 1, \end{cases} \tag{7}$$

where

$$\mu_1 = \sqrt[q_1]{\varrho_1^{q_1}\lambda_{f_1}^{q_1} + q_1\lambda\gamma_{S_1}\varpi_1^{q_1} - q_1\lambda\delta_{S_1} + q_1\lambda\varpi_1\varrho_1^{q_1-1}\lambda_{f_1}^{q_1-1} + q_1\lambda\varpi_1\tau_1^{q_1-1} + c_{q_1}\lambda^{q_1}\varpi_1^{q_1}},$$
$$\mu_2 = \sqrt[q_2]{\varrho_2^{q_2}\lambda_{f_2}^{q_2} + q_2\rho\gamma_{S_2}\pi_2^{q_2} - q_2\rho\delta_{S_2} + q_2\rho\pi_2\varrho_2^{q_2-1}\lambda_{f_2}^{q_2-1} + q_2\rho\pi_2\tau_2^{q_2-1} + c_{q_2}\rho^{q_2}\pi_2^{q_2}},$$

and for the case where q_i $(i = 1, 2)$ are even natural numbers, in addition to (7)*, the following conditions hold:*

$$\begin{cases} q_i\delta_i < 1 + (q_i\xi_i + c_{q_i})\lambda_{f_i}^{q_i}, \\ q_1\lambda\delta_{S_1} < \varrho_1^{q_1}\lambda_{f_1}^{q_1} + q_1\lambda\gamma_{S_1}\varpi_1^{q_1} + q_1\lambda\varpi_1\varrho_1^{q_1-1}\lambda_{f_1}^{q_1-1} + q_1\lambda\varpi_1\tau_1^{q_1-1} + c_{q_1}\lambda^{q_1}\varpi_1^{q_1}, \\ q_2\rho\delta_{S_2} < \varrho_2^{q_2}\lambda_{f_2}^{q_2} + q_2\rho\gamma_{S_2}\pi_2^{q_2} + q_2\rho\pi_2\varrho_2^{q_2-1}\lambda_{f_2}^{q_2-1} + q_2\rho\pi_2\tau_2^{q_2-1} + c_{q_2}\rho^{q_2}\pi_2^{q_2}, \end{cases}$$

where c_{q_i} $(i = 1, 2)$ are two constants guaranteed by Lemma 2.1, then, the iterative sequences $\{(x_n, y_n)\}_{n=0}^{\infty}$, $\{(u_n, v_n)\}_{n=0}^{\infty}$ and $\{(w_n, t_n)\}_{n=0}^{\infty}$ generated by Algorithm 3.4 converge strongly to (x, y), (u, v) and (w, t), respectively, and (x, y, u, v, w, t) is a solution of the SGMNVI (2)*.*

Proof. By using (5), Lemma 2.16 and the assumptions, it yields

$$\begin{aligned} \|x_{n+1} - x_n\|_1 &= \|(1-\alpha_1)x_n + \alpha_1\Big(x_n - f_1(x_n) + R_{M_1,\lambda}^{P_1,\eta_1}[P_1(f_1(x_n)) \\ &\quad - \lambda(S_1(p_1(x_n), v_n) + Q_1(w_n, t_n))]\Big) + \alpha_1 e_n + r_n \\ &\quad - (1-\alpha_1)x_{n-1} - \alpha_1\Big(x_{n-1} - f_1(x_{n-1}) + R_{M_1,\lambda}^{P_1,\eta_1}[P_1(f_1(x_{n-1})) \\ &\quad - \lambda(S_1(p_1(x_{n-1}), v_{n-1}) + Q_1(w_{n-1}, t_{n-1}))]\Big) - \alpha_1 e_{n-1} - r_{n-1}\|_1 \\ &\le (1-\alpha_1)\|x_n - x_{n-1}\|_1 + \alpha_1\Big(\|x_n - x_{n-1} - (f_1(x_n) - f_1(x_{n-1}))\|_1 \\ &\quad + \|R_{M_1,\lambda}^{P_1,\eta_1}[P_1(f_1(x_n)) - \lambda(S_1(p_1(x_n), v_n) + Q_1(w_n, t_n))] \\ &\quad - R_{M_1,\lambda}^{P_1,\eta_1}[P_1(f_1(x_{n-1})) - \lambda(S_1(p_1(x_{n-1}), v_{n-1}) \\ &\quad + Q_1(w_{n-1}, t_{n-1}))]\|_1\Big) + \alpha_1\|e_n - e_{n-1}\|_1 + \|r_n - r_{n-1}\|_1 \\ &\le (1-\alpha_1)\|x_n - x_{n-1}\|_1 + \alpha_1\|x_n - x_{n-1} - (f_1(x_n) - f_1(x_{n-1}))\|_1 \\ &\quad + \frac{\alpha_1\tau_1^{q_1-1}}{\theta_1}\|P_1(f_1(x_n)) - \lambda\Big(S_1(p_1(x_n), v_n) + Q_1(w_n, t_n)\Big) \\ &\quad - P_1(f_1(x_{n-1})) + \lambda\Big(S_1(p_1(x_{n-1}), v_{n-1}) + Q_1(w_{n-1}, t_{n-1})\Big)\|_1 \\ &\quad + \alpha_1\|e_n - e_{n-1}\|_1 + \|r_n - r_{n-1}\|_1 \\ &\le (1-\alpha_1)\|x_n - x_{n-1}\|_1 + \alpha_1\|x_n - x_{n-1} - (f_1(x_n) - f_1(x_{n-1}))\|_1 \\ &\quad + \frac{\alpha_1\tau_1^{q_1-1}}{\theta_1}\Big(\|P_1(f_1(x_n)) - P_1(f_1(x_{n-1})) - \lambda\Big(S_1(p_1(x_n), v_n) \\ &\quad - S_1(p_1(x_{n-1}), v_n)\Big)\|_1 + \lambda\|S_1(p_1(x_{n-1}), v_n) - S_1(p_1(x_{n-1}), v_{n-1})\|_1 \\ &\quad + \lambda\|Q_1(w_n, t_n) - Q_1(w_{n-1}, t_{n-1})\|_1\Big) + \alpha_1\|e_n - e_{n-1}\|_1 + \|r_n - r_{n-1}\|_1. \end{aligned} \tag{8}$$

Since f_1 is (ξ_1, δ_1)-relaxed cocoercive and λ_{f_1}-Lipschitz continuous, invoking Lemma 2.1, there exists a constant $c_{q_1} > 0$ such that for each $n \in \mathbb{N}$,

$$\begin{aligned} &\|x_n - x_{n-1} - (f_1(x_n) - f_1(x_{n-1}))\|_1^{q_1} \\ &\le \|x_n - x_{n-1}\|_1^{q_1} - q_1\langle f_1(x_n) - f_1(x_{n-1}), J_{q_1}(x_n - x_{n-1})\rangle_1 + c_{q_1}\|f_1(x_n) - f_1(x_{n-1})\|_1^{q_1} \\ &\le \|x_n - x_{n-1}\|_1^{q_1} - q_1(-\xi_1\|f_1(x_n) - f_1(x_{n-1})\|_1^{q_1} + \delta_1\|x_n - x_{n-1}\|_1^{q_1}) + c_{q_1}\|f_1(x_n) - f_1(x_{n-1})\|_1^{q_1} \\ &\le \Big(1 - q_1\delta_1 + (q_1\xi_1 + c_{q_1})\lambda_{f_1}^{q_1}\Big)\|x_n - x_{n-1}\|_1^{q_1}, \end{aligned}$$

which implies that

$$\|x_n - x_{n-1} - (f_1(x_n) - f_1(x_{n-1}))\|_1 \leq \sqrt[q_1]{1 - q_1\delta_1 + (q_1\xi_1 + c_{q_1})\lambda_{f_1}^{q_1}}\|x_n - x_{n-1}\|_1. \tag{9}$$

Owing to the fact that S_1 is $(\gamma_{S_1}, \delta_{S_1})$-relaxed η_1-cocoercive and ϖ_1-Lipschitz continuous with respect to p_1 in the first argument, η_1 is τ_1-Lipschitz continuous, P_1 is ϱ_1-Lipschitz continuous and f_1 is λ_{f_1}-Lipschitz continuous, utilizing Lemma 2.1, we get

$$\begin{aligned}
&\|P_1(f_1(x_n)) - P_1(f_1(x_{n-1})) - \lambda\Big(S_1(p_1(x_n), v_n) - S_1(p_1(x_{n-1}), v_n)\Big)\|_1^{q_1} \\
&\leq \|P_1(f_1(x_n)) - P_1(f_1(x_{n-1}))\|_1^{q_1} - q_1\lambda\langle S_1(p_1(x_n), v_n) - S_1(p_1(x_{n-1}), v_n), \\
&\quad J_{q_1}(\eta_1(x_n, x_{n-1}))\rangle_1 - q_1\lambda\langle S_1(p_1(x_n), v_n) - S_1(p_1(x_{n-1}), v_n), \\
&\quad J_{q_1}\Big(P_1(f_1(x_n)) - P_1(f_1(x_{n-1}))\Big) - J_{q_1}(\eta_1(x_n, x_{n-1}))\rangle_1 \\
&\quad + c_{q_1}\lambda^{q_1}\|S_1(p_1(x_n), v_n) - S_1(p_1(x_{n-1}), v_n)\|_1^{q_1} \\
&\leq \varrho_1^{q_1}\lambda_{f_1}^{q_1}\|x_n - x_{n-1}\|_1^{q_1} - q_1\lambda\Big(-\gamma_{S_1}\|S_1(p_1(x_n), v_n) - S_1(p_1(x_{n-1}), v_n)\|_1^{q_1} \\
&\quad + \delta_{S_1}\|x_n - x_{n-1}\|_1^{q_1}\Big) + q_1\lambda\langle S_1(p_1(x_n), v_n) - S_1(p_1(x_{n-1}), v_n), J_{q_1}(\eta_1(x_n, x_{n-1})) \\
&\quad - J_{q_1}\Big(P_1(f_1(x_n)) - P_1(f_1(x_{n-1}))\Big)\rangle_1 + c_{q_1}\lambda^{q_1}\varpi_1^{q_1}\|x_n - x_{n-1}\|_1^{q_1} \\
&\leq \varrho_1^{q_1}\lambda_{f_1}^{q_1}\|x_n - x_{n-1}\|_1^{q_1} + q_1\lambda\gamma_{S_1}\varpi_1^{q_1}\|x_n - x_{n-1}\|_1^{q_1} - q_1\lambda\delta_{S_1}\|x_n - x_{n-1}\|_1^{q_1} \\
&\quad + q_1\lambda\|S_1(p_1(x_n), v_n) - S_1(p_1(x_{n-1}), v_n)\|_1\Big(\|J_{q_1}(\eta_1(x_n, x_{n-1}))\|_1 \\
&\quad + \|J_{q_1}\Big(P_1(f_1(x_n)) - P_1(f_1(x_{n-1}))\Big)\|_1\Big) + c_{q_1}\lambda_n^{q_1}\varpi_1^{q_1}\|x_n - x_{n-1}\|_1^{q_1} \\
&\leq \Big(\varrho_1^{q_1}\lambda_{f_1}^{q_1} + q_1\lambda\gamma_{S_1}\varpi_1^{q_1} - q_1\lambda\delta_{S_1} + c_{q_1}\lambda^{q_1}\varpi_1^{q_1}\Big)\|x_n - x_{n-1}\|_1^{q_1} \\
&\quad + q_1\lambda\varpi_1\|x_n - x_{n-1}\|_1\Big(\|\eta_1(x_n, x_{n-1})\|_1^{q_1-1} + \|P_1(f_1(x_n)) - P_1(f_1(x_{n-1}))\|_1^{q_1-1}\Big) \\
&= \Big(\varrho_1^{q_1}\lambda_{f_1}^{q_1} + q_1\lambda\gamma_{S_1}\varpi_1^{q_1} - q_1\lambda\delta_{S_1} + c_{q_1}\lambda^{q_1}\varpi_1^{q_1}\Big)\|x_n - x_{n-1}\|_1^{q_1} \\
&\quad + q_1\lambda\varpi_1\|x_n - x_{n-1}\|_1\Big(\tau_1^{q_1-1}\|x_n - x_{n-1}\|_1^{q_1-1} + \varrho_1^{q_1-1}\lambda_{f_1}^{q_1-1}\|x_n - x_{n-1}\|_1^{q_1-1}\Big) \\
&= \Big(\varrho_1^{q_1}\lambda_{f_1}^{q_1} + q_1\lambda\gamma_{S_1}\varpi_1^{q_1} - q_1\lambda\delta_{S_1} + q_1\lambda\varpi_1\varrho_1^{q_1-1}\lambda_{f_1}^{q_1-1} + q_1\lambda\varpi_1\tau_1^{q_1-1} + c_{q_1}\lambda^{q_1}\varpi_1^{q_1}\Big)\|x_n - x_{n-1}\|_1^{q_1},
\end{aligned}$$

from which we deduce that for each $n \in \mathbb{N}$,

$$\begin{aligned}
&\|P_1(f_1(x_n)) - P_1(f_1(x_{n-1})) - \lambda_n\Big(S_1(p_1(x_n), v_n) - S_1(p_1(x_{n-1}), v_n)\Big)\|_1 \\
&\leq \mu_1\|x_n - x_{n-1}\|_1,
\end{aligned} \tag{10}$$

where

$$\mu_1 = \sqrt[q_1]{\varrho_1^{q_1}\lambda_{f_1}^{q_1} + q_1\lambda\gamma_{S_1}\varpi_1^{q_1} - q_1\lambda\delta_{S_1} + q_1\lambda\varpi_1\varrho_1^{q_1-1}\lambda_{f_1}^{q_1-1} + q_1\lambda\varpi_1\tau_1^{q_1-1} + c_{q_1}\lambda^{q_1}\varpi_1^{q_1}}.$$

In virtue of the facts that S_1 is ϖ_2-Lipschitz continuous with respect to p_1 in the second argument and F_2 is D_2-Lipschitz continuous with constant $\lambda_{D_{F_2}}$, by using (5), it follows that

$$\begin{aligned}
\|S_1(p_1(x_{n-1}), v_n) - S_1(p_1(x_{n-1}), v_{n-1})\|_1 &\leq \varpi_2\|v_n - v_{n-1}\|_2 \\
&\leq \varpi_2(1 + n^{-1})D_2(F_2(y_n), F_2(y_{n-1})) \\
&\leq \varpi_2\lambda_{D_{F_2}}(1 + n^{-1})\|y_n - y_{n-1}\|_2.
\end{aligned} \tag{11}$$

Taking into account that Q_1 is $(\lambda_{Q_1}, \lambda'_{Q_1})$-mixed Lipschitz continuous in the first and second arguments, by (5) and the facts that the mapping T_i is D_j-Lipschitz continuous with constant $\lambda_{D_{T_i}}$ for $i \in \{1, 2\}$ and

$j \in \{1,2\}\backslash\{i\}$, we obtain

$$\begin{aligned}\|Q_1(w_n,t_n) - Q_1(w_{n-1},t_{n-1})\|_1 &\le \lambda_{Q_1}\|w_n - w_{n-1}\|_2 + \lambda_{Q_1}\|t_n - t_{n-1}\|_1 \\ &\le \lambda_{Q_1}(1+n^{-1})D_2(T_1(y_n),T_1(y_{n-1})) \\ &\quad + \lambda_{Q_1}(1+n^{-1})D_1(T_2(x_n),T_2(x_{n-1})) \\ &\le \lambda_{Q_1}\lambda_{D_{T_1}}(1+n^{-1})\|y_n - y_{n-1}\|_2 \\ &\quad + \lambda_{Q_1}\lambda_{D_{T_2}}(1+n^{-1})\|x_n - x_{n-1}\|_1.\end{aligned} \tag{12}$$

Combining (8)–(12), we derive that for each $n \in \mathbb{N}$,

$$\begin{aligned}\|x_{n+1} - x_n\|_1 &\le (1-\alpha_1)\|x_n - x_{n-1}\|_1 + \alpha_1 \sqrt[q_1]{1 - q_1\delta_1 + (q_1\xi_1 + c_{q_1})\lambda_{f_1}^{q_1}}\|x_n - x_{n-1}\|_1 \\ &\quad + \frac{\alpha_1\tau_1^{q_1-1}}{\theta_1}\Big(\mu_1\|x_n - x_{n-1}\|_1 + \lambda\varpi_2\lambda_{D_{F_2}}(1+n^{-1})\|y_n - y_{n-1}\|_2 \\ &\quad + \lambda\lambda_{Q_1}\lambda_{D_{T_1}}(1+n^{-1})\|y_n - y_{n-1}\|_2 + \lambda\lambda_{Q_1}\lambda_{D_{T_2}}(1+n^{-1})\|x_n - x_{n-1}\|_1\Big) \\ &\quad + \alpha_1\|e_n - e_{n-1}\|_1 + \|r_n - r_{n-1}\|_1 \\ &= (1-\alpha_1)\|x_n - x_{n-1}\|_1 + \alpha_1 \sqrt[q_1]{1 - q_1\delta_1 + (q_1\xi_1 + c_{q_1})\lambda_{f_1}^{q_1}} \\ &\quad + \frac{\alpha_1\tau_1^{q_1-1}}{\theta_1}\Big(\mu_1 + \lambda\lambda_{Q_1}\lambda_{D_{T_2}}(1+n^{-1})\Big)\|x_n - x_{n-1}\|_1 \\ &\quad + \frac{\alpha_1\tau_1^{q_1-1}\lambda(\varpi_2\lambda_{D_{F_2}} + \lambda_{Q_1}\lambda_{D_{T_1}})(1+n^{-1})}{\theta_1}\|y_n - y_{n-1}\|_2 \\ &\quad + \alpha_1\|e_n - e_{n-1}\|_1 + \|r_n - r_{n-1}\|_1 \\ &= \Lambda_1(n)\|x_n - x_{n-1}\|_1 + \Gamma_1(n)\|y_n - y_{n-1}\|_2 + \alpha_1\|e_n - e_{n-1}\|_1 + \|r_n - r_{n-1}\|_1,\end{aligned} \tag{13}$$

where for each $n \in \mathbb{N}$,

$$\begin{aligned}\Lambda_1(n) &= 1 - \alpha_1 + \alpha_1 \sqrt[q_1]{1 - q_1\delta_1 + (q_1\xi_1 + c_{q_1})\lambda_{f_1}^{q_1}} + \frac{\alpha_1\tau_1^{q_1-1}}{\theta_1}(\mu_1 + \lambda\lambda_{Q_1}\lambda_{D_{T_2}}(1+n^{-1})), \\ \Gamma_1(n) &= \frac{\alpha_1\tau_1^{q_1-1}\lambda(\varpi_2\lambda_{D_{F_2}} + \lambda_{Q_1}\lambda_{D_{T_1}})(1+n^{-1})}{\theta_1}.\end{aligned}$$

In a similar manner, employing (5) and the assumptions, one can obtain

$$\begin{aligned}\|y_{n+1} - y_n\|_2 &\le \Lambda_2(n)\|x_n - x_{n-1}\|_1 + \Gamma_2(n)\|y_n - y_{n-1}\|_2 \\ &\quad + \alpha_2\|l_n - l_{n-1}\|_2 + \|k_n - k_{n-1}\|_2,\end{aligned} \tag{14}$$

where for each $n \in \mathbb{N}$,

$$\begin{aligned}\Lambda_2(n) &= \frac{\alpha_2\tau_2^{q_2-1}\rho(\pi_1\lambda_{D_{F_1}} + \lambda_{Q_2}\lambda_{D_{T_2}})(1+n^{-1})}{\theta_2}, \\ \Gamma_2(n) &= 1 - \alpha_2 + \alpha_2 \sqrt[q_2]{1 - q_2\delta_2 + (q_2\xi_2 + c_{q_2})\lambda_{f_2}^{q_2}} + \frac{\alpha_2\tau_2^{q_2-1}}{\theta_2}(\mu_2 + \rho\lambda_{Q_2}\lambda_{D_{T_1}}(1+n^{-1})), \\ \mu_2 &= \sqrt[q_2]{\varrho_2^{q_2}\lambda_{f_2}^{q_2} + q_2\rho\gamma_{S_2}\pi_2^{q_2} - q_2\rho\delta_{S_2} + q_2\rho\pi_2\varrho_2^{q_2-1}\lambda_{f_2}^{q_2-1} + q_2\rho\pi_2\tau_2^{q_2-1} + c_{q_2}\rho^{q_2}\pi_2^{q_2}}.\end{aligned}$$

Let us now define a norm $\|.\|_*$ on $X_1 \times X_2$ by

$$\|(u,v)\|_* = \|u\|_1 + \|v\|_2, \quad \forall (u,v) \in X_1 \times X_2.$$

It is easy to see that $(X_1 \times X_2, \|.\|_*)$ is a Banach space. Then by using (13) and (14), and picking $\alpha = \alpha_1 + \alpha_2$, we obtain

$$\begin{aligned}\|(x_{n+1}, y_{n+1}) - (x_n, y_n)\|_* &= \|x_{n+1} - x_n\|_1 + \|y_{n+1} - y_n\|_2 \\ &\leq (\Lambda_1(n) + \Lambda_2(n))\|x_n - x_{n-1}\|_1 + (\Gamma_1(n) + \Gamma_2(n))\|y_n - y_{n-1}\|_2 \\ &\quad + (\alpha_1 + \alpha_2)(\|e_n - e_{n-1}\|_1 + \|l_n - l_{n-1}\|_2) \\ &\quad + \|r_n - r_{n-1}\|_1 + \|k_n - k_{n-1}\|_2 \\ &\leq \vartheta(n)\|(x_n, y_n) - (x_{n-1}, y_{n-1})\|_* + \alpha\|(e_n, l_n) - (e_{n-1}, l_{n-1})\|_* \\ &\quad + \|(r_n, k_n) - (r_{n-1}, k_{n-1})\|_*,\end{aligned} \tag{15}$$

where for each $n \in \mathbb{N}$, $\vartheta(n) = \max\{\Lambda_1(n) + \Lambda_2(n), \Gamma_1(n) + \Gamma_2(n)\}$. In the light of the facts that $\Lambda_i(n) \to \Lambda_i$ and $\Gamma_i(n) \to \Gamma_i$, as $n \to \infty$, where

$$\begin{aligned}\Lambda_1 &= 1 - \alpha_1 + \alpha_1 \sqrt[q_1]{1 - q_1\delta_1 + (q_1\xi_1 + c_{q_1})\lambda_{f_1}^{q_1}} + \frac{\alpha_1\tau_1^{q_1-1}}{\theta_1}(\mu_1 + \lambda\lambda_{Q_1}\lambda_{D_{T_2}}), \\ \Lambda_2 &= \frac{\alpha_2\tau_2^{q_2-1}\rho(\pi_1\lambda_{D_{F_1}} + \lambda_{Q_2}\lambda_{D_{T_2}})}{\theta_2}, \ \Gamma_1 = \frac{\alpha_1\tau_1^{q_1-1}\lambda(\varpi_2\lambda_{D_{F_2}} + \lambda_{Q_1}\lambda_{D_{T_1}})}{\theta_1}, \\ \Gamma_2 &= 1 - \alpha_2 + \alpha_2 \sqrt[q_2]{1 - q_2\delta_2 + (q_2\xi_2 + c_{q_2})\lambda_{f_2}^{q_2}} + \frac{\alpha_2\tau_2^{q_2-1}}{\theta_2}(\mu_2 + \rho\lambda_{Q_2}\lambda_{D_{T_1}}),\end{aligned}$$

we deduce that $\vartheta(n) \to \vartheta$, as $n \to \infty$, where $\vartheta = \max\{\Lambda_1 + \Lambda_2, \Gamma_1 + \Gamma_2\}$. Clearly, with the help of (7) we infer that $\vartheta \in (0,1)$, and so there exists $\hat{\vartheta} \in (0,1)$ (take $\hat{\vartheta} = \frac{\vartheta+1}{2} \in (\vartheta, 1)$) and $n_0 \in \mathbb{N}$ such that $\vartheta(n) \leq \hat{\vartheta}$, for all $n \geq n_0$. Then, for all $n > n_0$, by (15), it follows that

$$\begin{aligned}\|(x_{n+1}, y_{n+1}) - (x_n, y_n)\|_* &\leq \hat{\vartheta}\|(x_n, y_n) - (x_{n-1}, y_{n-1})\|_* + \alpha\|(e_n, l_n) - (e_{n-1}, l_{n-1})\|_* \\ &\quad + \|(r_n, k_n) - (r_{n-1}, k_{n-1})\|_* \\ &\leq \hat{\vartheta}[\hat{\vartheta}\|(x_{n-1}, y_{n-1}) - (x_{n-2}, y_{n-2})\|_* + \alpha\|(e_{n-1}, l_{n-1}) - (e_{n-2}, l_{n-2})\|_* \\ &\quad + \|(r_{n-1}, k_{n-1}) - (r_{n-2}, k_{n-2})\|_*] \\ &\quad + \alpha\|(e_n, l_n) - (e_{n-1}, l_{n-1})\|_* + \|(r_n, k_n) - (r_{n-1}, k_{n-1})\|_* \\ &= \hat{\vartheta}^2\|(x_{n-1}, y_{n-1}) - (x_{n-2}, y_{n-2})\|_* + \alpha(\hat{\vartheta}\|(e_{n-1}, l_{n-1}) - (e_{n-2}, l_{n-2})\|_* \\ &\quad + \|(e_n, l_n) - (e_{n-1}, l_{n-1})\|_*) + \hat{\vartheta}\|(r_{n-1}, k_{n-1}) - (r_{n-2}, k_{n-2})\|_* \\ &\quad + \|(r_n, k_n) - (r_{n-1}, k_{n-1})\|_* \\ &\leq \ldots \\ &\leq \hat{\vartheta}^{n-n_0}\|(x_{n_0+1}, y_{n_0+1}) - (x_{n_0}, y_{n_0})\|_* \\ &\quad + \alpha\sum_{j=1}^{n-n_0}\hat{\vartheta}^{j-1}\|(l_{n-(j-1)}, l_{n-(j-1)}) - (e_{n-j}, l_{n-j})\|_* \\ &\quad + \sum_{j=1}^{n-n_0}\hat{\vartheta}^{j-1}\|(r_{n-(j-1)}, k_{n-(j-1)}) - (r_{n-j}, k_{n-j})\|_*.\end{aligned} \tag{16}$$

Making use of (16), for any $m \geq n > n_0$, we obtain

$$\begin{aligned}\|(x_m, y_m) - (x_n, y_n)\|_* &\leq \sum_{i=n}^{m-1} \|(x_{i+1}, y_{i+1}) - (x_i, y_i)\|_* \\ &\leq \sum_{i=n}^{m-1} \hat{\vartheta}^{i-n_0} \|(x_{n_0+1}, y_{n_0+1}) - (x_{n_0}, y_{n_0})\|_* \\ &\quad + \alpha \sum_{i=n}^{m-1} \sum_{j=1}^{i-n_0} \hat{\vartheta}^{j-1} \|(e_{i-(j-1)}, l_{i-(j-1)}) - (e_{i-j}, l_{i-j})\|_* \\ &\quad + \sum_{i=n}^{m-1} \sum_{j=1}^{i-n_0} \hat{\vartheta}^{j-1} \|(r_{i-(j-1)}, k_{i-(j-1)}) - (r_{i-j}, k_{i-j})\|_*. \end{aligned} \tag{17}$$

Since $\hat{\vartheta} < 1$, (6) and (17) guarantee that $\|(x_m, y_m) - (x_n, y_n)\|_* \to 0$, as $n \to \infty$, and so $\{(x_n, y_n)\}_{n=0}^{\infty}$ is a Cauchy sequence in $X_1 \times X_2$. In view of the completeness of $X_1 \times X_2$, there exists $(x, y) \in X_1 \times X_2$ such that $(x_n, y_n) \to (x, y)$, as $n \to \infty$. By (5) and in virtue of the facts that for each $i \in \{1, 2\}$, the mapping F_i is D_i-Lipschitz continuous with constant $\lambda_{D_{F_i}}$, and the mapping T_i is D_j-Lipschitz continuous with constant $\lambda_{D_{T_i}}$ for $j \in \{1, 2\}\backslash\{i\}$, we get

$$\begin{aligned}\|u_{n+1} - u_n\|_1 &\leq (1 + (1+n)^{-1}) D_1(F_1(x_{n+1}), F_1(x_n)) \leq (1 + (1+n)^{-1}) \lambda_{D_{F_1}} \|x_{n+1} - x_n\|_1, \\ \|v_{n+1} - v_n\|_2 &\leq (1 + (1+n)^{-1}) D_2(F_2(y_{n+1}), F_2(y_n)) \leq (1 + (1+n)^{-1}) \lambda_{D_{F_2}} \|y_{n+1} - y_n\|_2, \\ \|w_{n+1} - w_n\|_2 &\leq (1 + (1+n)^{-1}) D_2(T_1(y_{n+1}), T_1(y_n)) \leq (1 + (1+n)^{-1}) \lambda_{D_{T_1}} \|y_{n+1} - y_n\|_2, \\ \|t_{n+1} - t_n\|_1 &\leq (1 + (1+n)^{-1}) D_1(T_2(x_{n+1}), T_2(x_n)) \leq (1 + (1+n)^{-1}) \lambda_{D_{T_2}} \|x_{n+1} - x_n\|_1. \end{aligned}$$

The above relations imply that the sequences $\{u_n\}_{n=0}^{\infty}, \{t_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}, \{w_n\}_{n=0}^{\infty}$ are also Cauchy in X_1 and X_2, respectively. Thus, there are $u, t \in X_1$ and $v, w \in X_2$ such that $u_n \to u$, $t_n \to t$, $v_n \to v$ and $w_n \to w$, as $n \to \infty$. We now show that $u \in F_1(x)$. Since for each $n \geq 0$, $u_n \in F_1(x_n)$, applying (5) and considering the fact that F_1 is D_1-Lipschitz continuous with constant $\lambda_{D_{F_1}}$, we have

$$\begin{aligned} d_1(u, F_1(x)) &= \inf\{\|u - z\| : z \in F_1(x)\} \\ &\leq \|u - u_n\| + d_1(u_n, F_1(x)) \\ &\leq \|u - u_n\| + D_1(F_1(x_n), F_1(x)) \\ &\leq \|u - u_n\| + \lambda_{D_{F_1}} \|x_n - x\|, \end{aligned}$$

where d_1 is the metric induced by the norm $\|.\|_1$ in X_1. The right-hand side of the above inequality tends to zero, as $n \to \infty$. Since $F_1(x)$ is closed, we deduce that $u \in F_1(x)$. In a similar fashion to the preceding analysis, one can show that $v \in F_2(y)$, $w \in T_1(y)$ and $t \in T_2(x)$. Owing to the facts that the mappings $R_{M_1,\lambda}^{P_1,\eta_1}, R_{M_1,\lambda}^{P_1,\eta_1}, P_i, S_i, Q_i, f_i$ and p_i $(i = 1, 2)$ are continuous, it follows from (5) and (8) that

$$\begin{cases} f_1(x) = R_{M_1,\lambda}^{P_1,\eta_1}[P_1(f_1(x)) - \lambda(S_1(p_1(x), v) + Q_1(w, t))], \\ f_2(y) = R_{M_2,\rho}^{P_2,\eta_2}[P_2(f_2(y)) - \rho(S_2(u, p_2(y)) + Q_2(t, w))]. \end{cases}$$

Now, Lemma 3.1 guarantees that (x, y, u, v, t, w) is a solution of the SGMNVI (2). This completes the proof. □

We obtain the following corollary as a direct consequence of the above theorem immediately.

Corollary 3.11. *Assume that, for each $i \in \{1, 2\}$, X_i is a q_i-uniformly smooth Banach space with $q_i > 1$, $\eta_i : X_i \times X_i \to X_i$ is a τ_i-Lipschitz continuous mapping and $P_i : X_i \to X_i$ is a θ_i-strongly η_i-accretive and ϱ_i-Lipschitz continuous mapping. Let $M : X_1 \to 2^{X_1}$ be a P_1-η_1-accretive mapping and $N : X_2 \to 2^{X_2}$ be a P_2-η_2-accretive mapping. Let $f : X_1 \to X_1$ be a δ_1-strongly accretive and λ_f-Lipschitz continuous mapping, $g : X_2 \to X_2$ be a δ_2-strongly accretive* *and λ_g-Lipschitz continuous mapping. Suppose that the mapping $S : X_1 \times X_2 \to X_1$ is δ_S-strongly η_1-accretive and λ_{S_p}-Lipschitz continuous with respect to p in the first argument and λ_{S_2}-Lipschitz continuous with respect to p in the second argument, and the mapping $T : X_2 \times X_2 \to X_2$ is δ_T-strongly η_2-accretive and λ_{T_d}-Lipschitz continuous with respect to d in the second argument and λ_{T_1}-Lipschitz continuous with respect to d in the first argument. Assume that the mapping $E : X_1 \to CB(X_1)$ is D_1-Lipschitz continuous with constant λ_{D_E} and the mapping $F : X_2 \to CB(X_2)$ is D_2-Lipschitz continuous with constant λ_{D_F}. If there exist two constants $\lambda, \rho > 0$ such that*

$$\begin{cases} 1 - \alpha_1 + \alpha_1 \sqrt[q]{1 - q\delta_1 + c_q\lambda_f^q} + \frac{\alpha_1\tau_1^{q-1}}{\theta_1}\theta' + \frac{\alpha_2\tau_2^{q-1}\rho\lambda_{T_1}\lambda_{D_E}}{\theta_2} < 1, \\ 1 - \alpha_2 + \alpha_2 \sqrt[q]{1 - q\delta_2 + c_q\lambda_g^q} + \frac{\alpha_2\tau_2^{q-1}}{\theta_2}\theta'' + \frac{\alpha_1\tau_1^{q-1}\lambda\lambda_{S_2}\lambda_{D_F}}{\theta_1} < 1, \end{cases} \tag{18}$$

where

$$\theta' = \sqrt[q]{\varrho_1^q\lambda_f^q - q\lambda\delta_S + q\lambda\lambda_{S_p}\varrho_1^{q-1}\lambda_f^{q-1} + q\lambda\lambda_{S_p}\tau_1^{q-1} + c_q\lambda^q\lambda_{S_p}^q},$$
$$\theta'' = \sqrt[q]{\varrho_2^q\lambda_g^q - q\rho\delta_T + q\rho\lambda_{T_d}\varrho_2^{q-1}\lambda_g^{q-1} + q\rho\lambda_{T_d}\tau_2^{q-1} + c_q\rho^q\lambda_{T_d}^q},$$

and for the case where q is an even natural number, in addition to (18)*, the following conditions hold:*

$$\begin{cases} q\delta_1 < 1 + c_q\lambda_f^q,\ q\delta_2 \le 1 + c_q\lambda_g^q, \\ q\lambda\delta_S < \varrho_1^q\lambda_f^q + q\lambda\lambda_{S_p}\varrho_1^{q-1}\lambda_f^{q-1} + q\lambda\lambda_{S_p}\tau_1^{q-1} + c_q\lambda^q\lambda_{S_p}^q, \\ q\rho\delta_T < \varrho_2^q\lambda_g^q + q\rho\lambda_{T_d}\varrho_2^{q-1}\lambda_g^{q-1} + q\rho\lambda_{T_d}\tau_2^{q-1} + c_q\rho^q\lambda_{T_d}^q, \end{cases}$$

where c_q is a constant guaranteed by Lemma 2.1. Then the iterative sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ generated by Algorithm 3.5 converge strongly to x, y, u and v, respectively, and (x, y, u, v) is a solution of the system (3).

4. Remarks on $H(.,.)$-η-cocoercive mappings

In the present section, the notion of $H(.,.)$-η-cocoercive operator and the results in related to it, introduced and studied in [3] are investigated and analyzed, and some remarks on $H(.,.)$-η-cocoercive operators are stated. We also show that one can obtain the results given in [3] using the results derived in Section 3.

Definition 4.1. [3, Definition 2.4] *Let X be a q-uniformly smooth Banach space with $q > 1$. A multi-valued mapping $M : X \to 2^X$ is said to be η-cocoercive* (or γ-η-cocoercive), if there exists a constant $\gamma > 0$ such that

$$\langle u - v, J_q(\eta(x, y))\rangle \ge \gamma\|u - v\|^q, \quad \forall x, y \in X, u \in M(x), v \in M(y).$$

Obviously, for a given vector-valued mapping $\eta : X \times X \to X$, every η-cocoercive multi-valued mapping is η-accretive, but the coverse is not in general true. The following example illustrates that for given constant $\gamma > 0$ and a vector-valued mapping $\eta : X \times X \to X$, an η-accretive multi-valued mapping is not γ-η-cocoercive necessarily.

Example 4.2. Let $D_n(\mathbb{R})$ be the same as in Example 2.6 and let the mappings $M : D_n(\mathbb{R}) \to 2^{D_n(\mathbb{R})}$ and $\eta : D_n(\mathbb{R}) \times D_n(\mathbb{R}) \to D_n(\mathbb{R})$ be defined by

$$M(A) = \begin{cases} \{E_{ij} - E_{kk} : i, j = 1, 2, \ldots, n\}, & A = E_{kk}, \\ \gamma A + E_{kk}, & A \neq E_{kk}, \end{cases}$$

and

$$\eta(A, B) = \begin{cases} Q, & A, B \neq E_{kk}, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

for all $A = \begin{pmatrix} a_{ij} \end{pmatrix}, B = \begin{pmatrix} b_{ij} \end{pmatrix} \in D_n(\mathbb{R})$, where $Q = \begin{pmatrix} q_{ij} \end{pmatrix}$ is an $n \times n$ matrix with the entries

$$q_{ij} = \begin{cases} \alpha_i(b_{ii} - a_{ii}), & i = j, \\ 0, & i \neq j, \end{cases}$$

$\alpha_i (i = 1, 2, \dots, n), \gamma \in \mathbb{R}$ are arbitrary but fixed constants such that for each $i \in \{1, 2, \dots, n\}$, $\gamma < 0 < \alpha_i$, $\mathbf{0}$ is the zero $n \times n$ matrix, and $E_{i,j}, E_{kk}$ are the same as in Example 2.6.

Then for any $A = \begin{pmatrix} a_{ij} \end{pmatrix}, B = \begin{pmatrix} b_{ij} \end{pmatrix} \in D_n(\mathbb{R})$, $A \neq B \neq E_{kk}$, taking into account that $\gamma < 0 < \alpha_i$ for each $i \in \{1, 2, \dots, n\}$, it follows that

$$\begin{aligned} \langle M(A) - M(B), J_2(\eta(A, B)) \rangle &= \langle M(A) - M(B), \eta(A, B) \rangle \\ &= tr(\gamma(A - B)Q) = \sum_{i=1}^{n} -\gamma\alpha_i(b_{ii} - a_{ii})^2 > 0. \end{aligned} \tag{19}$$

In the meanwhile, for each of the cases when $A \neq B = E_{kk}$, $B \neq A = E_{kk}$ and $A = B = E_{kk}$, thanks to the fact that $\eta(A, B) = \mathbf{0}$, we deduce that

$$\langle u - v, J_2(\eta(A, B)) \rangle = 0, \quad \forall u \in M(A), v \in M(B).$$

Consequently, M is an η-accretive mapping. Furthermore, for any $A = \begin{pmatrix} a_{ij} \end{pmatrix}, B = \begin{pmatrix} b_{ij} \end{pmatrix} \in D_n(\mathbb{R})$, we obtain

$$\|A - B\|^2 = \langle A - B, A - B \rangle = tr((A - B)(A - B)) = \sum_{i=1}^{n} (a_{ii} - b_{ii})^2. \tag{20}$$

Letting $\varrho = \max\{\alpha_i : i = 1, 2, \dots, n\}$ and making use of (19) and (20), for any $A = \begin{pmatrix} a_{ij} \end{pmatrix}, B = \begin{pmatrix} b_{ij} \end{pmatrix} \in D_n(\mathbb{R})$, $A \neq B \neq E_{kk}$, it yields

$$\langle M(A) - M(B), J_2(\eta(A, B)) \rangle = \sum_{i=1}^{n} -\gamma\alpha_i(b_{ii} - a_{ii})^2 \leq -\gamma\varrho \sum_{i=1}^{n} (a_{ii} - b_{ii})^2 = -\gamma\varrho\|A - B\|^2,$$

and so M is not μ-η-cocoercive for all $\mu > -\gamma\varrho$.

Definition 4.3. [3, Definition 2.3] *Let X be a q-uniformly smooth Banach space with $q > 1$. Let $A, B : X \to X$, $H : X \times X \to X$, $\eta : X \times X \to X$ be the mappings and $J_q : X \to 2^{X^*}$ be the generalized duality mapping. Then*

(i) *$H(A, .)$ is said to be μ-η-cocoercive with respect to A if there exists a constant $\mu > 0$ such that*

$$\langle H(Ax, u) - H(Ay, u), J_q(\eta(x, y)) \rangle \geq \mu\|Ax - Ay\|^q, \quad \forall x, y, u \in X;$$

(ii) *$H(., B)$ is said to be γ-relaxed η-cocoercive (also referred to as γ-η-relaxed cocoercive, see, [3]) if there exists a constant $\gamma > 0$ such that*

$$\langle H(u, Bx) - H(u, By), J_q(\eta(x, y)) \rangle \geq -\gamma\|Bx - By\|^q, \quad \forall x, y, u \in X;$$

(iii) *$H(A, .)$ is said to be r_1-Lipschitz continuous with respect to A if there exists a constant $r_1 > 0$ such that*

$$\|H(Ax, u) - H(Ay, u)\| \leq r_1\|x - y\|, \quad \forall x, y, u \in X;$$

(iv) *$H(., B)$ is said to be r_2-Lipschitz continuous with respect to B if there exists a constant $r_2 > 0$ such that*

$$\|H(u, Bx) - H(u, By)\| \leq r_2\|x - y\|, \quad \forall x, y, u \in X.$$

In related to Definition 4.3, the authors [3] presented a Matlab programme and claimed that $H(., .)$ is $\frac{1}{3}$-η-cocoercive with respect to A and $\frac{1}{2}$-relaxed η-cocoercive with respect to B.

Example 4.4. Let $X = \mathbb{R}^2$ with usual inner product, and let $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $A(x_1, x_2) = (x_1, 3x_2)$ and $B(y_1, y_2) = (-y_1, -y_1 - y_2)$, for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Let $H(A, B), \eta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $H(Ax, By) = Ax + By$ and $\eta(x, y) = x - y$ for all $x, y \in \mathbb{R}^2$. The Hilbert space $\mathbb{R}^2$ is a 2-uniformly smooth Banach space due to the fact that it is finite dimensional. Then, for all $x, y, u \in \mathbb{R}^2$, we obtain

$$\begin{aligned}\langle H(Ax, u) - H(Ay, u), J_2(\eta(x, y))\rangle &= \langle Ax - Ay, x - y\rangle \\ &= \langle (x_1, 3x_2) - (y_1, 3y_2), (x_1 - y_1, x_2 - y_2)\rangle \\ &= \langle (x_1 - y_1, 3(x_2 - y_2)), (x_1 - y_1, x_2 - y_2)\rangle \\ &= (x_1 - y_1)^2 + 3(x_2 - y_2)^2\end{aligned}$$

and

$$\begin{aligned}\|Ax - Ay\|^2 = \langle Ax - Ay, Ax - Ay\rangle &= \langle (x_1 - y_1, 3(x_2 - y_2)), (x_1 - y_1, 3(x_2 - y_2))\rangle \\ &= (x_1 - y_1)^2 + 9(x_2 - y_2)^2 \\ &\leq 3(x_1 - y_1)^2 + 9(x_2 - y_2)^2 \\ &= 3\langle H(Ax, u) - H(Ay, u), J_2(\eta(x, y))\rangle,\end{aligned}$$

which implies that

$$\langle H(Ax, u) - H(Ay, u), J_2(\eta(x, y))\rangle \geq \frac{1}{3}\|Ax - Ay\|^2,$$

that is, $H(.,.)$ is $\frac{1}{3}$-η-cocoercive with respect to A. The authors claimed that $H(.,.)$ is $\frac{1}{2}$-relaxed η-cocoercive with respect to B. A careful checking illustrates that this fact is not true in general. In fact, in the light of Definition 4.6, $H(.,.)$ is $\frac{1}{2}$-relaxed η-cocoercive with respect to B if and only if

$$\langle H(u, Bx) - H(u, By), J_2(\eta(x, y))\rangle \geq -\frac{1}{2}\|Bx - By\|^2, \quad \forall x, y, u \in \mathbb{R}^2.$$

In view of the definitions of the mappings H, η and B, for all $x = (x_1, x_2), y = (y_1, y_2), u \in \mathbb{R}^2$, it yields

$$\begin{aligned}\langle H(u, Bx) - H(u, By), J_2(\eta(x, y))\rangle &= \langle Bx - By, x - y\rangle \\ &= \langle (y_1 - x_1, y_1 - x_1 + y_2 - x_2), (x_1 - y_1, x_2 - y_2)\rangle \\ &= -(x_1 - y_1)^2 - (x_1 - y_1)(x_2 - y_2) - (x_2 - y_2)^2 \\ &= -\{(x_1 - y_1)^2 + (x_1 - y_1)(x_2 - y_2) + (x_2 - y_2)^2\}\end{aligned}$$

and

$$\begin{aligned}\|Bx - By\|^2 &= \langle Bx - By, Bx - By\rangle \\ &= \langle (y_1 - x_1, y_1 - x_1 + y_2 - x_2), (y_1 - x_1, y_1 - x_1 + y_2 - x_2)\rangle \\ &= 2(y_1 - x_1)^2 + 2(y_1 - x_1)(y_2 - x_2) + (y_2 - x_2)^2 \\ &\leq 2\{(y_1 - x_1)^2 + (y_1 - x_1)(y_2 - x_2) + (y_2 - x_2)^2\} \\ &= -2\langle H(u, Bx) - H(u, By), J_2(\eta(x, y))\rangle,\end{aligned}$$

whence we deduce that

$$\langle H(u, Bx) - H(u, By), J_2(\eta(x, y))\rangle \leq -\frac{1}{2}\|Bx - By\|^2, \quad \forall x, y, u \in \mathbb{R}^2.$$

The preceding inequality shows that contrary to the claim in [3], $H(.,.)$ is not $\frac{1}{2}$-relaxed η-cocoercive with respect to B necessarily.

Proposition 4.5. *Let X be a q-uniformly smooth Banach space with $q > 1$, and let $A, B : X \to X$ and $H, \eta : X \times X \to X$ be the mappings. Suppose further that the mapping $P : X \times X \to X$ is defined by $P(x) = H(Ax, Bx)$, for all $x \in X$. Then, the following assertions hold:*

(i) *If the mapping $H(A,B)$ is μ-η-cocoercive with respect to A and γ-relaxed η-cocoercive with respect to B, the mapping A is α-expansive and B is β-Lipschitz continuous, $\mu > \gamma$ and $\alpha > \beta$, then P is $(\mu\alpha^q - \gamma\beta^q)$-strongly η-accretive and hence it is strictly η-accretive.*

(ii) *If $H(A,B)$ is r_1-Lipschitz continuous with respect to A and r_2-Lipschitz continuous with respect to B, then P is $(r_1 + r_2)$-Lipschitz continuous.*

Proof. (i) Owing to the fact that the mapping $H(A,B)$ is μ-η-cocoercive with respect to A and γ-relaxed η-cocoercive with respect to B, the mapping A is α-expansive and B is β-Lipschitz continuous, $\mu > \gamma$ and $\alpha > \beta$, for all $x, y \in X$, we obtain

$$\begin{aligned}\langle P(x) - P(y), \eta(x,y)\rangle &= \langle H(Ax,Bx) - H(Ay,By), \eta(x,y)\rangle \\ &= \langle H(Ax,Bx) - H(Ay,Bx), \eta(x,y)\rangle \\ &\quad + \langle H(Ay,Bx) - H(Ay,By), \eta(x,y)\rangle \\ &\geq \mu\|Ax - Ay\|^q - \gamma\|Bx - By\|^q \\ &\geq \mu\alpha^q\|x-y\|^q - \gamma\beta^q\|x-y\|^q \\ &= (\mu\alpha^q - \gamma\beta^q)\|x-y\|^q.\end{aligned}$$

Since $\mu > \gamma$, $\alpha > \beta$ and $q > 1$, the preceding inequality guarantees that P is $(\mu\alpha^q - \gamma\beta^q)$-strongly η-accretive. Now, the fact that P is strictly η-accretive is straightforward.

(ii) Relying on the fact that $H(A,B)$ is r_1-Lipschitz continuous and r_2-Lipschitz continuous with respect to A and B, respectively, it follows that for all $x, y \in X$,

$$\begin{aligned}\|P(x) - P(y)\| &= \|H(Ax,Bx) - H(Ay,By)\| \\ &\leq \|H(Ax,Bx) - H(Ay,Bx)\| \\ &\quad + \|H(Ay,Bx) - H(Ay,By)\| \\ &\leq (r_1 + r_2)\|x-y\|,\end{aligned}$$

that is, P is $(r_1 + r_2)$-Lipschitz continuous. This completes the proof. □

Ahmad et al. [3] introduced and studied a class of accretive mappings the so-called $H(.,.)$-η-cocoercive mappings as a generalization of P-η-accretive (or (H,η)-accretive) and $H(.,.)$-accretive mappings as follows.

Definition 4.6. [3, Definition 2.6] *Let X be a q-uniformly smooth Banach space with $q > 1$. Let $A, B : X \to X$, $H : X \times X \to X$, $\eta : X \times X \to X$ be the mappings. Then a multi-valued mapping $M : X \to 2^X$ is said to be $H(.,.)$-η-cocoercive with respect to the mappings A and B if M is η-cocoercive and $(H(A,B) + \lambda M)(X) = X$, for all $\lambda > 0$.*

From Definition 4.6 and in the light of the mentioned arguments, it follows that every $H(.,.)$-η-cocoercive mapping is actually a P-η-accretive mapping. In fact, by defining the mapping $P : X \to X$ as $P(x) = H(Ax,Bx)$, for all $x \in X$, and in view of the fact that every η-cocercive mapping is η-accretive, we deduce that the class of $H(.,.)$-η-cocoercive mappings coicides exactly with the class of P-η-accretive mappings and is not new. In other words, Definition 4.6 is actually the same Definition 2.5 and is not a new one.

In order to define the proximal mapping associated with the $H(.,.)$-η-cocoercive mappings, Ahmad et al. [3] presented the following theorem which states conditions under which the mapping $(H(A,B)+\lambda M)^{-1}$ is single-valued for every $\lambda > 0$.

Theorem 4.7. [3, Theorem 2.7] *Let X be a q-uniformly smooth Banach space with $q > 1$. Let $H(A,B)$ be μ-η-cocoercive with respect to A and γ-relaxed η-cocoercive with respect to B, A be α-expansive, B be β-Lipschit continuous, $\mu > \gamma$ and $\alpha > \beta$. Let M be an $H(.,.)$-η-cocoercive mapping with respect to A and B. Then the mapping $(H(A,B) + \lambda M)^{-1}$ is single-valued for every real constant $\lambda > 0$.*

Proof. Define $P : X \to X$ by $P(x) = H(Ax, Bx)$, for all $x \in X$. Thanks to the assumptions and by means of Proposition 4.5(i), we deduce that P is a strictly η-accretive mapping. Furthermore, M is a P-η-accretive mapping. We note that all the conditions of Lemma 2.1 hold. In accordance with Lemma 2.1, the mapping $(P + \lambda)^{-1} = (H(A, B) + \lambda)^{-1}$ is single-valued for every $\lambda > 0$. This gives the desired result. □

Based on Theorem 4.7, the authors [3] defined the proximal mapping $R^{H(.,.)-\eta}_{\lambda,M}$ associated with the $H(.,.)$-η-cocoercive mapping M as follows.

Definition 4.8. [3, Definition 2.8] *Let X be a q-uniformly smooth Banach space with $q > 1$. Let $H(A, B)$ be μ-η-cocoercive with respect to A and γ-relaxed η-cocoercive with respect to B. Suppose that A is α-expansive, B is β-Lipschitz continuous and $\mu > \gamma$, $\alpha > \beta$. Let M be an $H(.,.)$-η-cocoercive mapping with respect to A and B. Then the proximal mapping $R^{H(.,.)-\eta}_{\lambda,M} : X \to X$ is defined by*

$$R^{H(.,.)-\eta}_{\lambda,M}(u) = (H(A, B) + \lambda M)^{-1}(u), \quad \forall u \in X.$$

Remark 4.9. (i) In Theorem 4.7, the necessary and sufficient conditions for the mapping $(H(.,.) + \lambda M)^{-1}$ to be single-valued for every $\lambda > 0$, are stated. In the light of the mentioned theorem, and by comparing it with Definition 4.8, it should be pointed out that the τ-Lipschitz continuity condition of the mapping $\eta : X \times X \to X$, mentioned in the context of Definition 2.8 of [3] is extra and must be deleted, as we have done in Definition 4.8.

(ii) By defining $P : X \to X$ as $P(x) = H(Ax, Bx)$, for all $x \in X$, in virtue of the assumptions of Definition 4.8 and by using Proposition 4.5(i), P is a strictly η-accretive mapping and M is a P-η-accretive mapping. Regarding to Definition 2.14, for any constant $\lambda > 0$, the P-η-resolvent operator $R^{P,\eta}_{M,\lambda} : X \to X$ associated with P, η, M and λ, for any $x \in X$ is defined as follows:

$$R^{P,\eta}_{M,\lambda}(u) = R^{H(.,.)-\eta}_{\lambda,M}(u) = (P + \lambda M)^{-1}(u) = (H(A, B) + \lambda M)^{-1}(u), \quad \forall u \in X,$$

that is, Definition 4.8 is actually the same Definition 2.14 and is not a new one.

In Theorem 2.9 of [3], the authors proved the Lipschitz continuity of the resolvent operator $R^{H(.,.)-\eta}_{\lambda,M}$ and calculated its Lipschitz constant under some appropriate conditions as follows.

Theorem 4.10. [3, Theorem 2.9] *Let X be a q-uniformly smooth Banach space with $q > 1$. Let $H(A, B)$ be μ-η-cocoercive with respect to A, γ-relaxed η-cocoercive with respect to B, A be α-expansive, B be β-Lipschitz continuous, η be τ-Lipschitz continuous and $\mu > \gamma$, $\alpha > \beta$. Let M be an $H(.,.)$-η-cocoercive mapping with respect to A and B. Then the resolvent operator $R^{H(.,.)-\eta}_{\lambda,M} : X \to X$ is $\frac{\tau^{q-1}}{\mu\alpha^q-\gamma\beta^q}$-Lipschitz continuous, that is,*

$$\|R^{H(.,.)-\eta}_{\lambda,M}(u) - R^{H(.,.)-\eta}_{\lambda,M}(v)\| \leq \frac{\tau^{q-1}}{\mu\alpha^q - \gamma\beta^q}\|u - v\|, \quad \forall u \in X. \tag{21}$$

Proof. Let $P : X \to X$ be defined by $P(x) = H(Ax, Bx)$, for all $x \in X$. By utilizing the assumptions and Proposition 4.5(i), we conclude that P is $(\mu\alpha^q - \gamma\beta^q)$-strongly η-accretive. Furthermore, M is a P-η-accretive mapping. Then all the conditions of Lemma 2.16 hold. Therefore, by picking $\theta = \mu\alpha^q - \gamma\beta^q$, Lemma 2.16 implies that the resolvent operator $R^{P,\eta}_{M,\lambda} = R^{H(.,.)-\eta}_{\lambda,M} : X \to X$ is $\frac{\tau^{q-1}}{\theta}$-Lipschitz continuous, i.e., (21) holds. The proof is finished. □

Let X_1 and X_2 be two q-uniformly smooth Banach spaces with $q > 1$ and let $A_1, B_1 : X_1 \to X_1$, $A_2, B_2 : X_2 \to X_2$, $H_1, \eta_1 : X_1 \times X_1 \to X_1$ and $H_2, \eta_2 : X_2 \times X_2 \to X_2$ be the mappings. Recently, Ahmad et al. [3] considered and studied the system (4) when M and N are $H_1(A_1, B_1)$-η_1-cocoercive and $H_2(A_2, B_2)$-η_2-cocoercive mappings, respectively. With the goal of constructing an iterative algorithm for approximating a solution of the system (4) involving $H_i(A_i, B_i)$-η_i-cocoercive mappings ($i = 1, 2$), they presented a characterization of its solution as follows.

Lemma 4.11. [3, Lemma 3.1] *Let X_1 and X_2 be two q-uniformly smooth Banach spaces with $q > 1$. Let $f, p, A_1, B_1 : X_1 \to X_1$, $g, d, A_2, B_2 : X_2 \to X_2$, $S : X_1 \times X_2 \to X_1$, $T : X_1 \times X_2 \to X_2$, $H_1, \eta_1 : X_1 \times X_1 \to X_1$ and $H_2, \eta_2 : X_2 \times X_2 \to X_2$ be the mappings. Let, for each $i \in \{1,2\}$, $H_i(A_i, B_i)$ be μ_i-η_i-cocoercive with respect to A_i and γ_i-relaxed η_i-cocoercive with respect to B_i, A_i be α_i-expansive, B_i be β_i-Lipschitz continuous, $\mu_i > \gamma_i$ and $\alpha_i > \beta_i$. Let $E : X_1 \to CB(X_1)$, $F : X_2 \to CB(X_2)$, $M : X_1 \to 2^{X_1}$ and $N : X_2 \to 2^{X_2}$ be the multi-valued mappings such that M is an $H_1(A_1, B_1)$-η_1-cocoercive mapping and $N : X_2 \to 2^{X_2}$ is an $H_2(A_2, B_2)$-η_2-cocoercive mapping. Then, $(x, y, u, v) \in X_1 \times X_2 \times E(x) \times E(y)$ is a solution of the system* (4) (*involving $H_i(.,.)$-η_i-cocoercive mappings*) *if and only if (x, y, u, v) satisfies*

$$\begin{cases} f(x) = R^{H_1(.,.)-\eta_1}_{\lambda,M}[H_1(A_1(f(x)), B_1(f(x))) - \lambda S(p(x), v)], \\ g(y) = R^{H_2(.,.)-\eta_2}_{\rho,N}[H_2(A_2(g(y)), B_2(g(y))) - \rho T(u, d(y))], \end{cases}$$

where $\lambda, \rho > 0$ are two constants.

Proof. Assume that for each $i \in \{1,2\}$, $P_i : X_i \to X_i$ is defined by $P_i(x) = H_i(A_i x, B_i x)$, for all $x \in X_i$. The assumptions and Proposition 4.5(i) imply that P_i is a strictly η_i-accretive for $i = 1, 2$, M is a P_1-η_1-accretive mapping and N is a P_2-η_2-accretive mapping. Then, all the conditions of Lemma 3.2 hold, and so the assertion follows by Lemma 3.2 immediately. □

In the light of Remark 4.9, it is worth mentioning that the τ_1-Lipschitz continuity and τ_2-Lipschitz continuity conditions of the mappings η_1 and η_2, respectively, mentioned in the context of Lemma 3.1 of [3] are extra and must be deleted, as we have done in the context of Lemma 4.11. In view of the proof of Lemma 4.11, it must be remarked that contrary to the claim of the authors in [3], the characterization of the solution for the system (4) involving $H_i(.,.)$-η_i-cocoercive mappings ($i = 1, 2$), presented in Lemma 4.11 is actually the same characterization of the solution for the system (4) involving P_i-η_i-accretive mappings presented in Lemma 3.2, and is not a new one.

Utilizing Lemma 4.11, Ahmad et al. [3] suggested an iterative algorithm for solving the system (4) involving $H_i(.,.)$-η_i-cocoercive mappings ($i = 1, 2$) as follows.

Algorithm 4.12. [3, Algorithm 3.3] *Let $X_1, X_2, f, g, S, T, p, d, A_1, B_1, A_2, B_2, H_1, H_2, \eta_1, \eta_2$, E, F, M and N be the same as in Lemma 4.11. For any given $(x_0, y_0) \in X_1 \times X_2$, $u_0 \in E(x_0)$, $v_0 \in F(y_0)$, compute the sequences $\{(x_n, y_n)\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ by the following iterative schemes:*

$$x_{n+1} = (1 - t_1)x_n + t_1\Big[x_n - f(x_n) + R^{H_1(.,.)-\eta_1}_{\lambda,M}[H_1(A_1(f(x_n)), B_1(f(x_n))) - \lambda S(p(x_n), v_n)]\Big], \tag{22}$$

$$y_{n+1} = (1 - t_2)y_n + t_2\Big[y_n - g(y_n) + R^{H_2(.,.)-\eta_2}_{\rho,N}[H_2(A_2(g(y_n)), B_2(g(y_n))) - \rho T(u_n, d(y_n))]\Big], \tag{23}$$

where $t_1, t_2 \in (0, 1]$ are two parameters and $\lambda, \rho > 0$ are two constants, $n = 0, 1, 2, \ldots$ and we choose $u_{n+1} \in E(x_n)$, $v_{n+1} \in F(y_{n+1})$ such that

$$\begin{cases} \|u_{n+1} - u_n\| \le D(E(x_{n+1}), E(x_n)), \\ \|v_{n+1} - v_n\| \le D(F(y_{n+1}), F(y_n)). \end{cases} \tag{24}$$

By a careful reading Algorithm 4.12, we found that the sequences $\{(x_n, y_n)\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ generated by Algorithm 4.12 are not well defined necessarily. In fact, for any given $(x_0, y_0) \in X_1 \times X_1$, $u_0 \in E(x_0)$, $v_0 \in F(y_0)$, the authors computed x_1 and y_1 by means of the iterative schemes (22) and (23), respectively, and then they claimed that one can choose $u_1 \in E(x_1)$ and $v_1 \in F(y_1)$ such that the following relations hold:

$$\begin{cases} \|u_1 - u_0\| \le D(E(x_1), E(x_0)), \\ \|v_1 - v_0\| \le D(F(y_1), F(y_0)). \end{cases} \tag{25}$$

In the light of Lemma 3.3, if X is a metric space and $T : X \to CB(X)$ is a multi-valued mapping, then for any $\varepsilon > 0$ and for any given $x, y \in X$, $u \in T(x)$, there exists $v \in T(y)$ such that

$$d(u, v) \le (1 + \varepsilon)D(T(x), T(y)).$$

However, for any given $x, y \in X$, $u \in T(x)$, there may not be a point $v \in T(y)$ such that $d(u, v) \leq D(T(x), T(y))$. In support of this fact, the following example is provided.

Example 4.13. Consider $X = l^{\infty}(\mathbb{Z}) = \{z = \{z_n\}_{n=-\infty}^{\infty} | \sup\limits_{n \in \mathbb{Z}} |z_n| < \infty, z_n \in \mathbb{C}\}$, the Banach space consisting of all bounded complex sequences $z = \{z_n\}_{n=-\infty}^{\infty}$ with the supremum norm $\|z\|_{\infty} = \sup\limits_{n \in \mathbb{Z}} |z_n|$. Any element $z = \{z_n\}_{n=-\infty}^{\infty} = \{x_n + iy_n\}_{n=-\infty}^{\infty} \in l^{\infty}(\mathbb{Z})$ can be written as follows:

$$\begin{aligned} z = & \sum_{\sigma \in \{\pm 1, \pm 3, \dots\}} \Big[(\dots, 0, \dots, 0, x_{2\sigma-1} + iy_{2\sigma-1}, 0, x_{2\sigma+1} + iy_{2\sigma+1}, 0, \dots) \\ & + (\dots, 0, \dots, 0, x_{2\sigma} + iy_{2\sigma}, 0, x_{2\sigma+2} + iy_{2\sigma+2}, 0, \dots)\Big] \\ = & \sum_{\sigma \in \{\pm 1, \pm 3, \dots\}} \Big[\frac{y_{2\sigma-1} + y_{2\sigma+1} - i(x_{2\sigma-1} + x_{2\sigma+1})}{2} \omega_{2\sigma-1, 2\sigma+1} \\ & + \frac{y_{2\sigma-1} - y_{2\sigma+1} - i(x_{2\sigma-1} - x_{2\sigma+1})}{2} \omega'_{2\sigma-1, 2\sigma+1} + \frac{y_{2\sigma} + y_{2\sigma+2} - i(x_{2\sigma} + x_{2\sigma+2})}{2} \omega_{2\sigma, 2\sigma+2} \\ & + \frac{y_{2\sigma} - y_{2\sigma+2} - i(x_{2\sigma} - x_{2\sigma+2})}{2} \omega'_{2\sigma, 2\sigma+2}\Big], \end{aligned}$$

where for each $\sigma \in \{\pm 1, \pm 3, \dots\}$, $\omega_{2\sigma-1, 2\sigma+1} = (\dots, 0, \dots, 0, i_{2\sigma-1}, 0, i_{2\sigma+1}, 0, \dots)$, i in the $(2\sigma - 1)$th and $(2\sigma + 1)$th positions and 0's elsewhere, $\omega'_{2\sigma-1, 2\sigma+1} = (\dots, 0, \dots, 0, i_{2\sigma-1}, 0, -i_{2\sigma+1}, 0, \dots)$, i and $-i$ at the $(2\sigma-1)$th and $(2\sigma+1)$th coordinates, and all other coordinates are zero, $\omega_{2\sigma, 2\sigma+2} = (\dots, 0, \dots, 0, i_{2\sigma}, 0, i_{2\sigma+2}, 0, \dots)$, i at the (2σ)th and $(2\sigma + 2)$th places, respectively, and 0's everywhere else, and $\omega'_{2\sigma, 2\sigma+2} = (\dots, 0, \dots, 0, i_{2\sigma}, 0, -i_{2\sigma+2}, 0, \dots)$, i and $-i$ at the (2σ)th and $(2\sigma + 2)$th coordinates, respectively, and all other coordinates are zero. Therefore, the set

$$\mathfrak{B} = \Big\{\omega_{2\sigma-1, 2\sigma+1}, \omega'_{2\sigma-1, 2\sigma+1}, \omega_{2\sigma, 2\sigma+2}, \omega'_{2\sigma, 2\sigma+2} : \sigma = \pm 1, \pm 3, \dots\Big\}$$

spans the Banach space $l^{\infty}(\mathbb{Z})$. It is easy to show that the set $\mathfrak{B}$ is linearly independent and so it is a Schauder basis for the Banach space $l^{\infty}(\mathbb{Z})$. Define the multi-valued mapping $T : X \to CB(X)$ by

$$T(x) = \begin{cases} \{\{\frac{\xi}{\beta^{n^p!} \sqrt[n^q!+2]{n^{\gamma}!}} i\}_{n=-\infty}^{\infty}, \omega'_{2\sigma-1, 2\sigma+1}, \omega_{2\sigma, 2\sigma+2} : \sigma = \pm 1, \pm 3, \dots\}, & x \neq \omega_{2r-1, 2r+1}, \\ \{\omega_{2\sigma-1, 2\sigma+1}, \omega'_{2\sigma, 2\sigma+2} : \sigma = \pm 1, \pm 3, \dots\}, & x = \omega_{2r-1, 2r+1}, \end{cases}$$

where $\xi \in [-1, 0)$ and $\beta > 1$ are arbitrary but fixed real numbers, p, q and γ are arbitrary but fixed even natural numbers, and $r \in \{\pm 1, \pm 3, \dots\}$ is chosen arbitrarily but fixed. Take $\omega_{2r-1, 2r+1} \neq x \in X$ arbitrarily, $y = \omega_{2r-1, 2r+1}$ and $u = \{\frac{\xi}{\beta^{n^p!} \sqrt[n^q!+2]{n^{\gamma}!}} i\}_{n=-\infty}^{\infty}$. If $a = \{\frac{\xi}{\beta^{n^p!} \sqrt[n^q!+2]{n^{\gamma}!}} i\}_{n=-\infty}^{\infty}$, then in view of the fact that $\xi < 0$, for any $\sigma \in \{\pm 1, \pm 3, \dots\}$, it yields

$$\begin{aligned} d(a, \omega_{2\sigma-1, 2\sigma+1}) = & \|\{\frac{\xi}{\beta^{n^p!} \sqrt[n^q!+2]{n^{\gamma}!}} i\}_{n=-\infty}^{\infty} - \omega_{2\sigma-1, 2\sigma+1}\|_{\infty} \\ = & \sup\{|\frac{\xi}{\beta^{n^p!} \sqrt[n^q!+2]{n^{\gamma}!}}|, |\frac{\xi}{\beta^{(2\sigma-1)^p!} \sqrt[(2\sigma-1)^q!+2]{(2\sigma-1)^{\gamma}!}} - 1|, \\ & |\frac{\xi}{\beta^{(2\sigma+1)^p!} \sqrt[(2\sigma+1)^q!+2]{(2\sigma+1)^{\gamma}!}} - 1| : n \in \mathbb{Z}, n \neq 2\sigma - 1, 2\sigma + 1\} \\ = & \begin{cases} |\frac{\xi}{\beta^{(2\sigma-1)^p!} \sqrt[(2\sigma-1)^q!+2]{(2\sigma-1)^{\gamma}!}} - 1|, & \text{if } \sigma \in \{2m + 1 | m \in \mathbb{N} \cup \{0\}\}, \\ |\frac{\xi}{\beta^{(2\sigma+1)^p!} \sqrt[(2\sigma+1)^q!+2]{(2\sigma+1)^{\gamma}!}} - 1|, & \text{if } \sigma \in \{-(2m + 1) | m \in \mathbb{N} \cup \{0\}\}, \end{cases} \\ = & \begin{cases} 1 - \frac{\xi}{\beta^{(2\sigma-1)^p!} \sqrt[(2\sigma-1)^q!+2]{(2\sigma-1)^{\gamma}!}}, & \text{if } \sigma \in \{2m + 1 | m \in \mathbb{N} \cup \{0\}\}, \\ 1 - \frac{\xi}{\beta^{(2\sigma+1)^p!} \sqrt[(2\sigma+1)^q!+2]{(2\sigma+1)^{\gamma}!}}, & \text{if } \sigma \in \{-(2m + 1) | m \in \mathbb{N} \cup \{0\}\}, \end{cases} \end{aligned}$$

and

$$\begin{aligned}d(a,\omega'_{2\sigma,2\sigma+2}) &= \|\{\frac{\xi}{\beta^{n^p!}\sqrt[n^q!+2]{n^\gamma!}}i\}_{n=-\infty}^{\infty}-\omega'_{2\sigma,2\sigma+2}\|_\infty \\ &= \sup\{|\frac{\xi}{\beta^{n^p!}\sqrt[n^q!+2]{n^\gamma!}}|,|\frac{\xi}{\beta^{(2\sigma)^p!}\sqrt[(2\sigma)^q!+2]{(2\sigma)^\gamma!}}-1|, \\ &\quad |\frac{\xi}{\beta^{(2\sigma+2)^p!}\sqrt[(2\sigma+2)^q!+2]{(2\sigma+2)^\gamma!}}+1| : n\in\mathbb{Z}, n\neq 2\sigma, 2\sigma+2\} \\ &= |\frac{\xi}{\beta^{(2\sigma)^p!}\sqrt[(2\sigma)^q!+2]{(2\sigma)^\gamma!}}-1| = 1-\frac{\xi}{\beta^{(2\sigma)^p!}\sqrt[(2\sigma)^q!+2]{(2\sigma)^\gamma!}}.\end{aligned}$$

Since $\xi\in[-1,0)$, we infer that

$$d(a,T(y)) = \inf_{b\in T(y)} d(a,b) \quad = \inf\Big\{1-\frac{\xi}{\beta^{(2\sigma+\mu)^p!}\sqrt[(2\sigma+\mu)^q!+2]{(2\sigma+\mu)^\gamma!}} : \mu=0,\pm1; \sigma=\pm1,\pm3,\dots\Big\} = 1.$$

For the case when $a=\omega'_{2s-1,2s+1}$ for some $s\in\{\pm1,\pm3,\dots\}$, then for each $\sigma\in\{\pm1,\pm3,\dots\}$, we obtain

$$d(a,\omega_{2\sigma-1,2\sigma+1}) = \begin{cases} \|\omega'_{2s-1,2s+1}-\omega_{2s-1,2s+1}\|_\infty, & \sigma=s, \\ \|\omega'_{2s-1,2s+1}-\omega_{2\sigma-1,2\sigma+1}\|_\infty, & \sigma\neq s, \end{cases} = \begin{cases} 2, & \sigma=s, \\ 1, & \sigma\neq s, \end{cases}$$

and $d(a,\omega'_{2\sigma,2\sigma+2}) = \|\omega'_{2s-1,2s+1}-\omega'_{2\sigma,2\sigma+2}\|_\infty = 1$. Thus, $d(a,T(y)) = \inf_{b\in T(y)} d(a,b) = 1$.

If $a=\omega_{2t,2t+2}$ for some $t\in\{\pm1,\pm3,\dots\}$, in virtue of the facts that for each $\sigma\in\{\pm1,\pm3,\dots\}$,

$$d(a,\omega_{2\sigma-1,2\sigma+1}) = \|\omega_{2t,2t+2}-\omega_{2\sigma-1,2\sigma+1}\|_\infty = 1$$

and

$$d(a,\omega'_{2\sigma,2\sigma+2}) = \begin{cases} \|\omega_{2t,2t+2}-\omega'_{2t,2t+2}\|_\infty, & \sigma=t, \\ \|\omega_{2t,2t+2}-\omega'_{2\sigma,2\sigma+2}\|_\infty, & \sigma\neq t, \end{cases} = \begin{cases} 2, & \sigma=t, \\ 1, & \sigma\neq t, \end{cases}$$

we deduce that $d(a,T(y)) = \inf_{b\in T(y)} d(a,b) = 1$. Consequently, $\sup_{a\in T(x)} d(a,T(y)) = 1$.

If $b=\omega_{2k-1,2k+1}$ for some $k\in\{\pm1,\pm3,\dots\}$, due to the fact that $\xi\in[-1,0)$, it follows that

$$\begin{aligned}&d(\{\frac{\xi}{\beta^{n^p!}\sqrt[n^q!+2]{n^\gamma!}}i\}_{n=-\infty}^{\infty},\omega_{2k-1,2k+1}) \\ &= \|\{\frac{\xi}{\beta^{n^p!}\sqrt[n^q!+2]{n^\gamma!}}i\}_{n=-\infty}^{\infty}-\omega_{2k-1,2k+1}\|_\infty \\ &= \sup\{|\frac{\xi}{\beta^{n^p!}\sqrt[n^q!+2]{n^\gamma!}}|,|\frac{\xi}{\beta^{(2k-1)^p!}\sqrt[(2k-1)^q!+2]{(2k-1)^\gamma!}}-1|, \\ &\quad |\frac{\xi}{\beta^{(2k+1)^p!}\sqrt[(2k+1)^q!+2]{(2k+1)^\gamma!}}-1| : n\in\mathbb{Z}, n\neq 2k-1,2k+1\} \\ &= \begin{cases} |\frac{\xi}{\beta^{(2k-1)^p!}\sqrt[(2k-1)^q!+2]{(2k-1)^\gamma!}}-1|, & \text{if } k\in\{2m+1|m\in\mathbb{N}\cup\{0\}\}, \\ |\frac{\xi}{\beta^{(2k+1)^p!}\sqrt[(2k+1)^q!+2]{(2k+1)^\gamma!}}-1|, & \text{if } k\in\{-(2m+1)|m\in\mathbb{N}\cup\{0\}\}, \end{cases} \\ &= \begin{cases} 1-\frac{\xi}{\beta^{(2k-1)^p!}\sqrt[(2k-1)^q!+2]{(2k-1)^\gamma!}}, & \text{if } k\in\{2m+1|m\in\mathbb{N}\cup\{0\}\}, \\ 1-\frac{\xi}{\beta^{(2k+1)^p!}\sqrt[(2k+1)^q!+2]{(2k+1)^\gamma!}}, & \text{if } k\in\{-(2m+1)|m\in\mathbb{N}\cup\{0\}\}, \end{cases}\end{aligned}$$

and for each $\sigma \in \{\pm 1, \pm 3, \dots\}$,

$$d(\omega'_{2\sigma-1,2\sigma+1}, \omega_{2k-1,2k+1}) = \begin{cases} \|\omega'_{2k-1,2k+1} - \omega_{2k-1,2k+1}\|_\infty, & \sigma = k, \\ \|\omega'_{2\sigma-1,2\sigma+1} - \omega_{2k-1,2k+1}\|_\infty, & \sigma \neq k, \end{cases} = \begin{cases} 2, & \sigma = k, \\ 1, & \sigma \neq k, \end{cases}$$

and

$$d(\omega_{2\sigma,2\sigma+2}, \omega_{2k-1,2k+1}) = \|\sigma_{2\sigma,2\sigma+2} - \sigma_{2k-1,2k+1}\|_\infty = 1.$$

In the light of these facts and considering the fact that $\xi < 0$, we conclude that

$$d(T(x), b) = \inf_{a \in T(x)} d(a, b) = 1.$$

In the case where $b = \omega'_{2j,2j+2}$ for some $j \in \{\pm 1, \pm 3, \dots\}$, owing to the fact that $\xi \in [-1, 0)$, we get

$$\begin{aligned} d(\{\frac{\xi}{\beta^{n^p!} \sqrt[n^q!+2]{n^\gamma!}} i\}_{n=-\infty}^\infty, \omega'_{2j,2j+2}) = \|\{\frac{\xi}{\beta^{n^p!} \sqrt[n^q!+2]{n^\gamma!}} i\}_{n=-\infty}^\infty - \omega'_{2j,2j+2}\|_\infty \\ = \sup\{|\frac{\xi}{\beta^{n^p!} \sqrt[n^q!+2]{n^\gamma!}}|, |\frac{\xi}{\beta^{(2j)^p!} \sqrt[(2j)^q!+2]{(2j)^\gamma!}} - 1|, \\ |\frac{\xi}{\beta^{(2j+2)^p!} \sqrt[(2j+2)^q!+2]{(2j+2)^\gamma!}} + 1| : n \in \mathbb{Z}, n \neq 2j, 2j+2\} \\ = |\frac{\xi}{\beta^{(2j)^p!} \sqrt[(2j)^q!+2]{(2j)^\gamma!}} - 1| = 1 - \frac{\xi}{\beta^{(2j)^p!} \sqrt[(2j)^q!+2]{(2j)^\gamma!}}, \end{aligned}$$

and for each $\sigma \in \{\pm 1, \pm 3, \dots\}$,

$$d(\omega'_{2\sigma-1,2\sigma+1}, \omega'_{2j,2j+2}) = \|\omega'_{2\sigma-1,2\sigma+1} - \omega'_{2j,2j+2}\|_\infty = 1$$

and

$$d(\omega_{2\sigma,2\sigma+2}, \omega'_{2j,2j+2}) = \begin{cases} \|\omega_{2j,2j+2} - \omega'_{2j,2j+2}\|_\infty, & \sigma = j, \\ \|\omega_{2\sigma,2\sigma+2} - \omega'_{2j,2j+2}\|_\infty, & \sigma \neq j, \end{cases} = \begin{cases} 2, & \sigma = j, \\ 1, & \sigma \neq j. \end{cases}$$

Since $\xi < 0$, we conclude that $d(T(x), b) = \inf_{a \in T(x)} d(a, b) = 1$. Accordingly, $\sup_{b \in T(y)} d(T(x), b) = 1$, and so

$$D(T(x), T(y)) = \max\left\{\sup_{a \in T(x)} d(a, T(y)), \sup_{b \in T(y)} d(T(x), b)\right\} = 1.$$

Taking into account that for each $\sigma \in \{\pm 1, \pm 3, \dots\}$,

$$\begin{aligned} &\|\{\frac{\xi}{\beta^{n^p!} \sqrt[n^q!+2]{n^\gamma!}} i\}_{n=-\infty}^\infty - \omega_{2\sigma-1,2\sigma+1}\|_\infty \\ &= \begin{cases} 1 - \frac{\xi}{\beta^{(2\sigma-1)^p!} \sqrt[(2\sigma-1)^q!+2]{(2\sigma-1)^\gamma!}} > 1, & \text{if } \sigma \in \{2m+1 | m \in \mathbb{N} \cup \{0\}\}, \\ 1 - \frac{\xi}{\beta^{(2\sigma+1)^p!} \sqrt[(2\sigma+1)^q!+2]{(2\sigma+1)^\gamma!}} > 1, & \text{if } \sigma \in \{-(2m+1) | m \in \mathbb{N} \cup \{0\}\}, \end{cases} \end{aligned}$$

and

$$\|\{\frac{\xi}{\beta^{n^p!} \sqrt[n^q!+2]{n^\gamma!}} i\}_{n=-\infty}^\infty - \omega'_{2\sigma,2\sigma+2}\|_\infty = 1 - \frac{\xi}{\beta^{(2\sigma)^p!} \sqrt[(2\sigma)^q!+2]{(2\sigma)^\gamma!}} > 1,$$

because $\xi \in [-1, 0)$, it follows that for any $v \in T(y)$, $d(u, v) = \|u - v\|_\infty > D(T(x), T(y))$.

It is worthwhile to stress that if $T(y)$ is compact then such a point v does exist. In fact, if $T : X \to C(X)$, where $C(X)$ is the family of all the nonempty compact subsets of X, then for any given $x, y \in X$, $u \in T(x)$, there exists $v \in T(y)$ such that $d(u, v) \leq D(T(x), T(y))$. In virtue of the above mentioned arguments, Algorithm 4.12 is not well-defined necessarily. We now present the correct version of Algorithm 4.12, only by editing (24) as follows.

Algorithm 4.14. *Let $X_1, X_2, f, g, p, d, S, T, A_1, B_1, A_2, B_2, H_1, H_2, \eta_1, \eta_2, E, F, M$ and N be the same as in Lemma 4.11. For any given $(x_0, y_0) \in X_1 \times X_2$, $u_0 \in E(x_0)$, $v_0 \in F(y_0)$, define the sequences $\{(x_n, y_n)\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ in the following way:*

$$x_{n+1} = (1 - t_1)x_n + t_1\Big[x_n - f(x_n) + R_{\lambda,M}^{H_1(.,.)-\eta_1}[H_1(A_1(f(x_n)), B_1(f(x_n))) - \lambda S(p(x_n), v_n)]\Big],$$
$$y_{n+1} = (1 - t_2)y_n + t_2\Big[y_n - g(y_n) + R_{\rho,N}^{H_2(.,.)-\eta_2}[H_2(A_2(g(y_n)), B_2(g(y_n))) - \rho T(u_n, d(y_n))]\Big],$$

where $t_1, t_2 \in (0, 1]$ are two parameters and $\lambda, \rho > 0$ are two constants, $n = 0, 1, 2, \ldots$ and we choose $u_{n+1} \in E(x_n)$, $v_{n+1} \in F(y_{n+1})$ such that

$$\begin{cases} \|u_{n+1} - u_n\| \leq (1 + (1 + n)^{-1})D(E(x_{n+1}, E(x_n)), \\ \|v_{n+1} - v_n\| \leq (1 + (1 + n)^{-1})D(F(y_{n+1}, F(y_n)). \end{cases}$$

By defining $P_i : X_i \to X_i$ as $P_i(x) = H_i(A_i x, B_i x)$, for $i = 1, 2$, and for all $x \in X_i$, and in the light of the conditions of Lemma 4.11, Proposition 4.5 implies that for $i = 1, 2$, P_i is a strictly η_i-accretive mapping, and M and N are P_1-η_1-accretive and P_2-η_2-accretive mappings, respectively. Then, by letting $\alpha_i = t_i$, for $i = 1, 2$, we observe that Algorithm 4.14 is actually the same Algorithm 3.5 and is not a new one.

In Theorem 3.4 of [3], the authors studied the convergence analysis of Algorithm 4.12 under some certain conditions. Taking into account that Algorithm 4.12 is not in general well defined, and Algorithm 4.14 is the correct version of Algorithm 4.12, we infer that the statement of [3, Theorem 3.4] is not true necessarily. In the following its correct version is provided.

Theorem 4.15. *Let X_1 and X_2 be two q-uniformly smooth Banach spaces with $q > 1$. Let $A_1, B_1, p : X_1 \to X_1$, $A_2, B_2, d : X_2 \to X_2$, $H_1 : X_1 \times X_2 \to X_1$, $H_2 : X_2 \times X_1 \to X_2$ be the mappings such that $H_1(A_1, B_1)$ is η_1-cocoercive with respect to A_1 with constant μ_1 and relaxed η_1-cocoercive with respect to B_1 with constant γ_1, A_1 is α_1-expansive, B_1 is β_1-Lipschitz continuous, $\alpha_1 > \beta_1$ and $\mu_1 > \gamma_1$; $H_2(A_2, B_2)$ is η_2-cocoercive with respect to A_2 with constant μ_2 and relaxed η_2-cocoercive with respect to B_2 with constant γ_2, A_2 is α_2-expansive, B_2 is β_2-Lipschitz continuous, $\alpha_2 > \beta_2$ and $\mu_2 > \gamma_2$. Assume that $\eta_1 : X_1 \times X_1 \to X_1$ is τ_1-Lipschitz continuous, $\eta_2 : X_2 \times X_2 \to X_2$ is τ_2-Lipschitz continuous, $f : X_1 \to X_1$ is strongly accretive with constant δ_1 and λ_f-Lipschitz continuous and $g : X_2 \to X_2$ is strongly accretive with constant δ_2 and λ_g-Lipschitz continuous. Let $S : X_1 \times X_2 \to X_1$ be strongly η_1-accretive with respect to p with constant λ_S and λ_{S_p}-Lipschitz continuous with respect to p in the first argument and λ_{S_2}-Lipschitz continuous in the second argument. Suppose that $T : X_1 \times X_2 \to X_2$ is strongly η_2-accretive with constant δ_T with respect to d and λ_{T_d}-Lipschitz continuous with respect to d in the second argument, and λ_{T_1}-Lipschitz continuous in the first argument. Let $E : X_1 \to CB(X_1)$ be D-Lipschitz continuous with constant λ_{D_E} and $F : X_2 \to CB(X_2)$ be D-Lipschitz continuous with constant λ_{D_F}. Let $H_1(A_1, B_1)$ be r_1-Lipschitz continuous with respect to A_1 and r_2-Lipschitz continuous with respect to B_1, and $H_2(A_2, B_2)$ be r_3-Lipschitz continuous with respect to A_2 and r_4-Lipschitz continuous with respect to B_2. Suppose that $M : X_1 \to 2^{X_1}$ is $H_1(A_1, B_1)$-η_1-cocoercive and $N : X_2 \to 2^{X_2}$ is $H_2(A_2, B_2)$-η_2-cocoercive. If there exist positive constants ρ and λ such that*

$$1 - t_1 + t_1\sqrt[q]{1 - q\delta_1 + c_q\lambda_f^q} + \frac{t_1\tau_1^{q-1}}{\mu_1\alpha_1^q - \gamma_1\beta_1^q}\theta' + \frac{t_2\tau_2^{q-1}\rho\lambda_{T_1}\lambda_{D_E}}{\mu_2\alpha_2^q - \gamma_2\beta_2^q} < 1, \tag{26}$$

$$1 - t_2 + t_2\sqrt[q]{1 - q\delta_2 + c_q\lambda_g^q} + \frac{t_2\tau_2^{q-1}}{\mu_2\alpha_2^q - \gamma_2\beta_2^q}\theta'' + \frac{t_1\tau_1^{q-1}\lambda\lambda_{S_2}\lambda_{D_F}}{\mu_1\alpha_1^q - \gamma_1\beta_1^q} < 1, \tag{27}$$

where

$$\theta' = \sqrt[q]{(r_1+r_2)^q\lambda_f^q - q\lambda\delta_S + q\lambda\lambda_{S_p}(r_1+r_2)^{q-1}\lambda_f^{q-1} + q\lambda\lambda_{S_p}\tau_1^{q-1} + c_q\lambda_q\lambda_{S_p}^q},$$

$$\theta'' = \sqrt[q]{(r_3+r_4)^q\lambda_g^q - q\rho\delta_T + q\rho\lambda_{T_d}(r_3+r_4)^{q-1}\lambda_g^{q-1} + q\rho\lambda_{T_d}\tau_2^{q-1} + c_q\rho^q\lambda_{T_d}^q},$$

where in the case when q is an even natural number, in addition to (26) *and* (27)*, the following conditions hold:*

$$q\delta_1 < 1 + c_q\lambda_f^q,\ q\delta_2 \le 1 + c_q\lambda_g^q, \tag{28}$$

$$q\lambda\delta_S < (r_1+r_2)^q\lambda_f^q + q\lambda\lambda_{S_p}(r_1+r_2)^{q-1}\lambda_f^{q-1} + q\lambda\lambda_{S_p}\tau_2^{q-1} + c_q\lambda^q\lambda_{S_p}^q, \tag{29}$$

$$q\rho\delta_T < (r_3+r_4)^q\lambda_g^q + q\rho\lambda_{T_d}(r_3+r_4)^{q-1}\lambda_g^{q-1} + q\rho\lambda_{T_d}\tau_2^{q-1} + c_q\rho^q\lambda_{T_d}^q, \tag{30}$$

where c_q is a constant guaranteed by Lemma 2.1. Then, the iterative sequences $\{x_n\}_{n=0}^\infty$, $\{y_n\}_{n=0}^\infty$, $\{u_n\}_{n=0}^\infty$ and $\{v_n\}_{n=0}^\infty$ generated by Algorithm 4.14 converge strongly to x, y, u and v, respectively, and (x, y, u, v) is a solution of the system (4) (*involving $H_i(.,.)$-η_i-cocoercive mappings* $(i = 1, 2)$).

Proof. Let us define $P_i : X_i \to X_i$ as $P_i(x) = H_i(A_ix, B_ix)$, for each $i \in \{1, 2\}$ and $x \in X_i$. From the assumptions and Proposition 4.5, it follows that for each $i \in \{1, 2\}$, the mapping P_i is $(\mu_i\alpha_i^q - \gamma_i\beta_i^q)$-strongly η_i-accretive, P_1 is $(r_1 + r_2)$-Lipschitz continuous and P_2 is $(r_3 + r_4)$-Lipschitz continuous. Furthermore, M and N are P_1-η_1-accretive and P_2-η_2-accretive mappings, respectively. Taking $\varrho_1 = r_1 + r_2$, $\varrho_2 = r_3 + r_4$, $\theta_i = \mu_i\alpha_i^q - \gamma_i\beta_i^q$ and $\alpha_i = t_i$, for each $i \in \{1, 2\}$, we note that all the conditions of Corollary 3.11 hold. Now, the statement follows by utilizing the statement of Corollary 3.11 immediately. □

It should be pointed out that if q is an even natural number, then the positive constants ρ and λ, in addition to (5), must be also satisfied (28)–(30), as we have added the mentioned conditions to the conditions of Theorem 4.15. At the same time, there are some mistakes in (3.4) of [3]. In fact, in (3.4) of [3], λ_{f^q}, λ_{g^q} and τ^{q_1} must be replaced by λ_f^q, λ_g^q and τ_1^{q-1}, respectively, as we have done in (26) and (27).

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