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Strong Convergence of the Euler-Maruyama Method for the Generalized Stochastic Volterra Integral Equations Driven by Lévy Noise

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Abstract. In this paper, the theoretical and numerical analysis of the stochastic Volterra integral equations (SVIEs) driven by Lévy noise are considered. We investigate the existence, uniqueness, boundedness and Hölder continuity of the analytic solutions for SVIEs driven by Lévy noise. The Euler-Maruyama method for SVIEs driven by Lévy noise is proposed. The boundedness of the numerical solution is proved, and the strong convergence order is obtained. Some numerical examples are given to support the theoretical results.

1. Introduction

Stochastic Volterra integral equations (SVIEs) (see [11] and the references cited therein), as an extension of Volterra integral equations (VIEs) (see [20]) and stochastic differential equations (SDEs) (see [15] and the references cited therein), are used to model many problems in various application fields, such as biology, chemistry, physics, mathematical finance and optimal control theory. The theory of analytic solutions for SDEs have received a great deal of attentions in recent decades. However, there exist no explicit solutions for most SDEs , so numerical methods become appropriate choices (see, e.g., [6, 7, 15, 17, 19]).

Let us mention the recent interesting results of SVIEs. Mao studies the stability of stochastic Volterra integro-differential equations (SVIDEs) in [14]. Later, Mao and Riedle (see [16]) extend these results to the more generalized type of SVIDEs. Usually, such SVIEs do not possess any explicit solution and we have to resort to numerical methods to obtain their approximate solutions. Though there are numerous papers on numerical methods for stochastic differential equations (SDEs), however, there are only a few numerical results of SVIEs (see, e.g., [8, 9, 24, 25, 28] and the references cited therein). In 2020, we studied theoretical and numerical analysis of the Euler-Maruyama method for the generalized SVIDEs([27]).

Problems in economics, finance and many other branches of science are often affected by event-driven uncertainties. In finance, for example, the unpredictability of important events, like market crashes, central bank announcements, changes in credit ratings, defaults, etc., can have sudden and significant effects on stock price movements. So jumps have been studied extensively in modeling of such phenomenon([4, 5]

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and the references cited therein). A lot of progress has been made (see, e.g., [1–3, 13, 22, 26]) in the recent decades.

In this paper, we consider the following generalized SVIE driven by Lévy noise

$$\begin{aligned} Y(t) =&\varphi(t) + \int_0^t f\left(Y(t), \int_0^z k_1(z,s)Y(s)\,ds, \int_0^z \sigma_1(z,s)Y(s)\,dw(s)\right) dz \\ &+ \int_0^t g\left(Y(t), \int_0^z k_2(z,s)Y(s)\,ds, \int_0^z \sigma_2(z,s)Y(s)\,dw(s)\right) dw(z) \\ &+ \int_0^t \int_{\mathbf{Z}} \gamma\left(Y(t), \int_0^z k_3(z,s)Y(s)\,ds, \int_0^z \sigma_3(z,s)Y(s)\,dw(s), \xi\right) \tilde{N}(dz, d\xi) \end{aligned} \tag{1}$$

for $t \in [0,T]$, where $\varphi : [0,T] \to \mathbb{R}$ and $\|\varphi\|_\infty^2 = \max_{t\in[0,T]} |\varphi(t)|^2 < \infty$. Here $f : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $g : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $\gamma : \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbf{Z} \to \mathbb{R}$ are measurable functions. The kernels k_i: $D \to \mathbb{R}$ and σ_i: $D \to \mathbb{R}$ are continuous on $D := \{(t,s) : 0 \le s \le t \le T\}$. Set $\|k_i\|_\infty = \max_{(t,s)\in D} |k_i(t,s)|$ and $\|\sigma_i\|_\infty = \max_{(t,s)\in D} |\sigma_i(t,s)|$ for $i = 1, 2, 3$.

It is obviously that SVIE (1) is a generalization of SDEs and SVIEs, and also more extensions of SVIDEs in [14], [16] and [27]. To the best of the authors' knowledge, given some new difficulties of the multiple stochastic integral (see [12]) and Lévy noise, these are the first results in the literature for such generalized SVIE (1) driven by Lévy noise.

The outline of this paper is as follows: we will consider the existence, uniqueness, boundedness and Hölder continuity of the analytic solution of SVIE (1) in Section 2. The Euler-Maruyama method of SVIE (1) is proposed and its convergence order is established to be 1/2 in Section 3. Finally, we will give some numerical examples in Section 4 to illustrate the theoretical results of SVIE (1).

2. Theoretical analysis of SVIE driven by Lévy noise

Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P})$ denote a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions (i.e, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), and let $\mathbb{E}$ be the expectation corresponding to $\mathbb{P}$. A 1-dimensional Brownian motion defined on the probability space is denoted by $w(t)$ and $N(dt, d\xi)$ is a Poisson random measure defined on σ-finite measure space $(\mathbf{Z}, \mathcal{L}, \nu)$ with intensity measure $\nu \not\equiv 0$ (for the case when $\nu \equiv 0$, see [23]). Set $\tilde{N}(dt, d\xi) := N(dt, d\xi) - \nu(d\xi)dt$. Moreover, we assume that $w(t)$ is independent of $\tilde{N}(t,\cdot)$. The family of $\mathbb{R}$-valued $\mathcal{F}_t$-adapted processes $\{x(t)\}_{t\in[0,T]}$ such that $\mathbb{E}|x(t)|^p < \infty$ $(p \ge 1)$ is denoted by $\mathcal{L}^p([0,T];\mathbb{R})$. We denote by $\mathcal{M}^2([0,T];\mathbb{R})$ the family of processes $\{x(t)\}_{t\in[0,T]}$ in $\mathcal{L}^2([0,T];\mathbb{R})$ such that $\mathbb{E}\left[\int_0^T |x(t)|^2\,dt\right] < \infty$. For $a, b \in \mathbb{R}$, we use $a \vee b$ and $a \wedge b$ for $\max\{a,b\}$ and $\min\{a,b\}$, respectively. If G is a subset of Ω, its indicator function is denote by $\mathbf{1}_G$.

The assumptions are listed below.

(A1) There exists a positive constant K such that

$$\begin{aligned} &\left|f(x,y,z) - f(\bar{x},\bar{y},\bar{z})\right|^2 \vee \left|g(x,y,z) - g(\bar{x},\bar{y},\bar{z})\right|^2 \vee \int_{\mathbf{Z}} \left|\gamma(x,y,z,\xi) - \gamma(\bar{x},\bar{y},\bar{z},\xi)\right|^2 \nu(d\xi) \\ &\qquad \le K\left(|x-\bar{x}|^2 + \left|y-\bar{y}\right|^2 + |z-\bar{z}|^2\right) \end{aligned} \tag{2}$$

for $x, y, z, \bar{x}, \bar{y}, \bar{z} \in \mathbb{R}$.

It can be easily to check that Lipschitz condition **(A1)** implies the following linear growth condition **(A2)**.

(A2) For $x, y, z \in \mathbb{R}$, there is a positive constant $\bar{K}$ such that ′ has been added in **(A2)**

$$\left|f(x,y,z)\right|^2 \vee \left|g(x,y,z)\right|^2 \vee \int_{\mathbf{Z}} \left|\gamma(x,y,z,\xi)\right|^2 \nu(dz) \le \bar{K}\left(1 + |x|^2 + |y|^2 + |z|^2\right), \tag{3}$$

where $\bar{K} = 2\left(K \vee \left|f(0,0,0)\right|^2 \vee \left|g(0,0,0)\right|^2 \vee \int_{\mathbf{Z}} \left|\gamma(0,0,0,\xi)\right|^2 \nu(d\xi)\right)$.

(A3) There exists a positive constant $\grave{K}$, for $i = 1, 2, 3$, such that

$$\left|\varphi(t) - \varphi(\bar{t})\right|^2 \vee |k_i(t,s) - k_i(\bar{t},s)|^2 \le \grave{K}|t - \bar{t}|^2,$$
$$|\sigma_i(t,s) - \sigma_i(\bar{t},\bar{s})|^2 \le \grave{K}\left(|t - \bar{t}|^2 + |s - \bar{s}|^2\right)$$

for $x, y, z, \bar{x}, \bar{y}, \bar{z} \in \mathbb{R}$ and $t, s, \bar{t}, \bar{s} \in [0, T]$.

2.1. Existence and uniqueness of the analytic solution

The proof of the following lemma can be found in [10, 18, 21].

Lemma 2.1. *Let $r \ge 2$. There exists a constant c_1, depending only on r, such that for every real-valued, $\mathcal{B}[0,T] \bigotimes \mathcal{L}$ -measurable function g satisfying*

$$\int_0^T \int_{\mathbf{Z}} |g_t(z)|^r \nu(dz)dt < \infty$$

almost surely, the following estimate holds,

$$\mathbb{E} \sup_{0 \le t \le T} \left| \int_0^t \int_{\mathbf{Z}} g_t(\xi) \tilde{N}(dt, d\xi) \right|^r \le c_1 \mathbb{E} \left(\int_0^T \int_{\mathbf{Z}} |g_t(\xi)|^2 \nu(d\xi) dt \right)^{r/2} + c_1 \mathbb{E} \int_0^T \int_{\mathbf{Z}} |g_t(\xi)|^r \nu(d\xi) dt. \quad (4)$$

It is known that if $1 \le r \le 2$, then the second term in (4) *can be dropped.*

In order to prove the existence and uniqueness of the analytic solution of SVIE (1) under **(A1)**, we present the following lemma.

Lemma 2.2. *Under* **(A2)**, *if $Y(t)$ is a solution of SVIE* (1), *then*

$$\mathbb{E}|Y(t)|^2 \le K_0, \quad t \in [0, T], \quad (5)$$

where K_0 depends on k_1, k_2, k_3, σ_1, σ_2, σ_3, T, $\bar{K}$ and φ.

In particular, $Y(t)$ belongs to $\mathcal{M}^2([0,T]; \mathbb{R})$.

Proof. For any integer $n \ge 1$, define the stopping time

$$\tau_n = T \wedge \inf\{t \in [0,T] : |Y(t)| \ge n\}.$$

Is is easy to see that $\tau_n \to T$ a.s. by letting $n \to \infty$.

Define $Y_n(t) := Y(t \wedge \tau_n)$ for $t \in [0, T]$. Then one can prove that $Y_n(t)$ satisfies

$$\begin{aligned}
Y_n(t) =& \varphi(t) + \int_0^t f\Big(Y_n(z), \int_0^z k_1(z,s)Y_n(s)\mathbf{1}_{[0,\tau_n]}(s)\,ds, \int_0^z \sigma_1(z,s)Y_n(s)\mathbf{1}_{[0,\tau_n]}(s)\,dw(s)\Big)\mathbf{1}_{[0,\tau_n]}(z)\,dz \\
&+ \int_0^t g\Big(Y_n(z), \int_0^z k_2(z,s)Y_n(s)\mathbf{1}_{[0,\tau_n]}(s)\,ds, \int_0^z \sigma_2(z,s)Y_n(s)\mathbf{1}_{[0,\tau_n]}(s)\,dw(s)\Big)\mathbf{1}_{[0,\tau_n]}(z)\,dw(z), \\
&+ \int_0^t \int_{\mathbf{Z}} \gamma\Big(Y_n(z), \int_0^z k_3(z,s)Y_n(s)\mathbf{1}_{[0,\tau_n]}(s)\,ds, \\
&\qquad \int_0^z \sigma_3(z,s)Y_n(s)\mathbf{1}_{[0,\tau_n]}(s)\,dw(s), \xi\Big)\mathbf{1}_{[0,\tau_n]}(z)\tilde{N}(dz,d\xi) \quad t \in [0,T].
\end{aligned}$$

Applying the elementary inequality, Cauchy's inequality and Hölder's inequality, one obtains that for any $t \in [0, T]$,

$$|Y_n(t)|^2 \le 4|\varphi(t)|^2 + 4t \int_0^t \left| f\Big(Y_n(z), \int_0^z k_1(z,s)Y_n(s)\mathbf{1}_{[0,\tau_n]}(s)\,ds, \int_0^z \sigma_1(z,s)Y_n(s)\mathbf{1}_{[0,\tau_n]}(s)\,dw(s)\Big)\mathbf{1}_{[0,\tau_n]}(z) \right|^2 dz$$

$$+ 4\left|\int_0^t g\Big(Y_n(z), \int_0^z k_2(z,s)Y_n(s)\mathbf{1}_{[0,\tau_n]}(s)\,ds, \int_0^z \sigma_2(z,s)Y_n(s)\mathbf{1}_{[0,\tau_n]}(s)\,dw(s)\Big)\mathbf{1}_{[0,\tau_n]}(z)\,dw(z)\right|^2$$
$$+ 4\left|\int_0^t \int_{\mathbf{Z}} \gamma\Big(Y_n(z), \int_0^z k_3(z,s)Y_n(s)\mathbf{1}_{[0,\tau_n]}(s)\,ds, \int_0^z \sigma_3(z,s)Y_n(s)\mathbf{1}_{[0,\tau_n]}(s)\,dw(s), \xi\Big)\mathbf{1}_{[0,\tau_n]}(z)\tilde{N}(dz,d\xi)\right|^2.$$

Taking the expectation and using Hölder's inequality and Itô isometry, we have

$$\mathbb{E}\,|Y_n(t)|^2 \le 4\mathbb{E}|\varphi(t)|^2 + 4TB_1 + 4B_2 + 4B_3,$$

where

$$B_1 := \mathbb{E}\bigg[\int_0^t \left|f\Big(Y_n(z), \int_0^z k_1(z,s)Y_n(s)\mathbf{1}_{[0,\tau_n]}(s)\,ds, \int_0^z \sigma_1(z,s)Y_n(s)\mathbf{1}_{[0,\tau_n]}(s)\,dw(s)\Big)\mathbf{1}_{[0,\tau_n]}(z)\right|^2 dz\bigg],$$

$$B_2 := \mathbb{E}\bigg[\int_0^t \left|g\Big(Y_n(z), \int_0^z k_2(z,s)Y_n(s)\mathbf{1}_{[0,\tau_n]}(s)\,ds, \int_0^z \sigma_2(z,s)Y_n(s)\mathbf{1}_{[0,\tau_n]}(s)\,dw(s)\Big)\mathbf{1}_{[0,\tau_n]}(z)\right|^2 dz\bigg]$$

and

$$B_3 := \mathbb{E}\left|\int_0^t \int_{\mathbf{Z}} \gamma\Big(Y_n(z), \int_0^z k_3(z,s)Y_n(s)\mathbf{1}_{[0,\tau_n]}(s)\,ds, \int_0^z \sigma_3(z,s)Y_n(s)\mathbf{1}_{[0,\tau_n]}(s)\,dw(s), \xi\Big)\mathbf{1}_{[0,\tau_n]}(z)\tilde{N}(dz,d\xi)\right|^2.$$

Thus, one uses **(A2)** to estimate B_1 and B_2,

$$\begin{aligned} B_1 \le& \bar{K}\int_0^t \left[1 + \mathbb{E}|Y_n(z)|^2 + \mathbb{E}\left|\int_0^z k_1(z,s)Y_n(s)\,ds\right|^2 + \mathbb{E}\left|\int_0^z \sigma_1(z,s)Y_n(s)\,dw(s)\right|^2\right]dz,\\ B_2 \le& \bar{K}\int_0^t \left[1 + \mathbb{E}|Y_n(z)|^2 + \mathbb{E}\left|\int_0^z k_2(z,s)Y_n(s)\,ds\right|^2 + \mathbb{E}\left|\int_0^z \sigma_2(z,s)Y_n(s)\,dw(s)\right|^2\right]dz. \end{aligned}$$

Due to Lemma 2.1 and **(A2)**, B_3 can be estimate as follows

$$\begin{aligned} B_3 \le& c_1\mathbb{E}\Big(\int_0^T \int_{\mathbf{Z}} \left|\gamma\Big(Y_n(z), \int_0^z k_3(z,s)Y_n(s)\mathbf{1}_{[0,\tau_n]}(s)\,ds, \int_0^z \sigma_3(z,s)Y_n(s)\mathbf{1}_{[0,\tau_n]}(s)\,dw(s), \xi\Big)\mathbf{1}_{[0,\tau_n]}(z))\right|^2 \nu(d\xi)dt\Big)\\ \le& \bar{K}c_1\int_0^T \left[1 + \mathbb{E}|Y_n(z)|^2 + \mathbb{E}\left|\int_0^z k_3(z,s)Y_n(s)\,ds\right|^2 + \mathbb{E}\left|\int_0^z \sigma_3(z,s)Y_n(s)\,dw(s)\right|^2\right]dz. \end{aligned}$$

Hölder's inequality and Itô isometry imply, for $i = 1, 2, 3$,

$$\mathbb{E}\left|\int_0^z k_i(z,s)Y_n(s)\,ds\right|^2 \le z\int_0^z \|k_i\|_\infty^2 \mathbb{E}|Y_n(s)|^2\,ds \le \|k_i\|_\infty^2 T\int_0^t \mathbb{E}|Y_n(s)|^2\,ds$$

and

$$\mathbb{E}\left|\int_0^z \sigma_i(z,s)Y_n(s)\,dw(s)\right|^2 \le \int_0^z \|\sigma_i\|_\infty^2 \mathbb{E}|Y_n(s)|^2\,ds \le \|\sigma_i\|_\infty^2 \int_0^t \mathbb{E}|Y_n(s)|^2\,ds.$$

Consequently,

$$\mathbb{E}|Y_n(t)|^2 \le 4\mathbb{E}|\varphi(t)|^2 + 4\bar{K}T\,(T + 1 + c_1) + 4\bar{K}\,(C_{10}T + C_{11})\int_0^t \mathbb{E}|Y_n(s)|^2\,ds,$$

where $C_{10} := 1 + \|k_1\|_\infty^2 T^2 + \|\sigma_1\|_\infty^2 T$ and $C_{11} := 1 + c_1 + \|k_2\|_\infty^2 T^2 + \|\sigma_2\|_\infty^2 T + c_1\|k_3\|_\infty^2 T^2 + c_1\|\sigma_3\|_\infty^2 T$.

Gronwall's inequality yields that

$$\mathbb{E}\,|Y_n(t)|^2 \leq \left[4\|\varphi\|_\infty^2 + 4\bar{K}\,(T+1+c_1)\,T\right]\exp\left(4C_1 T\right) =: K_0,$$

where $C_1 := \bar{K}\,(C_{10}T + C_{11})$.

Thus, we have

$$\mathbb{E}\,|Y(t\wedge\tau_n)|^2 \leq K_0.$$

Consequently, (5) follows by letting $n\to\infty$. □

Theorem 2.3. *Under* **(A1)***, there exists a unique solution* $Y(t)$ *to SVIE* (1) *in* $\mathcal{M}^2([0,T];\mathbb{R})$ *and*

$$\mathbb{E}|Y(t)|^2 \leq C$$

for $t\in[0,T]$*, where* $C := 3\,(1+C_1T)\,\|\varphi\|_\infty^2 \exp\left(3C_1T\right)$*. Here* C_1 *is defined in the proof of Lemma 2.2.*

Proof. We divide the proof into two steps.

Step 1: Uniqueness. Let $Y(t)$ and $\bar{Y}(t)$ be two solutions of SVIE (1). Due to Lemma 2.2, both of them belong to $\mathcal{M}^2([0,T];\mathbb{R})$.

Applying Hölder's inequality, Itô isometry, Lemma 3.1 and **(A1)**, using the same way as the proof of Lemma 2.2, one can show that

$$\begin{aligned}\mathbb{E}\left|Y(t)-\bar{Y}(t)\right|^2 \leq& 3T\int_0^t \bar{K}\Bigg[\mathbb{E}\left|Y(s)-\bar{Y}(s)\right|^2 + \mathbb{E}\left|\int_0^z k_1(z,s)\,(Y(s)-\bar{Y}(s))\,ds\right|^2\\ &\qquad + \mathbb{E}\left|\int_0^z \sigma_1(z,s)\,(Y(s)-\bar{Y}(s))\,dw(s)\right|^2\Bigg]dz\\ &+3\int_0^t \bar{K}\Bigg[\mathbb{E}\left|Y(s)-\bar{Y}(s)\right|^2 + E\left|\int_0^z k_2(z,s)\,(Y(s)-\bar{Y}(s))\,ds\right|^2\\ &\qquad + \mathbb{E}\left|\int_0^z \sigma_2(z,s)\,(Y(s)-\bar{Y}(s))\,dw(s)\right|^2\Bigg]dz\\ &+3\int_0^t \bar{K}c_1\Bigg[\mathbb{E}\left|Y(s)-\bar{Y}(s)\right|^2 + \mathbb{E}\left|\int_0^z k_3(z,s)\,(Y(s)-\bar{Y}(s))\,ds\right|^2\\ &\qquad + \mathbb{E}\left|\int_0^z \sigma_3(z,s)\,(Y(s)-\bar{Y}(s))\,dw(s)\right|^2\Bigg]dz.\end{aligned}$$

Noting that, for $i=1,2,3$,

$$\begin{aligned}\int_0^t \mathbb{E}\left|\int_0^z k_i(z,s)\,(Y(s)-\bar{Y}(s))\,ds\right|^2 dz \leq& \int_0^t \left[z\mathbb{E}\left(\int_0^z \left|k_i(z,s)\,(Y(s)-\bar{Y}(s))\right|^2 ds\right)\right]dz\\ \leq& t^2\|k_i\|_\infty^2 \int_0^t \mathbb{E}\left|Y(s)-\bar{Y}(s)\right|^2 ds\end{aligned}$$

and

$$\begin{aligned}\int_0^t \mathbb{E}\left|\int_0^z \sigma_i(z,s)\,(Y(s)-\bar{Y}(s))\,dw(s)\right|^2 dz \leq& \int_0^t \mathbb{E}\left(\int_0^z \left|\sigma_i(z,s)\,(Y(s)-\bar{Y}(s))\right|^2 ds\right)dz\\ \leq& t\|\sigma_i\|_\infty^2 \int_0^t \mathbb{E}\left|Y(s)-\bar{Y}(s)\right|^2 ds,\end{aligned}$$

we get

$$\mathbb{E}\left|Y(t)-\bar{Y}(t)\right|^2 \le 3C_2\int_0^t \mathbb{E}\left|Y(s)-\bar{Y}(s)\right|^2\, ds,$$

where $C_2 := K(C_{10}T + C_{11})$.

Gronwall's inequality implies that

$$\mathbb{E}\left|Y(t)-\bar{Y}(t)\right|^2 = 0.$$

Hence, $Y(t) = \bar{Y}(t)$ for all $t \in [0,T]$ almost surely. The uniqueness has been proved.

Step 2: Existence. Set $Y_0(t) = \varphi(t)$ and define the Picard iterations

$$\begin{aligned} Y_n(t) =&\varphi(t) + \int_0^t f\left(Y_{n-1}(z), \int_0^z k_1(z,s)Y_{n-1}(s)\,ds, \int_0^z \sigma_1(z,s)Y_{n-1}(s)\,dw(s)\right) dz \\ &+ \int_0^t g\left(Y_{n-1}(z), \int_0^z k_2(z,s)Y_{n-1}(s)\,ds, \int_0^z \sigma_2(z,s)Y_{n-1}(s)\,dw(s)\right) dw(z) \\ &+ \int_0^t \int_{\mathbf{Z}} \gamma\Big(Y_{n-1}(z), \int_0^z k_3(z,s)Y_{n-1}(s)\,ds, \int_0^z \sigma_3(z,s)Y_{n-1}(s)\,dw(s), \xi\Big)\tilde{N}(dz,d\xi) \end{aligned} \tag{6}$$

for $t \in [0,T]$ and $n = 1,2,\ldots$. It is easy to see that $Y_0(\cdot) \in \mathcal{M}^2([0,T];\mathbb{R})$ and by induction we also have $Y_n(\cdot) \in \mathcal{M}^2([0,T];\mathbb{R})$.

Similarly to the proof of Lemma 2.2, we have

$$\mathbb{E}|Y_n(t)|^2 \le 4|\varphi(t)|^2 + 4C_1\int_0^t \mathbb{E}|Y_{n-1}(z)|^2\,dz,$$

where C_1 is defined in the proof of Lemma 2.2.

Hence for any $n \ge 1$, we get

$$\begin{aligned} \max_{1\le k\le n}\mathbb{E}|Y_k(t)|^2 \le& 4\|\varphi\|_\infty^2 + 4C_1\int_0^t \max_{1\le k\le n}\mathbb{E}|Y_{k-1}(z)|^2\,dz \\ \le& 4\|\varphi\|_\infty^2 + 4C_1\int_0^t \left[\mathbb{E}|\varphi(T)|^2 + \max_{1\le k\le n}\mathbb{E}|Y_k(z)|^2\right] dz \\ \le& 4\left(1+C_1T\right)\|\varphi\|_\infty^2 + 4C_1\int_0^t \left[\max_{1\le k\le n}\mathbb{E}|Y_k(z)|^2\right] dz. \end{aligned}$$

Gronwall's inequality yields that

$$\max_{1\le k\le n}\mathbb{E}|Y_k(t)|^2 \le 4\left(1+C_1T\right)\|\varphi\|_\infty^2 \exp\left(4C_1T\right) =: C.$$

Since k is arbitrary, one can get

$$\mathbb{E}|Y_n(t)|^2 \le C \ \text{ for } \ t \in [0,T], \ \ n \ge 1. \tag{7}$$

Noting that

$$\begin{aligned} |Y_1(t) - Y_0(t)|^2 =& \left|Y_1(t) - \varphi(0)\right|^2 \\ \le& 3\left|\int_0^t f\left(Y_0(z), \int_0^z k_1(z,s)Y_0(s)\,ds, \int_0^z \sigma_1(z,s)Y_0(s)\,dw(s)\right) dz\right|^2 \end{aligned}$$

$$+ 3\left|\int_0^t g\left(Y_0(z), \int_0^z k_2(z,s)Y_0(s)\,ds, \int_0^z \sigma_2(z,s)Y_0(s)\,dw(s)\right) dw(z)\right|^2$$
$$+ 3\left|\int_0^t \int_{\mathbf{Z}} \gamma\left(Y_0(z), \int_0^z k_3(z,s)Y_0(s)\,ds, \int_0^z \sigma_3(z,s)Y_0(s)\,dw(s), \xi\right)\tilde{N}(dz,d\xi)\right|^2$$

and taking the expectation, we have

$$\mathbb{E}\,|Y_1(t) - Y_0(t)|^2 \leq 3C_1T\|\varphi\|_\infty^2 \leq C_0.$$

We claim that for $n \geq 0$,

$$\mathbb{E}\,|Y_{n+1}(t) - Y_n(t)|^2 \leq \frac{C_0\,(3C_2t)^n}{n!}, \tag{8}$$

where C_2 is defined in the step 1. Then, inductively, (8) need to be still holds for $n+1$. Noting that

$$\begin{aligned}|Y_{n+2}(t) - Y_{n+1}(t)|^2 \leq & 3\left|\int_0^t f\left(Y_{n+1}(z), \int_0^z k_1(z,s)Y_{n+1}(s)\,ds, \int_0^z \sigma_1(z,s)Y_{n+1}(s)\,dw(s)\right) dz\right.\\ &\left. - \int_0^t f\left(Y_n(z), \int_0^z k_1(z,s)Y_n(s)\,ds, \int_0^z \sigma_1(z,s)Y_n(s)\,dw(s)\right) dz\right|^2\\ &+ 3\left|\int_0^t g\left(Y_{n+1}(z), \int_0^z k_2(z,s)Y_{n+1}(s)\,ds, \int_0^z \sigma_2(z,s)Y_{n+1}(s)\,dw(s)\right) dw(z)\right.\\ &\left. - \int_0^t g\left(Y_n(z), \int_0^z k_2(z,s)Y_n(s)\,ds, \int_0^z \sigma_2(z,s)Y_n(s)\,dw(s)\right) dw(z)\right|^2\\ &+ 3\left|\int_0^t \int_{\mathbf{Z}} \gamma\left(Y_{n+1}(z), \int_0^z k_3(z,s)Y_{n+1}(s)\,ds, \int_0^z \sigma_3(z,s)Y_{n+1}(s)\,dw(s), \xi\right)\tilde{N}(dz,d\xi)\right.\\ &\left. - \int_0^t \int_{\mathbf{Z}} \gamma\left(Y_n(z), \int_0^z k_3(z,s)Y_n(s)\,ds, \int_0^z \sigma_3(z,s)Y_n(s)\,dw(s), \xi\right)\tilde{N}(dz,d\xi)\right|^2\end{aligned} \tag{9}$$

and using (8) and similarly to the proof of Lemma 2.2, we derive that

$$\begin{aligned}\mathbb{E}\,|Y_{n+2}(t) - Y_{n+1}(t)|^2 \leq & 3C_2 \int_0^t \mathbb{E}\,|Y_{n+1}(z) - Y_n(z)|^2\,dz\\ \leq & 3C_2 \int_0^t \frac{C_0\,(3C_2z)^n}{n!}\,dz\\ = & \frac{C_0\,(3C_2t)^{n+1}}{(n+1)!}.\end{aligned}$$

Namely, (8) holds for any n. Therefore,

$$\mathbb{E}\,|Y_{n+1}(t) - Y_n(t)|^2 \leq \frac{C_0\,(3C_2T)^n}{n!}.$$

Applying Chebyshev's inequality, we have

$$P\left\{|Y_{n+1}(t) - Y_n(t)|^2 > \frac{1}{3^n}\right\} \leq \frac{C_0\,(3C_2T)^n}{n!}.$$

Since $\sum_{n=0}^{\infty} \frac{C_0\,(3C_2T)^n}{n!} < \infty$, by the Borel-Cantelli lemma, for almost all $\omega \in \Omega$, there exists a positive integer $n_0 = n_0(\omega)$ such that

$$|Y_{n+1}(t) - Y_n(t)|^2 \leq \frac{1}{3^n} \quad \text{for} \quad n \geq n_0.$$

It follows that, with probability 1, the partial sums

$$Y_0(t) + \sum_{k=0}^{n-1} [Y_{k+1}(t) - Y_k(t)] = Y_n(t)$$

are convergent uniformly for $t \in [0, T]$. Let $Y(t)$ be the limiting. It is easy to see that $Y(t)$ is continuous and $\mathcal{F}_t$-adapted. Moreover, we see from (8) that for any t, $\{Y_n(t)\}_{n\geq 1}$ is a Cauchy sequence in $\mathcal{L}^2([0, T]; \mathbb{R})$. Consequently, $Y_n(t) \to Y(t)$ in $\mathcal{L}^2([0, T]; \mathbb{R})$. Thus,

$$\mathbb{E}|Y(t)|^2 \leq C \text{ for } t \in [0, T],$$

where C depends on k_1, k_2, k_3, σ_1, σ_2, σ_3, T, $\bar{K}$ and φ, which yields $Y(\cdot) \in \mathcal{M}^2([0, T]; \mathbb{R})$. It remains to show that $Y(t)$ satisfies SVIE (1).

Since

$$\begin{aligned}
&3\mathbb{E}\bigg| \int_0^t f\left(Y_n(z), \int_0^z k_1(z,s)Y_n(s)\,ds, \int_0^z \sigma_1(z,s)Y_n(s)\,dw(s)\right) dz \\
&\quad - \int_0^t f\left(Y(z), \int_0^z k_1(z,s)Y(s)ds, \int_0^z \sigma_1(z,s)Y(s)dw(s)\right) dz\bigg|^2 \\
&+ 3\mathbb{E}\bigg| \int_0^t g\left(Y_n(z), \int_0^z k_2(z,s)Y_n(s)\,ds, \int_0^z \sigma_2(z,s)Y_n(s)\,dw(s)\right) dw(z) \\
&\quad - \int_0^t g\left(Y(z), \int_0^z k_2(z,s)Y(s)\,ds, \int_0^z \sigma_2(z,s)Y(s)\,dw(s)\right) dw(z)\bigg|^2 \\
&+ 3\mathbb{E}\bigg| \int_0^t \int_{\mathbf{Z}} \gamma\Big(Y_n(z), \int_0^z k_3(z,s)Y_n(s)\,ds, \int_0^z \sigma_3(z,s)Y_n(s)\,dw(s), \xi\Big)\tilde{N}(dz, d\xi) \\
&\quad - \int_0^t \int_{\mathbf{Z}} \gamma\Big(Y(z), \int_0^z k_3(z,s)Y(s)\,ds, \int_0^z \sigma_3(z,s)Y(s)\,dw(s), \xi\Big)\tilde{N}(dz, d\xi)\bigg|^2 \\
\leq & C_2 \int_0^T \mathbb{E}\,|Y_n(s) - Y(s)|^2\,ds \to 0 \text{ as } n \to \infty,
\end{aligned}$$

letting $n \to \infty$ in (6), we get

$$\begin{aligned}
Y(t) =& \varphi(t) + \int_0^t f\left(Y(z), \int_0^z k_1(z,s)Y(s)\,ds, \int_0^z \sigma_1(z,s)Y(s)\,dw(s)\right) dz \\
&+ \int_0^t g\left(Y(z), \int_0^z k_2(z,s)Y(s)\,ds, \int_0^z \sigma_2(z,s)Y(s)\,dw(s)\right) dw(z) \\
&+ \int_0^t \int_{\mathbf{Z}} \gamma\Big(Y(z), \int_0^z k_3(z,s)Y(s)\,ds, \int_0^z \sigma_3(z,s)Y(s)\,dw(s), \xi\Big)\tilde{N}(dz, d\xi).
\end{aligned}$$

The proof is complete. □

2.2. Hölder continuity of the analytic solutions

As mentioned in [15] for SDEs, the analytic solution is Hölder continuous with exponent $\upsilon = \frac{1}{2}$. Now we prove that this property is inherited by the analytic solution of SVIEs (1).

Theorem 2.4. *Under* **(A1)***, the solution* $Y(t)$ *is Hölder continuous with exponent* $\upsilon = \frac{1}{2}$.

Proof. For $0 < r < t$,

$$Y(t) - Y(r) = \int_r^t f\Big(Y(z), \int_0^z k_1(z,s)Y(s)\,ds, \int_0^z \sigma_1(z,s)Y(s)\,dw(s)\Big)\,dz + \int_r^t g\Big(Y(z), \int_0^z k_2(z,s)Y(s)\,ds, \int_0^z \sigma_2(z,s)Y(s)\,dw(s)\Big)\,dw(z) + \int_r^t \int_{\mathbf{Z}} \gamma\Big(Y(z), \int_0^z k_3(z,s)Y(s)\,ds, \int_0^z \sigma_3(z,s)Y(s)\,dw(s), \xi\Big)\tilde{N}(dz, d\xi).$$

Taking expectation, we have

$$\mathbb{E}\,|Y(t) - Y(r)|^2 \le D_1 + D_2 + D_3,$$

where

$$D_1 := 3\mathbb{E}\left|\int_r^t f\Big(Y(z), \int_0^z k_1(z,s)Y(s)\,ds, \int_0^z \sigma_1(z,s)Y(s)\,dw(s)\Big)\,dz\right|^2,$$

$$D_2 := 3\mathbb{E}\left|\int_r^t g\Big(Y(z), \int_0^z k_2(z,s)Y(s)\,ds, \int_0^z \sigma_2(z,s)Y(s)\,dw(s)\Big)\,dw(z)\right|^2$$

and

$$D_3 := 3\mathbb{E}\left|\int_r^t \int_{\mathbf{Z}} \gamma\Big(Y(z), \int_0^z k_3(z,s)Y(s)\,ds, \int_0^z \sigma_3(z,s)Y(s)\,dw(s), \xi\Big)\tilde{N}(dz, d\xi)\right|^2.$$

By the application of Hölder's inequality and **(A2)**, we have

$$\begin{aligned} D_1 \le& 3\,(t-r)\int_r^t \bar{K}\left[1 + \mathbb{E}|Y(z)|^2 + \mathbb{E}\left|\int_0^z k_1(z,s)Y(s)\,ds\right|^2 + \mathbb{E}\left|\int_0^z \sigma_1(z,s)Y(s)\,dw(s)\right|^2\right]dz \\ \le& 3\bar{K}\,(t-r)\left[(t-r) + \int_0^t \mathbb{E}|Y(s)|^2\,ds + T^2\|k_1\|_\infty \int_0^t \mathbb{E}|Y(s)|^2\,ds + T\|\sigma_1\|_\infty \int_0^t \mathbb{E}|Y(s)|^2\,ds\right] \\ \le& 3\bar{K}\,(t-r)\,C_3', \end{aligned}$$

where $C_3' := T + TC_{10}C$.

Using Itô isometry, Lemma 2.1 and **(A2)**, one obtains the following estimate

$$\begin{aligned} D_2 + D_3 \le& 3\int_r^t \bar{K}\left[1 + \mathbb{E}|Y(z)|^2 + \mathbb{E}\left|\int_0^z k_2(z,s)Y(s)\,ds\right|^2 + \mathbb{E}\left|\int_0^z \sigma_2(z,s)Y(s)\,dw(s)\right|^2\right]dz \\ &+ 3c_1\int_r^t \bar{K}\left[1 + \mathbb{E}|Y(z)|^2 + \mathbb{E}\left|\int_0^z k_3(z,s)Y(s)\,ds\right|^2 + \mathbb{E}\left|\int_0^z \sigma_3(z,s)Y(s)\,dw(s)\right|^2\right]dz \\ \le& 3\int_r^t \bar{K}\Big[1 + c_1 + (1+c_1)E|Y(z)|^2 + T^2\|k_2\|_\infty \int_0^t \mathbb{E}|Y(s)|^2\,ds \\ &+ T\|\sigma_2\|_\infty \int_0^t E|Y(s)|^2\,ds + T^2c_1\|k_3\|_\infty \int_0^t E|Y(s)|^2\,ds + T\|\sigma_3\|_\infty \int_0^t \mathbb{E}|Y(s)|^2\,ds\Big]dz \\ \le& 3\bar{K}\,(t-r)\,C_3'', \end{aligned}$$

where $C_3'' := 1 + c_1 + C_{11}C$.

Therefore, we have

$$\mathbb{E}\,|Y(t) - Y(r)|^2 \le 3\bar{K}\,(t-r)\,C_3' + 3\bar{K}\,(t-r)\,C_3'' = C_3\,(t-r)\,,$$

where $C_3 := 3\bar{K}\left(C_3' + C_3''\right)$, which depends on $k_1, k_2, k_3, \sigma_1, \sigma_2, \sigma_3, T, \bar{K}$ and K. Thus $Y(t)$ is Hölder continuous with exponent $1/2$ on $[0, T]$. □

Remark 2.5. *Let* $\gamma = 0$, *then* (1) *is reduced to the following SVIDE*

$$\begin{aligned} X(t) =& \varphi(t) + \int_0^t f\left(Y(t), \int_0^z k_1(z,s)Y(s)\,ds, \int_0^z \sigma_1(z,s)Y(s)\,dw(s)\right) dz \\ &+ \int_0^t g\left(Y(t), \int_0^z k_2(z,s)Y(s)\,ds, \int_0^z \sigma_2(z,s)Y(s)\,dw(s)\right) dw(z) \end{aligned} \tag{10}$$

which was considered in [27].

In SVIE (1), *let* $k_i(t,s) = \sigma_i(t,s) = 0$, *where* $i = 1, 2, 3$ *and* $\gamma = 0$. *Then SVIE* (1) *is reduced to the following SDE*

$$dY(t) = f\,(Y(t))\;dt + g\,(Y(t))\;dw(t), \tag{11}$$

which has been studied in [15]. We observe that the result obtained in Theorem 2.4 is consistent with that in [15].

3. The Euler-Maruyama method

Denote $I_h := \{t_n := nh : n = 0, 1, \ldots, S\ (t_S = T)\}$ which is a given mesh on $I = [0, T]$. For $n = 0, 1, \ldots, S-1$, define

$$\begin{aligned} X_{n+1} =& \varphi(t_n) + hf\left(X_n, \sum_{l=0}^{n-1}\int_{t_l}^{t_{l+1}} k_1(t_n,s)X_l\,ds, \sum_{l=0}^{n-1}\sigma_1(t_n,t_l)X_l\Delta w_l\right) \\ &+ g\left(X_n, \sum_{l=0}^{n-1}\int_{t_l}^{t_{l+1}} k_2(t_n,s)X_l\,ds, \sum_{l=0}^{n-1}\sigma_2(t_n,t_l)X_l\Delta w_l\right)\Delta w_n \\ &+ \int_{t_n}^{t_{n+1}}\int_{\mathbf{Z}}\gamma\Big(X_n, \sum_{l=0}^{n-1}\int_{t_l}^{t_{l+1}} k_3(t_n,s)X_l\,ds, \sum_{l=0}^{n-1}\sigma_3(t_n,t_l)X_l\Delta w_l, \xi\Big)\tilde{N}(dz,d\xi) \end{aligned} \tag{12}$$

with initial data $X_0 = \varphi(0)$, where $t_n = nh$ and $\Delta w_n = w(t_{n+1}) - w(t_n)$.

(12) can be rewritten as the following form:

$$\begin{aligned} X_{n+1} =& \varphi(0) + \sum_{r=0}^{n} hf\left(X_r, \sum_{l=0}^{r-1}\int_{t_l}^{t_{l+1}} k_1(t_r,s)X_l\,ds, \sum_{l=0}^{r-1}\sigma_1(t_r,t_l)X_l\Delta w_l\right) \\ &+ \sum_{r=0}^{n} g\left(X_r, \sum_{l=0}^{r-1}\int_{t_l}^{t_{l+1}} k_2(t_r,s)X_l\,ds, \sum_{l=0}^{r-1}\sigma_2(t_r,t_l)X_l\Delta w_l\right)\Delta w_r \\ &+ \sum_{r=0}^{n}\int_{t_r}^{t_{r+1}}\int_{\mathbf{Z}}\gamma\Big(X_r, \sum_{l=0}^{r-1}\int_{t_l}^{t_{l+1}} k_3(t_r,s)X_l\,ds, \sum_{l=0}^{r-1}\sigma_3(t_r,t_l)X_l\Delta w_l, \xi\Big)\tilde{N}(dz,d\xi) \end{aligned} \tag{13}$$

by induction.

3.1. Boundedness of the numerical solution

In order to deal with the difficulties caused by multiple random integrals and lévy noise, we propose the following lemma and obtain the boundedness of the numerical solution to generalized SVIE (1).

Lemma 3.1. *Assume that* **(A2)** *holds. Let* $\{X_n\}$ *be the numerical solution of the Euler-Maruyama method* (12). *Denote*

$$F_2 := \mathbb{E}\left|\sum_{r=0}^{n} g\left(X_r, \sum_{l=0}^{r-1}\int_{t_l}^{t_{l+1}} k_2(t_r,s)X_l\,ds, \sum_{l=0}^{r-1}\sigma_2(t_r,t_l)X_l\Delta w_l\right)\Delta w_r\right|^2$$

and

$$F_3 := \mathbb{E}\left|\sum_{r=0}^{n}\int_{t_r}^{t_{r+1}}\int_{\mathbf{Z}}\gamma\Big(X_r, \sum_{l=0}^{r-1}\int_{t_l}^{t_{l+1}} k_3(t_r,s)X_l\,ds, \sum_{l=0}^{r-1}\sigma_3(t_r,t_l)X_l\Delta w_l, \xi\Big)\tilde{N}(dz,d\xi)\right|^2.$$

Then

$$F_2 + F_3 \leq (1+c_1)\bar{K}T + \bar{K}C_{11}h\sum_{l=0}^{n}\mathbb{E}|X_l|^2,$$

where C_{11} *is defined in the proof of Lemma 2.2.*

Proof. By Lemma 3.1 of [27], one has

$$\begin{aligned}F_2 \leq & h\bar{K}\sum_{r=0}^{n}\left(1+\mathbb{E}|X_r|^2 + \mathbb{E}\left|\sum_{l=0}^{r-1}\int_{t_l}^{t_{l+1}} k_2(t_r,s)X_l\,ds\right|^2 + \mathbb{E}\left|\sum_{l=0}^{r-1}\sigma_2(t_r,t_l)X_l\Delta w_l\right|^2\right)\\ \leq & h\bar{K}\left[(n+1) + \sum_{r=0}^{n}\mathbb{E}|X_r|^2 + \|k_2\|_\infty^2 T^2\sum_{l=0}^{n}\mathbb{E}|X_l|^2 + \|\sigma_2\|_\infty^2 T\sum_{l=0}^{n}\mathbb{E}|X_l|^2\right].\end{aligned}$$

Using Lemma 2.1, we obtain

$$\begin{aligned}F_3 \leq & \sum_{r=0}^{n}\int_{t_r}^{t_{r+1}} c_1\mathbb{E}\left|\int_{\mathbf{Z}}\gamma\Big(X_r, \sum_{l=0}^{r-1}\int_{t_l}^{t_{l+1}} k_3(t_r,s)X_l\,ds, \sum_{l=0}^{r-1}\sigma_3(t_r,t_l)X_l\Delta w_l, \xi\Big)\gamma(d\xi)\right|^2 dz\\ \leq & c_1\bar{K}\sum_{r=0}^{n}\int_{t_r}^{t_{r+1}}\left[1+\mathbb{E}|X_r|^2 + \mathbb{E}\left|\sum_{l=0}^{r-1}\int_{t_l}^{t_{l+1}} k_3(t_r,s)X_l\,ds\right|^2 + \mathbb{E}\left|\sum_{l=0}^{r-1}\sigma_2(t_r,t_l)X_l\Delta w_l\right|^2\right]dz\\ \leq & c_1\bar{K}\left[T + h\sum_{r=0}^{n}\mathbb{E}|X_r|^2 + h\|k_3\|_\infty^2 T^2\sum_{l=0}^{n}\mathbb{E}|X_l|^2 + h\|\sigma_3\|_\infty^2 T\sum_{l=0}^{n}\mathbb{E}|X_l|^2\right]\end{aligned}$$

This completes the proof. □

Theorem 3.2. *Assume that* **(A2)** *holds. Let* $\{X_n\}$ *be the numerical solution of the Euler-Maruyama method* (12). *Then there exists a positive constant* M_0 *such that*

$$\mathbb{E}|X_n|^2 \leq M_0, \tag{14}$$

where M_0 *depends on* k_1, k_2, k_3, σ_1, σ_2, σ_3, T, φ *and* $\bar{K}$, *but not on* h.

Proof. Taking the expectation of (13) and using the elementary inequality, for all $0 \le t_{n+1} \le T$, we get

$$\begin{aligned}
\mathbb{E}|X_{n+1}|^2 \le & 4\mathbb{E}|\varphi(0)|^2 + 4\mathbb{E}\left|\sum_{r=0}^{n} hf\left(X_r, \sum_{l=0}^{r-1}\int_{t_l}^{t_{l+1}} k_1(t_r,s)X_l\,ds, \sum_{l=0}^{r-1}\sigma_1(t_r,t_l)X_l\Delta w_l\right)\right|^2 \\
& + 4\mathbb{E}\left|\sum_{r=0}^{n} g\left(X_r, \sum_{l=0}^{r-1}\int_{t_l}^{t_{l+1}} k_2(t_r,s)X_l\,ds, \sum_{l=0}^{r-1}\sigma_2(t_r,t_l)X_l\Delta w_l\right)\Delta w_r\right|^2 \\
& + 4\mathbb{E}\left|\sum_{r=0}^{n}\int_{t_r}^{t_{r+1}}\int_{\mathbf{Z}}\gamma\Big(X_r, \sum_{l=0}^{r-1}\int_{t_l}^{t_{l+1}} k_3(t_r,s)X_l\,ds, \sum_{l=0}^{r-1}\sigma_3(t_r,t_l)X_l\Delta w_l, \xi\Big)\tilde{N}(dz,d\xi)\right|^2 \\
\le & 4\mathbb{E}|\varphi(0)|^2 + 4F_1 + 4F_2 + 4F_3,
\end{aligned}$$

where

$$F_1 := \mathbb{E}\left|\sum_{r=0}^{n} hf\left(X_r, \sum_{l=0}^{r-1}\int_{t_l}^{t_{l+1}} k_1(t_r,s)X_l\,ds, \sum_{l=0}^{r-1}\sigma_1(t_r,t_l)X_l\Delta w_l\right)\right|^2$$

and F_2, F_3 are defined in Lemma 3.1. The Cauchy inequality, **(A2)** and the discrete Hölder's inequality imply that

$$\begin{aligned}
F_1 \le & (n+1)\sum_{r=0}^{n}\mathbb{E}\left|hf\left(X_r, \sum_{l=0}^{r-1}\int_{t_l}^{t_{l+1}} k_1(t_r,s)X_l\,ds, \sum_{l=0}^{r-1}\sigma_1(t_r,t_l)X_l\Delta w_l\right)\right|^2 \\
\le & (n+1)\,h^2\bar{K}\sum_{r=0}^{n}\left(1 + \mathbb{E}|X_r|^2 + \mathbb{E}\left|\sum_{l=0}^{r-1}\int_{t_l}^{t_{l+1}} k_1(t_r,s)X_l\,ds\right|^2 + \mathbb{E}\left|\sum_{l=0}^{r-1}\sigma_1(t_r,t_l)X_l\Delta w_l\right|^2\right).
\end{aligned} \tag{15}$$

Applying Hölder's inequality and Minkiskii's inequality, one obtains

$$\begin{aligned}
\sum_{r=0}^{n}\mathbb{E}\left|\sum_{l=0}^{r-1}\int_{t_l}^{t_{l+1}} k_1(t_r,s)X_l\,ds\right|^2 \le & \sum_{r=0}^{n}\left(\sum_{l=0}^{r-1}\left|\int_{t_l}^{t_{l+1}} k_1(t_r,s)\,ds\right|^2\sum_{l=0}^{r-1}\mathbb{E}|X_l|^2\right) \\
\le & \sum_{r=0}^{n}\left(\sum_{l=0}^{r-1} h^2\|k_1\|_\infty^2\sum_{l=0}^{r-1}\mathbb{E}|X_l|^2\right) \\
\le & h^2\|k_1\|_\infty^2\sum_{r=0}^{n} r\sum_{l=0}^{r-1}\mathbb{E}|X_l|^2 \\
\le & h^2\|k_1\|_\infty^2\,(n+1)^2\sum_{l=0}^{n}\mathbb{E}|X_l|^2 \\
\le & \|k_1\|_\infty^2 T^2\sum_{l=0}^{n}\mathbb{E}|X_l|^2.
\end{aligned} \tag{16}$$

Similarly to (16), we get

$$\begin{aligned}
\sum_{r=0}^{n}\mathbb{E}\left|\sum_{l=0}^{r-1}\sigma_1(t_r,t_l)X_l\Delta w_l\right|^2 \le & \sum_{r=0}^{n}\left(\sum_{l=0}^{r-1}|\sigma_1(t_r,t_l)|^2\,\mathbb{E}|X_l|^2|\Delta w_l|^2\right) \\
\le & h\sum_{r=0}^{n}\left(\sum_{l=0}^{r-1}\|\sigma_1\|_\infty^2\mathbb{E}|X_l|^2\right)
\end{aligned}$$

$$\leq \|\sigma_1\|_\infty^2 T \sum_{l=0}^{n} \mathbb{E}|X_l|^2. \tag{17}$$

Substituting (16) and (17) into (15), we derive

$$F_1 \leq \bar{K}T^2 + \bar{K}TC_{10}h \sum_{l=0}^{n} \mathbb{E}|X_l|^2. \tag{18}$$

One uses Lemma 3.1 and obtains the following

$$\mathbb{E}|X_{n+1}|^2 \leq 4\left(|\varphi(0)|^2 + \bar{K}(1+c_1)T + \bar{K}T^2\right) + 4\bar{K}\left(C_{10}T + C_{11}\right)h \sum_{l=0}^{n} \mathbb{E}|X_l|^2. \tag{19}$$

The discrete Gronwall inequality implies

$$\mathbb{E}|X_{n+1}|^2 \leq M_0,$$

where $M_0 := 4\left(|\varphi(0)|^2 + \bar{K}(1+c_1)T + \bar{K}T^2\right)\exp\left[4\bar{K}\left(C_{10}T + C_{11}\right)T\right]$. Here C_{10} and C_{11} are defined in the proof of Lemma 2.2. □

3.2. *Convergence of the Euler-Maruyama method*

Now we introduce the approximate time continuous interpolation of discrete numerical approximation, and then obtain the convergence results of Euler-Maruyama method (12).

Define

$$s_h := t_n \ \text{ and } \ X_h(s) := X_n, \ \text{ for } \ s \in [t_n, t_{n+1})$$

with $0 \leq n \leq S-1$. Let $X(t)$ be the continuous form of X_n with $X(t_n) = X_n$, i.e.,

$$\begin{aligned} X(t) =&\varphi(t) + \int_{t_n}^{t} f\left(X_h(z_h), \int_0^{z_h} k_1(z_h, s)X_h(s)\,ds, \int_0^{z_h} \sigma_1(z_h, s_h)X_h(s)\,dw(s)\right) dz \\ &+ \int_{t_n}^{t} g\left(X_h(z_h), \int_0^{z_h} k_2(z_h, s)X_h(s)\,ds, \int_0^{z_h} \sigma_2(z_h, s_h)X_h(s)\,dw(s)\right) dw(z) \\ &+ \int_{t_n}^{t}\int_{\mathbf{Z}} \gamma\left(X_h(z_h), \int_0^{z_h} k_3(z_h, s)X_h(s)\,ds, \int_0^{z_h} \sigma_3(z_h, s_h)X_h(s)\,dw(s), \xi\right)\tilde{N}(dz, d\xi) \\ =&\varphi(0) + \int_0^{t} f\left(X_h(z_h), \int_0^{z_h} k_1(z_h, s)X_h(s)\,ds, \int_0^{z_h} \sigma_1(z_h, s_h)X_h(s)\,dw(s)\right) dz \\ &+ \int_0^{t} g\left(X_h(z_h), \int_0^{z_h} k_2(z_h, s)X_h(s)\,ds, \int_0^{z_h} \sigma_2(z_h, s_h)X_h(s)\,dw(s)\right) dw(z) \\ &+ \int_0^{t}\int_{\mathbf{Z}} \gamma\left(X_h(z_h), \int_0^{z_h} k_3(z_h, s)X_h(s)\,ds, \int_0^{z_h} \sigma_3(z_h, s_h)X_h(s)\,dw(s), \xi\right)\tilde{N}(dz, d\xi), \end{aligned}$$

where $t \in [t_n, t_{n+1})$ with $0 \leq n \leq S-1$.

Lemma 3.3. *Let* $\{X_n\}$ *be the numerical solution of the Euler-Maruyama method* (12). *Under* **(A2)** *and* **(A3)**, *then there exists a positive constant such that*

$$\mathbb{E}\,|X(t) - X_n|^2 \leq M_1 h \tag{20}$$

with M_1 *depends on* $k_1, k_2, k_3, \sigma_1, \sigma_2, \sigma_3, T, \varphi, \grave{K}$ *and* $\bar{K}$ *but not on* h.

Proof. We observe that

$$\mathbb{E}\,|X(t)-X_n|^2 \le 4|\varphi(t)-\varphi(t_n)|^2 + 4G_1 + 4G_2 + 4G_3,$$

where

$$G_1 := \mathbb{E}\left|\int_{t_n}^{t} f\left(X_n, \int_0^{t_n} k_1(t_n,s)X_h(s)\,ds, \int_0^{t_n} \sigma_1(t_n,s_h)X_h(s)\,dw(s)\right) dz\right|^2,$$

$$G_2 := \mathbb{E}\left|\int_{t_n}^{t} g\left(X_n, \int_0^{t_n} k_2(t_n,s)X_h(s)\,ds, \int_0^{t_n} \sigma_2(t_n,s_h)X_h(s)\,dw(s)\right) dw(z)\right|^2$$

and

$$G_3 := \mathbb{E}\left|\int_{t_n}^{t}\int_{\mathbf{Z}} \gamma\Big(X_n, \int_0^{t_n} k_3(t_n,s)X_h(s)\,ds, \int_0^{t_n} \sigma_3(t_n,s_h)X_h(s)\,dw(s), \xi\Big)\tilde{N}(dz,d\xi)\right|^2.$$

Applying (**A2**) and Hölder's inequality, one gets

$$\begin{aligned}
G_1 \le & h\int_{t_n}^{t} \mathbb{E}\left|f\left(X_n, \int_0^{t_n} k_1(t_n,s)X_h(s)\,ds, \int_0^{t_n} \sigma_1(t_n,s_h)X_h(s)\,dw(s)\right)\right|^2 dz\\
\le & h\int_{t_n}^{t} \bar{K}\left[1+\mathbb{E}|X_n|^2 + \mathbb{E}\left|\int_0^{t_n} k_1(t_n,s)X_h(s)\,ds\right|^2 + \mathbb{E}\left|\int_0^{t_n} \sigma_1(t_n,s_h)X_h(s)\,dw(s)\right|^2\right] dz\\
\le & h^2\bar{K}\left[1+\mathbb{E}|X_n|^2 + \int_0^{t_n} \|k_1\|_\infty^2\,ds \int_0^{t_n} \mathbb{E}|X_h(s)|^2\,ds + \int_0^{t_n} \|\sigma_1\|_\infty^2 \mathbb{E}|X_h(s)|^2\,ds\right]\\
\le & h^2\bar{K} + h^2\bar{K}C_{10}M_0.
\end{aligned}$$

Using (**A2**), Lemma 3.1 and Itô isometry, one obtains that

$$\begin{aligned}
G_2+G_3 = & \int_{t_n}^{t} \mathbb{E}\left|g\left(X_n, \int_0^{t_n} k_2(t_n,s)X_h(s)\,ds, \int_0^{t_n} \sigma_2(t_n,s_h)X_h(s)\,dw(s)\right)\right|^2 dz\\
& + c_1\int_{t_n}^{t}\int_{\mathbf{Z}} \mathbb{E}\left|\gamma\Big(X_n, \int_0^{t_n} k_3(t_n,s)X_h(s)\,ds, \int_0^{t_n} \sigma_3(t_n,s_h)X_h(s)\,dw(s), \xi\Big)\right|^2 \nu(d\xi)dz\\
\le & \int_{t_n}^{t} \bar{K}\left[1+\mathbb{E}|X_n|^2 + \mathbb{E}\left|\int_0^{t_n} k_2(t_n,s)X_h(s)\,ds\right|^2 + \mathbb{E}\left|\int_0^{t_n} \sigma_2(t_n,s_h)X_h(s)\,dw(s)\right|^2\right] dz\\
& + c_1\int_{t_n}^{t} \bar{K}\left[1+\mathbb{E}|X_n|^2 + \mathbb{E}\left|\int_0^{t_n} k_3(t_n,s)X_h(s)\,ds\right|^2 + \mathbb{E}\left|\int_0^{t_n} \sigma_3(t_n,s_h)X_h(s)\,dw(s)\right|^2\right] dz\\
\le & h\bar{K}\Bigg[1+c_1+(1+c_1)\mathbb{E}|X_n|^2 + \int_0^{t_n} \|k_2\|_\infty^2\,ds \int_0^{t_n} \mathbb{E}|X_h(s)|^2\,ds + \int_0^{t_n} \|\sigma_2\|_\infty^2 \mathbb{E}|X_h(s)|^2\,ds\\
& + \int_0^{t_n} \|k_3\|_\infty^2\,ds \int_0^{t_n} \mathbb{E}|X_h(s)|^2\,ds + \int_0^{t_n} \|\sigma_3\|_\infty^2 \mathbb{E}|X_h(s)|^2\,ds\Bigg]\\
\le & (1+c_1)h\bar{K} + h\bar{K}C_{11}M_0.
\end{aligned}$$

By **(A3)**, we have

$$|\varphi(t)-\varphi(t_n)|^2 \le \grave{K}h^2.$$

Consequently, we have

$$\mathbb{E}\,|X(t)-X_n|^2 \le M_1 h,$$

where

$$M_1 := 4\grave{K}T + 4\bar{K}\,(T+1+c_1) + 4\bar{K}\,(C_{10}T + C_{11})\,M_0.$$

□

Lemma 3.4. *Let* $\{X_n\}$ *be the numerical solution of the Euler-Maruyama method* (12)*. Under* **(A1)** *and* **(A3)***, then there exist* K_2 *and* K_2' *such that*

$$H_2 + H_3 \le (1+c_1)K_2 h + K_2' \int_0^t \mathbb{E}\,|Y(s)-X(s)|^2\,ds,$$

where

$$\begin{aligned} H_2 :=& \mathbb{E}\Big|\int_0^t g\Big(Y(z), \int_0^z k_2(z,s)Y(s)\,ds, \int_0^z \sigma_2(z,s)Y(s)\,dw(s)\Big)\,dw(z) \\ &- g\Big(X_h(z_h), \int_0^{z_h} k_2(z_h,s)X_h(s)\,ds, \int_0^{z_h} \sigma_2(z_h,s_h)X_h(s)\,dw(s)\Big)\,dw(z)\Big|^2 \end{aligned}$$

and

$$\begin{aligned} H_3 :=& \mathbb{E}\Big|\int_0^t \int_{\mathbf{Z}} \gamma\Big(Y(z), \int_0^z k_3(z,s)Y(s)\,ds, \int_0^z \sigma_3(z,s)Y(s)\,dw(s), \xi\Big)\tilde{N}(dz,d\xi) \\ &- \int_0^t \int_{\mathbf{Z}} \gamma\Big(X_h(z_h), \int_0^{z_h} k_3(z_h,s)X_h(s)\,ds, \int_0^{z_h} \sigma_3(z_h,s_h)X_h(s)\,dw(s), \xi\Big)\tilde{N}(dz,d\xi)\Big|^2 \end{aligned}$$

with K_2 *and* K_2' *depend on* k_2, σ_2, k_3, σ_3, T, φ, K *and* $\grave{K}$*, but not on* h.

Proof. By the use of **(A1)**, Lemma 3.1, Hölder's inequality and Itô isometry, we get

$$\begin{aligned} H_2 + H_3 \le& \int_0^t \mathbb{E}\Big|g\Big(Y(z), \int_0^z k_2(z,s)Y(s)\,ds, \int_0^z \sigma_2(z,s)Y(s)\,dw(s)\Big) \\ &\qquad - g\Big(X_h(z_h), \int_0^{z_h} k_2(z_h,s)X_h(s)\,ds, \int_0^{z_h} \sigma_2(z_h,s_h)X_h(s)\,dw(s)\Big)\Big|^2 dz \\ &+ c_2 \int_0^t \int_{\mathbf{Z}} \mathbb{E}\Big|\gamma\Big(Y(z), \int_0^z k_3(z,s)Y(s)\,ds, \int_0^z \sigma_3(z,s)Y(s)\,dw(s), \xi\Big) \\ &\qquad - \gamma\Big(X_h(z_h), \int_0^{z_h} k_3(z_h,s)X_h(s)\,ds, \int_0^{z_h} \sigma_3(z_h,s_h)X_h(s)\,dw(s), \xi\Big)\Big|^2 \nu(d\xi)dz \\ \le& K \int_0^t \Big[(1+c_1)\mathbb{E}\,|Y(z)-X_h(z_h)|^2 + \mathbb{E}\Big|\int_0^z k_2(z,s)Y(s)\,ds - \int_0^{z_h} k_2(z_h,s)X_h(s)\,ds\Big|^2 \\ &\qquad + \mathbb{E}\Big|\int_0^z \sigma_2(z,s)Y(s)\,dw(s) - \int_0^{z_h} \sigma_2(z_h,s_h)X_h(s)\,dw(s)\Big|^2 \\ &\qquad + c_1\mathbb{E}\Big|\int_0^z k_3(z,s)Y(s)\,ds - \int_0^{z_h} k_3(z_h,s)X_h(s)\,ds\Big|^2 \end{aligned}$$

$$
\begin{aligned}
&+ c_1\mathbb{E}\left|\int_0^z \sigma_3(z,s)Y(s)\,dw(s) - \int_0^{z_h} \sigma_3(z_h,s_h)X_h(s)\,dw(s)\right|^2 \Bigg] dz \\
\leq& 2K\int_0^t \Bigg[(1+c_1)\mathbb{E}\,|Y(z)-X(z)|^2 + (1+c_1)\mathbb{E}\,|X(z)-X_h(z_h)|^2 + \mathbb{E}\left|\int_{z_h}^z k_2(z_h,s)X_h(s)\,ds\right|^2 \\
&+ \mathbb{E}\left|\int_0^z (k_2(z,s)Y(s) - k_2(z_h,s)X_h(s))\;ds\right|^2 + \mathbb{E}\left|\int_{z_h}^z \sigma_2(z_h,s_h)X_h(s)\,dw(s)\right|^2 \\
&+ \mathbb{E}\left|\int_0^z (\sigma_2(z,s)Y(s) - \sigma_2(z_h,s_h)\,X_h(s))\,dw(s)\right|^2 + c_1\mathbb{E}\left|\int_{z_h}^z k_3(z_h,s)X_h(s)\,ds\right|^2 \\
&+ c_1\mathbb{E}\left|\int_0^z (k_3(z,s)Y(s) - k_3(z_h,s)X_h(s))\;ds\right|^2 + c_1\mathbb{E}\left|\int_{z_h}^z \sigma_3(z_h,s_h)X_h(s)\,dw(s)\right|^2 \\
&+ c_1\mathbb{E}\left|\int_0^z (\sigma_3(z,s)Y(s) - \sigma_3(z_h,s_h)\,X_h(s))\,dw(s)\right|^2 \Bigg] dz.
\end{aligned}
$$

Using Hölder's inequality, Itô isometry, Theorem 3.2 and Lemma 3.3, for $i = 2,3$, we have

$$
\begin{aligned}
\int_0^t \mathbb{E}\left|\int_{z_h}^z k_i(z_h,s)X_h(s)\,ds\right|^2\,dz \leq& \int_0^t \left(\int_{z_h}^z |k_i(z_h,s)|^2\;ds \int_{z_h}^z \mathbb{E}|X_h(s)|^2\,ds\right) dz \\
\leq& h^2T\|k_i\|_\infty^2 M_0,
\end{aligned}
$$

$$
\begin{aligned}
\int_0^t \mathbb{E}\left|\int_{z_h}^z \sigma_i(z_h,s_h)X_h(s)\,dw(s)\right|^2\,dz \leq& \int_0^t \left(\int_{z_h}^z |\sigma_i(z_h,s_h)|^2\,\mathbb{E}|X_h(s)|^2\,ds\right) dz \\
\leq& M_0\int_0^t\left(\int_{z_h}^z \|\sigma_i\|_\infty^2\,ds\right) dz \\
\leq& hT\|\sigma_i\|_\infty^2 M_0,
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^t \mathbb{E}\left|\int_0^z (k_i(z,s)Y(s) - k_i(z_h,s)X_h(s))\;ds\right|^2\,dz \\
\leq& 2\int_0^t\int_0^z |k_i(z,s)-k_i(z_h,s)|^2\;ds\int_0^z \mathbb{E}|X_h(s)|^2\,ds\,dz + 4\int_0^t\int_0^z |k_i(z,s)|^2\,ds\int_0^z \mathbb{E}\,|Y(s)-X(s)|^2\;ds\,dz \\
&+ 4\int_0^t\int_0^z |k_i(z,s)|^2\,ds\int_0^z \mathbb{E}\,|X(s)-X_h(s)|^2\;ds\,dz \\
\leq& 2T^3\grave{K}h^2M_0 + 4T^3\|k_i\|_\infty^2 M_1 h + 4T^2\|k_i\|_\infty^2\int_0^t \mathbb{E}\,|Y(s)-X(s)|^2\;ds
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^t \mathbb{E}\left|\int_0^z (\sigma_i(z,s)Y(s) - \sigma_i(z_h,s_h)X_h(s))\;dw(s)\right|^2\,dz \\
\leq& \int_0^t\int_0^z \mathbb{E}\,|\sigma_i(z,s)Y(s) - \sigma_i(z_h,s_h)X_h(s)|^2\;ds\,dz \\
\leq& 2\int_0^t\int_0^z |\sigma_i(z,s)-\sigma_i(z_h,s_h)|^2\,\mathbb{E}|X_h(s))|^2\,ds\,dz + 4\int_0^t\int_0^z \mathbb{E}\,|\sigma_i(z,s)|^2\,\mathbb{E}\,|Y(s)-X(s))|^2\;ds\,dz
\end{aligned}
$$

$$+ 4\int_0^t \int_0^z \mathbb{E}\,|\sigma_i(z,s)|^2\, \mathbb{E}\,|X(s) - X_h(s)|^2 \; ds\, dz$$
$$\leq 4T^2 \grave{K} h^2 M_0 + 4T^2 \|\sigma_i\|_\infty^2 M_1 h + 4T\|\sigma_i\|_\infty^2 \int_0^t \mathbb{E}\,|Y(s) - X(s)|^2 \; ds.$$

Consequently, we get

$$H_2 + H_3 \leq (1 + c_1)K_2 h + K_2' \int_0^t \mathbb{E}\,|Y(s) - X(s)|^2 \; ds,$$

where

$$K_2 := 2KT\Big\{ \Big[2T\,(T + 2(1 + c_1))\,\grave{K}h + T(\|k_2\|_\infty^2 + c_1\|k_3\|_\infty^2)h + \|\sigma_2\|_\infty^2 + c_1\|\sigma_3\|_\infty^2\Big] M_0 T$$
$$+ \Big[1 + 4T^2(\|k_2\|_\infty^2 + \|k_3\|_\infty^2) + 4(T\|\sigma_2\|_\infty^2 + c_1\|\sigma_3\|_\infty^2)\Big] M_1 \Big\}$$

and $K_2' := 2KT\left[1 + c_1 + 4T^2(\|k_2\|_\infty^2 + \|k_3\|_\infty^2) + 4T(\|\sigma_2\|_\infty^2 + \|\sigma_3\|_\infty^2)\right].$ □

Theorem 3.5. *Under* **(A1)** *and* **(A3)**, *for* $i = 1, 2, 3$. *Let* $X(t)$ *and* $Y(t)$ *be the numerical solution of the Euler-Maruyama method and the analytic solution of SVIE* (1), *respectively. Then there exists a positive constant* M_2 *such that*

$$\mathbb{E}\,|X(t) - Y(t)|^2 \leq M_2 h, \tag{21}$$

where M_2 *depends on* k_1, k_2, k_3, σ_1, σ_2, σ_3, T, φ, K, $\bar{K}$ *and* $\grave{K}$ *but not on* h.

Proof. Applying **(A1)**, Hölder's inequality and the Itô isometry, one has

$$\mathbb{E}\,|Y(t) - X(t)|^2 \leq 3H_1 + 3H_2 + 3H_3,$$

where

$$H_1 := \mathbb{E}\bigg| \int_0^t f\Big(Y(z), \int_0^z k_1(z,s)Y(s)\, ds, \int_0^z \sigma_1(z,s)Y(s)\, dw(s)\Big)\, dz$$
$$- \int_0^t f\Big(X_h(z_h), \int_0^{z_h} k_1(z_h,s)X_h(s)\, ds, \int_0^{z_h} \sigma_1(z_h,s_h)X_h(s)\, dw(s)\Big)\, dz\bigg|^2$$

and H_2, H_3 are defined in Lemma 3.4.

By **(A1)**, Cauchy inequality, Hölder's inequality and Itô isometry, one obtains

$$H_1 \leq T\int_0^t \mathbb{E}\bigg| f\Big(Y(z), \int_0^z k_1(z,s)Y(s)\, ds, \int_0^z \sigma_1(z,s)Y(s)\, dw(s)\Big)$$
$$- f\Big(X_h(z_h), \int_0^{z_h} k_1(z_h,s)X_h(s)\, ds, \int_0^{z_h} \sigma_1(z_h,s_h)X_h(s)\, dw(s)\Big)\bigg|^2 dz$$
$$\leq KT\int_0^t \bigg[\mathbb{E}\,|Y(z) - X_h(z_h)|^2 + \mathbb{E}\bigg|\int_0^z k_1(z,s)Y(s)\, ds - \int_0^{z_h} k_1(z_h,s)X_h(s)\, ds\bigg|^2$$
$$+ \mathbb{E}\bigg|\int_0^z \sigma_1(z,s)Y(s)\, dw(s) - \int_0^{z_h} \sigma_1(z_h,s_h)X_h(s)\, dw(s)\bigg|^2\bigg] dz$$

$$\leq 2KT\int_0^t\Bigg[\mathbb{E}\,|Y(z)-X(z)|^2+\mathbb{E}\,|X(z)-X_h(z_h)|^2+\mathbb{E}\left|\int_{z_h}^z k_1(z_h,s)X_h(s)\,ds\right|^2$$
$$+\,\mathbb{E}\left|\int_0^z (k_1(z,s)Y(s)-k_1(z_h,s)X_h(s))\;ds\right|^2+\mathbb{E}\left|\int_{z_h}^z \sigma_1(z_h,s_h)X_h(s)\,dw(s)\right|^2$$
$$+\,\mathbb{E}\left|\int_0^z (\sigma_1(z,s)Y(s)-\sigma_1(z_h,s_h)X_h(s))\;dw(s)\right|^2\Bigg]dz.$$

Using Hölder's inequality, Itô isometry, Theorem 3.2 and Lemma 3.3, we have

$$\int_0^t \mathbb{E}\,|X(z)-X_h(z_h)|^2\;dz\leq TM_1h,$$

$$\begin{aligned}\int_0^t \mathbb{E}\left|\int_{z_h}^z k_1(z_h,s)X_h(s)\,ds\right|^2\;dz &\leq \int_0^t\left(\int_{z_h}^z |k_1(z_h,s)|^2\;ds\int_{z_h}^z \mathbb{E}|X_h(s)|^2\,ds\right)dz\\ &\leq \int_0^t \left(h^2\|k_1\|_\infty^2M_0\right)\,dz\\ &\leq h^2\|k_1\|_\infty^2TM_0,\end{aligned}$$

$$\begin{aligned}\int_0^t \mathbb{E}\left|\int_{z_h}^z \sigma_1(z_h,s_h)X_h(s)\,dw(s)\right|^2\;dz &\leq \int_0^t\left(\int_{z_h}^z |\sigma_1(z_h,s_h)|^2\,\mathbb{E}|X_h(s)|^2\,ds\right)dz\\ &\leq M_0\int_0^t \|\sigma_1\|_\infty^2h\,dz\\ &\leq Th\|\sigma_1\|_\infty^2M_0,\end{aligned}$$

$$\begin{aligned}&\int_0^t \mathbb{E}\left|\int_0^z (k_1(z,s)Y(s)-k_1(z_h,s)X_h(s))\;ds\right|^2\;dz\\ \leq&2\int_0^t\int_0^z |k_1(z,s)|^2\;ds\int_0^z \mathbb{E}\,|Y(s)-X_h(s)|^2\;ds\,dz+2\int_0^t\int_0^z |k_1(z,s)-k_1(z_h,s)|^2\;ds\int_0^z \mathbb{E}|X_h(s)|^2\,ds\,dz\\ \leq&2\int_0^t\int_0^z |k_1(z,s)-k_1(z_h,s)|^2\;ds\int_0^z \mathbb{E}|X_h(s)|^2\,ds\,dz+4\int_0^t\int_0^z |k_1(z,s)|^2\;ds\int_0^z \mathbb{E}\,|Y(s)-X(s)|^2\;ds\,dz\\ &+4\int_0^t\int_0^z |k_1(z,s)|^2\;ds\int_0^z \mathbb{E}\,|X(s)-X_h(s)|^2\;ds\,dz\\ \leq&2T^3\grave{K}h^2M_0+4T^3\|k_1\|_\infty^2M_1h+4T^2\|k_1\|_\infty^2\int_0^t \mathbb{E}\,|Y(s)-X(s)|^2\;ds\end{aligned}$$

and

$$\begin{aligned}&\int_0^t \mathbb{E}\left|\int_0^z (\sigma_1(z,s)Y(s)-\sigma_1(z_h,s_h)X_h(s))\;dw(s)\right|^2\;dz\\ \leq&\int_0^t\int_0^z \mathbb{E}\,|\sigma_1(z,s)Y(s)-\sigma_1(z_h,s_h)X_h(s)|^2\;ds\,dz\\ \leq&2\int_0^t\int_0^z \mathbb{E}\,|\sigma_1(z,s)|^2\,E\,|Y(s)-X_h(s)|^2\;ds\,dz+2\int_0^t\int_0^z \mathbb{E}\,|\sigma_1(z,s)-\sigma_1(z_h,s_h)|^2\,\mathbb{E}|X_h(s)|^2\,ds\,dz\end{aligned}$$

$$\leq 2\int_0^t\int_0^z \mathbb{E}\left|\sigma_1(z,s)-\sigma_1(z_h,s_h)\right|^2 \mathbb{E}|X_h(s)|^2\,ds\,dz + 4\int_0^t\int_0^z \mathbb{E}\left|\sigma_1(z,s)\right|^2 \mathbb{E}\left|Y(s)-X(s)\right|^2\,ds\,dz$$
$$+ 4\int_0^t\int_0^z \mathbb{E}\left|\sigma_1(z,s)\right|^2 \mathbb{E}\left|X(s)-X_h(s)\right|^2\,ds\,dz$$
$$\leq 4\grave{K}T^2h^2M_0 + 4T^2\|\sigma_1\|_\infty^2 M_1 h + 4T\|\sigma_1\|_\infty^2 \int_0^t \mathbb{E}\left|Y(s)-X(s)\right|^2\,ds.$$

Thus

$$H_1 \leq K_1Th + K_1'T\int_0^t \mathbb{E}\left|Y(s)-X(s)\right|^2\,ds,$$

where

$$K_1 := 2KT^2\left\{\left[2T(T+2)\grave{K}h + \|k_1\|_\infty^2 h + \|\sigma_1\|_\infty^2\right]M_0 + \left[1+4T^2\|k_1\|_\infty^2 + 4T\|\sigma_1\|_\infty^2\right]M_1\right\}$$

and $K_1' := 2K\left[1+4T^2\|k_1\|_\infty^2 + 4T\|\sigma_1\|_\infty^2\right]$.

Together with Lemma 3.4, we derive

$$\mathbb{E}\left|Y(t)-X(t)\right|^2 \leq 3(K_1T+K_2)h + 3\left(K_1'T+K_2'\right)\int_0^t \mathbb{E}\left|Y(s)-X(s)\right|^2\,ds.$$

Hence by Gronwall's equality, we have

$$\mathbb{E}\left|Y(t)-X(t)\right|^2 \leq M_2h,$$

where

$$M_2 := 3(K_1T+K_2)\exp\left[3\left(K_1'T+K_2'\right)T\right].$$

□

Remark 3.6. *Due to Theorem 3.5, we observe that the strong convergence order of the Euler-Maruyama approximate solution for SDE* (11) *is* 1/2 *under* **(A1)**, *which is consistent with* [15]. *We also find that the strong convergence order of the Euler-Maruyama approximate solution for SVIDE* (10) *is* 1/2 *under* **(A1)**, *which is consistent with* [27].

4. Numerical experiments

In this section, we support the results obtained in Theorem 3.5 numerically with some examples. We use discrete Brownian paths over $[0,1]$ with $\Delta t = 2^{-12}$ and take the numerical solution with $h = \Delta t$ to be an approximation of the analytic solution and compare this with the numerical approximation using $h = 2^5\Delta t$, $h = 2^6\Delta t$, $h = 2^7\Delta t$ and $h = 2^8\Delta t$ over $M = 4000$ sample paths. Here the mean-square error is denoted as follows:

$$Error_h := \left(\frac{1}{M}\sum_{i=1}^{M}\left|Y_h^i(T)-Y_{\Delta t}^i(T)\right|^2\right)^{1/2}, \tag{22}$$

where $Y_h^i(T)$ denotes the numerical solution of Euler-Maruyama method along the ith sample path at $t = T$ with stepsize h, and the strong convergence order is defined numerically by

$$Order = \log\frac{Error_h}{Error_{h/2}} / \log(2).$$

Consider the following SVIE:

$$Y(t) = \int_0^t \left[aY(z) + b\cos\left(\int_0^z k_1(z,s)Y(s)\,ds \right) + c\sin\left(\int_0^z \sigma_1(z,s)Y(s)\,dw(s) \right) \right] dz$$
$$+ \int_0^t \left[lY(z) + m\sin\left(\int_0^z k_2(z,s)Y(s)\,ds \right) + n\cos\left(\int_0^z \sigma_2(z,s)Y(s)\,dw(s) \right) \right] dw(z)$$
$$+ \int_0^t \int_{\mathbf{Z}} \left[pY(z) + q\cos\left(\int_0^z k_3(z,s)Y(s)\,ds \right) + u\cos\left(\int_0^z \sigma_3(z,s)Y(s)\,dw(s) \right) \right] \xi\tilde{N}(dz,d\xi) \quad (23)$$

with initial data $\varphi(t) = 1$ and $\lambda = 0.5$.

Firstly, we discuss two particular types.

Example 4.1. *In* (23)*, we take* $a = c = l = n = T = 1$*,* $b = m = p = q = u = 0$*,* $k_1(t,s) = k_2(t,s) \equiv 1$ *and the following two cases of* $\sigma_1(t,s)$ *and* $\sigma_2(t,s)$*:*

Case 1: $\sigma_1(t,s) = \sigma_2(t,s) = t - s + 1$*;*

Case 2: $\sigma_1(t,s) = t - s$ *and* $\sigma_2(t,s) = \sin(2t - s)$*.*

Table 1: Strong convergence order for Example 4.1.

	Case 1		Case 2	
stepsize	Error	order	Error	order
$2^5\Delta t$	0.4310	-	0.3946	-
$2^6\Delta t$	0.3055	0.4956	0.2836	0.4765
$2^7\Delta t$	0.2245	0.4445	0.2022	0.4881
$2^8\Delta t$	0.1557	0.5279	0.1404	0.5262

Example 4.2. *In* (23)*, we take* $a = b = l = m = T = 1$*,* $c = n = p = q = u = 0$ *and the following two cases of* $k_1(t,s)$*,* $k_2(t,s)$*,* $\sigma_1(t,s)$ *and* $\sigma_2(t,s)$*:*

Case 1: *The simplest convolution kernels* $k_1(t,s) = k_2(t,s) \equiv 1$ *and convolution kernels* $\sigma_1(t,s) = \sigma_2(t,s) = t - s + 1$*;*

Case 2: *The convolution kernels* $k_1(t,s) = t - s$*,* $k_2(t,s) = \sin(t - s)$*,* $\sigma_1(t,s) = \cos(t - s)$ *and* $\sigma_2(t,s) = t - s + 1$*.*

Table 2: Strong convergence order for Example 4.3.

	Case 1		Case 2	
stepsize	Error	order	Error	order
$2^5\Delta t$	0.4411	-	0.4207	-
$2^6\Delta t$	0.3168	0.4775	0.2988	0.4936
$2^7\Delta t$	0.2284	0.4720	0.2193	0.4463
$2^8\Delta t$	0.1585	0.5271	0.1508	0.5403

Observe Table 2, we can see that the Euler-Maruyama method of the generalized SVIEs is convergent of order 1/2 which is consist with that of [27].

secondly, we consider a more generalized type.

Example 4.3. *In* (23)*, we take* $a = b = c = l = m = m = T = p = q = u = 1$ *and the following three cases of* $k_1(t,s)$*,* $k_2(t,s)$*,* $k_3(t,s)$*,* $\sigma_1(t,s)$*,* $\sigma_2(t,s)$ *and* $\sigma_3(t,s)$*:*

Case 1: *The simplest convolution kernels* $k_1(t,s) = k_2(t,s) = k_3(t,s) \equiv 1$ *and convolution kernels* $\sigma_1(t,s) = \sigma_3(t,s) = t - s$ *and* $\sigma_2(t,s) = t - s + 1$*;*

Case 2: *The convolution kernels* $k_1(t,s) = k_3(t,s) = t - s$, $k_2(t,s) = \sin(t-s)$, $\sigma_1(t,s) = \sigma_3(t,s) = \cos(t-s)$ *and* $\sigma_2(t,s) = t - s + 1$;

Case 3: *The generalized kernels* $k_1(t,s) = ts$, $k_2(t,s) = \cos ts$, $k_3(t,s) = 0$ *and* $\sigma_1(t,s) = \sigma_3(t,s) = \cos(2t-s) - \cos s$ *and* $\sigma_2(t,s) = \sin(2t-s)$;

Table 3: Strong convergence order for Example 4.3.

	Case 1		Case 2		Case 3	
stepsize	Error	order	Error	order	Error	order
$2^5\Delta t$	0.5175	-	0.5222	-	0.3587	-
$2^6\Delta t$	0.3749	0.4651	0.3842	0.4427	0.2636	0.4444
$2^7\Delta t$	0.2725	0.4602	0.2759	0.4777	0.1882	0.4861
$2^8\Delta t$	0.1930	0.4977	0.1942	0.5073	0.1278	0.5584

The strong convergence results of the Euler-Maruyama method of Example 4.3 are shown in Table 3. From this table, we can see that the Euler-Maruyama method of the generalized SVIEs driven by lévy noise is convergent of order 1/2.

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