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On Estimates for the Fourier-Bessel Transform in the Space $L^p(\mathbb{R}^2_+, x^{2\alpha_1+1}y^{2\alpha_2+1}dxdy)$

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Abstract. In this paper, we prove two estimates useful in applications for the Fourier-Bessel transform in the space $L^p(\mathbb{R}^2_+, x^{2\alpha_1+1}y^{2\alpha_2+1}dxdy)$, $(1 < p \leq 2)$, as applied to some classes of functions characterized by a generalized modulus of continuity.

1. Introduction and preliminaries

In [2], Abilov and Kerimov proved two estimates for the Fourier-Bessel transform in the space $L^2(\mathbb{R}^2_+)$ characterized by the generalized modulus of continuity. In this paper, we prove of these estimates in the space $L^p(\mathbb{R}^2_+, x^{2\alpha_1+1}y^{2\alpha_2+1}dxdy)$, $(1 < p \leq 2)$. We point out that similar results have been established in the context of Bessel transform in the space $L^p(\mathbb{R}^+)$, for the Dunkl transform, for the Cherednik-Opdam transform, for the Fourier transform and etc (for example see [3–6]).

Assume that $L^p(\mathbb{R}^2_+) = L^p(\mathbb{R}^2_+, x^{2\alpha_1+1}y^{2\alpha_2+1}dxdy)$, $(1 < p \leq 2$ and $\alpha_1, \alpha_2 > -\frac{1}{2})$, is the space of p-power integrable two-variables functions $f : \mathbb{R}^2_+ \to \mathbb{R}$ with the norm

$$\|f\|_p = \|f\|_{L^p(\mathbb{R}^2_+)} = \left(\iint\limits_{\mathbb{R}^2_+} |f(x,y)|^p x^{2\alpha_1+1}y^{2\alpha_2+1}dxdy\right)^{\frac{1}{p}}$$

For $\alpha > -\frac{1}{2}$, we introduce the normalized spherical Bessel function j_α defined by

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\alpha+1)}(\frac{z}{2})^{2k}, z \in \mathbb{C},$$

where $\Gamma(x)$ is the gamma-function.

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From [1], we have

$$1 - j_\alpha(u) = O(1),\ u \geq 1. \tag{1}$$

$$1 - j_\alpha(u) = O(u^2),\ 0 \leq u \leq 1. \tag{2}$$

$$j_\alpha(u) = O(u^{-\alpha-\frac{1}{2}}) \tag{3}$$

Definition 1.1. *The Fourier-Bessel transform for two-variable functions is defined on* $\mathrm{L}^1(\mathbb{R}_+^2)$ *by*

$$\widehat{f}(\xi,\eta) = \iint\limits_{\mathbb{R}_+^2} f(x,y) j_{\alpha_1}(\xi x) j_{\alpha_2}(\eta y) x^{2\alpha_1+1} y^{2\alpha_2+1} dxdy$$

Proposition 1.2. *Let f be in* $\mathrm{D}_*(\mathrm{R}^2)$*, then we have inversion formula*

$$f(x,y) = \frac{1}{2^{2(\alpha_1+\alpha_2)}\Gamma^2(\alpha_1+1)\Gamma^2(\alpha_2+1)} \iint\limits_{\mathbb{R}_+^2} \widehat{f}(\xi,\eta) j_{\alpha_1}(\xi x) j_{\alpha_2}(\eta y) \xi^{2\alpha_1+1}\eta^{2\alpha_2+1} d\xi d\eta,$$

where $\mathrm{D}_*(\mathrm{R}^2)$ *the space of* C^∞*-function on* $\mathbb{R}^2$*, with compact support and even with respect to each variable.*

The Fourier-Bessel transform above extends to a bounded linear map $f \to \widehat{f}$ from $\mathrm{L}^p(\mathbb{R}_+^2)$ to $\mathrm{L}^q(\mathbb{R}_+^2)$. We have the Hausdorff Young inequality

$$\|\widehat{f}\|_q \leq A\|f\|_p,\ \forall f \in \mathrm{L}^p(\mathbb{R}_+^2) \tag{4}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and A is a positive constant.

In $\mathrm{L}^p(\mathbb{R}_+^2)$, consider the following generalized translation operator defined by

$$\mathrm{T}_h f(x,y) = c_{\alpha_1,\alpha_2} \iint\limits_{[0,\pi]^2} f(\sqrt{x^2+h^2-2xh\cos u}, \sqrt{y^2+h^2-2yh\cos v}) \sin^{2\alpha_1}(u)\sin^{2\alpha_2}(v) du dv,$$

wich corresponds to the Bessel operator for two-variable functions

$$\mathrm{D} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{2\alpha_1+1}{x}\frac{\partial}{\partial x} + \frac{2\alpha_2+1}{y}\frac{\partial}{\partial y}$$

and with

$$c_{\alpha_1,\alpha_2} = \frac{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}{\pi\Gamma(\alpha_1+\frac{1}{2})\Gamma(\alpha_2+\frac{1}{2})}$$

We note the important property of the Fourier-Bessel transform: If $f \in \mathrm{L}^p(\mathbb{R}_+^2)$

$$(\widehat{\mathrm{D}f})(\xi,\eta) = -(\xi^2+\eta^2)\widehat{f}(\xi,\eta) \tag{5}$$

The following relation connect the generalized translation operator and the Fourier-Bessel transform

$$(\widehat{\mathrm{T}_h f})(\xi,\eta) = j_{\alpha_1}(\xi h) j_{\alpha_2}(\eta h)\widehat{f}(\xi,\eta) \tag{6}$$

Note some its propreties (see [1, 7])

1. T_h is a linear operator
2. $\mathrm{T}_h j_{\alpha_1}(\lambda x) j_{\alpha_2}(\mu y) = j_{\alpha_1}(\lambda h) j_{\alpha_2}(\mu h) j_{\alpha_1}(\lambda x) j_{\alpha_2}(\mu y)$
3. $\|\mathrm{T}_h f - f\|_p \to 0$ as $h \to 0^+$

The first-and higher order finite differences of $f(x, y)$ as defined as follows

$$\Delta_h f(x, y) = \mathrm{T}_h f(x, y) - f(x, y) = (\mathrm{T}_h - \mathrm{I}) f(x, y)$$

$$\Delta_h^k f(x, y) = \Delta_h(\Delta_h^{k-1} f(x, y)) = (\mathrm{T}_h - \mathrm{I})^k f(x, y) \tag{7}$$

where I is the identity operator in the space $\mathrm{L}^p(\mathbb{R}_+^2)$ and $k = 1, 2, ...$

The kth-order generalized modulus of continuity of a function $f \in \mathrm{L}^p(\mathbb{R}_+^2)$ is defined as

$$\Omega_k(f, \delta) = \sup_{0<h\leq\delta} \|\Delta_h^k f(x, y)\|_p$$

Denote by $\mathbb{L}_r^p(\mathrm{D})$ $(r = 0, 1, ...)$ the class of functions $\mathrm{L}^p(\mathbb{R}_+^2)$ having generalized partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, ...$ in the sense of Levitan (see [8]) that satisfy the condition $\mathrm{D}^r f \in \mathbb{L}_r^p(\mathrm{D})$.

Let $W_{p,\phi}^{r,k}(\mathrm{D})$, $(r = 1, 2, ..; k = 1, 2, ...)$ denote the class of functions $f \in \mathrm{L}^p(\mathbb{R}_+^2)$ for which $\mathrm{D}^r f \in \mathrm{L}^p(\mathbb{R}_+^2)$ and

$$\Omega_k(\mathrm{D}^r f, \delta) = O(\phi(\delta^k)),$$

where $\phi(t)$ is a nonnegative function defined on $[0, \infty)$. Moreover, for the Bessel operator we have

$$\mathrm{D}^0 f = f, \ \mathrm{D}^r f = \mathrm{D}(\mathrm{D}^{r-1} f), \ r = 1, 2, ..$$

2. Estimates for the Fourier-Bessel transform for two-variable functions

In this section, we estimate the integral

$$\int\int_{\xi^2+\eta^2\geq N^2} |\widehat{f}(\xi, \eta)|^q \xi^{2\alpha_1+1} \eta^{2\alpha_2+1} d\xi d\eta$$

in some classes of two-variable functions.

Lemma 2.1. *For* $f \in \mathrm{L}^p(\mathbb{R}_+^2)$

$$\iint_{\mathbb{R}_+^2} |\widehat{f}(\xi, \eta)|^q (\xi^2 + \eta^2)^{qr} |1 - j_{\alpha_1}(\xi h) j_{\alpha_2}(\eta h)|^{qk} \xi^{2\alpha_1+1} \eta^{2\alpha_2+1} d\xi d\eta \leq A^q \|\Delta_h^k \mathrm{D}^r f(x, y)\|_p^q$$

Proof. From formula (5), we obtain

$$\widehat{D^r} f(\xi, \eta) = (-1)^r (\xi^2 + \eta 2)^r \widehat{f}(\xi, \eta) \tag{8}$$

We use the formulas (6) and (8), we conclude

$$(\widehat{\mathrm{T}_h^i \mathrm{D}^r f})(\xi, \eta) = (-1)^r j_{\alpha_1}(\xi h) j_{\alpha_2}(\eta h) (\xi^2 + \eta^2)^r \widehat{f}(\xi, \eta), \ 1 \leq i \leq k. \tag{9}$$

It followos from the definition of finite defference (7) and formula (9) the image $\Delta_h^k \mathrm{D}^r f(x, y)$ under the Fourier-Bessel transform has the forme

$$\widehat{\Delta_h^k \mathrm{D}^r} f(\xi, \eta) = (-1)^r (\xi^2 + \eta^2)^r (j_{\alpha_1}(\xi h) j_{\alpha_2}(\eta h) - 1)^k \widehat{f}(\xi, \eta)$$

then , using the Hausdorff Young inequality (4), we have the result. □

Theorem 2.2. *For functions $f(x,y) \in L^p(\mathbb{R}^2_+)$ in the space $W^{r,k}_{p,\phi}(D)$*

$$\sup_{W^{r,k}_{p,\phi}(D)} \int\int_{\xi^2+\eta^2\geq N^2} |\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1} d\xi d\eta = O\left(N^{-2rq}\phi^q((\frac{c}{N})^q)\right)$$

where $r = 0,1,...;\ k = 1,2,...;\ c > 0$ is a fixed contant, ϕ is any nonegative function defined on $[0,\infty)$ and $\frac{1}{p}+\frac{1}{q} = 1$.

Proof. Let $f \in W^{r,k}_{p,\phi}(D)$. Taking in to account the Hölder inequality

$$\begin{aligned}
&\int\int_{\xi^2+\eta^2\geq N^2} |\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1} d\xi d\eta - \int\int_{\xi^2+\eta^2\geq N^2} j_{\alpha_1}(\xi h) j_{\alpha_2}(\eta h)|\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1} d\xi d\eta \\
= &\int\int_{\xi^2+\eta^2\geq N^2} (1 - j_{\alpha_1}(\xi h) j_{\alpha_2}(\eta h))|\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1} d\xi d\eta \\
= &\int\int_{\xi^2+\eta^2\geq N^2} (1 - j_{\alpha_1}(\xi h) j_{\alpha_2}(\eta h))|\widehat{f}(\xi,\eta)|^{q-\frac{1}{k}} |\widehat{f}(\xi,\eta)|^{\frac{1}{k}} \xi^{2\alpha_1+1}\eta^{2\alpha_2+1} d\xi d\eta \\
\leq &\left(\int\int_{\xi^2+\eta^2\geq N^2} |\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1} d\xi d\eta\right)^{\frac{qk-1}{qk}} \\
\times &\left(\int\int_{\xi^2+\eta^2\geq N^2} |1 - j_{\alpha_1}(\xi h) j_{\alpha_2}(\eta h)|^{qk}|\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1} d\xi d\eta\right)^{\frac{1}{qk}} \\
= &\left(\int\int_{\xi^2+\eta^2\geq N^2} |\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1} d\xi d\eta\right)^{\frac{qk-1}{qk}} \\
\times &\left(\int\int_{\xi^2+\eta^2\geq N^2} \frac{1}{(\xi^2+\eta 2)^{qr}}(\xi^2+\eta 2)^{qr}|1 - j_{\alpha_1}(\xi h) j_{\alpha_2}(\eta h)|^{qk}|\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1} d\xi d\eta\right)^{\frac{1}{qk}} \\
\leq &N^{-\frac{2r}{k}}\left(\int\int_{\xi^2+\eta^2\geq N^2} |\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1} d\xi d\eta\right)^{\frac{qk-1}{qk}} \\
\times &\left(\int\int_{\xi^2+\eta^2\geq N^2} (\xi^2+\eta 2)^{qr}|1 - j_{\alpha_1}(\xi h) j_{\alpha_2}(\eta h)|^{qk}|\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1} d\xi d\eta\right)^{\frac{1}{qk}}
\end{aligned}$$

From Lemma 2.1, we have the inequality

$$\iint_{\mathbb{R}^2_+} |\widehat{f}(\xi,\eta)|^q(\xi^2+\eta^2)^{qr}|1 - j_{\alpha_1}(\xi h) j_{\alpha_2}(\eta h)|^{qk} \xi^{2\alpha_1+1}\eta^{2\alpha_2+1} d\xi d\eta \leq A^q \|\Delta^k_h D^r f(x,y)\|^q_p$$

Thus

$$\begin{aligned}
&\int\int_{\xi^2+\eta^2\geq N^2} |\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1}d\xi d\eta \\
\leq &\int\int_{\xi^2+\eta^2\geq N^2} j_{\alpha_1}(\xi h) j_{\alpha_2}(\eta h)|\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1}d\xi d\eta \\
+ &A^{\frac{1}{k}} N^{-\frac{2r}{k}} \left(\int\int_{\xi^2+\eta^2\geq N^2} |\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1}d\xi d\eta\right)^{\frac{qk-1}{qk}} \|\Delta_h^k \mathrm{D}^r f(x,y)\|_p^{\frac{1}{k}}
\end{aligned}$$

Now we estimate the integral

$$\mathrm{I} = \int\int_{\xi^2+\eta^2\geq N^2} j_{\alpha_1}(\xi h) j_{\alpha_2}(\eta h)|\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1}d\xi d\eta$$

which is devided into two

$$\mathrm{I} = \iint_{B_1} + \iint_{B_2}$$

where $B_1 = \{(\xi,\eta);\ \xi^2+\eta^2 \geq N^2;\ \xi \geq \eta\}$ and $B_1 = \{(\xi,\eta);\ \xi^2+\eta^2 \geq N^2;\ \xi < \eta\}$.
Combining this with (1) gives

$$\mathrm{I} = O\left(\iint_{B_1} |j_{\alpha_1}(\xi h)||\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1}d\xi d\eta + \iint_{B_2} |j_{\alpha_2}(\eta h)||\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1}d\xi d\eta\right)$$

It followos that from (3) that

$$j_{\alpha_1}(\xi h) = O\left((\xi h)^{-\alpha_1-\frac{1}{2}}\right);\ j_{\alpha_2}(\xi h) = O\left((\xi h)^{-\alpha_2-\frac{1}{2}}\right)$$

Therefore

$$\mathrm{I} = O\left(h^{-\alpha_1-\frac{1}{2}} \iint_{B_1} \xi^{-\alpha_1-\frac{1}{2}}|\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1}d\xi d\eta + h^{-\alpha_2-\frac{1}{2}} \iint_{B_2} \eta^{-\alpha_2-\frac{1}{2}}|\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1}d\xi d\eta\right)$$

Then

$$\begin{aligned}
\mathrm{I} &= O\left(N^{-\alpha_1-\frac{1}{2}}\right) h^{-\alpha_1-\frac{1}{2}} \iint_{B_1} |\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1}d\xi d\eta \\
&+ O\left(N^{-\alpha_2-\frac{1}{2}}\right) h^{-\alpha_2-\frac{1}{2}} \iint_{B_2} |\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1}d\xi d\eta
\end{aligned}$$

Now let $h = \frac{c}{N}$, where $c > 0$ is an arbitray constant, then

$$\mathrm{I} = O\left(max(c^{-\alpha_1-\frac{1}{2}}, c^{-\alpha_2-\frac{1}{2}})\right) \int\int_{\xi^2+\eta^2\geq N^2} |\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1}d\xi d\eta$$

We obtain

$$\int\int_{\xi^2+\eta^2\geq N^2} |\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1}d\xi d\eta \leq A^{\frac{1}{k}}N^{-\frac{2r}{k}}\left(\int\int_{\xi^2+\eta^2\geq N^2} |\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1}d\xi d\eta\right)^{\frac{qk-1}{qk}}$$

$$\times \quad \|\Delta_h^k \mathrm{D}^r f(x,y)\|_p^{\frac{1}{k}} + O\left(max(c^{-\alpha_1-\frac{1}{2}}, c^{-\alpha_2-\frac{1}{2}})\right)\int\int_{\xi^2+\eta^2\geq N^2} |\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1}d\xi d\eta$$

Now choosing the necessary constant $c > 0$, such that $1 - A\left(max(c^{-\alpha_1-\frac{1}{2}}, c^{-\alpha_2-\frac{1}{2}})\right) \geq \frac{1}{2}$, where A is a positive contant.

$$\int\int_{\xi^2+\eta^2\geq N^2} |\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1}d\xi d\eta$$

$$= \quad O(N^{-\frac{2r}{k}})\left(\int\int_{\xi^2+\eta^2\geq N^2} |\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1}d\xi d\eta\right)^{\frac{qk-1}{qk}} \|\Delta_h^k \mathrm{D}^r f(x,y)\|_p^{\frac{1}{k}}$$

It followos that

$$\left(\int\int_{\xi^2+\eta^2\geq N^2} |\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1}d\xi d\eta\right)^{\frac{1}{qk}} = O(N^{-\frac{2r}{k}})\|\Delta_h^k \mathrm{D}^r f(x,y)\|_p^{\frac{1}{k}}$$

and this ends the proof. □

Corollary 2.3. *Let $f(x,y) \in W^{r,k}_{p,t^\nu}(\mathrm{D})$, $(\nu > 0)$, then*

$$\int\int_{\xi^2+\eta^2\geq N^2} |\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1}d\xi d\eta = O\left(N^{-2rq-qk\nu}\right)$$

where $r = 0, 1, ...;$ $k = 1, 2, ...$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $f \in W^{r,k}_{p,t^\nu}(\mathrm{D})$ and $\phi(t) = t^\nu$. Then from Theorem 2.2, we have

$$\int\int_{\xi^2+\eta^2\geq N^2} |\widehat{f}(\xi,\eta)|^q \xi^{2\alpha_1+1}\eta^{2\alpha_2+1}d\xi d\eta = O\left(N^{-2rq-qk\nu}\right)$$

Thus, the proof is finished. □

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