Filomat 36:2 (2022), 669–682
https://doi.org/10.2298/FIL2202669Q

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Approximation by a Generalized Szász-Bézier Operators

Qiulan Qi[a], Dandan Guo[a]

[a] *School of Mathematical Sciences, Hebei Normal University, Shijiazhuang, 050024, P.R. China*

Abstract. The application of Bézier type operators is very extensive and has attracted people's attention. In the year 2017, Ren established a generalized Bernstein-Bézier type operators acting on $C[0,1]$. Inspired by this, in this paper, a generalized Szász-Bézier type operators, with Gamma function defined on the positive semi-axis, is extended. Then, the equivalent theorem and the Voronovskaja type asymptotic formulas are also obtained.

1. Introduction

The approximation properties of the classical Szász operators $S_n(f;x)$ were widely investigated in the literature[1−6]. During the past thirty years, the Bézier basis function was extensively used to construct various generalizations of many classical approximation processes[7−14]. In 2017, Ren et al[14] introduced a generalized Bernstein-Bézier type operators acting on $C[0,1]$, have been considered in connection with Beta function, and obtained a Jackson type direct theorem. In order to get the Bernstein type inverse theorem, in [15], we introduced a kind of Bernstein-Bézier operators with parameters. This paper is concerned with generalized Szász-Bézier type operators acting on functions defined on the positive semi-axis, with Gamma function. The Szász-Bézier operators are defined as follows:[11]

$$S_{n,\alpha}(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\cdot\left(J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)\right), \tag{1.1}$$

where $\alpha \geq 1$, $J_{n,k}(x) = \sum_{j=k}^{\infty} s_{n,j}(x), k = 0,1,...,s_{n,j}(x) = \frac{(nx)^j}{j!}e^{-nx}$, $J_{n,k}(x)$ is the Szász-Bézier basis function. Obviously, when $\alpha = 1$, $S_{n,\alpha}(f;x)$ becomes $S_n(f;x)$, and for $x \in [0,\infty)$, one has[11] $1 = J_{n,0}(x) \geq J_{n,1}(x) \geq ...J_{n,k}(x) \geq J_{n,k+1}(x) \geq ... \geq 0, J_{n,k}(x) - J_{n,k+1}(x) = s_{n,k}(x)$.
In this paper, we are going to study a new kind of Szász type operators for $f(x) \in C_B[0,\infty)$ as follows:

$$D_{n,\beta}(f;x) = f(0)s_{n,0}(x) + \sum_{k=1}^{\infty} s_{n,k}(x)T_{n,k}^{(\beta)}(f), \tag{1.2}$$

2020 *Mathematics Subject Classification*. Primary 41A10; Secondary 41A25, 41A36
Keywords. Szász-Bézier type operators, approximation theorem, the Cauchy-Schwarz inequality, Voronovskaja type asymptotic formula
Received: 18 February 2021; Accepted: 13 July 2021
Communicated by Miodrag Spalević
Research supported by NSF of China(11871191), Science and Technology Project of Hebei Education Department(ZD2019053), Science Foundation of Hebei Normal University(L2020Z03).
Email addresses: qiqiulan@hebtu.edu.cn (Qiulan Qi), 2728561580@qq.com (Dandan Guo)

where $0 \le \beta \le 1$,

$$T_{n,k}^{(\beta)}(f) = \frac{n^k}{\Gamma(k)} \int_0^\infty t^{k-1} e^{-nt} f\left(\beta t + (1-\beta)\frac{k}{n}\right) dt,$$

$\Gamma(\cdot)$ is the Gamma function. When $\beta = 0$, $D_{n,\beta}(f;x)$ becomes $S_n(f;x)$.
We will also study a generalized Szász-Bézier-type operators for $f(x) \in C_B[0,\infty)$ as follows:

$$D_{n,\beta}^{(\alpha)}(f;x) = f(0)G_{n,0}^{(\alpha)}(x) + \sum_{k=1}^{\infty} G_{n,k}^{(\alpha)}(x) T_{n,k}^{(\beta)}(f), \tag{1.3}$$

where $0 \le \beta \le 1, \alpha \ge 1, G_{n,k}^{(\alpha)}(x) = J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)$, $J_{n,k}(x)$ and $T_{n,k}^{(\beta)}(f)$ are defined as above.
The operators $D_{n,\beta}^{(\alpha)}(f;x)$ are bounded and positive on $C_B[0,\infty)$. When $\alpha = 1$, $D_{n,\beta}^{(\alpha)}(f;x)$ becomes $D_{n,\beta}(f;x)$. When $\beta = 0$, $D_{n,\beta}^{(\alpha)}(f;x)$ becomes $S_{n,\alpha}(f;x)$.
The goal of the paper is to investigate the rate of convergence. Direct and inverse theorems are proved using Ditzian-Totik modulus of smoothness. The Voronovskaja type asymptotic formulas are also obtained.

Remark 1 Throughout this paper, M is a positive constant independent of n and x, the value of M may be different in different places.

Remark 2 In this paper, for $f(x) \in C_B[0,\infty)$:=$\{f : f$ is continuous and bounded on $[0,\infty)\}$, the norm of $f(x)$ is defined as $\|f\| = \max\{|f(x)| : x \in [0,\infty)\}$.

Remark 3[3]
(1)$S_n(1;x) = 1$;
(2)$S_n(t;x) = x$;
(3)$S_n(t^2;x) = x^2 + \frac{x}{n}$;
(4)$S_n(t^3;x) = x^3 + \frac{3x^2}{n} + \frac{x}{n^2}$;
(5)$S_n(t^4;x) = x^4 + \frac{6x^3}{n} + \frac{7x^2}{n^2} + \frac{x}{n^3}$.

2. Estimates of the moments

By the definition of $T_{n,k}^{(\beta)}(f), D_{n,\beta}(f;x)$, Remark 3 and using the integral by parts, we have Lemma 2.1, Lemma 2.2 and Lemma 2.3. Here we omit the details.

Lemma 2.1 For $T_{n,k}^{(\beta)}(t^i)$, $i = 0,1,2,3,4, 0 \le \beta \le 1$, we have
(1)$T_{n,k}^{(\beta)}(1) = 1$;
(2)$T_{n,k}^{(\beta)}(t) = \frac{k}{n}$;
(3)$T_{n,k}^{(\beta)}(t^2) = \frac{k^2}{n^2} + \frac{\beta^2 k}{n^2}$;
(4)$T_{n,k}^{(\beta)}(t^3) = \frac{k^3}{n^3} + \frac{3\beta^2 k^2}{n^3} + \frac{2\beta^3 k}{n^3}$;
(5)$T_{n,k}^{(\beta)}(t^4) = \frac{k^4}{n^4} + \frac{6\beta^2 k^3}{n^4} + \frac{(3\beta^4+8\beta^3)k^2}{n^4} + \frac{6\beta^4 k}{n^4}$.

Lemma 2.2 For $D_{n,\beta}((t-x)^i;x)$, $i = 2,4, 0 \le \beta \le 1$, we have

$$(1)D_{n,\beta}((t-x)^2;x) = \frac{1+\beta^2}{n}x;$$

$$(2)D_{n,\beta}((t-x)^4;x) = \frac{3+6\beta^2+3\beta^4}{n^2}x^2 + \frac{1+6\beta^2+8\beta^3+9\beta^4}{n^3}x.$$

Lemma 2.3
(1)$\frac{1}{n}\sum_{k=1}^{\infty} J_{n,k}(x) = x$;

(2) $\frac{1}{n^2}\sum_{k=1}^{\infty} kJ_{n,k}(x) = \frac{x^2}{2} + \frac{x}{n}$;
(3) $\frac{1}{n^3}\sum_{k=1}^{\infty} k^2 J_{n,k}(x) = \frac{x^3}{3} + \frac{3x^2}{2n} + \frac{x}{n^2}$;
(4) $\frac{1}{n^4}\sum_{k=1}^{\infty} k^3 J_{n,k}(x) = \frac{x^4}{4} + \frac{2x^3}{n} + \frac{7x^2}{2n^2} + \frac{x}{n^3}$.

Lemma 2.4 Let $\alpha \geq 1$, we have
(1) $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{\infty} J_{n,k}^{\alpha}(x) = x$;
(2) $\lim_{n\to\infty} \frac{1}{n^2}\sum_{k=1}^{\infty} kJ_{n,k}^{\alpha}(x) = \frac{x^2}{2}$;
(3) $\lim_{n\to\infty} \frac{1}{n^3}\sum_{k=1}^{\infty} k^2 J_{n,k}^{\alpha}(x) = \frac{x^3}{3}$;
(4) $\lim_{n\to\infty} \frac{1}{n^4}\sum_{k=1}^{\infty} k^3 J_{n,k}^{\alpha}(x) = \frac{x^4}{4}$.
Proof (1) For $\varepsilon > 0, \delta > 0$, there exists a positive integer $N = N(\varepsilon, \delta)$, since

$$\lim_{n\to\infty} J_{n,k}(x) = \begin{cases} 1, & for \quad k \leq n(x-\varepsilon); \\ 0, & for \quad k \geq n(x+\varepsilon), \end{cases}$$

we see that

$$\begin{cases} 0 \leq 1 - J_{n,k}^{\alpha-1}(x) < \delta, & for \quad k \leq n(x-\varepsilon); \\ 0 \leq J_{n,k}(x) < \delta, & for \quad k \geq n(x+\varepsilon). \end{cases}$$

Since $\sum_{k=1}^{\infty} J_{n,k}(x)[1 - J_{n,k}^{\alpha-1}(x)]$ is convergent, then for $\delta > 0$, there exists an enough big K, such that $\sum_{k=K}^{\infty} J_{n,k}(x)[1 - J_{n,k}^{\alpha-1}(x)] < \delta$.
By Lemma 2.3 (1),

$$\begin{aligned} 0 \leq x - \frac{1}{n}\sum_{k=1}^{\infty} J_{n,k}^{\alpha}(x) &= \frac{1}{n}\sum_{k=1}^{\infty} J_{n,k}(x)\left[1 - J_{n,k}^{\alpha-1}(x)\right] \\ &= \frac{1}{n}\left[\sum_{k\leq n(x-\varepsilon)} + \sum_{k\geq n(x+\varepsilon)} + \sum_{n(x-\varepsilon)<k<n(x+\varepsilon)}\right]. \end{aligned}$$

With the last three terms denoted by $\sum_1, \sum_2, \sum_3$ respectively. For enough big n, the following estimates are easily obtained

$$\begin{aligned} 0 &\leq \textstyle\sum_1 &&\leq \frac{\delta}{n}\sum_{k\leq n(x-\varepsilon)} J_{n,k}(x) \leq \frac{\delta}{n}\cdot nx = \delta x, \\ 0 &\leq \textstyle\sum_2 &&\leq \frac{\delta}{n}\sum_{k\geq n(x+\varepsilon)} J_{n,k}(x)[1 - J_{n,k}^{\alpha-1}(x)] < \delta, \\ 0 &\leq \textstyle\sum_3 &&\leq \frac{1}{n}\sum_{n(x-\varepsilon)<k<n(x+\varepsilon)} = \frac{1}{n}\cdot 2n\varepsilon = 2\varepsilon. \end{aligned}$$

Hence $x - \frac{1}{n}\sum_{k=1}^{\infty} J_{n,k}^{\alpha}(x) \to 0$, we get Lemma 2.4(1).
(2) By Lemma 2.3 (2), one has

$$\frac{x^2}{2} - \frac{1}{n^2}\sum_{k=1}^{\infty} kJ_{n,k}^{\alpha}(x) = \frac{1}{n^2}\sum_{k=1}^{\infty} kJ_{n,k}(x) - \frac{x}{n} - \frac{1}{n^2}\sum_{k=1}^{\infty} kJ_{n,k}^{\alpha}(x). \tag{2.1}$$

From Lemma 2.3 (2), for $\varepsilon > 0$, there exists $K' \in N$, for $k > K'$, such that

$$\left|\frac{1}{n^2}\sum_{k=K'}^{\infty} kJ_{n,k}(x)\right| < \varepsilon. \tag{2.2}$$

From Lemma 2.4 (1), there exists $N' \in N$, for $n > N'$, such that

$$\left| x - \frac{1}{n}\sum_{k=1}^{\infty} J_{n,k}^{\alpha}(x) \right| < \varepsilon. \tag{2.3}$$

For a fixed $x \in [0,\infty)$, choosing $N = \max\{K', N'\}$, we write

$$\left| \frac{x^2}{2} - \frac{1}{n^2}\sum_{k=1}^{\infty} kJ_{n,k}^{\alpha}(x) \right| \le \left| \frac{1}{n}\sum_{k=1}^{N-1} J_{n,k}(x) - \frac{1}{n}\sum_{k=1}^{N-1} J_{n,k}^{\alpha}(x) \right| + \frac{1}{n^2}\sum_{k=N}^{\infty} kJ_{n,k}(x) + \frac{x}{n}. \tag{2.4}$$

Combining Lemma 2.3 (1) and (2.1) ∼ (2.4), we have get Lemma 2.4(2).
Similarly, we can obtain Lemma 2.4(3), (4) by some computations.
Noting $J_{n,0}^{\alpha}(x) = 1, \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) = 1$ and Lemma 2.4, by simple calculation, one can get the following Lemma 2.5.

Lemma 2.5 Let $\alpha \ge 1, 0 \le \beta \le 1$, we have
(1)$D_{n,\beta}^{(\alpha)}(1;x) = 1$;
(2) $\lim_{n\to\infty} D_{n,\beta}^{(\alpha)}(t;x) = x$;
(3) $\lim_{n\to\infty} D_{n,\beta}^{(\alpha)}(t^2;x) = x^2$;
(4) $\lim_{n\to\infty} D_{n,\beta}^{(\alpha)}(t^3;x) = x^3$;
(5) $\lim_{n\to\infty} D_{n,\beta}^{(\alpha)}(t^4;x) = x^4$.

Lemma 2.6 Let $\alpha \ge 1, 0 \le \beta \le 1, \varphi^2(x) = x$, we have

$$(1) D_{n,\beta}^{(\alpha)}((t-x)^2;x) \le \frac{\alpha}{n}(1+\beta^2)\varphi^2(x);$$

$$(2) D_{n,\beta}^{(\alpha)}((t-x)^4;x) \le \frac{\alpha}{n^2}\left[\left(3+6\beta^2+3\beta^4\right)\varphi^4(x) + \frac{1+6\beta^2+8\beta^3+9\beta^4}{n}\varphi^2(x)\right].$$

Proof Using the mean value theorem for differential calculus, for $x \in [0,\infty)$, $\alpha \ge 1$, $k = 0,1,\ldots$, we have $0 \le G_{n,k}^{(\alpha)}(x) \le \alpha s_{n,k}(x)$. Since

$$\begin{aligned} D_{n,\beta}^{(\alpha)}((t-x)^2;x) &= x^2 G_{n,0}^{(\alpha)}(x) + \sum_{k=1}^{\infty} G_{n,k}^{(\alpha)}(x) T_{n,k}^{(\beta)}((t-x)^2) \\ &\le \alpha \cdot \left[x^2 s_{n,0}(x) + \sum_{k=1}^{\infty} s_{n,k}(x) T_{n,k}^{(\beta)}((t-x)^2) \right] \\ &= \alpha \cdot D_{n,\beta}((t-x)^2;x) = \frac{\alpha}{n}(1+\beta^2)\varphi^2(x), \end{aligned}$$

we get Lemma 2.6(1). Similarly, we can obtain Lemma 2.6(2), we omit the details.

Remark 4 By the Korovkin theorem[1,3] and Lemma 2.5, the following result follows immediately:
For $f(x) \in C_B[0,\infty)$, the functions $D_{n,\beta}^{(\alpha)}(f;x)$ converge to $f(x)$ on $[0,\infty)$.

3. Direct Theorems

For $f(x) \in C_B[0,\infty)$, $\varphi(x) = \sqrt{x}$, $0 \le \lambda \le 1$, let[1,3]

$$\omega_{\varphi^\lambda}(f;t) = \sup_{0<h\le t} \sup_{x \pm \frac{h\varphi^\lambda(x)}{2} \in [0,\infty)} \left| f\left(x + \frac{h\varphi^\lambda(x)}{2}\right) - f\left(x - \frac{h\varphi^\lambda(x)}{2}\right) \right|,$$

be the Ditzian-Totik modulus, and

$$K_{\varphi^\lambda}(f;t) = \inf_{g\in W_\lambda[0,\infty)} \left\{\|f-g\| + t\|\varphi^\lambda g'\|\right\},$$

be the corresponding K-functional, here $W_\lambda = \{g|g \in A.C._{loc.}[0,\infty), \|\varphi^\lambda g'\| < \infty\}$. It is well know that[1,3]

$$K_{\varphi^\lambda}(f;t) \sim \omega_{\varphi^\lambda}(f;t).$$

Theorem 3.1 For $f \in C_B[0,\infty)$, $\alpha \geq 1$, $\varphi(x) = \sqrt{x}$ and $0 \leq \beta \leq 1, 0 \leq \lambda \leq 1$, then we have

$$|D_{n,\beta}^{(\alpha)}(f;x) - f(x)| \leq M\omega_{\varphi^\lambda}\left(f; \frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\right).$$

Proof Let $g \in W_\lambda$, then

$$\left|D_{n,\beta}^{(\alpha)}(f;x) - f(x)\right| \leq \left|D_{n,\beta}^{(\alpha)}(f-g;x)\right| + \left|f(x) - g(x)\right| + \left|D_{n,\beta}^{(\alpha)}(g;x) - g(x)\right|.$$

Since $g(t) = \int_x^t g'(u)du + g(x)$, $D_{n,\beta}^{(\alpha)}(1;x) = 1$, we know

$$|D_{n,\beta}^{(\alpha)}(g;x) - g(x)| \leq \|\varphi^\lambda g'\| \cdot D_{n,\beta}^{(\alpha)}\left(\left|\int_x^t \varphi^{-\lambda}(u)du\right|;x\right),$$

and by the Hölder inequality, we get

$$\left|\int_x^t \varphi^{-\lambda}(u)du\right| \leq 2^\lambda \varphi^{-\lambda}(x) \cdot |t-x|. \tag{3.1}$$

Thus,

$$\left|D_{n,\beta}^{(\alpha)}(g;x) - g(x)\right| \leq 2\|\varphi^\lambda g'\| \cdot \varphi^{-\lambda}(x) \cdot D_{n,\beta}^{(\alpha)}(|t-x|;x).$$

Combining Lemma 2.6 (2), using the Cauchy-Schwarz inequality, we have

$$\left|D_{n,\beta}^{(\alpha)}(g;x) - g(x)\right| \leq 2\sqrt{(1+\beta^2)\alpha} \cdot \|\varphi^\lambda g'\| \cdot \frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}.$$

By the definition of $D_{n,\beta}^{(\alpha)}(f;x)$ and Lemma 2.5 (1), we have $|D_{n,\beta}^{(\alpha)}(f;x)| \leq \|f\|$, so

$$\left|D_{n,\beta}^{(\alpha)}(f;x) - f(x)\right| \leq 2\|f-g\| + 2\sqrt{(1+\beta^2)\alpha} \cdot \|\varphi^\lambda g'\| \cdot \frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}.$$

Taking infimum on the right hand side over all $g \in W_\lambda$, one can obtain

$$\left|D_{n,\beta}^{(\alpha)}(f;x) - f(x)\right| \leq MK_{\varphi^\lambda}\left(f; \frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\right) \leq M\omega_{\varphi^\lambda}\left(f; \frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\right).$$

Theorem 3.2 For $f'(x)$ is continuous and bounded on $[0,\infty)$, and $\alpha \geq 1, 0 \leq \beta \leq 1$, $\varphi(x) = \sqrt{x}$, then

$$\left|D_{n,\beta}^{(\alpha)}(f;x) - f(x)\right| \leq \sqrt{\frac{(1+\beta^2)\alpha}{n}} \cdot \left\{\|f'\| + \omega\left(f'; \frac{1}{\sqrt{n}}\right) \cdot \left(1 + \sqrt{(1+\beta^2)\alpha} \cdot \varphi(x)\right)\right\} \cdot \varphi(x).$$

Proof For $\delta > 0$, $t, x \in [0,\infty)$, $|t-x| < \delta$, by the Taylor's expansion, we get

$$\left|f(t) - f(x) - f'(x)(t-x)\right| \leq \left|\int_x^t \left|f'(u) - f'(x)\right| du\right| \leq \omega(f';\delta) \cdot \left(|t-x| + \delta^{-1}(t-x)^2\right),$$

applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left|D_{n,\beta}^{(\alpha)}(f(t)-f(x)-f'(x)(t-x));x)\right| \\
\leq\ & \omega(f';\delta)\cdot\left(D_{n,\beta}^{(\alpha)}(|t-x|;x\right)+\delta^{-1}D_{n,\beta}^{(\alpha)}\left((t-x)^2);x\right) \\
\leq\ & \omega(f';\delta)\cdot\left[\sqrt{D_{n,\beta}^{(\alpha)}(1;x)}+\delta^{-1}\sqrt{D_{n,\beta}^{(\alpha)}((t-x)^2;x)}\right]\cdot\sqrt{D_{n,\beta}^{(\alpha)}((t-x)^2;x)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left|D_{n,\beta}^{(\alpha)}(f;x)-f(x)\right| \\
\leq\ & \|f'\|\cdot D_{n,\beta}^{(\alpha)}(|t-x|;x)+\omega(f';\delta)\cdot\left[1+\delta^{-1}\sqrt{D_{n,\beta}^{(\alpha)}((t-x)^2;x)}\right]\cdot\sqrt{D_{n,\beta}^{(\alpha)}((t-x)^2;x)}.
\end{aligned}$$

Taking $\delta=\frac{1}{\sqrt{n}}$, by Lemma 2.6 (1), we can obtain the desired results.

4. Inverse Theorem

Lemma 4.1 Let $f\in C_B[0,\infty)$, $\varphi(x)=\sqrt{x}$, $\alpha\geq 1, 0\leq\beta\leq 1, 0\leq\lambda\leq 1$, we have

$$\left|\varphi^{\lambda}(x)\cdot\left(D_{n,\beta}^{(\alpha)}(f;x)\right)'\right|\leq 9\alpha\varphi^{\lambda-1}(x)\sqrt{n}\|f\|.$$

Proof We write

$$\left(D_{n,\beta}^{(\alpha)}(f;x)\right)'=f(0)\left(G_{n,0}^{(\alpha)}(x)\right)'+\left(\sum_{k=1}^{\infty}G_{n,k}^{(\alpha)}(x)T_{n,k}^{(\beta)}(f)\right)'=R_1+R_2, \tag{4.1}$$

and will estimate R_1 and R_2, respectively. Noting that $J'_{n,0}(x)=0$, we have

$$|R_1|=\left|n\alpha f(0)\cdot(1-e^{-nx})^{\alpha-1}\cdot e^{-nx}\right|\leq\frac{n\alpha|f(0)|}{e^{nx}}.$$

For a fixed $x\in[0,+\infty)$, $\lim_{n\to\infty}\frac{\sqrt{nx}}{e^{nx}}=0$, one may say $\frac{\sqrt{nx}}{e^{nx}}\leq 1$, then

$$\varphi^{\lambda}(x)\cdot|R_1|\leq\frac{\sqrt{nx}\cdot x^{\frac{\lambda-1}{2}}}{e^{nx}}\cdot\alpha\sqrt{n}\cdot\|f\|\leq\alpha\varphi^{\lambda-1}(x)\sqrt{n}\cdot\|f\|. \tag{4.2}$$

$$R_2=\alpha\sum_{k=1}^{\infty}T_{n,k}^{(\beta)}(f)\left\{\left[J_{n,k}^{\alpha-1}(x)-J_{n,k+1}^{\alpha-1}(x)\right]J'_{n,k+1}(x)+J_{n,k}^{\alpha-1}(x)\cdot s'_{n,k}(x)\right\}.$$

For $k=0,1,2,3...$, $1=J_{n,0}(x)\geq J_{n,1}(x)\geq...\geq J_{n,k}(x)\geq J_{n,k+1}(x)\geq...\geq 0$, and $J'_{n,0}(x)=0$, $J'_{n,k}(x)=ns_{n,k-1}(x)\geq 0$,

$$\left|T_{n,k}^{(\beta)}(f)\right|=\left|\frac{n^k}{\Gamma(k)}\int_0^{\infty}t^{k-1}e^{-nt}\cdot f\left(\beta t+(1-\beta)\frac{k}{n}\right)dt\right|\leq\|f\|,$$

we have

$$|R_2|\leq\alpha\|f\|\left(\sum_{k=1}^{\infty}\left[J_{n,k}^{\alpha-1}(x)-J_{n,k+1}^{\alpha-1}(x)\right]J'_{n,k+1}(x)+\sum_{k=1}^{\infty}J_{n,k}^{\alpha-1}(x)\cdot\left|s'_{n,k}(x)\right|\right)=\alpha\|f\|(V_1+V_2). \tag{4.3}$$

Noting that $J'_{n,1}(x) > 0, J'_{n,0}(x) = 0$, we have

$$\begin{aligned} V_1 &\le \sum_{k=0}^{\infty} J_{n,k}^{\alpha-1}(x) J'_{n,k+1}(x) - \left[\sum_{k=0}^{\infty} [J_{n,k+1}^{\alpha-1}(x) J'_{n,k+1}(x) - J_{n,1}^{\alpha-1}(x) J'_{n,1}(x) \right] \\ &= \sum_{k=0}^{\infty} J_{n,k}^{\alpha-1}(x) J'_{n,k}(x) - \sum_{k=0}^{\infty} J_{n,k}^{\alpha-1}(x) s'_{n,k}(x) - \sum_{k=0}^{\infty} J_{n,k+1}^{\alpha-1}(x) J'_{n,k+1}(x) \\ &\quad +(1-e^{-nx})^{\alpha-1} \cdot ne^{-nx} \\ &\le -J_{n,0}^{\alpha-1}(x) s'_{n,0}(x) + \sum_{k=1}^{\infty} J_{n,k}^{\alpha-1}(x) \cdot \left| s'_{n,k}(x) \right| + ne^{-nx}, \end{aligned}$$

thus,

$$V_1 \le V_2 + \frac{2n}{e^{nx}}. \tag{4.4}$$

For $x > 0$, $s'_{n,k}(x) = \frac{n}{\varphi^2(x)} \left[\frac{k}{n} - x \right] \cdot s_{n,k}(x)$, combining the fact that $J_{n,1}^{\alpha-1}(x) = (1-e^{-nx})^{\alpha-1}, s'_{n,1} = ne^{-nx}(1-nx)$, we write that

$$\begin{aligned} \varphi^\lambda(x) V_2 &\le ne^{-nx}(1+nx) \cdot \varphi^\lambda(x) + \sum_{k=2}^{\infty} J_{n,k}^{\alpha-1}(x) \cdot \left| s'_{n,k}(x) \right| \cdot \varphi^\lambda(x) \\ &\le \frac{\sqrt{n} \cdot \sqrt{nx} \cdot x^{\frac{\lambda-1}{2}}}{e^{nx}} + \frac{\sqrt{n} \cdot (nx)^{\frac{3}{2}} \cdot x^{\frac{\lambda-1}{2}}}{e^{nx}} + n \sum_{k=2}^{\infty} \left| \frac{k}{n} - x \right| \cdot s_{n,k}(x) \cdot \varphi^{\lambda-2}(x) \\ &\le 2\sqrt{n} \cdot \varphi^{\lambda-1}(x) + n\varphi^{\lambda-2}(x) \cdot \left(S_n((t-x)^2; x) \right)^{\frac{1}{2}}, \end{aligned}$$

then,

$$\varphi^\lambda(x) V_2 \le 3\varphi^{\lambda-1}(x)\sqrt{n}, \tag{4.5}$$

and $\varphi^\lambda(x) V_1 \le 3\varphi^{\lambda-1}(x)\sqrt{n} + 2\sqrt{n} \cdot \varphi^{\lambda-1}(x) \cdot \frac{\sqrt{nx}}{e^{nx}} \le 5\varphi^{\lambda-1}(x)\sqrt{n}$.
So

$$\varphi^\lambda(x)|R_2| \le \alpha\|f\| \cdot \varphi^\lambda(x)\,(V_1 + V_2) \le 8\alpha\sqrt{n}\|f\| \cdot \varphi^{\lambda-1}(x). \tag{4.6}$$

From (4.1)-(4.6), the desired result follows.

Lemma 4.2 Let $f \in W_\lambda$, $\varphi(x) = \sqrt{x}$, $\alpha \ge 1, 0 \le \beta \le 1, 0 \le \lambda \le 1$, we have

$$\left| \varphi^\lambda(x) \cdot \left(D_{n,\beta}^{(\alpha)}(f;x) \right)' \right| \le 38\alpha\|\varphi^\lambda f'\|.$$

Proof Since $f(x)\left(D_{n,\beta}^{(\alpha)}(1;x) \right)' = 0$, we get

$$\left(D_{n,\beta}^{(\alpha)}(f;x) \right)' = [f(0) - f(x)] \left(G_{n,0}^{(\alpha)}(x) \right)' + \sum_{k=1}^{\infty} \left[T_{n,k}^{(\beta)}(f) - f(x) \right] \left[J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x) \right]',$$

we write

$$\left(D_{n,\beta}^{(\alpha)}(f;x) \right)' = H_1 + H_2, \tag{4.7}$$

and will estimate H_1 and H_2 respectively. First, from (3.1), we have

$$\varphi^\lambda(x)|H_1| \le \varphi^\lambda(x) \cdot \|\varphi^\lambda f'\| \cdot 2^\lambda \cdot x^{1-\frac{\lambda}{2}} \cdot \frac{\alpha n}{e^{nx}} \le 2\alpha\|\varphi^\lambda f'\|. \tag{4.8}$$

Next,

$$\begin{aligned} H_2 &= \sum_{k=1}^{\infty} \frac{n^k}{\Gamma(k)} \int_0^{\infty} t^{k-1}e^{-nt} \cdot \left[f\left(\beta t + (1-\beta)\frac{k}{n}\right) - f(x) \right] dt \cdot \left[J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x) \right]' \\ &= \sum_{k=1}^{\infty} \frac{n^k}{\Gamma(k)} \int_0^{\infty} t^{k-1}e^{-nt} \cdot \int_x^{\beta t+(1-\beta)\frac{k}{n}} f'(u)du \cdot dt \cdot \left[J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x) \right]'. \end{aligned}$$

From (3.1), we have

$$\begin{aligned} \varphi^{\lambda}(x) \left| \int_x^{\beta t+(1-\beta)\frac{k}{n}} f'(u)du \right| &\le \varphi^{\lambda}(x) \|\varphi^{\lambda} f'\| \cdot \left| \int_x^{\beta t+(1-\beta)\frac{k}{n}} \frac{1}{\varphi^{\lambda}(u)} du \right| \\ &\le 2\|\varphi^{\lambda} f'\| \cdot \left| \beta t + (1-\beta)\frac{k}{n} - x \right|, \end{aligned}$$

then,

$$\begin{aligned} \varphi^{\lambda}(x)|H_2| &\le 2\|\varphi^{\lambda} f'\| \cdot \sum_{k=1}^{\infty} \frac{n^k}{\Gamma(k)} \int_0^{\infty} t^{k-1}e^{-nt} \cdot \left| \beta t + (1-\beta)\frac{k}{n} - x \right| dt \cdot \left| \left[J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x) \right]' \right| \\ &= 2\alpha\|\varphi^{\lambda} f'\| \cdot \left\{ \sum_{k=1}^{\infty} \frac{n^k}{\Gamma(k)} \int_0^{\infty} t^{k-1}e^{-nt} \cdot \left| \beta t + (1-\beta)\frac{k}{n} - x \right| dt \cdot \left[J_{n,k}^{\alpha-1}(x) - J_{n,k+1}^{\alpha-1}(x) \right] \right. \\ &\quad \left. \times \left| J_{n,k+1}'(x) \right| + \sum_{k=1}^{\infty} \frac{n^k}{\Gamma(k)} \int_0^{\infty} t^{k-1}e^{-nt} \cdot \left| \beta t + (1-\beta)\frac{k}{n} - x \right| dt \cdot J_{n,k}^{\alpha-1}(x) \left| s_{n,k}'(x) \right| \right\}. \end{aligned}$$

Write

$$\varphi^{\lambda}(x)|H_2| \le 2\alpha\|\varphi^{\lambda} f'\| \cdot (A + B). \tag{4.9}$$

We will estimate A and B on E_n^C and E_n respectively.
(I). For $x \in E_n^c = [0, \frac{1}{n})$:

$$\begin{aligned} B &\le \sum_{k=1}^{\infty} \frac{n^k}{\Gamma(k)} \int_0^{\infty} t^{k-1}e^{-nt} \cdot \left| \beta t + (1-\beta)\frac{k}{n} - x \right| dt \cdot J_{n,k}^{\alpha-1}(x) \cdot \left| n(s_{n,k-1}(x) - s_{n,k}(x)) \right| \\ &\le \sum_{k=1}^{\infty} \frac{n^k}{\Gamma(k)} \int_0^{\infty} t^{k-1}e^{-nt} \cdot \left| \beta t + (1-\beta)\frac{k}{n} - x \right| dt \cdot n s_{n,k-1}(x) \\ &\quad + \sum_{k=1}^{\infty} \frac{n^k}{\Gamma(k)} \int_0^{\infty} t^{k-1}e^{-nt} \cdot \left| \beta t + (1-\beta)\frac{k}{n} - x \right| dt \cdot n s_{n,k}(x), \end{aligned}$$

we write

$$B \le L_1 + L_2. \tag{4.10}$$

Since $D_{n,\beta}((t-x)^2; x) = \frac{(1+\beta^2)x}{n}$, by the Cauchy-Schwarz inequality, we get

$$L_2 \le n \left(D_{n,\beta}((t-x)^2; x) \right)^{\frac{1}{2}} \le \sqrt{1+\beta^2}. \tag{4.11}$$

Using the fact that $\Gamma(j+1) = j\Gamma(j)$, and for $j \ge 1$

$$\begin{aligned} &\int_0^{\infty} t^j e^{-nt} \cdot \left| \beta t + (1-\beta)\frac{j}{n} - x \right| dt \\ \le\ &\frac{j}{n} \int_0^{\infty} e^{-nt} t^{j-1} \cdot \left| \beta t + (1-\beta)\frac{j}{n} - x \right| dt + \frac{\beta}{n} \int_0^{\infty} e^{-nt} \cdot t^j dt, \end{aligned}$$

then

$$\begin{aligned}
L_1 &\le \sum_{j=0}^{\infty}\frac{n^{j+1}}{\Gamma(j+1)}\int_0^{\infty}t^je^{-nt}\cdot\left|\beta t+(1-\beta)\frac{j}{n}-x\right|dt\cdot ns_{n,j}(x)\\
&\quad+\sum_{j=0}^{\infty}\frac{n^{j+1}}{\Gamma(j+1)}\int_0^{\infty}t^je^{-nt}\cdot\frac{1-\beta}{n}dt\cdot ns_{n,j}(x)\\
&= \frac{n}{\Gamma(1)}\int_0^{\infty}t^0e^{-nt}\cdot\left|\beta t+(1-\beta)\frac{1}{n}-x-(1-\beta)\frac{1}{n}\right|dt\cdot ns_{n,0}(x)\\
&\quad+\sum_{j=1}^{\infty}\frac{n^{j+1}}{\Gamma(j+1)}\int_0^{\infty}t^je^{-nt}\cdot\left|\beta t+(1-\beta)\frac{j}{n}-x\right|dt\cdot ns_{n,j}(x)\\
&\quad+\sum_{j=0}^{\infty}\frac{n^{j+1}}{\Gamma(j+1)}\int_0^{\infty}t^je^{-nt}\cdot\frac{1-\beta}{n}dt\cdot ns_{n,j}(x)\\
&\le F_{n,1}^{(\beta)}(t;x)\cdot\frac{n}{e^{nx}}+\frac{nx}{e^{nx}}+\frac{1-\beta}{e^{nx}}+\sum_{j=1}^{\infty}\frac{n^j}{\Gamma(j)}\int_0^{\infty}t^{j-1}e^{-nt}\cdot\left|\beta t+(1-\beta)\frac{j}{n}-x\right|dt\cdot ns_{n,j}(x)\\
&\quad+\sum_{j=0}^{\infty}\frac{n^{j+1}}{\Gamma(j+1)}\cdot\frac{\beta}{n}\cdot\int_0^{\infty}t^je^{-nt}dt\cdot ns_{n,j}(x)+\sum_{j=0}^{\infty}\frac{n^{j+1}}{\Gamma(j+1)}\cdot\frac{1-\beta}{n}\cdot\int_0^{\infty}t^je^{-nt}dt\cdot ns_{n,j}(x)\\
&\le \frac{1}{e^{nx}}+\frac{nx}{e^{nx}}+\frac{1-\beta}{e^{nx}}+\sum_{j=1}^{\infty}\frac{n^j}{\Gamma(j)}\int_0^{\infty}t^{j-1}e^{-nt}\cdot\left|\beta t+(1-\beta)\frac{j}{n}-x\right|dt\cdot ns_{n,j}(x)\\
&\quad+\sum_{j=0}^{\infty}\frac{n^{j+1}}{\Gamma(j+1)}\cdot\frac{1}{n}\int_0^{\infty}t^je^{-nt}\cdot dt\cdot ns_{n,j}(x).
\end{aligned}$$

we get

$$L_1\le 3+L_2+1\le 4+\sqrt{1+\beta^2}, \tag{4.12}$$

from (4.10)-(4.12), we know $B\le 4+2\sqrt{1+\beta^2}$.
Noting that $J'_{n,0}(x)=0$, for $x\in[0,\frac{1}{n})$, one has

$$\begin{aligned}
A &= \sum_{k=1}^{\infty}\frac{n^k}{\Gamma(k)}\int_0^{\infty}t^{k-1}e^{-nt}\cdot\left|\beta t+(1-\beta)\frac{k}{n}-x\right|dt\cdot J_{n,k}^{\alpha-1}(x)J'_{n,k}(x)\\
&\quad-\sum_{k=1}^{\infty}\frac{n^k}{\Gamma(k)}\int_0^{\infty}t^{k-1}e^{-nt}\cdot\left|\beta t+(1-\beta)\frac{k}{n}-x\right|dt\cdot J_{n,k}^{\alpha-1}(x)s'_{n,k}(x)\\
&\quad-\sum_{k=1}^{\infty}\frac{n^k}{\Gamma(k)}\int_0^{\infty}t^{k-1}e^{-nt}\cdot\left|\beta t+(1-\beta)\frac{k}{n}-x\right|dt\cdot J_{n,k+1}^{\alpha-1}(x)\cdot J'_{n,k+1}(x)\\
&\le L_3+B-\sum_{k=1}^{\infty}\frac{n^k}{\Gamma(k)}\int_0^{\infty}t^{k-1}e^{-nt}\cdot\left|\beta t+(1-\beta)\frac{k}{n}-x\right|dt\cdot J_{n,k+1}^{\alpha-1}(x)\cdot J'_{n,k+1}(x),
\end{aligned}$$

and

$$\begin{aligned}
L_3 &= \sum_{j=0}^{\infty} \frac{n^{j+1}}{\Gamma(j+1)} \int_0^{\infty} t^j e^{-nt} \cdot \left|\beta t + (1-\beta)\frac{j+1}{n} - x\right| dt \cdot J_{n,j+1}^{\alpha-1}(x) J'_{n,j+1}(x) \\
&\le \sum_{j=0}^{\infty} \frac{n^{j+1}}{\Gamma(j+1)} \int_0^{\infty} t^j e^{-nt} \cdot \left|\beta t + (1-\beta)\frac{j}{n} - x\right| dt \cdot J_{n,j+1}^{\alpha-1}(x) J'_{n,j+1}(x) \\
&\quad + \sum_{j=0}^{\infty} \frac{n^{j+1}}{\Gamma(j+1)} \int_0^{\infty} t^j e^{-nt} \cdot \frac{1-\beta}{n} dt \cdot J_{n,j+1}^{\alpha-1}(x) J'_{n,j+1}(x) \\
&= L_4 + \sum_{j=0}^{\infty} \frac{n^{j+1}}{\Gamma(j+1)} \int_0^{\infty} t^j e^{-nt} \cdot \frac{1-\beta}{n} dt \cdot J_{n,j+1}^{\alpha-1}(x) J'_{n,j+1}(x),
\end{aligned}$$

$$\begin{aligned}
L_4 &\le \sum_{j=0}^{\infty} \frac{n^j}{\Gamma(j)} \int_0^{\infty} e^{-nt} \cdot t^{j-1} \cdot \left|\beta t + (1-\beta)\frac{j}{n} - x\right| dt \cdot J_{n,j+1}^{\alpha-1}(x) J'_{n,j+1}(x) \\
&\quad + \sum_{j=0}^{\infty} \frac{n^{j+1}}{\Gamma(j+1)} \cdot \frac{\beta}{n} \int_0^{\infty} e^{-nt} \cdot t^j dt \cdot J_{n,j+1}^{\alpha-1}(x) J'_{n,j+1}(x) \\
&\le \frac{1}{\Gamma(0)} \int_0^{\infty} e^{-nt} \cdot t^{0-1} \cdot \left|\beta t - x\right| dt \cdot J_{n,1}^{\alpha-1}(x) J'_{n,1}(x) \\
&\quad + \sum_{j=1}^{\infty} \frac{n^j}{\Gamma(j)} \int_0^{\infty} e^{-nt} \cdot t^{j-1} \cdot \left|\beta t + (1-\beta)\frac{j}{n} - x\right| dt \cdot J_{n,j+1}^{\alpha-1}(x) J'_{n,j+1}(x) \\
&\quad + \sum_{j=0}^{\infty} \frac{n^{j+1}}{\Gamma(j+1)} \cdot \frac{\beta}{n} \int_0^{\infty} e^{-nt} \cdot t^j dt \cdot J_{n,j+1}^{\alpha-1}(x) J'_{n,j+1}(x),
\end{aligned}$$

thus, we have that

$$\begin{aligned}
L_3 &\le \left(F_{n,0}^{(\beta)}(t;x) + x\right) \cdot J'_{n,1}(x) + \sum_{j=1}^{\infty} \frac{n^j}{\Gamma(j)} \int_0^{\infty} t^{j-1} e^{-nt} \cdot \left|\beta t + (1-\beta)\frac{j}{n} - x\right| dt \cdot J_{n,j+1}^{\alpha-1}(x) J'_{n,j+1}(x) \\
&\quad + \sum_{j=0}^{\infty} \frac{1}{n} \cdot J_{n,j+1}^{\alpha-1}(x) J'_{n,j+1}(x) \\
&\le \frac{nx}{e^{nx}} + \sum_{j=1}^{\infty} \frac{n^j}{\Gamma(j)} \int_0^{\infty} t^{j-1} e^{-nt} \cdot \left|\beta t + (1-\beta)\frac{j}{n} - x\right| dt \cdot J_{n,j+1}^{\alpha-1}(x) J'_{n,j+1}(x) + \sum_{j=1}^{\infty} \frac{1}{n} J'_{n,j}(x) \\
&\le 1 + \sum_{j=1}^{\infty} \frac{n^j}{\Gamma(j) \cdot t} \int_0^{\infty} t^{j-1} e^{-nt} \cdot \left|\beta t + (1-\beta)\frac{j}{n} - x\right| dt \cdot J_{n,j+1}^{\alpha-1}(x) J'_{n,j+1}(x) + \sum_{j=1}^{\infty} s_{n,j-1} \\
&\le 2 + \sum_{j=1}^{\infty} \frac{n^j}{\Gamma(j) \cdot t} \int_0^{\infty} t^{j-1} e^{-nt} \cdot \left|\beta t + (1-\beta)\frac{j}{n} - x\right| dt \cdot J_{n,j+1}^{\alpha-1}(x) J'_{n,j+1}(x),
\end{aligned}$$

hence $A \le 2 + |B| \le 6 + 2\sqrt{1+\beta^2}$.
Combing A and B, we have,

$$\left|\varphi^{\lambda}(x) H_2\right| \le 2\alpha \|\varphi^{\lambda} f'\| \cdot (10 + 4\sqrt{1+\beta^2}) \le 36\alpha \|\varphi^{\lambda} f'\|. \tag{4.13}$$

From (4.7), (4.8), (4.13), for $x \in E_n^C$, we have $|\varphi^{\lambda}(x)\cdot(D_{n,\beta}^{(\alpha)}(f;x))'| \le 38\alpha\|\varphi^{\lambda} f'\|$.
(II). $x \in E_n = [\frac{1}{n}, +\infty)$: Noting $s'_{n,k}(x) = \frac{n}{\varphi^2(x)}[\frac{k}{n} - x]\cdot s_{n,k}(x)$, using the Cauchy-Schwarz inequality,

$$\begin{aligned}
B &\le \sum_{k=1}^{\infty}\frac{n^k}{\Gamma(k)}\int_0^{\infty} t^{k-1}e^{-nt}\cdot\left|\beta t + (1-\beta)\frac{k}{n} - x\right| dt\cdot|s'_{n,k}(x)| \\
&\le \left(\sum_{k=1}^{\infty}\frac{n^k}{\Gamma(k)}\int_0^{\infty} t^{k-1}e^{-nt}\cdot\left(\beta t + (1-\beta)\frac{k}{n} - x\right)^2 dt\cdot s_{n,k}(x)\right)^{\frac{1}{2}} \\
&\quad\times\left(\sum_{k=1}^{\infty} s_{n,k}(x)\left(\frac{k}{n} - x\right)^2\right)^{\frac{1}{2}}\cdot\frac{n}{\varphi^2(x)} \\
&\le \left(D_{n,\beta}((t-x)^2;x)\right)^{\frac{1}{2}}\cdot\left(S_n((t-x)^2;x)\right)^{\frac{1}{2}}\cdot\frac{n}{\varphi^2(x)} \\
&\le \frac{\sqrt{1+\beta^2}\cdot\varphi(x)}{\sqrt{n}}\cdot\frac{\varphi(x)}{\sqrt{n}}\cdot\frac{n}{\varphi^2(x)} = \sqrt{1+\beta^2}.
\end{aligned}$$

Using the same method as used in the case (I)$x \in [0, \frac{1}{n})$, we get $A \le 2 + B \le 2 + \sqrt{1+\beta^2}$, then

$$|\varphi^{\lambda}(x)H_2| \le 12\alpha\|\varphi^{\lambda} f'\|.$$

Hence, for $x \in E_n$, we have $|\varphi^{\lambda}(x)\cdot(D_{n,\beta}^{(\alpha)}(f;x))'| \le |\varphi^{\lambda}(x)H_1| + |\varphi^{\lambda}(x)H_2| \le 14\alpha\|\varphi^{\lambda} f'\|$.

Theorem 4.1 Let $f(x) \in C_B[0,\infty)$, $\varphi(x) = \sqrt{x}$, $0 \le \beta \le 1, 0 \le \gamma \le 1, 0 \le \lambda \le 1$, if

$$|D_{n,\beta}^{(\alpha)}(f;x) - f(x)| = O(n^{-\frac{\gamma}{2}}),\ \ \textit{one has}\ \ \omega_{\varphi^{\lambda}}(f;t) = O(t^{\gamma}).$$

Proof By the definition of the K-functional, for $g \in W_{\lambda}$,

$$\begin{aligned}
K_{\varphi^{\lambda}}(f;t) &\le \left\|f - D_{n,\beta}^{(\alpha)}(f;x)\right\| + t\left\|\varphi^{\lambda}(x)\left(D_{n,\beta}^{(\alpha)}(f;x)\right)'\right\| \\
&\le Mn^{-\frac{\gamma}{2}} + t\left(\|\varphi^{\lambda}(x)\left(D_{n,\beta}^{(\alpha)}((f-g);x)\right)'\| + \|\varphi^{\lambda}(x)(D_{n,\beta}^{(\alpha)}(g;x))'\|\right) \\
&\le Mn^{-\frac{\gamma}{2}} + t\sqrt{n}\left(\|f-g\| + \frac{1}{\sqrt{n}}\|\varphi^{\lambda} g'\|\right) \\
&\le M\left(n^{-\frac{\gamma}{2}} + t\sqrt{n}\cdot K_{\varphi^{\lambda}}(f;n^{-\frac{1}{2}})\right),
\end{aligned}$$

applying to the Berens-Lorentz Lemma[1], then $K_{\varphi^{\lambda}}(f;t) = O(t^{\gamma})$.
We know[1]: $\omega_{\varphi^{\lambda}}(f;t) \sim K_{\varphi^{\lambda}}(f;t)$, then one get Theorem 4.1.
From Theorem 3.4 and Theorem 4.1, we get the equivalent theorem.

Theorem 4.2 Let $f(x) \in C_B[0,\infty)$, $\varphi(x) = \sqrt{x}$, $\alpha \ge 1, 0 \le \beta \le 1, 0 \le \gamma \le 1, 0 \le \lambda \le 1$, we have

$$\left|D_{n,\beta}^{(\alpha)}(f;x) - f(x)\right| = O(n^{-\frac{\gamma}{2}}) \Leftrightarrow \omega_{\varphi^{\lambda}}(f;t) = O(t^{\gamma}).$$

5. Voronovskaja type theorem

In this section, we will first prove Voronovskaja type theorems for the operators $D_{n,\beta}^{(\alpha)}(f;x)$ by means of the Ditzian-Totik modulus of smoothness $\omega_{\varphi}(f;t)$.

Theorem 5.1 For any $f''(x)$ is continuous and bounded on $[0,\infty)$, $\alpha \geq 1$, $0 \leq \beta \leq 1$, and n sufficiently large, the following inequality holds:

$$\left|D_{n,\beta}^{(\alpha)}(f;x) - f(x) - E_n(x)f'(x) - \frac{1}{2}F_n(x)f''(x)\right| \leq \frac{M}{n}\varphi^2(x)\omega_\varphi\left(f'';n^{-\frac{1}{2}}\right),$$

where $E_n(x) = D_{n,\beta}^{(\alpha)}(t-x;x); F_n(x) = D_{n,\beta}^{(\alpha)}((t-x)^2;x)$.

Proof For $t, x \in [0,\infty)$, by the Taylor's expansion, we have

$$f(t) - f(x) - f'(x)(t-x) = \int_x^t (t-y)f''(y)dy + \frac{1}{2}(t-x)^2 f''(x),$$

we write $f(t) - f(x) - f'(x)(t-x) - \frac{1}{2}(t-x)^2 f''(x) = \int_x^t (t-y)[f''(y) - f''(x)]dy$.

Applying $D_{n,\beta}^{(\alpha)}(f;x)$ to both side of the above relation, we get

$$\left|D_{n,\beta}^{(\alpha)}(f;x) - f(x) - E_n(x)f'(x) - \frac{1}{2}F_n(x)f''(x)\right| \leq D_{n,\beta}^{(\alpha)}(f;x)\left(\left|\int_x^t |t-y| \cdot \left|f''(y) - f''(x)\right| dy\right|;x\right).$$

The quantity $\left|\int_x^t |t-y||f''(y) - f''(x)|dy\right|$ was estimated as follows:[16,p.337]

$$\left|\int_x^t |t-y| \cdot \left|f''(y) - f''(x)\right| dy\right| \leq 2\|f'' - h\|(t-x)^2 + 2\|\varphi h'\|\varphi^{-1}(x)|t-x|^3,$$

where $h \in W_1[0,\infty)$.

Using Lemma 2.6 it follows that there exists a constant $M > 0$ such that, for n sufficiently large,

$$D_{n,\beta}^{(\alpha)}\left((t-x)^2;x\right) \leq \frac{M}{2n}\varphi^2(x) \quad and \quad D_{n,\beta}^{(\alpha)}\left((t-x)^4;x\right) \leq \frac{M}{2n^2}\varphi^4(x),$$

applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\left|D_{n,\beta}^{(\alpha)}(f;x) - f(x) - E_n(x)f'(x) - \frac{1}{2}F_n(x)f''(x)\right| \\
\leq\ & 2\|f'' - h\|D_{n,\beta}^{(\alpha)}((t-x)^2;x) + 2\|\varphi h'\|\varphi^{-1}(x)D_{n,\beta}^{(\alpha)}((t-x)^3;x) \\
\leq\ & \frac{M}{n}\varphi^2(x)\|f'' - h\| + 2\|\varphi h'\|\varphi^{-1}(x) \cdot \left(D_{n,\beta}^{(\alpha)}((t-x)^2;x)\right)^{\frac{1}{2}} \cdot \left(D_{n,\beta}^{(\alpha)}((t-x)^4;x)\right)^{\frac{1}{2}} \\
\leq\ & \frac{M}{n}\varphi^2(x)\left\{\|f'' - h\| + n^{-\frac{1}{2}}\|\varphi h'\|\right\} \leq \frac{M}{n}\varphi^2(x)\omega_\varphi\left(f'';n^{-\frac{1}{2}}\right),
\end{aligned}$$

in the last inequality, we have used the relation of K-functional and the modulus[1,3].

Corollary 5.1 If $f''(x)$ is continuous and bounded on $[0,\infty)$, $\alpha \geq 1$, then

$$\lim_{n\to\infty} n\left\{D_{n,\beta}^{(\alpha)}(f;x) - f(x) - E_n(x)f'(x) - \frac{1}{2}F_n(x)f''(x)\right\} = 0,$$

where $E_n(x)$ and $F_n(x)$ are defined in Theorem 5.1.

The *Grüss* type approximation problem has been studied by many authors [17–20]. Next, we will provide a *Grüss*-Voronovskaja type theorem for the operators $D_{n,\beta}(f;x)$.

Theorem 5.2 If $f''(x), g''(x)$ is continuous and bounded on $[0,\infty)$, for each $x \in [0,\infty)$, we have

$$\lim_{n\to\infty} n\left\{D_{n,\beta}((fg);x) - D_{n,\beta}(f;x)D_{n,\beta}(g;x)\right\} = (1+\beta^2)xf'(x)g'(x).$$

Proof We write that

$$\begin{aligned}
& D_{n,\beta}((fg);x) - D_{n,\beta}(f;x)D_{n,\beta}(g;x) \\
= \; & D_{n,\beta}((fg);x) - f(x)g(x) - E_n(x)(fg)^{'}(x) - \frac{1}{2}F_n(x)(fg)^{''} \\
& -D_{n,\beta}(f;x)\cdot\left[D_{n,\beta}(g;x) - g(x) - E_n(x)g^{'}(x) - \frac{1}{2}F_n(x)\cdot g^{''}(x)\right] \\
& -g(x)\cdot\left[D_{n,\beta}(f;x) - f(x) - E_n(x)f^{'}(x) - \frac{1}{2}F_n(x)f^{''}(x)\right] \\
& +\frac{1}{2}F_n(x)\cdot\left[(fg)''(x) - g''(x)D_{n,\beta}(f;x) - g(x)f''(x)\right] \\
& +E_n(x)\cdot\left[(fg)'(x) - g'(x)D_{n,\beta}(f;x) - g(x)f'(x)\right].
\end{aligned}$$

From the definition of $E_n(x) = D_{n,\beta}(t-x;x), F_n(x) = D_{n,\beta}((t-x)^2;x)$, and the relation $(fg)'' = (f'g+fg')' = f''g + 2f'g' + fg'', \quad (fg)' = f'g + fg'$, we can express that

$$F_n(x)\cdot\left[(fg)''(x) - g''(x)D_{n,\beta}(f;x) - g(x)f''(x)\right] = F_n(x)\cdot\left[2f'(x)g'(x) + f(x)g''(x) - g''(x)D_{n,\beta}(f;x)\right];$$

$$\textit{and} \;\; E_n(x)\cdot\left[(fg)'(x) - g'(x)D_{n,\beta}(f;x) - g(x)f'(x)\right] = E_n(x)\cdot\left[f(x)g'(x) - g'(x)D_{n,\beta}(f;x)\right].$$

Because that $E_n(x) = D_{n,\beta}(t-x;x) = 0$ and Lemma 2.2, Lemma 2.5, combining the Korovkin Theorem[3] and Corollary 5.1, we get

$$\begin{aligned}
& \lim_{n\to\infty} n\left\{D_{n,\beta}((fg);x) - D_{n,\beta}(f;x)D_{n,\beta}(g;x)\right\} \\
= \; & 0+0+0+\lim_{n\to\infty} nf'(x)g'(x)F_n(x) + \frac{1}{2}\lim_{n\to\infty} ng''(x)\cdot\left[f(x) - D_{n,\beta}(f;x)\right]F_n(x) \\
& +\lim_{n\to\infty} ng'(x)\cdot\left[f(x) - D_{n,\beta}(f;x)\right]\cdot E_n(x) \\
= \; & \lim_{n\to\infty} nf'(x)g'(x)F_n(x) = f'(x)g'(x)\varphi^2(x)(1+\beta^2) = (1+\beta^2)xf'(x)g'(x).
\end{aligned}$$

Acknowledgments
We express our gratitude to the referees for their helpful suggestions.

References

[1] Z. Ditzian, V. Totik, Moduli of Smoothness, Spring-Verlag, New York, 1987.
[2] V. Totik, Uniform approximation by Szász-Mirakian operators, Acta Math. Acad. Sci. Hungar., 41(1983) 291–307.
[3] W. Chen, Approximation theory of operators, Xiamen University Press, 1989.
[4] D. Zhou, Weighted approximation by Szász-Mirakian operators, J. Approx. Theory, 76(1994) 393–402.
[5] S. G. Gal, Approximation with an arbitrary order by generalized Szász-Mirakian operators, Studia Univ. Babes-Bolyai Math., 59(1)(2014) 77–81.
[6] S. Guo, C. Li, et al. Pointwise estimate for Szász-type operators, J. Approx. Theory, 94(1998) 160–171.
[7] Z. Ye, X. Long, X. M. Zeng: Adjustment algorithms for Bézier curve and surface. In: International Conference on Computer Science and Education, (2010) 1712–1716.
[8] G. Chang, Generalized Bernstein-Bézier polynomial, J. Computer Math., 1(4)(1983) 322–327.
[9] X. M. Zeng, A. Piriou, On the rate of convergence of two Bernstein-Bézier type operators for bounded variation functions, J. Approx. Theory, 95(1998) 369–387.
[10] X. M. Zeng, V. Gupta, Rate of convergence of Baskakov-Bézier type operators for locally bounded functions, Comput. Math. Appl., 44(3)(2002) 1445–1453.
[11] X. M. Zeng, J. Yang, S. L. Zuo, Approximation of Pointwise of Szász-type operators for bounded functions, International Journal of Mathematics,International Journal of Mathematics, Game Theory and Algebra Volume 13(2)(2003) 191–197.
[12] S. Guo, Q. Qi, G. Liu, The central approximation theorem for Baskakov-Bézier operators, J. Approx. Theory, 147(1)(2007) 112–124.
[13] X. Deng, G. Wu, On approximation of Bernstein-Durrmeyer-Bézier operators in Orlicz spaces, Pure. Appl. Math., 31(3)(2015) 307–317.

[14] M. Y. Ren, X. M. Zeng, W. H. Zhang, Approximation of a kind of new Baskakov-Bézier type operators, J. Computational Analysis and Applications, 23(2)(2017) 355–364.
[15] Q. Qi, D. Guo, Approximation properties of a new Bernstein-Bézier operators with parameters, Acta Math. Sci., 41A(3)(2021) 583–594.
[16] Z. Finta, Remark on Voronovskaja theorem for q-Bernstein operators, Stud. Univ. Babes-Bolyai Math., 56(2)(2011) 335–339.
[17] A. Acu, H. Gonska, I. Rasa, *Grüss*-type and Ostrowski-type inequalities in approximation theory, Ukr. Math. J, 63(6)(2011) 843–864.
[18] T.Acar, Quantitative q-Voronovskaya and q-*Grüss*-Voronovskaya-type results for q-Szász operators, Georgian Math. J, 23(4)(2016) 459–468.
[19] T. Acar, A. Aral, I.Rasa, The new foms of Voronovskaya's theorem in weighted spaces, Positivity, 20(1)(2016) 25–40.
[20] A. Erencin, I. Rasa, Voronovskaya type theorems in weighted spaces, Numer. Funct. Anal. Optim, 37(12)(2016) 1517–1528.