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Image Deblurring Using a Projective Inertial Parallel Subgradient Extragradient-Line Algorithm of Variational Inequality Problems

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Abstract. In this paper, we introduce projective inertial parallel subgradient extragradient-line algorithm for solving variational inequalites of L-Lipschitz continuous and monotone mappings which L is unknown. We prove a strong convergence result under some mild conditions in Hilbert space. We also present some numerical examples in Euclidean space $\mathbb{R}^3$ compared with Parallel-Viscosity-Type Subgradient Extragradient-Line Method. Finally, we deblur the Grey and RGB images from common types of blur matrixes Gaussian blur, Out of focus blur and Motion blur using our proposed algorithm and show the better efficeincy when the number of types of blur matrixes is large.

1. Introduction and Definitions

We consider the classical variational inequality problem (VIP) in a real Hilbert space H which is to find a point $x^* \in C$ such that

$$\langle Bx^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{1}$$

where C is a nonempty closed and convex subset of H and $B : H \to H$ is a mapping. Let us denote the solution set of VIP (1) by VI(C,B). It is well known that x^* solves the VIP (1) if and only if x^* solves the fixed point equation

$$x^* = P_C(x^* - \lambda Bx^*), \quad \lambda > 0,$$

where λ is any positive real number. Many problems in the real-world can formulated in the form of the VIP (1), such as economics, engineering mechanics, signal processing, image recovery, transportation, and others (see, for example [3, 6, 9, 13, 14]). Several numerical methods have been constructed for solving variational inequalities and related optimization problems, in [1, 2, 15, 20, 22, 25–29] and the references cited therein.

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One of the important methods for solving VIP is the extragradient method (EGM) introduced by Korpelevich [10] in 1976 for solving the saddle point problems and then extended to solve the VIPs. The extragradient method was stated as follows:

$$\begin{cases} x_1 \in C, \\ y_k = P_C(x_k - \lambda B x_k), \\ x_{k+1} = P_C(x_k - \lambda B y_k), \end{cases} \tag{2}$$

where $\lambda \in (0, \frac{1}{L})$ and P_C denotes the metric projection from H onto C. This method converges if B is L-Lipschitz continuous and monotone operator.

In 2011, Censor et al. [27] improved the extragradient method (2) by introducing the subgradient extragradient method (SEGM), in which the second projection onto C is replaced by a projection onto a specific constructible half-space. Their algorithm is of the form

$$\begin{cases} x_1 \in C, \\ y_k = P_C(x_k - \lambda B x_k), \\ T_k = \{w \in H : \langle x_k - \lambda B x_k - y_k, w - y_k\rangle \leq 0\}, \\ x_{k+1} = P_{T_k}(x_k - \lambda B y_k), \end{cases} \tag{3}$$

where $\lambda \in (0, \frac{1}{L})$. The subgradient extragradient method for solving VIP (1) has received great attention by many authors (see, e.g., [7, 18] and the references therein).

Motivated and inspired by the results of Alvarez and Attouch in [8], and of Censor et al. in [27], used the inertial technique with the SEGM. The method have been called the inertial subgradient extragradient method, that is, for any initial $x_0, x_1 \in H$, the sequence $\{x_k\}$ is generated by

$$\begin{cases} w_k = x_k + \alpha_k(x_k - x_{k-1}), \\ y_k = P_C(w_k - \lambda B w_k), \\ T_k = \{x \in H | \langle w_k - \lambda B w_k - y_k, x - y_k\rangle \leq 0\}, \\ x_{k+1} = P_{T_k}(w_k - \lambda B y_k), \end{cases} \tag{4}$$

where $\lambda > 0$, $\alpha_k \geq 0$. Under suitable conditions, they proved the weak convergence of $\{x_k\}$ to an element of VI(C,B).

Recently, Censor, Gibali and Reich [23, 24] introduced the common solutions to variational inequality problem (CSVIP), which consists of finding common solutions to unrelated variational inequality. The general form of the CSVIP is the following: Let C be a nonempty closed and convex subset of H. Let $B_i : H \to H$, $i = 1, 2, ..., N$ be mappings. The CSVIP is to find $x^* \in C$ such that

$$\langle B_i x^*, x - x^*\rangle \geq 0, \quad \forall x \in C, i = 1, 2, ..., N. \tag{5}$$

If $N = 1$, CSVIP (5) becomes VIP (1).

Very recently, using a modified viscosity-type subgradient extragradient-line method, Suantai et al. [19] introduced the parallel viscosity-type subgradient extragradient-line method (PVSEGM) for solving the VIP. The strong convergence theorem was proved when each of the operator A_i is Lipschitz continuous monotone mapping that the Lipschitz constant is unknow. They introduced the following algorithm:

$$\begin{cases} x_1 \in H, \\ y_k^i = P_C(x_k - \lambda_k^i B_i x_k), \quad \lambda_k^i = \rho^{l_k^i}, \\ (l_k^i \text{ is the smallest nonegative integer } l^i \text{ such that } \lambda_k^i \|B_i x_k - B_i y_k^i\| \leq \mu \|x_k - y_k^i\|), \\ z_k^i = P_{T_k^i}(x_k - \lambda_k^i B y_k^i), \\ T_k^i = \{z \in H : \langle x_k - \lambda_k^i B_i x_k - y_k^i, z - y_k^i\rangle \leq 0\}, \\ x_{k+1} = \alpha_k^0 f(x_k) + \sum_{i=1}^{N} \alpha_k^i z_k^i, \quad k \geq 1, \end{cases} \tag{6}$$

where $\rho, \mu \in (0,1)$, $\{\alpha_k\}_{k=1}^{\infty} \subseteq (0,1)$, $\lim_{k\to\infty} \alpha_k^0 = 0$, $\sum_{k=1}^{\infty} \alpha_k^0 = \infty$, and they proved that the sequence $\{x_k\}_{k=1}^{\infty}$ generated by (6) converges strongly to $x^* \in$ VI(C,B). The benefit of the PVSEGM was presented to solve the problem of multiblur effects in an image restoration. As a result, the resulting image quality can be improved sharper by using the PVSEGM in the resolution of common resolution VIP problems.

Motivated and inspired by the works in the literature, we study strong convergence of the algorithm for solving common solution of variational inequality problem (5). The algorithm is generated by the hybrid inertial techniques and a parallel subgradient extragradient-line method. Several numerical experiments are implemented to support the theoretical results. Our numerical results have illustrated the better convergence of the new algorithms over the PVSEGM method of Suantai et al. [19]. Finally, we present a solution to the problem of multiblur effects in an image is solved by applying our algorithm.

2. Main result

In this section we present a new algorithm for solving the CSVIP (5). Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $B_i : H \to H$ be monotone and Lipschitz continuous on H with the constant L_i but L_i is unknown for all $i = 1, 2, ..., N$. Moreover, we denote $\Psi := \bigcap_{i=1}^{N} VI(C, B_i) \neq \emptyset$. Suppose $\{x_k\}_{k=1}^{\infty}$ is generated in the following algorithm:

Algorithm 2.1. Initialization: Given $\gamma > 0$, $\mu \in (0, 1)$. Let x_0, $x_1 \in H$ be arbitrary.

Iterative Steps: Calculate x_{k+1} as follows:
Step 1. Set $w_k = x_k + \alpha_k(x_k - x_{k-1})$ and compute

$$y_k^i = P_C(w_k - \tau_k^i B_i w_k), \quad \forall k \geq 1,$$

where $\tau_k^i = \gamma^{\ell_k^i}$ and ℓ_k^i is the smallest nonnegative integer such that

$$\tau_k^i \parallel B_i w_k - B_i y_k^i \parallel \leq \mu \parallel w_k - y_k^i \parallel . \tag{7}$$

Step 2. Compute

$$z_k^i = P_{T_k^i}(w_k - \tau_k^i B_i y_k^i),$$

where $T_k^i := \{x \in H | \langle w_k - \tau_k^i B_i w_k - y_k^i, x - y_k^i \rangle \leq 0\}$.
Step 3. Compute $\bar{z}_k$, i.e.

$$\bar{z}_k = argmax\{\|z_k^i - x_k\| : i = 1, 2, ..., N\}. \tag{8}$$

Step 4. Compute

$$x_{k+1} = P_{C_{k+1}} x_1,$$

where $C_{k+1} := \{z \in C_k : \|\bar{z}_k - z\| \leq \|w_k - z\|\}$.
Again set $k := k + 1$ and go to **Step 1.**

Theorem 2.2. *Assume that condition* $\sum_{k=1}^{\infty} \alpha_k \parallel x_k - x_{k-1} \parallel < \infty$ *holds. Then the sequences* $\{x_k\}$ *generated by Algorithm 2.1 converge strongly to* $z \in P_\Psi x_1$.

Proof **Claim 1.** We prove that the sequence $\{x_k\}$ is well defined and $\lim_{k\to\infty}\|x_k - x_1\|$ exists. Since $C = C_1$, C_1 is closed and convex. Assume that C_k is closed and convex. By Lemma 1.3 in [4] we have C_{k+1} is closed and convex. Let $x^* \in \Psi$ and $h_k^i = w_k - \tau_k^i B_i y_k^i$, $\forall k \geq 1$, $i = 1, 2, ..., N$, we have

$$\|z_k^i - x^*\|^2 = \|P_{T_k^i}(h_k^i) - h_k^i\|^2 + 2\langle P_{T_k^i}(h_k^i) - h_k^i, h_k^i - x^*\rangle + \|h_k^i - x^2\|. \tag{9}$$

Since $x^* \in \Psi \subseteq C \subseteq T_k^i$ and by the characterization of the metric projection $P_{T_k^i}$, we obtain

$$2\|h_k^i - P_{T_k^i}(h_k^i)\|^2 + 2\langle P_{T_k^i}(h_k^i) - h_k^i, h_k^i - x^*\rangle = 2\langle h_k^i - P_{T_k^i}(h_k^i), x^* - P_{T_k^i}(h_k^i)\rangle \leq 0. \tag{10}$$

This implies that

$$\|h_k^i - P_{T_k^i}(h_k^i)\|^2 + 2\langle P_{T_k^i}(h_k^i) - h_k^i, h_k^i - x^*\rangle \leq -\|h_k^i - P_{T_k^i}(h_k^i)\|^2. \tag{11}$$

By the inequalities (9), (10) and the definition of Algorithm 2.1, we obtain

$$\begin{aligned}\|z_k^i - x^*\|^2 &\leq \|h_k^i - x^*\|^2 - \|h_k^i - z_k^i\|^2\\ &= \|(w_k - x^*) - \tau_k^i B_i y_k^i\|^2 - \|(w_k - z_k^i) - \tau_k^i B_i y_k^i\|^2\\ &= \|w_k - x^*\|^2 - \|w_k - z_k^i\|^2 + 2\tau_k^i\langle x^* - z_k^i, B_i y_k^i\rangle. \end{aligned}\tag{12}$$

By the monotonicity of the operator B_i, we have

$$\begin{aligned}0 &\leq \langle B_i y_k^i - B_i x^*, y_k^i - x^*\rangle\\ &= \langle B_i y_k^i, y_k^i - x^*\rangle - \langle B_i x^*, y_k^i - x^*\rangle\\ &\leq \langle B_i y_k^i, y_k^i - x^*\rangle\\ &= \langle B_i y_k^i, y_k^i - z_k^i\rangle + \langle B_i y_k^i, z_k^i - x^*\rangle.\end{aligned}$$

Thus

$$\langle x^* - z_k^i, B_i y_k^i\rangle \leq \langle B_i y_k^i, y_k^i - z_k^i\rangle. \tag{13}$$

Using (13) in (12), we obtain

$$\begin{aligned}\|z_k^i - x^*\|^2 &\leq \|w_k - x^*\|^2 - \|w_k - z_k^i\|^2 + 2\tau_k^i\langle B_i y_k^i, y_k^i - z_k^i\rangle\\ &= \|w_k - x^*\|^2 - \|w_k - y_k^i\|^2 - \|y_k^i - z_k^i\|^2 + 2\langle w_k - \tau_k^i B_i y_k^i - y_k^i, z_k^i - y_k^i\rangle.\end{aligned}\tag{14}$$

Observe that

$$\begin{aligned}\langle w_k - \tau_k^i B_i y_k^i - y_k^i, z_k^i - y_k^i\rangle &= \langle w_k - \tau_k^i B_i w_k - y_k^i, z_k^i - y_k^i\rangle + \langle \tau_k^i B_i w_k - \tau_k^i B_i y_k^i, z_k^i - y_k^i\rangle\\ &\leq \langle \tau_k^i B_i w_k - \tau_k^i B_i y_k^i, z_k^i - y_k^i\rangle.\end{aligned}\tag{15}$$

Using the inequality (15) in (14) and the existence of the step size τ_k^i of Lemma 3.1 in [30] , we have

$$\begin{aligned}\|z_k^i - x^*\|^2 &\leq \|w_k - x^*\|^2 - \|w_k - y_k^i\|^2 - \|y_k^i - z_k^i\|^2 + 2\langle \tau_k^i B_i w_k - \tau_k^i B_i y_k^i, z_k^i - y_k^i\rangle\\ &\leq \|w_k - x^*\|^2 - \|w_k - y_k^i\|^2 - \|y_k^i - z_k^i\|^2 + 2\tau_k^i\|B_i w_k - B_i y_k^i\|\|z_k^i - y_k^i\|\\ &\leq \|w_k - x^*\|^2 - \|w_k - y_k^i\|^2 - \|y_k^i - z_k^i\|^2 + 2\mu\|w_k - y_k^i\|\|z_k^i - y_k^i\|\\ &\leq \|w_k - x^*\|^2 - \|w_k - y_k^i\|^2 - \|y_k^i - z_k^i\|^2 + \mu(\|w_k - y_k^i\|^2 + \|z_k^i - y_k^i\|^2)\\ &= \|w_k - x^*\|^2 - (1-\mu)(\|w_k - y_k^i\|^2 + \|y_k^i - z_k^i\|^2)\\ &\leq \|w_k - x^*\|^2.\end{aligned}\tag{16}$$

This implies that $\|\bar{z}_k - x^*\| \leq \|w_k - x^*\|$, so $x^* \in C_k$, $\forall k \in N$. This shows that $\{x_k\}$ is well-defined.
From $x_k = P_{C_k}x_1$ and $x_{k+1} \in C_k$, for all $k \geq 1$, we get

$$\|x_k - x_1\| \leq \|x_{k+1} - x_1\|, \quad \forall k \geq 1. \tag{17}$$

On the other hand, as $\Psi \subset C_k$, we obtain

$$\|x_k - x_1\| \leq \|x^* - x_1\|, \quad \forall k \geq 1. \tag{18}$$

From (17) and (18) that the sequence $\{x_k\}$ is bounded and nondecreasing. Therefore $\lim_{k\to\infty} \|x_k - x_1\|$ exists.

Claim 2. Show that $x_k \to v \in C$ as $k \to \infty$. For $m > r$, by the definition of C_r, since $x_m = P_{C_m}x_1 \in C_m \subset C_r$, so by the property of the metric projection P_{C_r} [11], we have

$$\|x_m - x_r\|^2 \leq \|x_m - x_1\|^2 - \|x_r - x_1\|^2.$$

Since $\lim_{r\to\infty} \|x_r - x_1\|$ exists, we have $\|x_m - x_r\| \to 0$, as $m, r \to \infty$ this means that $\{x_k\}$ is a Cauchy sequence. Hence, there exists $v \in C$ such that $x_k \to v$ as $k \to \infty$. Therefore,

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0. \tag{19}$$

Claim 3. Show that $\lim_{k\to\infty} \|z_k^i - x_k\| = \lim_{k\to\infty} \|x_k - y_k^i\| = \lim_{k\to\infty} \|y_k^i - z_k^i\| = 0, \forall i = 1, 2, ..., N.$ From the definition of C_k and $x_{k+1} \in C_{k+1} \subset C_k$, we have

$$\begin{aligned} \|\bar{z}_k - x_{k+1}\| &\leq \|x_{k+1} - w_k\| \\ &\leq \|x_{k+1} - x_k\| + \|x_k - w_k\| \\ &= \|x_{k+1} - x_k\| + \alpha_k\|x_k - x_{k-1}\|. \end{aligned}$$

From (19) and condition in Theorem 2.2, we obtain

$$\lim_{k\to\infty} \|\bar{z}_k - x_{k+1}\| = 0.$$

This the triangle inequality $\|\bar{z}_k - x_k\| \leq \|\bar{z}_k - x_{k+1}\| + \|x_{k+1} - x_k\|$ implies that

$$\lim_{k\to\infty} \|\bar{z}_k - x_k\| = 0. \tag{20}$$

From the definition of z_k^i and (20), we get

$$\lim_{k\to\infty} \|z_k^i - x_k\| = 0, \quad \forall i = 1, 2, ..., N. \tag{21}$$

From (16) and $w_k = x_k + \alpha_k(x_k - x_{k-1})$, we have

$$\begin{aligned} \|z_k^i - x^*\|^2 &\leq \|x_k + \alpha_k(x_k - x_{k-1}) - x^*\|^2 - (1-\mu)(\|x_k + \alpha_k(x_k - x_{k-1}) - y_k^i\|^2 + \|y_k^i - z_k^i\|^2) \\ &= \|(x_k - x^*) + \alpha_k(x_k - x_{k-1})\|^2 - (1-\mu)(\|(x_k - y_k^i) + \alpha_k(x_k - x_{k-1})\|^2 + \|y_k^i - z_k^i\|^2) \\ &\leq \|x_k - x^*\|^2 + 2\alpha_k\langle x_k - x_{k-1}, w_k - x^*\rangle \\ &\quad -(1-\mu)(\|x_k - y_k^i\|^2 + 2\alpha_k\langle x_k - x_{k-1}, w_k - y_k^i\rangle + \|y_k^i - z_k^i\|^2) \\ &= \|x_k - x^*\|^2 - (1-\mu)(\|x_k - y_k^i\|^2 + \|y_k^i - z_k^i\|^2) + 2\alpha_k\langle x_k - x_{k-1}, w_k - x^*\rangle \\ &\quad -2\alpha_k(1-\mu)\langle x_k - x_{k-1}, w_k - y_k^i\rangle. \end{aligned} \tag{22}$$

From (22), for each $c \in \Psi$, we obtain

$$\begin{aligned} (1-\mu)\|x_k - y_k^i\|^2 &\leq \|x_k - c\|^2 - \|z_k^i - c\|^2 + 2\alpha_k\langle x_k - x_{k-1}, w_k - c\rangle \\ &\quad -2\alpha_k(1-\mu)\langle x_k - x_{k-1}, w_k - y_k^i\rangle. \end{aligned} \tag{23}$$

From (21), (23) and the boundedness of $\{w_k\}$, $\{x_k\}$, $\{y_k^i\}$, $\{z_k^i\}$ and condition in Theorem 2.2, we have

$$\lim_{k\to\infty} \|x_k - y_k^i\| = 0, \quad \forall i = 1, 2, ..., N. \tag{24}$$

Using the equation (21) and (24), we obtain

$$\lim_{k\to\infty} \|y_k^i - z_k^i\| = 0, \quad \forall i = 1, 2, ..., N. \tag{25}$$

Claim 4. Show that $v \in \Psi$ and $v = P_\Psi x_1$. Now, $x_k - y_k^i \to 0$ implies that $y_k^i \to v$ and since $y_k^i \in C$, we then obtain $v \in C$. For all $x \in C$ and using the property of the projection P_C, we have (Since B_i is monotone)

$$\begin{aligned} 0 &\leq \langle y_k^i - w_k + \tau_k^i B_i w_k, x - y_k^i \rangle \\ &= \langle y_k^i - w_k, x - y_k^i \rangle + \langle \tau_k^i B_i w_k, x - x_k \rangle + \langle \tau_k^i B_i w_k, x_k^i - y_k^i \rangle \\ &\leq \langle y_k^i - x_k, x - x_k^i \rangle + \tau_k^i \langle B_i x, x - x_k \rangle + \tau_k^i \langle B_i x_k, x_k^i - y_k^i \rangle \\ &\quad + \langle \alpha_k (x_k - x_{k-1}), x - y_k^i \rangle + \tau_k^i \langle B_i \alpha_k (x_k - x_{k-1}), x - x_k \rangle \\ &\quad + \tau_k^i \langle B_i \alpha_k (x_k - x_{k-1}), x_k - y_k^i \rangle. \end{aligned} \tag{26}$$

Taking the limit as $k \to \infty$ in(26), we obtain (recall that $\inf_{k\geq 1} \tau_{k_r} > 0$ by Remark 3.2 in [21])

$$0 \leq \langle B_i v, x - v \rangle, \quad \forall x \in C.$$

This implies that $v \in VI(C, B_i)$ for all $i = 1, 2, ..., N$. It follows from (18) that, for $x^* \in \Psi$ $\|v - x_1\| \leq \|x^* - x_1\|$. This implies that $v = P_\Psi x_1$. The proof is completed.

We give the following numerical example to illustrate Theorem 2.2.

Example 2.3. Let $B_1, B_2, B_3 : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $B_1 x = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} x$, $B_2 x = \begin{pmatrix} 6 & -3 & 3 \\ -3 & 6 & -3 \\ 3 & -3 & 6 \end{pmatrix} x$ and $B_3 x = \begin{pmatrix} 10 & -5 & 5 \\ -5 & 10 & -5 \\ 5 & -5 & 10 \end{pmatrix} x$ for all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Let $C = \{x \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 \leq 4\}$. The stopping criterion is defined by $\|x_k - x_{k-1}\| < 10^{-5}$. We choose parameters

$$\alpha_k = \begin{cases} 0.2 & \text{if } x_k \neq x_{k-1} \text{ and } k \leq 1000 \\ \frac{1}{k^2 \|x_k - x_{k-1}\|} & \text{if } x_k \neq x_{k-1} \text{ and } k > 1000 \\ 0 & \text{Otherwise,} \end{cases}$$

$\gamma = 0.45$ and $\mu = 0.35$ for our algorithm, and choose $\alpha_k^0 = 1 - \frac{3k}{3k+1}$, $\alpha_k^1 = \frac{k}{3k+1}$, $\alpha_k^2 = \frac{k}{3k+1}$, $\alpha_k^3 = 1 - \alpha_k^0 - \alpha_k^1 - \alpha_k^2$ $\rho = 0.2$ and $\mu = 0.1$ for PVSEGM.

Table 1: Comparison of the number of iterations in Theorem 2.2 and Theorem 1 [19] of Example 2.3 by choosing $x_0 = (-2.15, -4.35, 1.12)$ and $x_1 = (6.13, -5.24, -1.19)$.

Inputting	Our Algorithm ($\alpha \neq 0$)		PVSEGM	
	CPU Time	Iter.No.	CPU Time	Iter.No.
B_1	0.0000122	47	0.000012	263
B_2	0.0000125	61	0.0000159	310
B_3	0.0000242	71	0.0000234	263
B_1, B_2	0.0000228	47	0.0000291	275
B_1, B_3	0.0000441	47	0.0000195	263
B_2, B_3	0.0000448	56	0.0000213	258
B_1, B_2, B_3	0.0000502	44	0.0000228	258

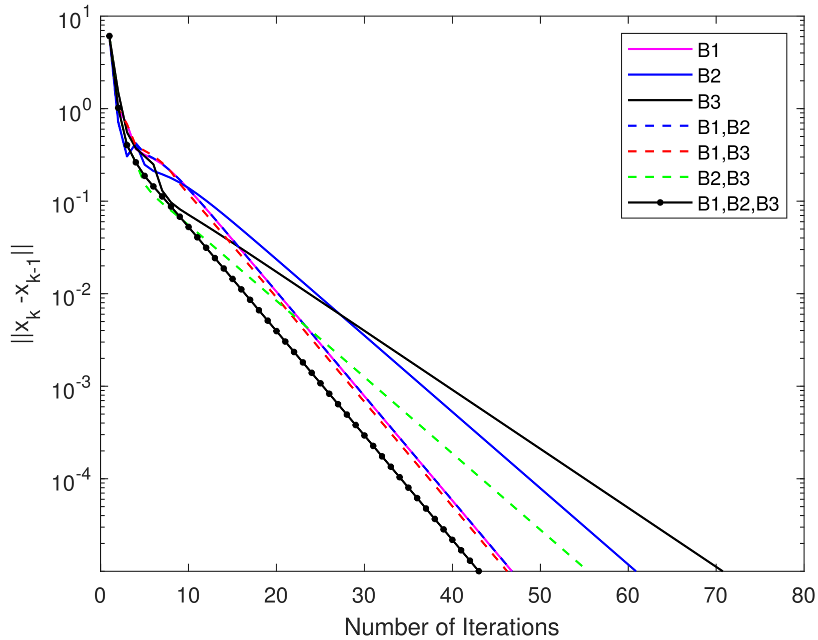


Figure 1: The error plotting $\|x_k - x_{k-1}\|$ for Table 1 in Example 2.3.

From Table 1 and Figure 1, we see that the common solution of two or more inputting B_i gives the number of iterations smaller than inputting one and the comparison between our Algorithm and PVSEGM and the inertial term ($\alpha_k \neq 0$) can speed up the convergence of the algorithm. We see that our Algorithm get the good CPU Time and number of iterations more than PVSEGM.

3. Application to image restoration problems

Image restoration is the process of recovering a degraded image that is blurred and noisy image. The image restoration problem can be formulated in the linear equation system as follows:

$$b = Ax + \omega, \tag{27}$$

where $x \in \mathbb{R}^{n\times 1}$ is an original image, $b \in \mathbb{R}^{m\times 1}$ is the unknown image, ω is additive noise and $A \in \mathbb{R}^{m\times n}$ is the blurring matrix. For solving the problem of image recovery (27) is an approximation of the original image x. In some case, finding $x = A^{-1}(b - \omega)$ maybe a hard task, thus finding the solution x by mean of convex minimization can overbear such hard, which is called a least squares (LS) problem as follows:

$$\min_x \frac{1}{2}\|b - Ax\|_2^2, \tag{28}$$

where$\|.\|$ is l_2-norm defined by $\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$. The solution of (28) can be approximated by many well known iteration methods [5, 12, 16, 17].

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|A_1x - b_1\|_2^2, \min_{x\in\mathbb{R}^n} \frac{1}{2}\|A_2x - b_2\|_2^2, ..., \min_{x\in\mathbb{R}^n} \frac{1}{2}\|A_Nx - b_N\|_2^2, \tag{29}$$

where x is the original true image, A_i is the blurred matrix, b_i is the blurred image by the blurred matrix A_i for all $i = 1, 2, ..., N$. For Algorithm 2.1 can apply to solve the problem (29), we know that $A_i^T(A_ix - b_i)$ is Lipschitz continuous for each $i = 1, 2, ..., N$. This algorithm is generated as follows:

Algorithm 3.1. Initialization: Given $\gamma > 0$, $\mu \in (0,1)$. Let $x_0, x_1 \in H$ be arbitrary.

Iterative Steps: Calculate x_{k+1} as follows:
Step 1. Set $w_k = x_k + \alpha_k(x_k - x_{k-1})$ and compute

$$y_k^i = P_C(w_k - \tau_k^i A_i^T(A_i w_k - b_i)), \quad \forall k \geq 1,$$

where $\tau_k^i = \gamma^{\ell_k^i}$ and ℓ_k^i is the smallest nonnegative integer such that

$$\tau_k^i \parallel A_i^T(A_i w_k - b_i) - A_i^T(A_i y_k^i - b_i) \parallel \leq \mu \parallel w_k - y_k^i \parallel . \tag{30}$$

Step 2. Compute

$$z_k^i = P_{T_k^i}(w_k - \tau_k^i A_i^T(A_i y_k^i - b_i)),$$

where $T_k^i := \{x \in H | \langle w_k - \tau_k^i A_i w_k - y_k^i, x - y_k^i \rangle \leq 0\}$.
Step 3. Compute $\bar{z}_k$, i.e.

$$\bar{z}_k = argmax\{\|z_k^i - x_k\| : i = 1, 2, ..., N\}. \tag{31}$$

Step 4. Compute

$$x_{k+1} = P_{C_{k+1}} x_1,$$

where $C_{k+1} := \{z \in C_k : \|\bar{z}_k - z\| \leq \|w_k - z\|\}$.
Again set $k := k + 1$ and go to **Step 1.**

We will present the advantages of our Algorithm 3.1 in images corrupted by the following three blur types:
(I) Gaussian blur of filter size 9×9 with standard deviation $\sigma = 4$ (blur matrix A_1).
(II) Out of focus blur (Disk) with radius $r = 6$ (blur matrix A_2).
(III) Motion blur specifying with motion length of 21 pixels (len = 21) and motion orientation 11° ($\theta = 11$) (blur matrix A_3).
We will show the original RGB and grey images in the following figure 2-3.

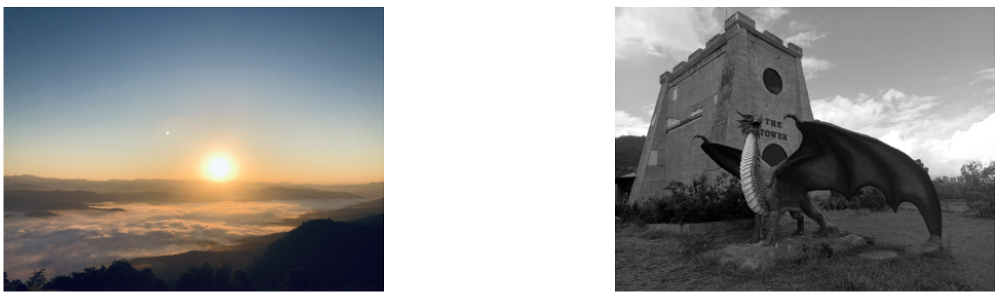


Figure 2-3: The matrix size of RGB and grey images are $277 \times 370 \times 3$ and 277×370, respectively.

Three different types of blurred RGB and grey images degraded by the blurring matrices A_1, A_2 and A_3 are shown in figures 4-9.

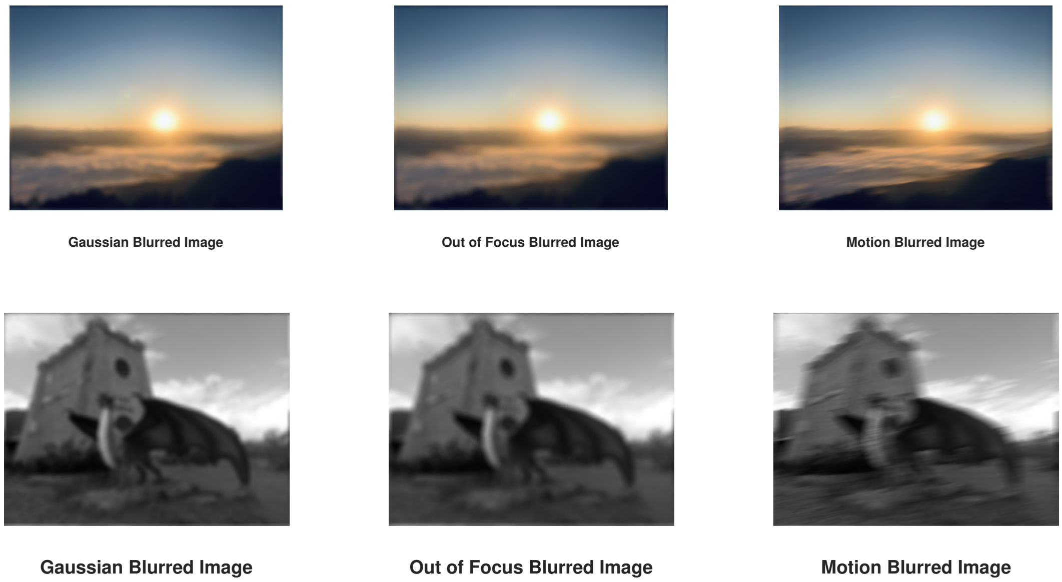


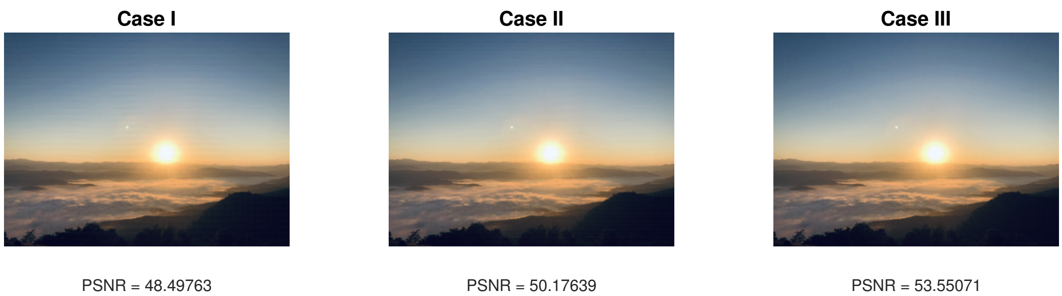
Figure 4-9: The degraded RGB and grey images by blurred matrices A_1, A_2 and A_3, respectively.

We apply the PVSEGM and our Algorithm 3.1 to obtain the solution of the deblurring problem (VIP) with the three blurring matrices A_1, A_2 and A_3. The results of the PVSEGM and our Algorithm 3.1 are considered in following seven cases:
Case I: Inputting A_1 on the PVSEGM and Algorithm 3.1,
Case II: Inputting A_2 on the PVSEGM and Algorithm 3.1,
Case III: Inputting A_3 on the PVSEGM and Algorithm 3.1,
Case IV: Inputting A_1 and A_2 on the PVSEGM and Algorithm 3.1,
Case V: Inputting A_1 and A_3 on the PVSEGM and Algorithm 3.1,
Case VI: Inputting A_2 and A_3 on the PVSEGM and Algorithm 3.1,
Case VII: Inputting A_1, A_2 and A_3 on the PVSEGM and Algorithm 3.1.
The following parameters are used for our algorithm:

$$\alpha_k = \begin{cases} 0.2 & \text{if } x_k \neq x_{k-1} \text{ and } k \leq 10,000 \\ \frac{1}{k^2\|x_k - x_{k-1}\|} & \text{if } x_k \neq x_{k-1} \text{ and } k > 10,000 \\ 0 & \text{Otherwise,} \end{cases}$$

$\gamma = 0.2$ and $\mu = 0.3$. We choose $\mu = 0.95$, $\rho = 0.5$, $\alpha_k^0 = 1 - \frac{3k}{3k+1}$, $\alpha_k^1 = \frac{k}{3k+1}$, $\alpha_k^2 = \frac{k}{3k+1}$ and $\alpha_k^3 = 1 - \alpha_k^0 - \alpha_k^1 - \alpha_k^2$ for PVSEGM.

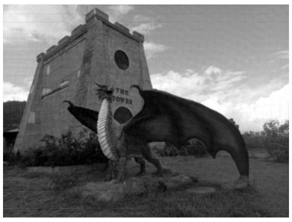
Figures 10-15 show the reconstructed RGB and grey images with 10000 iterations. It comprises RGB and gray image quality restored, and PSNR.



Case I

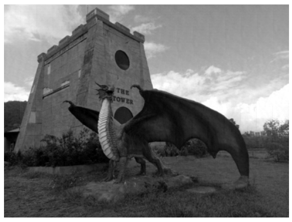
PSNR = 35.95345

Case II



PSNR = 37.92625

Case III



PSNR = 42.21131

Figure 10-15: The reconstructed RGB and grey images with their PSNR for Case I - Case III using the proposed our Algorithm 3.1 presented on 10000^{th} iterations, respectively.

Later in figures 16-21, we can see that the image quality restored using our algorithms for solving common problem resolution (VIP) problems with (N = 2) has been improved when compare with the previous results in Figures 10-15.

Case IV

PSNR = 57.88788

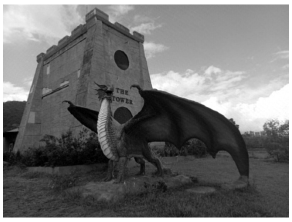
Case V

PSNR = 63.35813

Case VI

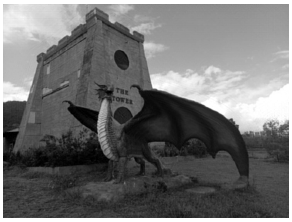
PSNR = 57.72298

Case V



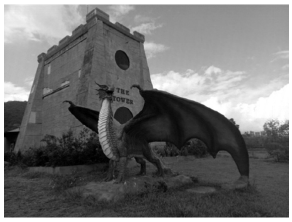
PSNR = 46.08089

Case IV



PSNR = 42.12156

Case VI



PSNR = 47.87718

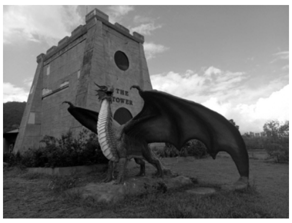
Figure 16-21: The reconstructed RGB and grey images with their PSNR for Case IV - Case VI using the proposed our Algorithm 3.1 presented on 10000^{th} iterations, respectively.

Finally, the common solution of the deblurring problem (VIP) with (N= 3) using the proposed algorithm was also tested (Inputting A_1, A_2 and A_3 in the proposed algorithm).

Case VII

PSNR = 72.89197

Case VII



PSNR = 50.73579

Figure 22-23: The reconstructed RGB and grey images with their PSNR for Case VII using the proposed our Algorithm 3.1 presented on 10000^{th} iterations, respectively.

Figure 22 and 23 show the reconstructed RGB and grey images with thousand iteration. The quality of the recovered RGB and grey images obtained by this algorithm were the highest compared to the previous two algorithms.

Figures 24-37 show the reconstructed RGB and grey images using the proposed algorithm in obtaining the common solution of the following problem with the same PSNR.

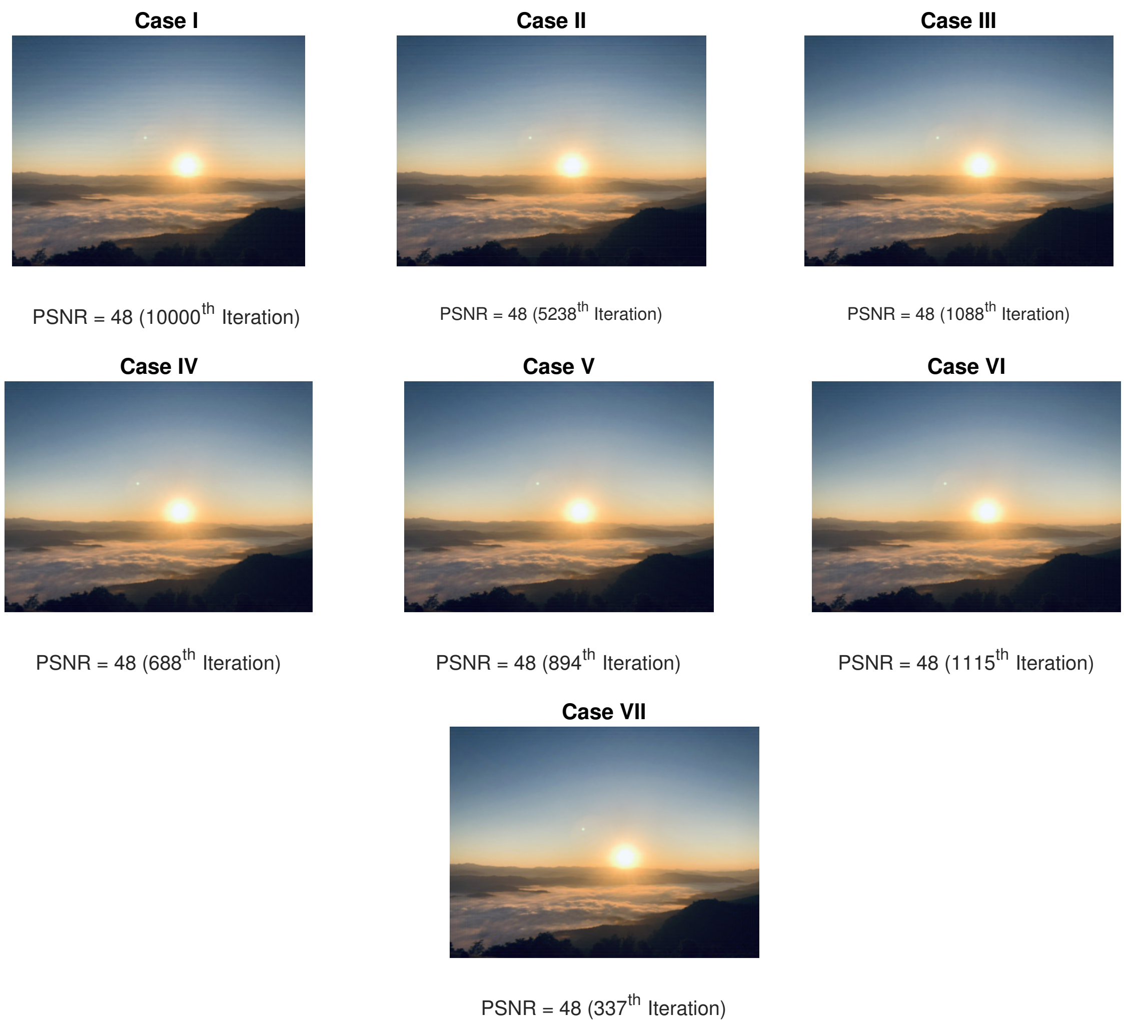


Figure 24-30: The reconstructed RGB images of all cases being used our Algorithm 3.1 with PSNR = 48.

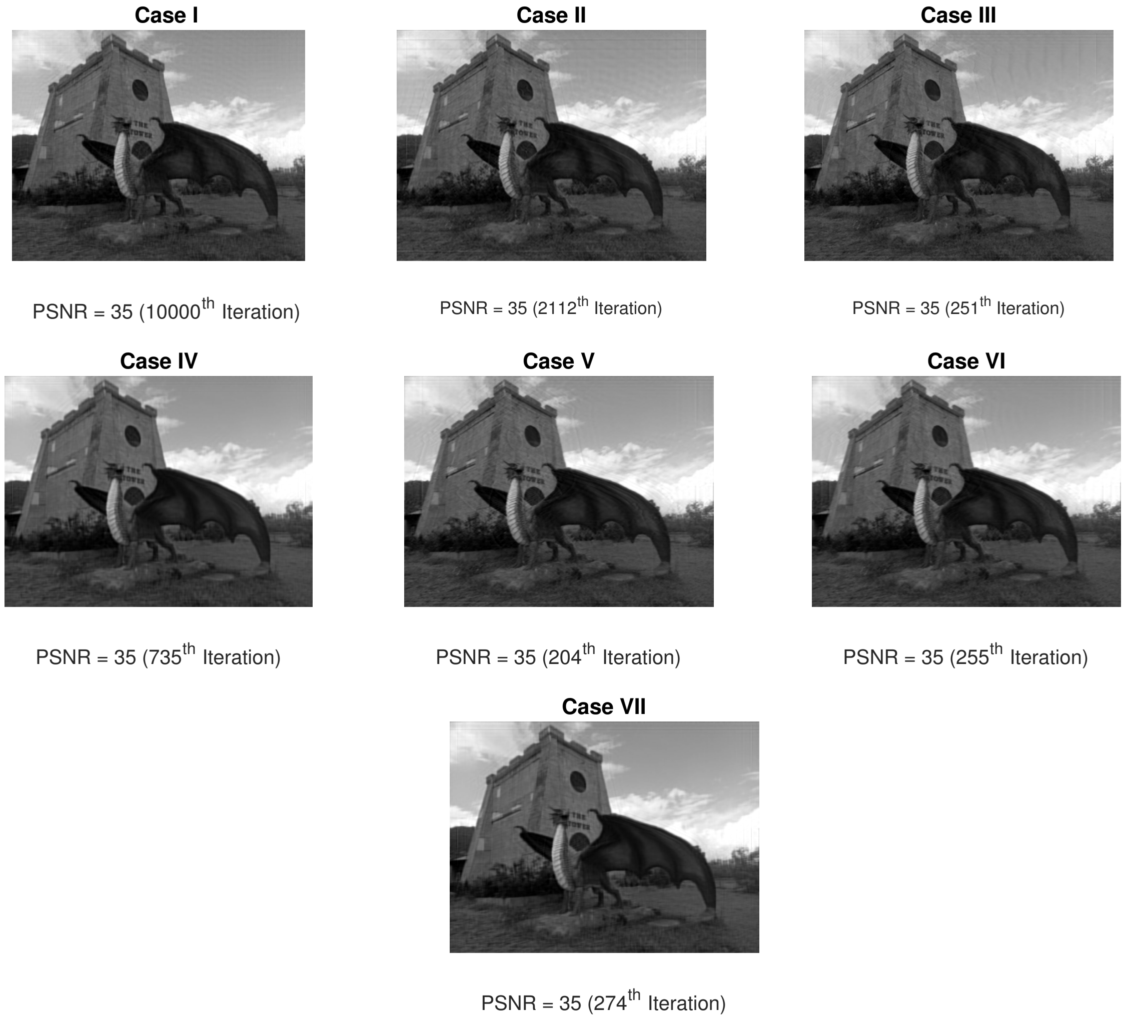


Figure 31-37: The reconstructed grey images of all cases being used our Algorithm 3.1 with PSNR = 35.

Table 2: Comparison of the number of iterations in RGB images.

Inputting	PSNR of 10000^{th}		Number of Iterations 43 PSNR	
	Our Algorithm	PVSEGM	Our Algorithm	PVSEGM
A_1	48.49763	45.10671	455^{th}	2743^{th}
A_2	50.17639	43.30487	1419^{th}	8691^{th}
A_3	53.55071	47.71041	147^{th}	821^{th}
A_1, A_2	57.88788	50.47587	203^{th}	873^{th}
A_1, A_3	63.35813	55.76737	138^{th}	375^{th}
A_2, A_3	57.72298	51.32098	146^{th}	585^{th}
A_1, A_2, A_3	72.89197	56.62458	136^{th}	400^{th}

From Table 2 we will compared our algorithm with PVSEGM, when PSNR of 10000^{th} and number of iterations 43 PSNR of RGB images. Moreover, the Cauchy error, the Image error and the peak signal-to-noise ratio (PSNR) for recovering processes of the degraded RGB images by using the proposed method within the first 10000^{th} iterations are shown in figures 38-40.

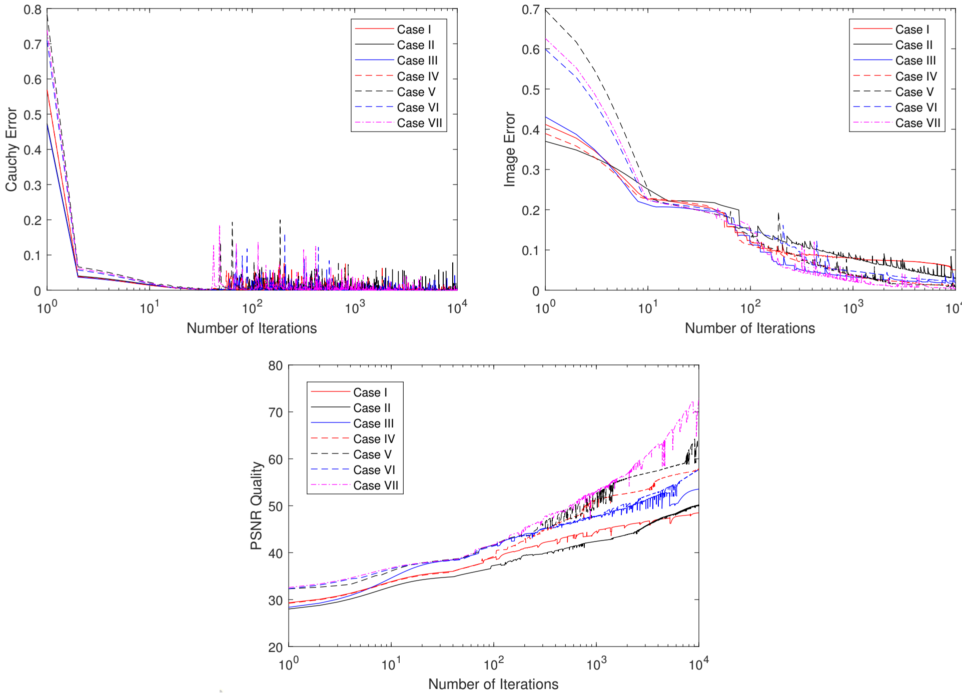
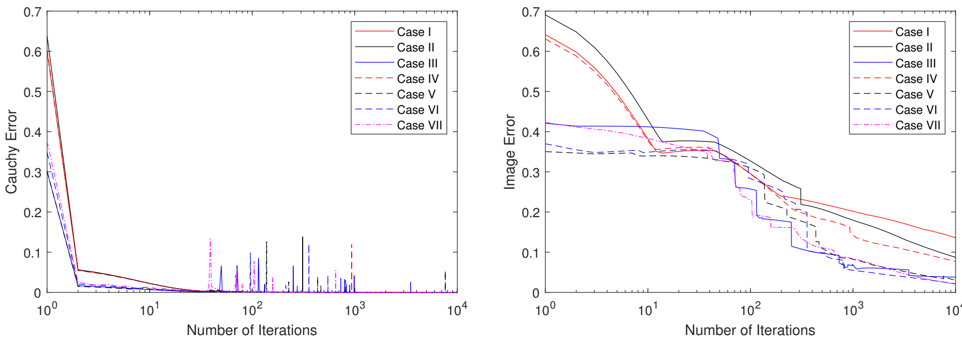


Figure 38-40: Cauchy error, Figure error and PSNR quality plots of the proposed iteration in all cases of RGB images.

Table 3: Comparison of the number of iterations in grey images.

Inputting	PSNR of 10000^{th}		Number of Iterations 34 PSNR	
	Our Algorithm	PVSEGM	Our Algorithm	PVSEGM
A_1	35.95345	34.42142	2184^{th}	7088^{th}
A_2	37.92625	35.31389	1110^{th}	5215^{th}
A_3	42.21131	37.58995	132^{th}	1901^{th}
A_1, A_2	42.12156	39.05422	461^{th}	1108^{th}
A_1, A_3	46.08089	42.89091	138^{th}	542^{th}
A_2, A_3	47.87718	42.95463	212^{th}	699^{th}
A_1, A_2, A_3	50.73579	45.43944	158^{th}	482^{th}

From Table 3 we will compared our algorithm with PVSEGM, when PSNR of 10000^{th} and number of iterations 34 PSNR of grey images. Moreover, the Cauchy error, the Image error and the peak signal-to-noise ratio (PSNR) for recovering processes of the degraded grey images by using the proposed method within the first 10000^{th} iterations are shown in figures 41-43.



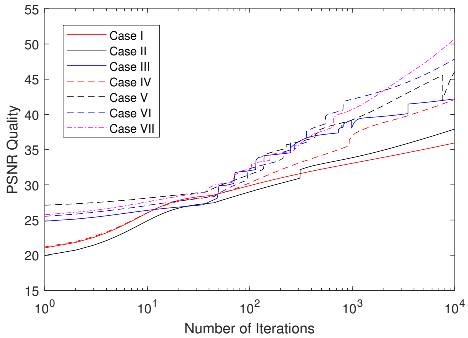


Figure 41-43: Cauchy error, Figure error and PSNR quality plots of the proposed iteration in all cases of grey images.

4. Conclusions

In this paper, we propose projective inertial parallel subgradient extragradient-line algorithm for solving variational inequalites. Under some suitable conditions imposed on parameters, we have proved the strong convergence of the algorithm. A numerical example illustrating the proposed algorithm performance in comparison with PVSEGM, see Table 1 and Figure 1. Our algorithm can solve image recovery under unknown situation of blur matrix type, to demonstrate the computational performance see in Figures 10-23 and Figures 24-37. Finally, we apply our proposed algorithm to recover RGB image, when PSNR of 10000^{th} and number of iterations 43 PSNR and grey image, when PSNR of 10000^{th} and number of iterations 34 PSNR compared to PVSEGM, see in Figures 38-43. Our algorithm is more efficient than PVSEGM see in Table 2 and 3.

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