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Intrinsic Strong Shape of Global Attractors

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Abstract. We present a proof of the strong shape theorem for global attractors in compact metric spaces using the intrinsic approach to strong shape from [18] which combines continuity over a covering and the corresponding homotopies of second order.

1. Introduction

Although shape and strong shape classification of metric compacta is the same i.e. two compacta have the same strong shape if and only if they have the same shape , for two compacta X and Y, in general, there are more strong shape morphisms than shape morphisms.

Example 1.1. Let S be a solenoid of Vietoris-van Dantzig. Any two maps $pt \to S$ define the same shape morphism, while there are uncountable different strong shape morphisms defined by the maps $pt \to S$.

One of the well known unsolved problems from open problems [11] is: Is every shape equivalence a strong shape equivalence?

The following main theorem of the paper, can be considered as a partial answer, since it is an improvement of the already known result: If M is a global attractor of a semi-dynamical system defined on a compact metric space X, then the inclusion $i : M \to X$ induces a shape equivalence.

Theorem 1.2. *Suppose $\Phi : X \times \mathbb{R}^+ \to X$ is a semi-dynamical system on a compact metric space X with a global attractor M. Then the inclusion $i : M \to X$ induces a strong shape equivalence.*

We shall present a proof of this theorem which is the main result of the paper using the intrinsic approach to strong shape from [18]. Our conclusion is that the intrinsic strong shape theory from [18] is a convenient framework to study the global properties which the attractor inherits from the phase space.

2. Introduction to intrinsic shape for compact metric spaces

The main references for theory of shape [3] and [9], use external spaces (polyhedra, ANR's) to describe the shape of a space with complicated local properties. The first intrinsic approach to shape theory (an approach without external spaces) for compact metric spaces is based on the concept of continuity over

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coverings (ϵ-continuity) and is given in [4] and [16]. Both papers use nets of functions to describe a shape morphism from X to Y. In [18] the author successfully defines the shape of metric compacta, using sequences of functions instead. This enables to define strong shape of metric compacta using only homotopies of second order. This is a significant simplification, since in the case of nets of functions one must consider homotopies of all orders ([1, 10]).

Let us start with some of the basic definitions from [18]:

Let X and Y be compact metric spaces. For collections $\mathcal{U}$ and $\mathcal{V}$ of subsets of X, we denote $\mathcal{U}$ refines $\mathcal{V}$ by $\mathcal{U} \prec \mathcal{V}$, i.e. each $U \in \mathcal{U}$ is contained in some $V \in \mathcal{V}$. By covering we understand a covering consisting of open sets.

Definition 2.1. Suppose $\mathcal{V}$ is a finite covering of Y . A function $f : X \to Y$ is $\mathcal{V}$-continuous at a point $x \in X$ if there exists a neighborhood U_x of x and $V \in \mathcal{V}$ such that:

$$f(U_x) \subseteq V.$$

A function $f : X \to Y$ is $\mathcal{V}$-continuous if it is $\mathcal{V}$-continuous at every point $x \in X$. In this case, the family of all U_x form a covering of X.

According to this, $f : X \to Y$ is $\mathcal{V}$-continuous if there exists a finite covering $\mathcal{U}$ of X such that for any $x \in X$ there exists a neighborhood U of x and $V \in \mathcal{V}$ such that $f(U) \subseteq V$. We denote shortly: there exists $\mathcal{U}$ such that $f(\mathcal{U}) \prec \mathcal{V}$.

If $f : X \to Y$ is $\mathcal{V}$-continuous, then $f : X \to Y$ is $\mathcal{W}$-continuous for any $\mathcal{W}$ such that $\mathcal{V} \prec \mathcal{W}$.

Let $\mathcal{V}$ be a finite covering of Y and $V \in \mathcal{V}$. The open set st(V) (star of V) is the union of all $W \in \mathcal{V}$ such that $W \cap V \neq \emptyset$. We form a new covering of Y, $\mathrm{st}(\mathcal{V}) = \{\mathrm{st}(V) | V \in \mathcal{V}\}$.

Definition 2.2. The functions $f, g : X \to Y$ are $\mathcal{V}$-homotopic if there exists a function $F : X \times I \to Y$ such that:

i) $F : X \times I \to Y$ is st($\mathcal{V}$)-continuous,

ii) $F : X \times I \to Y$ is $\mathcal{V}$-continuous at all points of $X \times \partial I$,

iii) $F(x, 0) = f(x)$, $F(x, 1) = g(x)$.

The relation of $\mathcal{V}$-homotopy is an equivalence relation and is denoted by $f \overset{\mathcal{V}}{\simeq} g$.

In order to define intrinsic shape for compact metric spaces, we introduce the key notion of cofinal sequence of finite coverings.

In a compact metric space there exists a sequence of finite coverings, $\mathcal{V}_1 \succ \mathcal{V}_2 \succ ...$ with the property that for any covering $\mathcal{V}$, there exists $n \in \mathbb{N}$, such that $\mathcal{V} \succ \mathcal{V}_n$. We call such a sequence-cofinal sequence of finite coverings.

This fact allows to work with proximate sequences instead of with proximate nets.

Definition 2.3. The sequence (f_n) of functions $f_n : X \to Y$ is a proximate sequence from X to Y if there exists a cofinal sequence of finite coverings of Y, $\mathcal{V}_1 \succ \mathcal{V}_2 \succ ...$ and for all indices , f_n and f_{n+1} are $\mathcal{V}_n$-homotopic. In this case we say that (f_n) is a proximate sequence over $(\mathcal{V}_n)$.

If (f_n) and (f'_n) are proximate sequences from X to Y, then there exists a cofinal sequence of finite coverings of Y, $\mathcal{V}_1 \succ \mathcal{V}_2 \succ ...$ such that (f_n) and (f'_n) are proximate sequences over $(\mathcal{V}_n)$.

Two proximate sequences (f_n) and $(f'_n) : X \to Y$ are homotopic if for some cofinal sequence of finite coverings $\mathcal{V}_1 \succ \mathcal{V}_2 \succ ...$ (f_n) and (f'_n) are proximate sequences over $(\mathcal{V}_n)$ and for all integers n, f_n and f'_n are $\mathcal{V}_n$-homotopic. We say that (f_n) and (f'_n) are homotopic over $(\mathcal{V}_n)$.

If $(f_n) : X \to Y$ is a proximate sequence over $(\mathcal{V}_n)$ and $(g_k) : Y \to Z$ is a proximate sequence over $(\mathcal{W}_k)$, then for a covering $\mathcal{W}_k$ of Z there exist a covering $\mathcal{V}_{n_k}$ of Y such that $g_k(\mathcal{V}_{n_k}) \prec \mathcal{W}_k$. Now, the composition of these two proximate sequences is the proximate sequence $(h_k) = (g_k f_{n_k}) : X \to Z$. This proximate sequence is unique up to homotopy.

Compact metric spaces and homotopy classes of proximate sequences $[(f_n)]$ form a category whose isomorphisms induce classifications which coincide with the standard shape classification, i.e., isomorphic spaces in this category have the same shape (for proof see [12]).

3. Basic notions and some examples about dynamical systems

The study of dynamical systems involves many areas of mathematics, most notably analysis and topology. The pioneering work of Poincare, later continued by Morse, Smale and Conley can be considered as landmarks in the study of dynamical systems through their phase portraits. This approach gave rise to a whole new branch, where tools like homotopy theory, later shape theory, and various types of homology, played a prominent role in the investigations of dynamical systems. The use of shape theory in the study of dynamical systems was initiated by Hastings in [7]. Other authors have shown how to apply shape theory to obtain global properties of attractors in the papers [2, 5, 6, 8, 15]. Shape theory was related with differential equations in [15] and it is the main tool used in [13, 14] to define a Conley index for discrete dynamical systems. Before proceeding further we will recall some elementary concepts and fix notations.

Definition 3.1. Let (X, d) be a given metric space. A flow in X is a continuous map $\Phi : X \times \mathbb{R} \to X$ such that satisfies the following two conditions:

$$\Phi(x, 0) = x,\ \Phi(\Phi(x, t), s) = \Phi(x, t + s) \text{ for all } x \in X \text{ and } t, s \in \mathbb{R}.$$

The triplet $(X, \mathbb{R}, \Phi)$ forms a dynamical system (flow) with phase map Φ and phase space X. If we replace the set $\mathbb{R}$ with $\mathbb{R}^+$ we get the corresponding notion of semi-dynamical system.

For every $t \in \mathbb{R}$ we will consider the transition map $\Phi_t : X \to X$ defined by $\Phi_t(x) = \Phi(x, t)$. A dynamical system $(X, \mathbb{R}, \Phi)$ is also denoted by $\{\Phi_t : X \to X \mid t \in \mathbb{R}^+\}$ or shortly $\{\Phi_t\}$.

Definition 3.2. We say that a given subset $M \subseteq X$ is invariant under the flow Φ if $\Phi(M, t) \subseteq M$, for all $t \in \mathbb{R}$. If we replace the set $\mathbb{R}$ with $\mathbb{R}^+$ or $\mathbb{R}^-$, we obtain the corresponding notions of positively and negatively invariant set.

Definition 3.3. A compact invariant set M is stable if every neighborhood U of M contains a positively invariant neighborhood V of M.

The trajectory of a point x is the set $\gamma(x) = \{\Phi(x, t) | t \in \mathbb{R}\}$. By replacing the set $\mathbb{R}$ with $\mathbb{R}^+ \cup \{0\}$ or $\mathbb{R}^- \cup \{0\}$ we obtain the corresponding notions of positive and negative semi trajectory. We denote by $\gamma^+(x)$ and $\gamma^-(x)$ correspondingly.

Many questions concerning flows involve their long term behavior. We introduce positive limit set of a given subset $M \subseteq X$ with the following:

$$\omega(M) = \{x \in X | \exists \textit{ sequence } (x_n) \textit{ in } M \textit{ and a sequence } t_n \to \infty, \textit{ such that } \Phi(x_n, t_n) \to x\}.$$

Analogous we define negative limit set $\alpha(M)$.

Definition 3.4. A set $M \subseteq X$ attracts a set $C \subseteq X$ if for every neighborhood U of M there exists $T \in \mathbb{R}$ such that $\Phi_t(C) \subseteq U$, for every $t \geq T$.

Definition 3.5. Let (Φ_t) be a semi-dynamical system on a metric space X. A compact invariant set $M \subseteq X$ is said to be a global attractor if it attracts all compact sets.

In a metric space a global attractor is stable.

We consider the most natural notion of attraction from the topological viewpoint i.e. the notion of compact attraction, which is more general than the notion of attraction of bounded sets considered very often in the literature of dynamical systems. This notion of bounded attraction is not only more restrictive, it is also non-topological, i.e., it depends on the metric considered in the phase space and it is not preserved by a change in the metric.

For example, if one considers the equation $y' = -y$ in $\mathbb{R}$, with the usual metric d , then the origin is a global attractor in the sense of compact attraction but it is not a global attractor in the sense of bounded attraction if we consider the equivalent metric $d_1 = \frac{d}{1+d}$.

The paper of Vaughan [20], shows that the difference between both definitions of attraction could be significant in the absence of local compactness of the phase space.

Example 3.6. Let us consider the following subset $Y = \bigcup_{n\in\mathbb{N}}[\frac{1}{2^{2n+1}}, \frac{1}{2^{2n}}] \subseteq [0,1]$ of the unit interval $I = [0,1]$ and let $X = [0,1]\times[0,1]$. We define the set $A = (\partial Y \times [0,1]) \cup (Y \times \{0\}) \cup (([0,1]\setminus Y)\times\{1\})$ where ∂Y denotes the boundary of Y in $[0,1]$. Let us also consider the function $f : [0,1] \to \mathbb{R}$ defined by:

$$f(x) = dist(x, \partial Y), \quad \text{for every } x \in [0,1]$$

Now let us define a semi flow on X by:

$$\Phi((x,y),t) = \begin{cases} (x, e^{-f(x)t}y),\ if\ (x,y) \in Y \times [0,1], \\ (x, 1 - e^{-f(x)t}(1-y)),\ otherwise. \end{cases}$$

Let us note that A is a global attractor for $\{\Phi_t\}$, connected, however with two path connected components $A_1 = \{0\}\times[0,1]$ and $A_2 = A \setminus A_1$. Never the less according to our main theorem the strong shape of A is trivial.

The following example shows that the main theorem does not hold for discrete dynamical systems:

Example 3.7. Let us consider the discrete semi-group $\{\Phi_n\}$ on a convex set $C \subseteq \mathbb{R}^2$ containing the unit circle and generated by the function $f(x,y) = (cos10x, sin10x)$, for all $(x,y) \in C$.

Let us set $A = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Since $f(\mathbb{R}^2) = A$ and $f(A) = A$, it is clear that A is the global attractor for $\{\Phi_n\}$. Since C is contractible its strong shape clearly differs from the circle A.

4. Intrinsic strong shape for compact metric spaces

In this section we define strong shape of a compact metric space and prove several results needed for the proof of the main theorem.

Definition 4.1. The sequence of pairs $(f_n, f_{n,n+1})$ of functions $f_n : X \to Y$ and $f_{n,n+1} : X \times I \to Y$ is a **strong proximate sequence** from X to Y if there exists a cofinal sequence of finite coverings, $\mathcal{V}_1 > \mathcal{V}_2 > ...$ of Y such that for each natural number n, $f_n : X \to Y$ is a $\mathcal{V}_n$- continuous function and $f_{n,n+1} : X \times I \to Y$ is a homotopy connecting $\mathcal{V}_n$-continuous functions f_n: $X \to Y$ and f_{n+1}: $X \to Y$.

We say that $(f_n, f_{n,n+1})$ is a strong proximate sequence over $(\mathcal{V}_n)$.

If $(f_n, f_{n,n+1})$ and $(f'_n, f'_{n,n+1})$ are strong proximate sequences from X to Y, then there exists a cofinal sequence of finite coverings $(\mathcal{V}_n)$ such that $(f_n, f_{n,n+1})$ and $(f'_n, f'_{n,n+1})$ are strong proximate sequences over $(\mathcal{V}_n)$.

Two strong proximate sequences $(f_n, f_{n,n+1})$: $X \to Y$ and $(f'_n, f'_{n,n+1})$: $X \to Y$ are **homotopic** if there exists a strong proximate sequence $(F_n, F_{n,n+1})$: $X \times I \to Y$ over $(\mathcal{V}_n)$ such that:

i) $F_n : X \times I \to Y$ is a $\mathcal{V}_n$-homotopy between $\mathcal{V}_n$-continuous maps f_n and f'_n

ii) $F_{n,n+1} : X \times I \times I \to Y$ is a $\mathrm{st}^2(\mathcal{V}_n)$-continuous function at all points from $X \times \partial I^2$ is $\mathrm{st}(\mathcal{V}_n)$-continuous and

$$F_{n,n+1}(x,t,0) = f_{n,n+1}(x,t), \qquad F_{n,n+1}(x,t,1) = f'_{n,n+1}(x,t).$$

Homotopy classes of strong proximate sequences are morphisms of strong shape category.

Theorem 4.2. *Suppose X is a compact metric space and A a subset of X. If there exists a strong proximate sequence $(r_n, r_{n,n+1}) : X \to A$ and a strong proximate sequence $(H_n, H_{n,n+1}) : X \times I \to X$ such that:*

$H_n(A \times I) \subseteq A$

$H_{n,n+1}(A \times I \times I) \subseteq A$

$$H_n(x,0) = x, \ \ H_n(x,1) = r_n(x) \tag{1}$$

$$H_{n,n+1}(x,t,0) = x, \ \ H_{n,n+1}(x,t,1) = r_{n,n+1}(x,t) \tag{2}$$

Then the inclusion $i : A \to X$ induces a strong shape equivalence.

Proof. We consider the strong proximate sequence $(i_n, i_{n,n+1}) : A \to X$ induced by the inclusion $i : A \to X$.

First note that $i_n r_n(x) = r_n(x)$ and by (1), H_n connects $i_n r_n$ and the identity map 1_X. Second, if we put $(i_n, i_{n,n+1})(r_n, r_{n,n+1}) = (h_n, h_{n,n+1})$, then

$$h_{n,n+1}(x,t) = \begin{cases} r_{n,n+1}(x, 2t), & \text{if } 0 \le t \le \frac{1}{2} \\ r_{n+1}(x), & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

If we define

$$H'_{n,n+1}(x,t,s) = \begin{cases} H_{n,n+1}(x, \frac{2t}{2-s}, s), & \text{if } 0 \le t \le \frac{2-s}{2} \\ H_{n+1}(x,s), & \text{if } \frac{2-s}{2} \le t \le 1 \end{cases}$$

the function is well defined and

$$H'_{n,n+1}(x,0,s) = H_n(x,s), \;\; H'_{n,n+1}(x,1,s) = H_{n+1}(x,s).$$

So, $(H_n, H'_{n,n+1}) : X \times I \to X$ is a strong proximate sequence. Since

$$H'_{n,n+1}(x,t,0) = x, \;\; H'_{n,n+1}(x,t,1) = h_{n,n+1}(x,t)$$

the strong proximate sequence $(H_n, H'_{n,n+1})$ is a homotopy connecting the strong proximate sequence $(h_n, h_{n,n+1})$ and the strong proximate sequence induced by the identity 1_X.

On the other hand, $r_n i_n(x) = r_n(x)$ and by (1), $H_n|A$ connects $r_n i_n : A \to A$ and the identity map 1_A.

Second, if we put $(r_n, r_{n,n+1})(i_n, i_{n,n+1}) = (h_n, h_{n,n+1})$, then $(h_n, h_{n,n+1}) : A \to A$ is defined by

$$h_{n,n+1}(x,t) = \begin{cases} r_n(x), & \text{if } 0 \le t \le \frac{1}{2} \\ r_{n,n+1}(x, 2t-1), & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

If we define $H''_{n,n+1} : A \times I \times I \to A$

$$H''_{n,n+1}(x,t,s) = \begin{cases} H_n(x,s), & \text{if } 0 \le t \le \frac{s}{2} \\ H_{n,n+1}(x, \frac{2t-s}{2-s}, s), & \text{if } \frac{s}{2} \le t \le 1 \end{cases}$$

then the function is well defined and

$$H''_{n,n+1}(x,0,s) = H_n(x,s), \;\; H''_{n,n+1}(x,1,s) = H_{n+1}(x,s).$$

So, $(H_n, H''_{n,n+1}) : A \times I \to A$ is a strong proximate sequence. Since

$$H''_{n,n+1}(x,t,0) = x, \;\; H''_{n,n+1}(x,t,1) = h_{n,n+1}(x,t)$$

the strong proximate sequence $(H_n, H''_{n,n+1})$ is a homotopy connecting the strong proximate sequence $(h_n, h_{n,n+1})$ and the strong proximate sequence induced by the identity 1_A. □

5. The proof of the main theorem

In this section we shall prove the main Theorem 5.1. This theorem is well known for shape (see[2]). Of course the proof given here for strong shape is also a proof for shape.

Theorem 5.1. *Suppose $\Phi : X \times \mathbb{R}^+ \to X$ is a semi-dynamical system on a compact metric space X with a global attractor M. Then the inclusion $i : M \to X$ induces a strong shape equivalence.*

To prove the above theorem in Lemma 5.5 we construct a strong proximate sequence $(r_n, r_{n,n+1}) : X \to M$ and in Lemma 5.6 we prove that it is a strong shape equivalence and that its strong shape inverse is the inclusion $i : M \to X$.

To start we need several lemmas about continuity over coverings.

A covering $\mathcal{V}$ of M in X is called regular if it satisfies the following conditions:

1) If $V \in \mathcal{V}$ then $V \cap M \neq \emptyset$

2) If $U, V \in \mathcal{V}$ and $U \cap V \neq \emptyset$, then $U \cap V \in \mathcal{V}$.

For a covering $\mathcal{V}$ of M we introduce the notation $|\mathcal{V}| = \bigcup_{V \in \mathcal{V}} V$.

For a finite regular covering $\mathcal{V}$ we define a function $r_{\mathcal{V}} : |\mathcal{V}| \to M$ in the following way:

For points $y \in M$ we put $r_{\mathcal{V}}(y) = y$.

For points $y \in |\mathcal{V}| \backslash M$, by induction we can choose the smallest member $V \in \mathcal{V}$ such that $y \in V$, then choose a fixed point $y_V \in V \cap M$ and put $r_{\mathcal{V}}(y) = y_V$.

The function $r_{\mathcal{V}}$ is $\mathcal{V}$-continuous. If $\mathcal{V} > \mathcal{W}$, then $r_{\mathcal{W}} : |\mathcal{W}| \to M$ and $r_{\mathcal{V}} : |\mathcal{W}| \to M$ (the restriction of $r_{\mathcal{V}} : |\mathcal{V}| \to M$ to $|\mathcal{W}|$) are $\mathcal{V}$-near.

Lemma 5.2. *If $\mathcal{V} > \mathcal{W}$, then $r_{\mathcal{W}} : |\mathcal{W}| \to M$ and $r_{\mathcal{V}} : |\mathcal{W}| \to M$ are $\mathcal{V}$-homotopic.*

Proof. $r_{\mathcal{W}} : |\mathcal{W}| \to M$ and $r_{\mathcal{V}} : |\mathcal{W}| \to M$ are $\mathcal{V}$-near and so $\mathcal{V}$-homotopic by a homotopy $r_{\mathcal{V}\mathcal{W}} : |\mathcal{W}| \times I \to M$ ([19]). □

Lemma 5.3. *$i \circ r_{\mathcal{V}} : |\mathcal{V}| \to |\mathcal{V}|$ and $1_{\mathcal{V}} : |\mathcal{V}| \to |\mathcal{V}|$ are $\mathcal{V}$-homotopic by a homotopy $R_{\mathcal{V}} : |\mathcal{V}| \times I \to |\mathcal{V}|$ such that $R_{\mathcal{V}}(x, t) = x$, for $x \in M$.*

Proof. The functions $r_{\mathcal{V}}$ and $1_{\mathcal{V}}$ are $\mathcal{V}$-near and so $\mathcal{V}$-homotopic by a homotopy $R_{\mathcal{V}}$ ([19]). From the definition of $R_{\mathcal{V}}$ in the proof, it is satisfied $R_{\mathcal{V}}(x, t) = x$, for $x \in M$. □

Lemma 5.4. *If $\mathcal{V} > \mathcal{W}$, then $R_{\mathcal{W}} : |\mathcal{W}| \times I \to |\mathcal{W}|$ and $R_{\mathcal{V}} : |\mathcal{W}| \times I \to |\mathcal{V}|$ (the restriction of $R_{\mathcal{V}} : |\mathcal{V}| \times I \to |\mathcal{V}|$ to $|\mathcal{W}| \times I$) are $\mathcal{V}$-homotopic by a homotopy $R_{\mathcal{V}\mathcal{W}} : |\mathcal{W}| \times I \times I \to |\mathcal{V}|$ such that $R_{\mathcal{V}\mathcal{W}}(x, z, 0) = x$.*

Proof. We define

$$R_{\mathcal{V}\mathcal{W}}(x, z, t) = \begin{cases} x, & \text{if } 0 \le t \le \frac{1}{2} \\ r_{\mathcal{V}\mathcal{W}}(x, z), & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

Then

$$\begin{aligned} R_{\mathcal{V}\mathcal{W}}(x, 0, t) &= \begin{cases} x, & \text{if } 0 \le t \le \frac{1}{2} \\ r_{\mathcal{V}\mathcal{W}}(x, 0), & \text{if } \frac{1}{2} \le t \le 1 \end{cases} \\ &= \begin{cases} x, & \text{if } 0 \le t \le \frac{1}{2} \\ r_{\mathcal{V}}(x), & \text{if } \frac{1}{2} \le t \le 1 \end{cases} \\ &= R_{\mathcal{V}}(x, t) \end{aligned}$$

Similarly,

$$R_{\mathcal{V}\mathcal{W}}(x,1,t) = R_{\mathcal{W}}(x,t).$$

Also

$$R_{\mathcal{V}\mathcal{W}}(x,z,1) = r_{\mathcal{V}\mathcal{W}}(x,z), \;\; R_{\mathcal{V}\mathcal{W}}(x,z,0) = x.$$

□

Now we construct $(r_n, r_{n,n+1}) : X \to M$, the induced strong shape morphism by a semi-flow. The construction is natural and shows that the intrinsic strong shape theory from [18] provides a natural tool to treat semi-dynamical systems in compact metric spaces.

First we choose a cofinal sequence of regular coverings $(\mathcal{V}_n)$ of the compact M. Since M is a global attractor there exists a sequence $t_n \to +\infty$ such that:

$$\Phi(X \times [t_n, \infty)) \subseteq |\mathcal{V}_n|$$

and we can choose $t_n \leq t_{n+1}$.

For $t \in [t_n, \infty)$, $\Phi(x,t) \in |\mathcal{V}_n|$ and the composition $r_{\mathcal{V}_n}\Phi(x,t)$ is defined.

The functions $f(x) = r_{\mathcal{V}_n}\Phi(x, t_{n+1})$ and $g(x) = r_{\mathcal{V}_{n+1}}\Phi(x, t_{n+1})$ are $\mathcal{V}_n$-homotopic by a homotopy given by $r_{\mathcal{V}_n\mathcal{V}_{n+1}}(\Phi(x, t_{n+1}), t)$ i.e.

$$r_{\mathcal{V}_n\mathcal{V}_{n+1}}(\Phi(x, t_{n+1}), 0) = r_{\mathcal{V}_n}\Phi(x, t_{n+1}), \;\; r_{\mathcal{V}_n\mathcal{V}_{n+1}}(\Phi(x, t_{n+1}), 1) = r_{\mathcal{V}_{n+1}}\Phi(x, t_{n+1}).$$

Lemma 5.5. *If the functions $r_n : X \to M$ are defined by $r_n(x) = r_{\mathcal{V}_n}\Phi(x, t_n)$, then there exists a strong proximate sequence $(r_n, r_{n,n+1}) : X \to M$.*

Proof. We define $r_{n,n+1} : X \times I \to M$ as the juxtaposition of homotopies:

$$r_{n,n+1}(x,t) = r_{\mathcal{V}_n}\Phi(x, (1-t)t_n + t t_{n+1}) * r_{\mathcal{V}_n\mathcal{V}_{n+1}}(\Phi(x, t_{n+1}), t)$$

Then

$$r_{n,n+1}(x,0) = r_{\mathcal{V}_n}\Phi(x, t_n) = r_n(x)$$

and

$$r_{n,n+1}(x,1) = r_{\mathcal{V}_{n+1}}\Phi(x, t_{n+1}) = r_{n+1}(x).$$

Let us note that $r_{n,n+1} : X \times I \to M$ is well defined since the homotopy $r_{\mathcal{V}_n}\Phi(x, (1-t)t_n + t t_{n+1})$ for $t = 1$ equals $r_{\mathcal{V}_n}\Phi(x, t_{n+1})$ and also $r_{\mathcal{V}_n\mathcal{V}_{n+1}}(\Phi(x, t_{n+1}), 0) = r_{\mathcal{V}_n}\Phi(x, t_{n+1})$. □

Lemma 5.6. *The strong proximate sequence $(r_n, r_{n,n+1}) : X \to M$ induces a strong shape equivalence.*

Proof. We construct a strong proximate sequences $(r_n, r_{n,n+1}) : X \to M$ and a strong proximate sequence $(H_n, H_{n,n+1}) : X \times I \to X$ such that:

$$H_n(M \times I) \subseteq M$$

$$H_{n,n+1}(M \times I \times I) \subseteq M$$

$$H_n(x,0) = x, \;\; H_n(x,1) = r_n(x) \tag{3}$$

$$H_{n,n+1}(x,t,0) = x, \;\; H_{n,n+1}(x,t,1) = r_{n,n+1}(x,t) \tag{4}$$

Then by Theorem 4.2 it will follow that $(r_n, r_{n,n+1}) : X \to M$ is a strong shape equivalence and $i : M \to X$ is the strong shape inverse of $(r_n, r_{n,n+1}) : X \to M$.

The construction of $(H_n, H_{n,n+1}) : X \times I \to X$

We define a strong proximate sequence $(H_n, H_{n,n+1}) : X \times I \to X$ by: $H_n(x,s) = R_{\mathcal{V}_n}(\Phi(x, s t_n), s)$.

Then

$$H_n(x,0) = R_{\mathcal{V}_n}(\Phi(x,0),0) = \Phi(x,0) = x$$
$$H_n(x,1) = R_{\mathcal{V}_n}(\Phi(x,t_n),1) = r_{\mathcal{V}_n}\Phi(x,t_n) = r_n(x)$$

and H_n is $\mathcal{V}_n$-continuous. Hence (3) holds.

Now we define $H_{n,n+1} : X \times I \times I \to X$ by juxtaposition $H_{n,n+1} = K_{n,n+1} * F_{n,n+1}$. The first half is defined by

$$K_{n,n+1}(x,t,s) = R_{\mathcal{V}_n}(\Phi(x,s((1-t)t_n + tt_{n+1})),s)$$

and the second half is defined by

$$F_{n,n+1}(x,t,s) = R_{\mathcal{V}_n\mathcal{V}_{n+1}}(\Phi(x,st_{n+1}),t,s).$$

Since

$$K_{n,n+1}(x,1,s) = R_{\mathcal{V}_n}(\Phi(x,st_{n+1}),s)$$
$$F_{n,n+1}(x,0,s) = R_{\mathcal{V}_n}(\Phi(x,st_{n+1}),s)$$

the juxtaposition is well defined. Moreover

$$H_{n,n+1}(x,0,s) = K_{n,n+1}(x,0,s) = R_{\mathcal{V}_n}(\Phi(x,st_n),s) = H_n(x,s)$$
$$H_{n,n+1}(x,1,s) = F_{n,n+1}(x,1,s) = R_{\mathcal{V}_{n+1}}(\Phi(x,st_{n+1}),s) = H_{n+1}(x,s).$$

Since

$$K_{n,n+1}(x,t,0) = R_{\mathcal{V}_n}(\Phi(x,0),0) = R_{\mathcal{V}_n}(x,0) = x$$
$$F_{n,n+1}(x,t,0) = R_{\mathcal{V}_n\mathcal{V}_{n+1}}(\Phi(x,0),t,0) = \Phi(x,0) = x$$

it follows

$$H_{n,n+1}(x,t,0) = K_{n,n+1}(x,t,0) * F_{n,n+1}(x,t,0) = x.$$

Since

$$\begin{aligned} K_{n,n+1}(x,t,1) &= R_{\mathcal{V}_n}(\Phi(x,(1-t)t_n + tt_{n+1}),1) \\ &= r_{\mathcal{V}_n}\Phi(x,(1-t)t_n + tt_{n+1}) \\ F_{n,n+1}(x,t,1) &= R_{\mathcal{V}_n\mathcal{V}_{n+1}}(\Phi(x,t_{n+1}),t,1) = r_{\mathcal{V}_n\mathcal{V}_{n+1}}(\Phi(x,t_{n+1}),t) \end{aligned}$$

from the definition of $r_{n,n+1}$ in the proof of lemma 5.5 it follows

$$\begin{aligned} H_{n,n+1}(x,t,1) &= K_{n,n+1}(x,t,1) * F_{n,n+1}(x,t,1) \\ &= r_{\mathcal{V}_n}\Phi(x,(1-t)t_n + tt_{n+1}) * r_{\mathcal{V}_n\mathcal{V}_{n+1}}(\Phi(x,t_{n+1}),t) \\ &= r_{n,n+1}(x,t) \end{aligned}$$

i.e. the required conditions from (4) $H_{n,n+1}(x,t,0) = x$, $H_{n,n+1}(x,t,1) = r_{n,n+1}(x,t)$ are satisfied. □

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