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A Study on I–Deferred Strongly Cesáro Summable and μ-Deferred I–Statistical Convergence for Complex Uncertain Sequences

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Abstract. In this study, we introduce the five different types of I–deferred strongly Cesàro summablity and μ-deferred I–statistical convergence (i.e., with respect to almost surely, uniformly almost surely, in measure, in distribution and in mean) of a complex uncertain sequence. We also introduce the spaces of such kind of sequences. Furthermore, some interesting properties of these definitions and inclusion relations between the spaces have been established under some conditions.

1. Introduction

In the fundamental theory of functional analysis, convergence of sequences is a key notion. It becomes more crucial notion in sequence spaces. Using the idea of natural density [2], Fast [9] and Steinhaus [28] founded a new type of convergence of a sequence called statistical convergence which is a generalization of usual convergence.

Definition 1.1. =A sequence $(y_j) \in \omega$ is said to be statistical convergent to k if $\forall\ \varepsilon > 0$, the natural density of the set $\{j \in \mathbb{N} : |y_j - k| \geq \varepsilon\}$ is equal to zero. Then, we say that $st-\lim(y_j)= k$.

Kostyrko et al. [14, 15] generalized notion of statistical convergence by defining ideal convergence (shortly I-convergence) of a sequence. I-convergence depends on ideal defined on the set $\mathbb{N}$.

Definition 1.2. ([14]) Let X be any set and $I \subset P(X)$, the power set of X, then I is said to be an ideal if
a. $\emptyset \in I$,
b. $G \cup H \in I, \forall\ G, H \in I$,
c. $\forall G \in I$ and $H \subset G$ then $H \in I$.

Ideal I is said to be non-trivial ideal if $I \neq 2^X$. Ideal I is said to be admissible ideal if $\{\{x\} : x \in X\} \subset I$.

Definition 1.3. ([14]) Let X be any set and $\mathcal{F}(I) \subset P(X)$, the power set of X, then $\mathcal{F}(I)$ is said to be filter if
a. $\emptyset \notin \mathcal{F}(I)$,
b. $G \cap H \in \mathcal{F}(I), \forall\ G, H \in \mathcal{F}(I)$,
c. $\forall G \in \mathcal{F}(I)$ and $H \supset G$ then $H \in \mathcal{F}(I)$.

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Proposition 1.4. ([14]) *Class $\mathcal{F}(I) = \{A \subset X : A = X\backslash B$, for some $B \in I\}$ is a filter on X, where $I \subset P(\mathbb{N})$ is a non trivial ideal. $\mathcal{F}(I)$ is known as the filter associated with the ideal I.*

Definition 1.5. ([14]) Let I is a non-trivial ideal defined on $\mathbb{N}$, a sequence $(y_j) \in \omega$ is said to be I–convergent to k if $\forall\ \varepsilon > 0$, the set

$$\{j \in \mathbb{N} :\mid y_j - k \mid\geq \varepsilon\} \in I.$$

Then, we write it as $I-\lim(y_j)= k$.

Šalát et al. [24, 25] and many other authors further studied the notion of I-convergence. Savas and Das [26] defined notion of statistical convergence via ideals.

Definition 1.6. ([26]) Let I is a non-trivial ideal defined on $\mathbb{N}$, a sequence $(y_j) \in \omega$ is said to be I–convergent to k if $\forall\ \varepsilon, \delta > 0$, the set

$$\left\{j \in \mathbb{N} : \frac{\text{card}(\{n \leq j :\mid y_j - k \mid\geq \varepsilon\})}{j} \geq \delta\right\} \in I.$$

Then, we say that I–st–$\lim(y_j)= k$.

Thereafter, strong Cesàro convergence for real sequences was defined by Hardy and Littlewood [10]. Let $p = (p_n), q = (q_n)$ is any pair of increasing sequences of non-negative integers s.t.

$$p_n < q_n \text{ and } \lim_{n\to\infty} q_n = \infty.$$

Agnew [1] defined deferred Cesàro mean of a sequence (y_j), which is defined as

$$(D_{p,q}y)_j = \frac{1}{q_n - p_n}\sum_{j=p_n+1}^{q_n} y_j,\ j = 1, 2, ... \tag{1}$$

The deferred density of $A \subset \mathbb{N}$ is defined by

$$D(A) = \lim_{n\to\infty} \frac{\text{card}(\{a : p_n < a \leq q_n,\ a \in A\})}{q_n - p_n}$$

provided that the limit exists.

Using deferred density notion Küçükaslan et al. [16] introduce the idea of deferred statistical convergence of sequence. After that Şengül et al. [27] defined I–deferred statistical convergence of a sequence.

Definition 1.7. ([27]) Let I is a non-trivial ideal defined on $\mathbb{N}$, a sequence $(y_j) \in \omega$ is said to be I–deferred statistical convergent to k if $\forall\ \varepsilon, \delta > 0$, the set

$$\left\{n \in \mathbb{N} : \frac{\text{card}(\{p_n < j \leq q_n :\mid y_j - k \mid\geq \varepsilon\})}{q_n - p_n} \geq \delta\right\} \in I.$$

Then, we write it as $DS_{p,q}(I)$–$\lim(y_j) = k$.

Recently, Et, et al. [8] introduced the concept of deferred strongly Cesàro summable and μ-deferred statistically convergent functions. Since a real sequence is a function from $\mathbb{N}$ to $\mathbb{R}$ and any function from $\mathbb{N}$ to $\mathbb{R}$ is always measurable as $P(\mathbb{N})$ forms a sigma algebra. So deferred strongly Cesàro summablity and μ-deferred statistically convergence for sequences are defined as:

Definition 1.8. ([12]) A sequence (y_j) is said to be deferred strongly Cesàro summable to $k \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{q_n - p_n}\sum_{j=p_n}^{q_n} |y_j - k| = 0.$$

Then, we say that $DC_{p,q}$–$\lim(y_j)= k$.

Let $X = \mathbb{N}$ and Σ be a sigma algebra of the subsets of X and μ be a sigma finite measure on Σ such that $\mu(X) = \infty$. Measure of any subset A of X which is in Σ will be denote by $\mu(A) := |A|$. Note that here $|A|$ denote the measure of set A, not the cardinality of the set A as it is in natural or deferred density.

The μ–deferred density of $A \subset \mathbb{N}$ is defined by

$$_{\mu}D(A) = \lim_{n\to\infty} \frac{|A \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} \tag{2}$$

provided that the limit exists, where $I^*_{p,q}(n) = [p_n, q_n] \cap \mathbb{N}$.

Definition 1.9. ([12]) A sequence $y = (y_j)$ is said to be μ-deferred statistically convergent to a real number k if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{|\{j \in \mathbb{N} : | y_j - k | \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} = 0 \ \text{(or } \lim_{n\to\infty} \frac{|\{j \in \mathbb{N} : | y_j - k | < \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} = 1).$$

Then, we say that $_{\mu}DS_{p,q}-\lim(y_j)= k$.

Quite recently, Khan et al. [12] defined the notion of I–deferred strongly Cesàro summable and μ-deferred I–statistically convergence and introduced their respective sequence spaces.

Definition 1.10. ([12]) A sequence $y = (y_j)$ is said to be I–deferred strongly Cesáro summable to a number $k \in \mathbb{R}$ if for all $\varepsilon > 0$, the following set

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n} \sum_{j=p_n}^{q_n} |y_j - k| \geq \varepsilon\right\}$$

belongs to I. Then, we write $DC^I_{p,q} - \lim y = k$.

Definition 1.11. ([12]) A sequence $y = (y_j)$ is said to be μ–deferred I–statistically convergent to a number $k \in \mathbb{R}$ if for all $\varepsilon,\delta > 0$, the following set

$$\left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : |y_j - k| \geq \varepsilon \text{ for some } k \in \mathbb{R}\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} \geq \delta\right\}$$

belongs to I. Then, we write $_{\mu}DS^I_{p,q} - \lim y = k$.

Uncertainty encounters in every real world problems. To know the procedure how to deal with these kind of problems plays a significant role. In 2007, Liu [17] gave a theory of uncertainty which is also refined by Liu [19] in 2009. To model human uncertainty, uncertainty theory of sequences has become a branch of mathematics [18]. Many authors studied the nature of sequence in an uncertain atmosphere. Liu [17] defined different types of convergence, viz. convergence with respect to almost surely, in distribution, in mean and in measure of a real uncertain sequence. Convergence w.r.t. uniformly almost surely of a real uncertain sequence was introduced by You [31]. He established the interrelations between the same and the existing convergence. Chen et al. [3] defined this different convergence for the sequence of complex uncertain variables. In 2017, a study of statistical convergence for complex uncertain sequence was given by Tripathy et al. [29]. Moreover, Kişi and Güler [11] defined λ-statistically convergence of a sequence of uncertain complex variables. Inspired by this, Kişi [13] introduced the notion of λ_I-statistical convergence of a sequence of uncertain complex variables and established some good results. Several researchers studied the charaterizations of different convergences of complex uncertain sequences [4–7, 20–23, 30]. In this study, we define the notion of strongly I–deferred Cesàro summable and μ-deferred I–statistically convergence with respect to almost surely, in distribution, in mean and in measure of a sequence of complex uncertain variables. We also introduced the spaces of these kind of sequences and established some inclusion relations related to these spaces.

Now recalling some existing definitions which play an important role in the forthcoming section of research paper.

Definition 1.12. ([17]) Let Γ be a non-empty set and $\mathcal{L}$ be a σ-algebra on Γ. An uncertain measure is a set function $\mathcal{M}$ on Γ which satisfies the following axioms:
(1) Normality Axiom: $\mathcal{M}\{\Gamma\} = 1$;
(2) Duality Axiom: $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$, for any $\Lambda \in \Gamma$;
(3) Subadditivity Axiom:

$$\mathcal{M}\Big\{\bigcup_{j=1}^{\infty}\Lambda_j\Big\} \leq \sum_{j=1}^{\infty}\mathcal{M}(\Lambda_j), \text{ for every countable sequence of } \{\Lambda_j\} \in \mathcal{L}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space and every element $\Lambda \in \mathcal{L}$ is called an event.
A product uncertain measure is defined as follows to find an uncertain measure of compound events,
(4) Product Axiom: Let $(\Gamma_n, \mathcal{L}_n, \mathcal{M}_n)$ be uncertainty spaces, for $n = 1, 2, 3, ...$ The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\Big\{\prod_{j=1}^{\infty}\Lambda_j\Big\} = \bigwedge_{j=1}^{\infty}\mathcal{M}(\Lambda_j),$$

where Λ_n are arbitrarily chosen events from Γ_n, for $n = 1, 2, 3, ...$, respectively.

Definition 1.13. ([17]) A complex uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to $\mathbb{C}$. i.e., for any Borel set A of complex numbers, the set $\{\xi \in A\} = \{\gamma \in \Gamma : \xi(\gamma \in A)\}$ is an event.

Definition 1.14. ([17]) The sequence (ξ_j) of complex uncertain variables is said to be convergent almost surely (a.s.) to ξ if $\exists\ \Lambda$ with $\mathcal{M}(\Lambda) = 1$ such that

$$\lim_{j\to\infty} \|\xi_j(\gamma) - \xi(\gamma)\| = 0, \forall\gamma \in \Lambda.$$

Definition 1.15. ([17]) The sequence (ξ_j) of complex uncertain variables is said to be convergent in measure to ξ if for all $\varepsilon > 0$,

$$\lim_{j\to\infty} \mathcal{M}(\|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon) = 0.$$

Definition 1.16. ([17]) The sequence (ξ_j) of complex uncertain variables is said to be convergent in mean to ξ if

$$\lim_{j\to\infty} E[\|\xi_j(\gamma) - \xi(\gamma)\|] = 0.$$

Definition 1.17. ([17]) Let $\varphi,\ \varphi_1,\ \varphi_2, ...$ are the complex uncertainty distributions of the complex uncertain variables $\xi,\ \xi_1,\ \xi_2, \ldots$, respectively. We say the sequence (ξ_j) of complex uncertain variables is said to be convergent in distribution to ξ if,

$$\lim_{j\to\infty} \varphi_j(a) = \varphi(a),$$

for all points a at which φ is continuous.

Definition 1.18. ([17]) The sequence (ξ_j) of complex uncertain variables is said to be convergent uniformly almost surely (u.a.s.) to ξ if $\exists$ a sequence (E_j) with $\mathcal{M}(E_j) \to 0$ such that (ξ_j) converges uniformly to ξ in $\Gamma - (E_j)$, for any fixed $j \in \mathbb{N}$.

2. Main results

Throughout the article, we consider $p = (p_n), q = (q_n)$ is any pair of increasing sequences of non-negative integers such that

$$p_n < q_n \text{ and } \lim_{n\to\infty} q_n = \infty. \tag{3}$$

and $I^*_{p,q}(n) = [p_n, q_n] \cap \mathbb{N}$, $I^*_q(n) = [1, q_n] \cap \mathbb{N}$ & $I^*(n) = [1, n] \cap \mathbb{N}$.

Like (2), we also consider that $\mu(A) =: |A|$ denote the sigma finite measure of set $A \subset \mathbb{N}$, not the cardinality of the set A as it is in natural or deferred density.

Definition 2.1. The sequence (ξ_j) of complex uncertain variables is said to be I–deferred strongly Cesàro summable almost surely (a.s.) to ξ if for all $\varepsilon > 0$, there exists an event Λ such that $\mathcal{M}(\Lambda) = 1$ then the following set

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n} \sum_{j=p_n}^{q_n} \|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon\right\} \in I, \text{ for every } \gamma \in \Lambda.$$

We write it as $DC^I_{p,q}(\Gamma_{a.s.}) - \lim(\xi_j) = \xi$. Space of all I–deferred strongly Cesáro summable complex uncertain sequences w.r.t. almost surely(a.s.) is denoted by $DC^I_{p,q}(\Gamma_{a.s.})$.

Definition 2.2. The sequence (ξ_j) of complex uncertain variables is said to be μ–deferred I–statistically convergent almost surely (a.s.) to ξ if for all $\varepsilon > 0$ and $\delta > 0$, there exists an event Λ such that $\mathcal{M}(\Lambda) = 1$ then the following set

$$\left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} \geq \delta\right\} \in I, \text{ for every } \gamma \in \Lambda.$$

We write it as ${}_{\mu}DS^I_{p,q}(\Gamma_{a.s.}) - \lim(\xi_j) = \xi$. We denote set of all μ–deferred I–statistically convergent complex uncertain sequences with respect to almost surely (a.s.) by ${}_{\mu}DS^I_{p,q}(\Gamma_{a.s.})$.

Example 2.3. Suppose the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\gamma_1, \gamma_2, \gamma_3, ...$ with

$$\mathcal{M}(\Lambda) = \begin{cases} \sup_{\gamma_j\in\Lambda} \frac{j}{(2j+1)}, & \text{if } \sup_{\gamma_j\in\Lambda} \frac{j}{(2j+1)} < \frac{1}{2}, \\ 1 - \sup_{\gamma_j\in\Lambda^c} \frac{j}{(2j+1)}, & \text{if } \sup_{\gamma_j\in\Lambda^c} \frac{j}{(2j+1)} < \frac{1}{2}, \\ \frac{1}{2}, & \text{otherwise} \end{cases}$$

and the complex uncertain variables are defined by

$$\xi_j(\gamma) = \begin{cases} ij, & \text{if } \gamma = \gamma_j, \\ 0, & \text{otherwise} \end{cases}$$

for $j = 1, 2, 3, ...$ and $\xi \equiv 0$, here "i" denote the imaginary unit. Clearly, by Definition 2.1 and 2.2, complex uncertain sequence (ξ_j) is I–deferred strongly Cesàro summable as well as μ–deferred I–statistically convergent almost surely (a.s.) to ξ.

Definition 2.4. The sequence (ξ_j) of complex uncertain variables is said to be I–deferred strongly Cesàro summable in measure to ξ if for all $\varepsilon > 0$ and $\delta > 0$, the following set

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n} \sum_{j=p_n}^{q_n} \mathcal{M}(\|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon) \geq \delta\right\} \in I.$$

We write it as $DC^I_{(p,q)}(\Gamma_{\mathcal{M}}) - \lim(\xi_j) = \xi$. Space of all I–deferred strongly Cesàro summable complex uncertain sequences in measure is denoted by $DC^I_{(p,q)}(\Gamma_{\mathcal{M}})$.

Definition 2.5. The sequence (ξ_j) of complex uncertain variables is said to be μ–deferred I–statistically convergent in measure to ξ if for all $\varepsilon > 0$, $\zeta > 0$ and $\delta > 0$, the following set

$$\left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \mathcal{M}(\|\xi_j(\gamma) - \xi(\gamma)\| \geq \zeta) \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} \geq \delta\right\} \in I.$$

We write it as ${}_{\mu}DS^I_{p,q}(\Gamma_{\mathcal{M}}) - \lim(\xi_j) = \xi$. Space of all μ–deferred I–statistically convergent complex uncertain sequences in measure is denoted by ${}_{\mu}DS^I_{p,q}(\Gamma_{\mathcal{M}})$.

Example 2.6. Suppose the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\gamma_1, \gamma_2, \gamma_3, ...$ with

$$\mathcal{M}(\Lambda) = \begin{cases} \sup_{\gamma_j \in \Lambda} \frac{1}{(j+1)}, & \text{if } \sup_{\gamma_j \in \Lambda} \frac{1}{(j+1)} < \frac{1}{2}, \\ 1 - \sup_{\gamma_j \in \Lambda^c} \frac{1}{(j+1)}, & \text{if } \sup_{\gamma_j \in \Lambda^c} \frac{1}{(j+1)} < \frac{1}{2}, \\ \frac{1}{2}, & \text{otherwise} \end{cases}$$

and the complex uncertain variables are defined by

$$\xi_j(\gamma) = \begin{cases} i(j+1), & \text{if } \gamma = \gamma_j, \\ 0, & \text{otherwise} \end{cases}$$

for $j = 1, 2, 3, ...$ and $\xi \equiv 0$. Here, by definition 2.4 and 2.5, complex uncertain sequence (ξ_j) is I–deferred strongly Cesàro summable as well as μ–deferred I–statistically convergent in measure to ξ.

Definition 2.7. The sequence (ξ_j) of complex uncertain variables is said to be I–deferred strongly Cesàro summable in mean to ξ if for all $\varepsilon > 0$, the following set

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n} \sum_{j=p_n}^{q_n} E[\|\xi_j(\gamma) - \xi(\gamma)\|] \geq \varepsilon\right\} \in I.$$

We write it as $DC^I_{(p,q)}(\Gamma_E) - \lim(\xi_j) = \xi$. Space of all I–deferred strongly Cesàro summable complex uncertain sequences in mean is denoted by $DC^I_{(p,q)}(\Gamma_E)$.

Definition 2.8. The sequence (ξ_j) of complex uncertain variables is said to be μ–deferred I–statistically convergent in mean to ξ if for all $\varepsilon > 0$, and $\delta > 0$, the following set

$$\left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : E[\|\xi_j(\gamma) - \xi(\gamma)\|] \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} \geq \delta\right\} \in I.$$

We write it as ${}_{\mu}DS^I_{p,q}(\Gamma_E) - \lim(\xi_j) = \xi$. Space of all μ–deferred I–statistically convergent complex uncertain sequences in mean is denoted by ${}_{\mu}DS^I_{p,q}(\Gamma_E)$.

Example 2.9. Suppose the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ with $\mathcal{M}$ be the Lebesgue measure and $\mathcal{L} = [0, 1]$. Then we define complex uncertain variable by

$$\xi_j(\gamma) = \begin{cases} i, & \text{if } \frac{k}{2^p} \leq \gamma \leq \frac{k+1}{2^p}, \\ 0, & \text{otherwise} \end{cases}$$

for all $j = 2^p + k$, where $p, k \in \mathbb{Z}$ and $\xi \equiv 0$. Calculation shows that complex uncertain sequence (ξ_j) is I–deferred strongly Cesàro summable as well as μ–deferred I–statistically convergent in mean to ξ.

Definition 2.10. Let $\varphi,\ \varphi_1,\ \varphi_2,...$ are the complex uncertainty distributions of the complex uncertain variables $\xi,\ \xi_1,\ \xi_2,\ldots$, respectively. We say the sequence (ξ_j) of complex uncertain variables is said to be I–deferred strongly Cesàro summable in distribution to ξ if for all $\varepsilon > 0$, the following set

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\sum_{j=p_n}^{q_n} \|\varphi_j(a) - \varphi(a)\| \geq \varepsilon\right\} \in I, \text{ for all points } a \text{ at which } \varphi \text{ is continuous.}$$

We write it as $DC^I_{p,q}(\Gamma_\varphi) - \lim(\xi_j) = \xi$. Space of all I–deferred strongly Cesàro summable complex uncertain sequences in distribution is denoted by $DC^I_{p,q}(\Gamma_\varphi)$.

Definition 2.11. Let $\varphi,\ \varphi_1,\ \varphi_2,...$ are the complex uncertainty distributions of the complex uncertain variables $\xi,\ \xi_1,\ \xi_2,\ldots$, respectively. We say the sequence (ξ_j) of complex uncertain variables is said to be μ–deferred I–statistically convergent in distribution to ξ if for all $\varepsilon > 0$ and $\delta > 0$, the following set

$$\left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \|\varphi_j(a) - \varphi(a)\| \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} \geq \delta\right\} \in I, \text{ for all points } a \text{ at which } \varphi \text{ is continuous.}$$

We write it as ${}_\mu DS^I_{p,q}(\Gamma_\varphi) - \lim(\xi_j) = \xi$. Space of all μ–deferred I–statistically convergent complex uncertain sequences in distribution is denoted by ${}_\mu DS^I_{p,q}(\Gamma_\varphi)$.

Example 2.12. Suppose the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with $\mathcal{M}(\gamma_1) = \mathcal{M}(\gamma_2) = \frac{1}{2}$ and we define a complex uncertain variable by

$$\xi(\gamma) = \begin{cases} i, & \text{if } \gamma = \gamma_1, \\ -i, & \text{if } \gamma = \gamma_2. \end{cases}$$

We also define $\xi_j = -\xi$, for $j = 1, 2, 3, ...$ Then, the distribution of ξ_j and ξ are same

$$\varphi_j(a) = \varphi_j(c + id) = \begin{cases} 0, & \text{if } c < 0, -\infty < d < \infty, \\ 0, & \text{if } c \geq 0, d < -1, \\ \frac{1}{2}, & \text{if } c \geq 0, -1 < d < 1, \\ 1, & \text{if } c \geq 0, d \geq 1. \end{cases}$$

Then, by Definition 2.10 and 2.11, complex uncertain sequence (ξ_j) is I–deferred strongly Cesàro summable as well as μ–deferred I–statistically convergent in distribution to ξ.

Definition 2.13. The sequence (ξ_j) of complex uncertain variables is said to be I–deferred strongly Cesàro summable with respect to uniformly almost surely (u.a.s.) to ξ if $\forall \varepsilon > 0$, $\exists \delta > 0$ and a sequence (E_j) with

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\sum_{j=p_n}^{q_n} |\mathcal{M}(E_j) - 0| \geq \varepsilon\right\} \in I$$

i.e., $DC^I_{p,q} - \lim \mathcal{M}(E_j) = 0$ such that

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\sum_{j=p_n}^{q_n} \|\xi_j(\gamma) - \xi(\gamma)\| \geq \delta\right\} \in I, \ \ \forall \gamma \in \Gamma - (E_j).$$

We write it as $DC^I_{p,q}(\Gamma_{u.a.s.}) - \lim(\xi_j) = \xi$. Space of all I–deferred strongly Cesàro summable complex uncertain sequences with respect to uniformly almost surely (u.a.s.) is denoted by $DC^I_{p,q}(\Gamma_{u.a.s.})$.

Definition 2.14. The sequence (ξ_j) of complex uncertain variables is said to be μ–deferred I–statistically convergent w.r.t. uniformly almost surely (u.a.s.) to ξ if $\forall \varepsilon, \zeta > 0$, $\exists \delta > 0$ and a sequence (E_j) with

$$\left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : |\mathcal{M}(E_j) - 0| \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} \geq \zeta\right\} \in I$$

i.e. ${}_{\mu}DS^I_{p,q} - \lim \mathcal{M}(E_j) = 0$ such that

$$\left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \|\xi_j(\gamma) - \xi(\gamma)\| \geq \delta\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} \geq \zeta\right\} \in I, \quad \forall \gamma \in \Gamma - (E_j).$$

We write it as ${}_{\mu}DS^I_{p,q}(\Gamma_{u.a.s.}) - \lim(\xi_j) = \xi$. Space of all μ–deferred I–statistically convergent complex uncertain sequences with respect to uniformly almost surely (u.a.s.) is denoted by ${}_{\mu}DS^I_{p,q}(\Gamma_{u.a.s.})$.

Example 2.15. Suppose the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$, where $\mathcal{L} = P(\Gamma)$, $\Gamma = \{\gamma_1, \gamma_2\gamma_3, \gamma_4\}$ and

$$\mathcal{M}(\Lambda) = \begin{cases} 0, & \text{if } \Lambda = \emptyset, \\ 1, & \text{if } \Lambda = \Gamma, \\ 0.6, & \text{if } \gamma_1 \in \Lambda, \\ 0.4, & \text{if } \gamma_1 \notin \Lambda. \end{cases}$$

We define a complex uncertain variables by

$$\xi_j(\gamma) = \begin{cases} i, & \text{if } \gamma = \gamma_1, \\ 2i, & \text{if } \gamma = \gamma_2, \\ 3i, & \text{if } \gamma = \gamma_3, \\ 4i, & \text{if } \gamma = \gamma_4, \\ 0, & \text{otherwise} \end{cases}$$

and $\xi \equiv 0$. Clearly, by Definition 2.13 and 2.14, complex uncertain sequence (ξ_j) is I–deferred strongly Cesàro summable as well as μ–deferred I–statistically convergent uniformly almost surely to ξ.

Theorem 2.16. *If a complex uncertain sequence (ξ_j) is I–deferred strongly Cesàro summable almost surely (a.s.) to ξ then (ξ_j) is μ–deferred I–statistically convergent almost surely to ξ.*

Proof. Let (ξ_j) be the complex uncertain sequence which is I–deferred strongly Cesàro summable almost surely (a.s.) to ξ then by definition 2.1 $\forall \varepsilon > 0$, $\exists$ an event Λ such that uncertain measure of λ is unity then the following set

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n} \sum_{j=p_n}^{q_n} \|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon\right\} \in I, \text{ for every } \gamma \in \Lambda. \tag{4}$$

Now, for any $\varepsilon > 0$, we have

$$\begin{aligned} \sum_{j=p_n}^{q_n} \|\xi_j(\gamma) - \xi(\gamma)\| &\geq \sum_{p_n, \|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon}^{q_n} \|\xi_j(\gamma) - \xi(\gamma)\| \\ &\geq |I^*_{p,q}(n)|\varepsilon \\ &\geq |\{j \in \mathbb{N} : \|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon\} \cap I^*_{p,q}(n)|\varepsilon. \end{aligned}$$

which implies that

$$\frac{1}{\varepsilon|I^*_{p,q}(n)|}\sum_{j=p_n}^{q_n}\|\xi_j(\gamma)-\xi(\gamma)\| \geq \frac{1}{|I^*_{p,q}(n)|}|\{j\in\mathbb{N}:\|\xi_j(\gamma)-\xi(\gamma)\|\geq\varepsilon\}\cap I^*_{p,q}(n)|.$$

Thus, for any $\delta > 0$ we get,

$$\left\{n\in\mathbb{N}:\frac{|\{j\in\mathbb{N}:\|\xi_j(\gamma)-\xi(\gamma)\|\geq\varepsilon\}\cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|}\geq\delta\right\}\subseteq\left\{n\in\mathbb{N}:\frac{1}{q_n-p_n}\sum_{j=p_n}^{q_n}\|\xi_j(\gamma)-\xi(\gamma)\|\geq\varepsilon\delta\right\}\in I$$

Hence, (ξ_j) is μ–deferred I–statistically convergent almost surely to ξ. □

Remark 2.17. Converse of Theorem 2.16 is not true in general.

Example 2.18. Let p_n and q_n are given as defined earlier in (3), Take an uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ to be $\{\gamma_1,\gamma_2,...\}$ with

$$\mathcal{M}(\Lambda)=\begin{cases}\sup_{\gamma_k\in\Lambda}\frac{1}{k+1}, & \text{if } \sup_{\gamma_k\in\Lambda}\frac{1}{k+1}<\frac{1}{2},\\ 1-\sup_{\gamma_k\in\Lambda^c}\frac{1}{k+1}, & \text{if } \sup_{\gamma_k\in\Lambda^c}\frac{1}{k+1}<\frac{1}{2},\\ \frac{1}{2}, & \text{otherwise .}\end{cases}$$

and the complex uncertain variables are defined by

$$\xi_j(\gamma=\gamma_k)=\begin{cases}(jk)i, & \text{if } q_n-\sqrt{q_n}\leq j\leq q_n,\\ 0, & \text{otherwise}\end{cases}$$

and $\xi\equiv 0$. Hence, for any small $\varepsilon>0$ and any $\gamma=\gamma_k$,

$$\frac{|\{j\in\mathbb{N}:\|\xi_j(\gamma)-\xi(\gamma)\|\geq\varepsilon\}\cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|}\leq\frac{\sqrt{q_n}}{|I^*_{p,q}(n)|}.$$

Thus for any $\delta>0$ we obtain we have

$$\left\{n\in\mathbb{N}:\frac{|\{j\in\mathbb{N}:\|\xi_j(\gamma)-\xi(\gamma)\|\geq\varepsilon\}\cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|}\geq\delta\right\}\subseteq\left\{n\in\mathbb{N}:\frac{\sqrt{q_n}}{|I^*_{p,q}(n)|}\geq\delta\right\}.$$

Since the set $\left\{n\in\mathbb{N}:\frac{\sqrt{q_n}}{|I^*_{p,q}(n)|}\geq\delta\right\}$ is finite, so belongs to I, therefore $(\xi_j)\in\ {}_{\mu}DS^I_{(p,q)}(\Gamma_{a.s.})$.
But for any $\gamma=\gamma_k$,

$$\frac{1}{q_n-p_n}\sum_{j=p_n}^{q_n}\|\xi_j(\gamma)-\xi(\gamma)\|=\frac{k}{q_n-p_n}\sum_{q_n-\sqrt{q_n}}^{q_n}|j|=\frac{k}{q_n-p_n}\frac{2q_n{}^{\frac{3}{2}}-(q_n-\sqrt{q_n})}{2}\leq\frac{k}{q_n-p_n}\frac{2q_n{}^{\frac{3}{2}}-p_n}{2}.$$

Thus we have

$$\left\{n\in\mathbb{N}:\frac{1}{q_n-p_n}\sum_{j=p_n}^{q_n}\|\xi_j(\gamma)-\xi(\gamma)\|\geq\frac{1}{4}\right\}\subseteq\left\{n\in\mathbb{N}:\frac{2q_n{}^{\frac{3}{2}}-p_n}{q_n-p_n}\geq\frac{1}{2k}\right\}=\{a,a+1,a+2,...\}$$

for some $a\in\mathbb{N}$ and so belongs to $\mathcal{F}(I)$ as I is an admissible ideal. Hence $(\xi_j)\notin DC^I_{(p,q)}(\Gamma_{a.s.})$.

Theorem 2.19. *If a complex uncertain sequence (ξ_j) is I–deferred strongly Cesàro summable in measure to ξ then (ξ_j) is μ–deferred I–statistically convergent in measure to ξ.*

Proof. Let (ξ_j) be the complex uncertain sequence which is I–deferred strongly Cesàro summable in measure to ξ. Then by Definition 2.4 for all $\varepsilon, \delta > 0$, we obtain

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n} \sum_{j=p_n}^{q_n} \mathcal{M}(\|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon) \geq \delta\right\} \in I. \tag{5}$$

Now, for any $\varepsilon, \delta > 0$, we have

$$\begin{aligned}\sum_{j=p_n}^{q_n} \mathcal{M}(\|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon) &\geq \sum_{p_n, \mathcal{M}(\|\xi_j(\gamma)-\xi(\gamma)\|\geq\varepsilon)\geq\delta}^{q_n} \mathcal{M}(\|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon)\\ &\geq |I^*_{p,q}(n)|\delta\\ &\geq |\{j \in \mathbb{N} : \mathcal{M}(\|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon) \geq \delta\} \cap I^*_{p,q}(n)|\delta,\end{aligned}$$

which implies,

$$\frac{1}{\delta|I^*_{p,q}(n)|} \sum_{j=p_n}^{q_n} \mathcal{M}(\|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon) \geq \frac{1}{|I^*_{p,q}(n)|} |\{j \in \mathbb{N} : \mathcal{M}(\|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon) \geq \delta\} \cap I^*_{p,q}(n)|.$$

Thus, for any $\zeta > 0$ we get,

$$\begin{aligned}&\left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \mathcal{M}(\|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon) \geq \delta\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} \geq \zeta\right\} \subseteq\\ &\qquad \left\{n \in \mathbb{N} : \frac{1}{q_n - p_n} \sum_{j=p_n}^{q_n} \mathcal{M}(\|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon) \geq \zeta\delta\right\} \in I.\end{aligned}$$

Hence, (ξ_j) is μ–deferred I–statistically convergent in measure to ξ. □

Now some theorems are given below without proof because by using same techniques as above, these theorems can be established.

Theorem 2.20. *If a complex uncertain sequence (ξ_j) is I–deferred strongly Cesàro summable in mean to ξ, then (ξ_j) is μ–deferred I–statistically convergent in mean to ξ.*

Theorem 2.21. *If a complex uncertain sequence (ξ_j) is I–deferred strongly Cesáro summable in distribution to ξ, then (ξ_j) is μ–deferred I–statistically convergent in distribution to ξ.*

Theorem 2.22. *If a complex uncertain sequence (ξ_j) is I–deferred strongly Cesàro summable uniformly almost surely to ξ, then (ξ_j) is μ–deferred I–statistically convergent uniformly almost surely to ξ.*

Definition 2.23. Let (ξ_j) be a sequence of complex uncertain variables. We say (ξ_j) is bounded in measure if $\forall\varepsilon > 0$, $\exists D > 0$ such that

$$\mathcal{M}(\|\xi_j\| \geq D) < \varepsilon.$$

We denote the set of such type of complex uncertain sequences by $\ell_\infty(\Gamma_{\mathcal{M}})$.

Definition 2.24. Let (ξ_j) be a sequence of complex uncertain variables. We say (ξ_j) is bounded in mean if

$$\sup_j E[\|\xi_j\|] \text{ is finite.}$$

We denote the set of such type of complex uncertain sequences by $\ell_\infty(\Gamma_E)$.

Definition 2.25. Let (ξ_j) be a sequence of complex uncertain variables and φ_j be the distribution for the complex uncertain variable ξ_j. Then we say (ξ_j) is bounded in distribution if

$$\sup_j \|\varphi_j(a)\|] \text{ is finite,}$$

where a is the point at which the distribution function is continuous. We denote the set of such type of complex uncertain sequences by $\ell_\infty(\Gamma_\varphi)$.

Definition 2.26. Let (ξ_j) be a sequence of complex uncertain variables. We say (ξ_j) is bounded in almost surely if $\forall \varepsilon > 0$ $\exists$ some event Λ with $\mathcal{M}(\Lambda) = 1$ such that

$$\sup_j \|\xi_j(\gamma)\|] \text{ is finite, } \forall \gamma \in \Lambda.$$

We denote the set of such type of complex uncertain sequences by $\ell_\infty(\Gamma_{a.s.})$.

Definition 2.27. *Let (ξ_j) be a sequence of complex uncertain variables. We say (ξ_j) is bounded w.r.t. uniformly almost surely if $\forall \varepsilon > 0$ $\exists$ sequence (E_j) with $\mathcal{M}(E_j) \to 0$ and*

$$\sup_j \|\xi_j(\gamma)\|] \textit{ is finite, } \forall \gamma \in \Lambda - (E_j).$$

We denote the set of such type of complex uncertain sequences by $\ell_\infty(\Gamma_{u.a.s.})$.

Theorem 2.28. *If $(\xi_j) \in l_\infty(\Gamma_{a.s.})$ then ${}_\mu DS^I_{(p,q)}(\Gamma_{a.s.}) \subset DC^I_{(p,q)}(\Gamma_{a.s.})$.*

Proof. Let a sequence $(\xi_j) \in \ell_\infty(\Gamma_{a.s.})$ such that ${}_\mu DS^I_{p,q}(\Gamma_{a.s.}) - \lim(\xi_j) = \xi$. Hence $\exists D > 0$ such that $\|\xi_j(\gamma) - \xi(\gamma)\| \leq D$ for all $j \in \mathbb{N}$ and for all γ.

For a given $\varepsilon > 0$, we have

$$\begin{aligned} &\tfrac{1}{q_n-p_n}\textstyle\sum_{j=p_n}^{q_n} \|\xi_j(\gamma) - \xi(\gamma)\| \\ &= \frac{1}{q_n - p_n} \sum_{p_n, \|\xi_j(\gamma)-\xi(\gamma)\| \geq \frac{\varepsilon}{2}}^{q_n} \|\xi_j(\gamma) - \xi(\gamma)\| + \frac{1}{q_n - p_n} \sum_{p_n, \|\xi_j(\gamma)-\xi(\gamma)\| < \frac{\varepsilon}{2}}^{q_n} \|\xi_j(\gamma) - \xi(\gamma)\| \\ &\leq \frac{D}{|I^*_{p,q}(n)|} |\{j \in \mathbb{N} : \|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon\} \cap I^*_{p,q}(n)| + \frac{\varepsilon}{2}. \end{aligned}$$

Therefore, we get

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n} \sum_{j=p_n}^{q_n} \|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon\right\} \subseteq \left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} \geq \frac{\varepsilon}{2D}\right\} \in I.$$

Hence $(\xi_j) \in DC^I_{(p,q)}(\Gamma_{a.s.})$ This concludes the theorem. □

We can easily prove the following results by the same method as above, hence the proof of the following Theorems are omitted.

Theorem 2.29. *If $(\xi_j) \in l_\infty(\Gamma_{\mathcal{M}})$ then ${}_\mu DS^I_{(p,q)}(\Gamma_{\mathcal{M}}) \subset DC^I_{(p,q)}(\Gamma_{\mathcal{M}})$.*

Theorem 2.30. *If $(\xi_j) \in l_\infty(\Gamma_E)$ then ${}_\mu DS^I_{(p,q)}(\Gamma_E) \subset DC^I_{(p,q)}(\Gamma_E)$.*

Theorem 2.31. *If $(\xi_j) \in l_\infty(\Gamma_\varphi)$ then ${}_\mu DS^I_{(p,q)}(\Gamma_\varphi) \subset DC^I_{(p,q)}(\Gamma_\varphi)$.*

Theorem 2.32. *If $(\xi_j) \in l_\infty(\Gamma_{u.a.s.})$ then ${}_\mu DS^I_{(p,q)}(\Gamma_{a.s.}) \subset DC^I_{(p,q)}(\Gamma_{u.a.s.})$.*

From Theorem 2.16 and 2.28, Theorem 2.19 and 2.29, Theorem 2.20 and 2.30, Theorem 2.21 and 2.31, Theorem 2.22 and 2.32, we have following result.

Theorem 2.33. *The following statements hold:*
a) ${}_{\mu}DS^{I}_{(p,q)}(\Gamma_{a.s.}) \cap \ell_{\infty}(\Gamma_{a.s.}) = DC^{I}_{(p,q)}(\Gamma_{a.s.}) \cap \ell_{\infty}(\Gamma_{a.s.})$.
b) ${}_{\mu}DS^{I}_{(p,q)}(\Gamma_{\mathcal{M}}) \cap \ell_{\infty}(\Gamma_{\mathcal{M}}) = DC^{I}_{(p,q)}(\Gamma_{\mathcal{M}}) \cap \ell_{\infty}(\Gamma_{\mathcal{M}})$.
c) ${}_{\mu}DS^{I}_{(p,q)}(\Gamma_{E}) \cap \ell_{\infty}(\Gamma_{E}) = DC^{I}_{(p,q)}(\Gamma_{E}) \cap \ell_{\infty}(\Gamma_{E})$.
d) ${}_{\mu}DS^{I}_{(p,q)}(\Gamma_{\varphi}) \cap \ell_{\infty}(\Gamma_{\varphi}) = DC^{I}_{(p,q)}(\Gamma_{\varphi}) \cap \ell_{\infty}(\Gamma_{\varphi})$.
e) ${}_{\mu}DS^{I}_{(p,q)}(\Gamma_{u.a.s.}) \cap \ell_{\infty}(\Gamma_{u.a.s.}) = DC^{I}_{(p,q)}(\Gamma_{u.a.s.}) \cap \ell_{\infty}(\Gamma_{u.a.s.})$.

Theorem 2.34. *The space of all bounded as well as μ–deferred I–statistically convergent complex uncertain sequences in measure i.e., ${}_{\mu}DS^{I}_{(p,q)}(\Gamma_{\mathcal{M}}) \cap \ell_{\infty}(\Gamma_{\mathcal{M}})$ is a closed subspace of $\ell_{\infty}(\Gamma_{\mathcal{M}})$.*

Proof. Let $\xi^{m} = (\xi^{m}_{j})_{m\in\mathbb{N}} \in {}_{\mu}DS^{I}_{(p,q)}(\Gamma_{\mathcal{M}}) \cap \ell_{\infty}(\Gamma_{\mathcal{M}})$ is a convergent sequence and convergent to $\xi = (\xi_{j}) \in \ell_{\infty}(\Gamma_{\mathcal{M}})$. Suppose ${}_{\mu}DS^{I}_{p,q}(\Gamma_{\mathcal{M}}) - \lim \xi^{m} = k_{m}$, for all $m \in \mathbb{N}$. Take a sequence (ε_{m}) such that $\varepsilon_{m} = \frac{\varepsilon}{2^{m}}$. Hence, for any $\varepsilon > 0$, $(\varepsilon_{m}) \to 0$. So choose a natural number m such that $\mathcal{M}(\|\xi - \xi^{m}\| \geq \zeta) < \frac{\varepsilon_{m}}{4}$. Suppose $\delta \in (0,1)$ then

$$A = \left\{ n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \mathcal{M}(\|\xi^{m}_{j} - k_{m}\| \geq \zeta) \geq \frac{\varepsilon_{m}}{4}\} \cap I^{*}_{p,q}(n)|}{|I^{*}_{p,q}(n)|} < \frac{\delta}{3} \right\} \in \mathcal{F}(I)$$

and

$$B = \left\{ n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \mathcal{M}(\|\xi^{m+1}_{j} - k_{m+1}\| \geq \zeta) \geq \frac{\varepsilon_{m+1}}{4}\} \cap I^{*}_{p,q}(n)|}{|I^{*}_{p,q}(n)|} < \frac{\delta}{3} \right\} \in \mathcal{F}(I).$$

Hence, $A \cap B \in \mathcal{F}(I)$. Therefore, $A \cap B$ can not be empty set so choose $n \in A \cap B$. then

$$\frac{|\{j \in \mathbb{N} : \mathcal{M}(\|\xi^{m}_{j} - k_{m}\| \geq \zeta) \geq \frac{\varepsilon_{m}}{4}\} \cap I^{*}_{p,q}(n)|}{|I^{*}_{p,q}(n)|} < \frac{\delta}{3}$$

and

$$\frac{|\{j \in \mathbb{N} : \mathcal{M}(\|\xi^{m+1}_{j} - k_{m+1}\| \geq \zeta) \geq \frac{\varepsilon_{m+1}}{4}\} \cap I^{*}_{p,q}(n)|}{|I^{*}_{p,q}(n)|} < \frac{\delta}{3}$$

Hence

$$\frac{|\{j \in \mathbb{N} : \mathcal{M}(\|\xi^{m}_{j} - k_{m}\| \geq \zeta) \geq \frac{\varepsilon_{m}}{4} \vee \mathcal{M}(\|\xi^{m+1}_{j} - k_{m+1}\| \geq \zeta) \geq \frac{\varepsilon_{m+1}}{4}\} \cap I^{*}_{p,q}(n)|}{|I^{*}_{p,q}(n)|} < \delta < 1.$$

So there exists a j such that $\mathcal{M}(\|\xi^{m}_{j} - k_{m}\| \geq \zeta) < \frac{\varepsilon_{m}}{4}$ and $\mathcal{M}(\|\xi^{m+1}_{j} - k_{m+1}\| \geq \zeta) < \frac{\varepsilon_{m+1}}{4}$.
So we have,

$$\begin{aligned}
&\mathcal{M}(\|k_{m} - k_{m+1}\| \geq \zeta) \\
&\leq \mathcal{M}(\|\xi^{m}_{j} - k_{m}\| \geq \zeta') + \mathcal{M}(\|\xi^{m}_{j} - \xi^{m+1}_{j}\| \geq \zeta') + \mathcal{M}(\|\xi^{m+1}_{j} - k_{m+1}\| \geq \zeta'), \text{ for some } \zeta' \leq \frac{\zeta}{3} \\
&\leq \mathcal{M}(\|\xi^{m}_{j} - k_{m}\| \geq \zeta') + \mathcal{M}(\|\xi^{m}_{j} - \xi\| \geq \zeta'') + \mathcal{M}(\|\xi^{m+1}_{j} - \xi\| \geq \zeta'') + \mathcal{M}(\|\xi^{m+1}_{j} - k_{m+1}\| \geq \zeta'), \text{ for some } \zeta'' \leq \frac{\zeta'}{2} \\
&\leq \frac{\varepsilon_{m}}{4} + \frac{\varepsilon_{m+1}}{4} + \frac{\varepsilon_{m}}{4} + \frac{\varepsilon_{m+1}}{4} \leq \varepsilon_{m}
\end{aligned}$$

Hence (k_{m}) is a Cauchy sequence in measure, so there exists k such that $k_{m} \to k$ in measure, as $m \to \infty$. Now we prove that $(\xi_{j}) \in {}_{\mu}DS^{I}_{(p,q)}(\Gamma_{\mathcal{M}})$ and ${}_{\mu}DS^{I}_{(p,q)}(\Gamma_{\mathcal{M}}) - \lim\ (\xi_{j}) = k$. For a given $\varepsilon, \zeta > 0$, choose $m \in \mathbb{N}$ such that

$\varepsilon_m < \frac{\varepsilon}{4}$, $\mathcal{M}(\|\xi_j^m - \xi\| \geq \zeta) < \frac{\varepsilon}{4}$, $\mathcal{M}(\|k_m - k\| \geq \zeta) < \frac{\varepsilon}{4}$. Then

$$\begin{aligned}
&\frac{|\{n\in\mathbb{N}:\mathcal{M}(\|\xi_j-k\|\geq\zeta)\geq\varepsilon\}\cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|}\\
\leq\ & \frac{|\{j \in \mathbb{N} : \mathcal{M}(\|\xi_j^m - k_m\| \geq \zeta') + \mathcal{M}(\|\xi_j - \xi_j^m\| \geq \zeta') + \mathcal{M}(\|k_m - k\| \geq \zeta') \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|}\\
\leq\ & \frac{|\{j \in \mathbb{N} : \mathcal{M}(\|\xi_j^m - k_m\| \geq \zeta') + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|}\\
\leq\ & \frac{|\{j \in \mathbb{N} : \mathcal{M}(\|\xi_j^m - k_m\| \geq \zeta') \geq \frac{\varepsilon}{2}\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|}.
\end{aligned}$$

Therefore, for any $\delta > 0$ we obtain

$$\begin{aligned}
&\left\{n \in \mathbb{N} : \frac{|\{j\in\mathbb{N}:\mathcal{M}(\|\xi_j-k\|\geq\zeta)\geq\varepsilon\}\cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} < \delta\right\}\\
&\supseteq \left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \mathcal{M}(\|\xi_j^m - k_m\| \geq \zeta') \geq \frac{\varepsilon}{2}\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} < \delta\right\} \in \mathcal{F}(I).
\end{aligned}$$

Consequently, we have

$$\left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \mathcal{M}(\|\xi_j - k\| \geq \zeta) \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} < \delta\right\} \in \mathcal{F}(I).$$

Hence, $(\xi_j) \in {}_{\mu}DS^I_{(p,q)}(\Gamma_{\mathcal{M}})$ and ${}_{\mu}DS^I_{(p,q)}(\Gamma_{\mathcal{M}}) - \lim\ (\xi_j) = k$. This concludes the result. □

Theorem 2.35. *The space of all bounded as well as μ–deferred I–statistically convergent complex uncertain sequences in mean i.e., ${}_{\mu}DS^I_{(p,q)}(\Gamma_E) \cap \ell_\infty(\Gamma_E)$ is a closed linear subspace of $\ell_\infty(\Gamma_E)$.*

Proof. In the proof of Theorem 2.34, take uncertain expected value operator. □

Theorem 2.36. *The space of all bounded as well as μ–deferred I–statistically convergent complex uncertain sequences in distribution i.e., ${}_{\mu}DS^I_{(p,q)}(\Gamma_\varphi) \cap \ell_\infty(\Gamma_\varphi)$ is a closed linear subspace of $\ell_\infty(\Gamma_\varphi)$.*

Proof. By taking complex uncertainty function in the proof of Theorem 2.34, the proof can be established. □

Theorem 2.37. *The space of all bounded as well as μ–deferred I–statistically convergent complex uncertain sequences with respect to uniformly almost surely i.e., ${}_{\mu}DS^I_{(p,q)}(\Gamma_{\mathcal{M}}) \cap \ell_\infty(\Gamma_{u.a.s.})$ is a closed subspace of $\ell_\infty(\Gamma_{u.a.s.})$.*

Proof. Let $\xi^m = (\xi_j^m)_{m\in\mathbb{N}} \in {}_{\mu}DS^I_{(p,q)}(\Gamma_{u.a.s.}) \cap \ell_\infty(\Gamma_{u.a.s.})$ is a convergent sequence and convergent to $\xi = (\xi_j) \in \ell_\infty(\Gamma_{u.a.s.})$. Suppose ${}_{\mu}DS^I_{p,q}(\Gamma_{u.a.s.}) - \lim \xi^m = k_m$, for all $m \in \mathbb{N}$. Take a sequence (ε_m) such that $\varepsilon_m = \frac{\varepsilon}{2^m}$. Hence, for any $\varepsilon > 0$, $(\varepsilon_m) \to 0$. So choose a natural number m s.t. $\|\xi_j^m(\gamma) - \xi_j(\gamma)\| < \frac{\varepsilon_m}{4}$. As ${}_{\mu}DS^I_{p,q}(\Gamma_{u.a.s.}) - \lim \xi^m = k_m$, there exists a sequence of events (E_j) such that ${}_{\mu}DS^I_{p,q} - \lim \mathcal{M}(E_j) = 0$. Let $\delta \in (0, 1)$ then

$$A = \left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \|\xi_j^m(\gamma) - k_m\| \geq \frac{\varepsilon_m}{4}\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} < \frac{\delta}{3}\right\} \in \mathcal{F}(I)$$

and

$$B = \left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \|\xi_j^{m+1}(\gamma) - k_{m+1}\| \geq \frac{\varepsilon_{m+1}}{4}\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} < \frac{\delta}{3}\right\} \in \mathcal{F}(I)$$

$\forall\gamma \in \Gamma - (E_j)$. Hence, $A \cap B \in \mathcal{F}(I)$. Therefore, $A \cap B$ can not be empty set so choose $n \in A \cap B$. then

$$\frac{|\{j \in \mathbb{N} : \|\xi_j^m(\gamma) - k_m\| \geq \frac{\varepsilon_m}{4}\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} < \frac{\delta}{3}$$

and

$$\frac{|\{j \in \mathbb{N} : \|\xi_j^{m+1}(\gamma) - k_{m+1}\| \geq \frac{\varepsilon_{m+1}}{4}\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} < \frac{\delta}{3}$$

Hence

$$\frac{|\{j \in \mathbb{N} : \|\xi_j^m(\gamma) - k_m\| \geq \frac{\varepsilon_m}{4} \vee \|\xi_j^{m+1}(\gamma) - k_{m+1}\| \geq \frac{\varepsilon_{m+1}}{4}\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} < \delta < 1.$$

So there exists a j such that $\|\xi_j^m(\gamma) - k_m\| < \frac{\varepsilon_m}{4}$ and $\|\xi_j^{m+1}(\gamma) - k_{m+1}\| < \frac{\varepsilon_{m+1}}{4}$.
So we have,

$$\begin{aligned}
\|k_m - k_{m+1}\| &\leq \|\xi_j^m(\gamma) - k_m\| + \|\xi_j^m(\gamma) - \xi_j^{m+1}(\gamma)\| + \|\xi_j^{m+1}(\gamma) - k_{m+1}\| \\
&\leq \|\xi_j^m(\gamma) - k_m\| + \|\xi_j^m(\gamma) - \xi(\gamma)\| + \|\xi_j^{m+1}(\gamma) - \xi(\gamma)\| + \|\xi_j^{m+1}(\gamma) - k_{m+1}\| \\
&\leq \frac{\varepsilon_m}{4} + \frac{\varepsilon_{m+1}}{4} + \frac{\varepsilon_m}{4} + \frac{\varepsilon_{m+1}}{4} \leq \varepsilon_m
\end{aligned}$$

Hence (k_m) is a Cauchy sequence, hence there exists k such that $(k_m) \to k$, as $m \to \infty$. Now we prove that $(\xi_j) \in {}_\mu DS^I_{(p,q)}(\Gamma_{u.a.s.})$ and ${}_\mu DS^I_{(p,q)}(\Gamma_{u.a.s.}) - \lim\ (\xi_j) = k$.
As ${}_\mu DS^I_{p,q}(\Gamma_{u.a.s.}) - \lim \xi^m = k_m$, for a given $\varepsilon > 0, \delta > 0$, there exists a sequence of events (E_j) such that ${}_\mu DS^I_{p,q} - \lim \mathcal{M}(E_j) = 0$ and

$$\left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \|\xi_j^m(\gamma) - k_m\| \geq \frac{\varepsilon}{2}\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} < \delta\right\} \in \mathcal{F}(I), \ \ \gamma \in \Gamma - (E_j).$$

Choose $m \in \mathbb{N}$ such that $\varepsilon_m < \frac{\varepsilon}{4}$, $\|\xi_j^m(\gamma) - \xi_j(\gamma)\| < \frac{\varepsilon}{4}$, $\|k_m - k\| < \frac{\varepsilon}{4}$. Then for any $\gamma \in \Gamma - (E_j)$ we get

$$\begin{aligned}
\frac{|\{n \in \mathbb{N} : \|\xi_j(\gamma) - k\| \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} &\leq \frac{|\{j \in \mathbb{N} : \|\xi_j^m(\gamma) - k_m\| + \|\xi_j(\gamma) - \xi_j^m(\gamma)\| + \|k_m - k\| \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} \\
&\leq \frac{|\{j \in \mathbb{N} : \|\xi_j^m(\gamma) - k_m\| + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} \\
&\leq \frac{|\{j \in \mathbb{N} : \|\xi_j^m(\gamma) - k_m\| \geq \frac{\varepsilon}{2}\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|}.
\end{aligned}$$

holds. Therefore, for any $\delta > 0$ we obtain

$$\left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \|\xi_j(\gamma) - k\| \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} < \delta\right\} \supseteq$$
$$\left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \|\xi_j^m(\gamma) - k_m\| \geq \frac{\varepsilon}{2}\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} < \delta\right\} \in \mathcal{F}(I).$$

Consequently, we have

$$\left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \|\xi_j(\gamma) - k\| \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} < \delta\right\} \in \mathcal{F}(I), \ \ \forall\gamma \in \Gamma - (E_j).$$

Hence, $(\xi_j) \in {}_\mu DS^I_{(p,q)}(\Gamma_{u.a.s.})$ and ${}_\mu DS^I_{(p,q)}(\Gamma_{u.a.s.}) - \lim\ (\xi_j) = k$. This concludes the result. □

By using the same techniques and methods as adopted above, next result can be proved.

Theorem 2.38. *The space of all bounded as well as μ–deferred I–statistically convergent complex uncertain sequences with respect to almost surely i.e., ${}_{\mu}DS^{I}_{(p,q)}(\Gamma_{a.s.}) \cap \ell_\infty(\Gamma_{a.s.})$ is a closed subspace of $\ell_\infty(\Gamma_{a.s.})$.*

Remark 2.39. By Theorem 2.33, 2.34, 2.35, 2.37, 2.38 and 2.36 we obtain:
a) Space $DC^{I}_{(p,q)}(\Gamma_{\mathcal{M}}) \cap \ell_\infty(\Gamma_{\mathcal{M}})$ is a closed subspace of $\ell_\infty(\Gamma_{\mathcal{M}})$.
b) Space $DC^{I}_{(p,q)}(\Gamma_{E}) \cap \ell_\infty(\Gamma_{E})$ is a closed subspace of $\ell_\infty(\Gamma_{E})$.
c) Space $DC^{I}_{(p,q)}(\Gamma_{u.a.s.}) \cap \ell_\infty(\Gamma_{u.a.s.})$ is a closed subspace of $\ell_\infty(\Gamma_{u.a.s.})$.
d) Space $DC^{I}_{(p,q)}(\Gamma_{a.s.}) \cap \ell_\infty(\Gamma_{a.s.})$ is a closed subspace of $\ell_\infty(\Gamma_{a.s.})$.
e) Space $DC^{I}_{(p,q)}(\Gamma_{\varphi}) \cap \ell_\infty(\Gamma_{\varphi})$ is a closed subspace of $\ell_\infty(\Gamma_{\varphi})$.

Theorem 2.40. *If $\liminf_n \frac{p_n}{q_n} \neq 1$, then ${}_{\mu}S^{I}(\Gamma_{\mathcal{M}}) \subset {}_{\mu}DS^{I}_{(p,q)}(\Gamma_{\mathcal{M}})$, where ${}_{\mu}S^{I}(\Gamma_{\mathcal{M}})$ denotes space of all I–${}_{\mu}$statistically convergent complex uncertain sequences in measure.*

Proof. Let $\liminf_n \frac{p_n}{q_n} = a(\neq 1)$, so there exists $b > 0$ such that $\frac{p_n}{q_n} \geq a + b$ for sufficiently large n. Hence we have

$$\frac{q_n - p_n}{q_n} \geq \frac{b}{a+b}$$

Suppose $(\xi_j) \in {}_{\mu}S^{I}(\Gamma_{\mathcal{M}})$. Then for all $\varepsilon, \zeta > 0$ and $\delta > 0$, the set

$$\left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \mathcal{M}(\|\xi_j - \xi\| \geq \zeta) \geq \varepsilon\} \cap I^*(n)|}{|I^*(n)|} \geq \delta\right\}$$

belongs to I. As $\lim_{n\to\infty}(q_n) = \infty$, we also have for any given $\varepsilon, \zeta > 0$ and $\delta > 0$, the set

$$\left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \mathcal{M}(\|\xi_j - \xi\| \geq \zeta) \geq \varepsilon\} \cap I^*_q(n)|}{|I^*_q(n)|} \geq \delta\right\} \in I.$$

For any pair of sequences (p, q) which satisfy (3), $I^*_q(n) \supset I^*_{p,q}(n)$ holds. Therefore we have

$$\begin{aligned}
\frac{|\{j \in \mathbb{N} : \mathcal{M}(\|\xi_j - \xi\| \geq \zeta) \geq \varepsilon\} \cap I^*_q(n)|}{|I^*_q(n)|} &\geq \frac{|\{j \in \mathbb{N} : \mathcal{M}(\|\xi_j - \xi\| \geq \zeta) \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_q(n)|} \\
&= \frac{|I^*_{p,q}(n)|}{|I^*_q(n)|} \frac{|\{j \in \mathbb{N} : \mathcal{M}(\|\xi_j - \xi\| \geq \zeta) \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} \\
&\geq \frac{b}{a+b} \frac{|\{j \in \mathbb{N} : \mathcal{M}(\|\xi_j - \xi\| \geq \zeta) \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|}
\end{aligned}$$

and so

$$\frac{a+b}{b} \frac{|\{j \in \mathbb{N} : \mathcal{M}(\|\xi_j - \xi\| \geq \zeta) \geq \varepsilon\} \cap I^*_q(n)|}{|I^*_q(n)|} \geq \frac{|\{j \in \mathbb{N} : \mathcal{M}(\|\xi_j - \xi\| \geq \zeta) \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|}.$$

Hence, for any $\delta > 0$ we obtain

$$\begin{aligned}
&\left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \mathcal{M}(\|\xi_j - \xi\| \geq \zeta) \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} \geq \delta\right\} \\
&\subseteq \left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \mathcal{M}(\|\xi_j - \xi\| \geq \zeta) \geq \varepsilon\} \cap I^*_q(n)|}{|I^*_q(n)|} \geq \frac{\delta b}{a+b}\right\} \in I.
\end{aligned}$$

Therefore $(\xi_j) \in {}_{\mu}DS^{I}_{(p,q)}(\Gamma_{\mathcal{M}})$. □

Remark 2.41. If $\liminf_n \frac{p_n}{q_n} \neq 1$, then ${}_{\mu}S^I(\Gamma_{\mathcal{M}}) \subset DC^I_{(p,q)}(\Gamma_{\mathcal{M}})$.

Theorem 2.42. *If $\liminf_n \frac{p_n}{q_n} \neq 1$, then $C^I(\Gamma_{\mathcal{M}}) \subset DC^I_{(p,q)}(\Gamma_{\mathcal{M}})$, where $C^I(\Gamma_{\mathcal{M}})$ denotes space of all I–Cesàro summable complex uncertain sequences in measure.*

Proof. Let $\liminf_n \frac{p_n}{q_n} = a(\neq 1)$, so there exists $b > 0$ such that $\frac{p_n}{q_n} \geq a + b$ for sufficiently large n, we have

$$\frac{q_n - p_n}{q_n} \geq \frac{b}{a+b}$$

Suppose $(\xi_j) \in C^I(\Gamma_{\mathcal{M}})$. Then for all $\varepsilon, \delta > 0$, the set

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{j=1}^{n} \mathcal{M}(\|\xi_j - \xi\| \geq \varepsilon) \geq \delta \right\}$$

belongs to I. As $\lim_{n\to\infty}(q_n) = \infty$, we also have for any given $\varepsilon, \delta > 0$, the set

$$\left\{ n \in \mathbb{N} : \frac{1}{q_n} \sum_{j=1}^{q_n} \mathcal{M}(\|\xi_j - \xi\| \geq \varepsilon) \geq \delta \right\} \in I.$$

For any pair of sequences (p, q) which satisfy (3),

$$\sum_{j=1}^{q_n} \mathcal{M}(\|\xi_j - \xi\| \geq \varepsilon) \quad \geq \quad \sum_{j=p_n}^{q_n} \mathcal{M}(\|\xi_j - \xi\| \geq \varepsilon)$$

holds and also implies that

$$\sum_{j=1}^{q_n} \mathcal{M}(\|\xi_j - \xi\| \geq \varepsilon) \geq \frac{q_n - p_n}{q_n} \frac{1}{q_n - p_n} \sum_{j=p_n}^{q_n} \mathcal{M}(\|\xi_j - \xi\| \geq \varepsilon) \geq \frac{b}{a+b} \frac{1}{q_n - p_n} \sum_{j=p_n}^{q_n} \mathcal{M}(\|\xi_j - \xi\| \geq \varepsilon)$$

and so

$$\frac{a+b}{b} \sum_{j=1}^{q_n} \mathcal{M}(\|\xi_j - \xi\| \geq \varepsilon) \geq \frac{1}{q_n - p_n} \sum_{j=p_n}^{q_n} \mathcal{M}(\|\xi_j - \xi\| \geq \varepsilon)$$

Hence, for any $\delta > 0$ we obtain

$$\left\{ n \in \mathbb{N} : \frac{1}{q_n - p_n} \sum_{j=p_n}^{q_n} \mathcal{M}(\|\xi_j - \xi\| \geq \varepsilon) \geq \delta \right\} \subseteq \left\{ n \in \mathbb{N} : \frac{1}{q_n} \sum_{j=1}^{q_n} \mathcal{M}(\|\xi_j - \xi\| \geq \varepsilon) \geq \frac{\delta b}{a+b} \right\} \in I.$$

Therefore $(\xi_j) \in DC^I_{(p,q)}(\Gamma_{\mathcal{M}})$. □

Remark 2.43. Since every complex uncertain convergent sequence in measure (ξ_j) belongs to ${}_{\mu}S^I(\Gamma_{\mathcal{M}})$ as well as $C^I(\Gamma_{\mathcal{M}})$. By Theorem 2.40 and 2.42 we obtain:

If a sequence (ξ_j) of complex uncertain variables is convergent in measure then $(\xi_j) \in {}_{\mu}DS^I_{(p,q)}(\Gamma_{\mathcal{M}}) \cap DC^I_{(p,q)}(\Gamma_{\mathcal{M}})$.

Theorem 2.44. *If $\liminf_n \frac{p_n}{q_n} \neq 1$, then ${}_{\mu}S^I(\Gamma_E) \subset {}_{\mu}DS^I_{(p,q)}(\Gamma_E)$, where ${}_{\mu}S^I(\Gamma_E)$ denotes space of all I–${}_{\mu}$statistically convergent complex uncertain sequences in mean.*

Proof. Take uncertain expected value operator in the proof of Theorem 2.40. □

Theorem 2.45. *If* $\liminf_n \frac{p_n}{q_n} \neq 1$, *then* $C^I(\Gamma_E) \subset DC^I_{(p,q)}(\Gamma_E)$, *where* $C^I(\Gamma_E)$ *denotes space of all I–Cesàro summable complex uncertain sequences in mean.*

Proof. Take uncertain expected value operator in the proof of Theorem 2.42. □

Theorem 2.46. *If* $\liminf_n \frac{p_n}{q_n} \neq 1$, *then* ${}_\mu S^I(\Gamma_\varphi) \subset {}_\mu DS^I_{(p,q)}(\Gamma_\varphi)$, *where* ${}_\mu S^I(\Gamma_\varphi)$ *denotes space of all I–*$_\mu$*statistically convergent complex uncertain sequences in distribution.*

Proof. In the proof of Theorem 2.40, take complex uncertainty function, the proof can be established similarly. □

Theorem 2.47. *If* $\liminf_n \frac{p_n}{q_n} \neq 1$, *then* $C^I(\Gamma_\varphi) \subset DC^I_{(p,q)}(\Gamma_\varphi)$, *where* $C^I(\Gamma_\varphi)$ *denotes space of all I–Cesàro summable complex uncertain sequences in distribution.*

Proof. In the proof of Theorem 2.42, take complex uncertainty function, the proof can be established. □

Theorem 2.48. *If* $\liminf_n \frac{p_n}{q_n} \neq 1$, *then* ${}_\mu S^I(\Gamma_{a.s.}) \subset {}_\mu DS^I_{(p,q)}(\Gamma_{a.s.})$, *where* ${}_\mu S^I(\Gamma_{a.s.})$ *denotes space of all I–*$_\mu$*statistically convergent complex uncertain sequences w.r.t. almost surely.*

Proof. Let $\liminf_n \frac{p_n}{q_n} = a(\neq 1)$. There exists $b > 0$ such that $\frac{p_n}{q_n} \geq a + b$ for sufficiently large n. So we have

$$\frac{q_n - p_n}{q_n} \geq \frac{b}{a+b}.$$

Suppose $(\xi_j) \in {}_\mu S^I(\Gamma_{a.s.})$. Then for all ε and $\delta > 0$, there exists an event Λ such that $\mathcal{M}(\Lambda) = 1$ so that

$$\left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon\} \cap I^*(n)|}{|I^*(n)|} \geq \delta\right\} \in I, \ \ \forall \gamma \in \Lambda.$$

As $\lim_{n\to\infty}(q_n) = \infty$, we also have for any given $\varepsilon, \delta > 0$, the set

$$\left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon\} \cap I^*_q(n)|}{|I^*_q(n)|} \geq \delta\right\} \in I, \ \ \forall \gamma \in \Lambda.$$

For any pair of sequences (p, q) which satisfy (3), $I^*_q(n) \supset I^*_{p,q}(n)$ holds. Therefore for any $\gamma \in \Lambda$, we have

$$\begin{aligned}
\frac{|\{j \in \mathbb{N} : \|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon\} \cap I^*_q(n)|}{|I^*_q(n)|} &\geq \frac{|\{j \in \mathbb{N} : \|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_q(n)|} \\
&= \frac{|I^*_{p,q}(n)|}{|I^*_q(n)|} \frac{|\{j \in \mathbb{N} : \|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} \\
&\geq \frac{b}{a+b} \frac{|\{j \in \mathbb{N} : \|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|}
\end{aligned}$$

and so

$$\frac{a+b}{b} \frac{|\{j \in \mathbb{N} : \|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon\} \cap I^*_q(n)|}{|I^*_q(n)|} \geq \frac{|\{j \in \mathbb{N} : \|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|}.$$

Hence, for any $\delta > 0$ we obtain

$$\begin{aligned}
&\left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \|\xi_j - \xi\| \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} \geq \delta\right\} \\
&\subseteq \left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \|\xi_j - \xi\| \geq \varepsilon\} \cap I^*_q(n)|}{|I^*_q(n)|} \geq \frac{\delta b}{a+b}\right\} \in I, \ \ \forall \gamma \in \Lambda.
\end{aligned}$$

Therefore $(\xi_j) \in {}_\mu DS^I_{(p,q)}(\Gamma_{a.s.})$. □

Remark 2.49. If $\liminf_n \frac{p_n}{q_n} \neq 1$, then ${}_{\mu}S^I(\Gamma_{a.s.}) \subset DC^I_{(p,q)}(\Gamma_{a.s.})$.

Theorem 2.50. *If* $\liminf_n \frac{p_n}{q_n} \neq 1$, *then* $C^I(\Gamma_{a.s.}) \subset DC^I_{(p,q)}(\Gamma_{a.s.})$, *where* $C^I(\Gamma_{a.s.})$ *denotes space of all I–Cesàro summable complex uncertain sequences with respect to almost surely.*

Proof. Let $\liminf_n \frac{p_n}{q_n} = a(\neq 1)$. There exists $b > 0$ such that $\frac{p_n}{q_n} \geq a + b$ for sufficiently large n. So we have

$$\frac{q_n - p_n}{q_n} \geq \frac{b}{a+b}$$

Suppose $(\xi_j) \in C^I(\Gamma_{a.s.})$. Then for all $\delta > 0$, there exists an event Λ such that $\mathcal{M}(\Lambda) = 1$ so that

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{j=1}^{n} \|\xi_j(\gamma) - \xi(\gamma)\| \geq \delta\right\} \in I, \ \ \forall \gamma \in \Lambda.$$

belongs to I. As $\lim_{n\to\infty}(q_n) = \infty$, we also have for any given $\delta > 0$, the set

$$\left\{n \in \mathbb{N} : \frac{1}{q_n}\sum_{j=1}^{q_n} \|\xi_j(\gamma) - \xi(\gamma)\| \geq \delta\right\} \in I, \ \ \forall \gamma \in \Lambda.$$

For any pair of sequences (p, q) which satisfy (3), for any $\gamma \in \Lambda$, the inequality

$$\sum_{j=1}^{q_n} \|\xi_j(\gamma) - \xi(\gamma)\| \ \ \geq \ \ \sum_{j=p_n}^{q_n} \|\xi_j(\gamma) - \xi(\gamma)\|$$

holds and also implies that

$$\sum_{j=1}^{q_n} \|\xi_j(\gamma) - \xi(\gamma)\| \geq \frac{q_n - p_n}{q_n}\frac{1}{q_n - p_n}\sum_{j=p_n}^{q_n} \|\xi_j(\gamma) - \xi(\gamma)\| \geq \frac{b}{a+b}\frac{1}{q_n - p_n}\sum_{j=p_n}^{q_n} \|\xi_j(\gamma) - \xi(\gamma)\|$$

and so

$$\frac{a+b}{b}\sum_{j=1}^{q_n} \|\xi_j(\gamma) - \xi(\gamma)\| \geq \frac{1}{q_n - p_n}\sum_{j=p_n}^{q_n} \|\xi_j(\gamma) - \xi(\gamma)\|.$$

Hence, for any $\delta > 0$ we obtain

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\sum_{j=p_n}^{q_n} \|\xi_j(\gamma) - \xi(\gamma)\| \geq \delta\right\} \subseteq \left\{n \in \mathbb{N} : \frac{1}{q_n}\sum_{j=1}^{q_n} \|\xi_j(\gamma) - \xi(\gamma)\| \geq \frac{\delta b}{a+b}\right\} \in I, \ \ \forall \gamma \in \Lambda.$$

Therefore $(\xi_j) \in DC^I_{(p,q)}(\Gamma_{a.s.})$. □

As every complex uncertain convergent sequence in measure (ξ_j) belongs to ${}_{\mu}S^I(\Gamma_{\mathcal{M}})$ as well as $C^I(\Gamma_{\mathcal{M}})$ so we have following remark which can be proved by using Theorem 2.48 and 2.50.

Remark 2.51. If a sequence (ξ_j) of complex uncertain variables is convergent almost surely then $(\xi_j) \in {}_{\mu}DS^I_{(p,q)}(\Gamma_{a.s.}) \cap DC^I_{(p,q)}(\Gamma_{a.s.})$.

Theorem 2.52. *If* $\liminf_n \frac{p_n}{q_n} \neq 1$, *then* ${}_{\mu}S^I(\Gamma_{u.a.s.}) \subset {}_{\mu}DS^I_{(p,q)}(\Gamma_{u.a.s.})$, *where* ${}_{\mu}S^I(\Gamma_{u.a.s.})$ *denotes space of all I–${}_{\mu}$statistically convergent complex uncertain sequences w.r.t. uniformly almost surely.*

Proof. Consider the events $\gamma \in \Gamma - (E_j)$, where sequence (E_j) be such that ${}_{\mu}DS^I_{p,q} - \lim \mathcal{M}(E_j) = 0$ and adapt the method which is used in the Theorem 2.48, the proof can be established easily. □

Theorem 2.53. *If* $\liminf_n \frac{p_n}{q_n} \neq 1$, *then* $C^I(\Gamma_{u.a.s.}) \subset DC^I_{(p,q)}(\Gamma_{u.a.s.})$, *where* $C^I(\Gamma_{u.a.s.})$ *denotes space of all I–Cesàro summable complex uncertain sequences w.r.t. uniformly almost surely.*

Proof. Consider the events $\gamma \in \Gamma - (E_j)$, where sequence (E_j) be such that ${}_{\mu}DS^I_{p,q} - \lim \mathcal{M}(E_j) = 0$ and adapt the method which is used in the Theorem 2.50, the proof can be established easily. □

Definition 2.54. Any two complex uncertain sequences (ξ_j) and (ζ_j) are said to be equivalent with respect to (p, q) if for any $\varepsilon > 0$, the following set

$$\left\{n \in \mathbb{N} : \frac{|M \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} \geq \varepsilon\right\} \in I,$$

where $M := \{j \in \mathbb{N} : \xi_j(\gamma) \neq \zeta_j(\gamma)\}$.

Theorem 2.55. *Let* (ξ_j) *and* (ζ_j) *are two equivalent complex uncertain sequences with respect to* (p, q) *then sequence* (ξ_j) *is μ–deferred I–statistically convergent almost surely implies that sequence* (ζ_j) *is μ–deferred I–statistically convergent almost surely.*

Proof. Let complex uncertain sequence (ξ_j) is μ–deferred I–statistically convergent almost surely. Then for all $\varepsilon, \delta > 0$ there exists an event Λ such that $\mathcal{M}(\Lambda) = 1$ so that

$$\left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} \geq \delta\right\} \in I, \quad \forall \gamma \in \Lambda.$$

Consider the set $M := \{j \in \mathbb{N} : \xi_j(\gamma) \neq \zeta_j(\gamma)\}$. So for any preassigned $\varepsilon > 0$ and $\gamma \in \Lambda$, we have

$$\{j \in \mathbb{N} : \|\zeta_j(\gamma) - \zeta(\gamma)\| \geq \varepsilon\} \cap I^*_{p,q}(n) = (\{j \in \mathbb{N} : \|\zeta_j(\gamma) - \zeta(\gamma)\| \geq \varepsilon\} \cap M_{p,q}) \cup (\{j \in \mathbb{N} : \|\zeta_j(\gamma) - \zeta(\gamma)\| \geq \varepsilon\} \cap M^c_{p,q})$$

where $M_{p,q} := I^*_{p,q}(n) \cap M$ and $M^c_{p,q} := I^*_{p,q}(n) \cap M^c$. Thus we have

$$\{j \in \mathbb{N} : \|\zeta_j(\gamma) - \zeta(\gamma)\| \geq \varepsilon\} \cap I^*_{p,q}(n) \subset (\{j \in \mathbb{N} : \|\zeta_j(\gamma) - \zeta(\gamma)\| \geq \varepsilon\} \cap M_{p,q}) \cup (\{j \in \mathbb{N} : \|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon\} \cap I^*_{p,q}(n))$$

Hence

$$\frac{|\{j \in \mathbb{N} : \|\zeta_j(\gamma) - \zeta(\gamma)\| \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} \leq \frac{|\{j \in \mathbb{N} : \|\zeta_j(\gamma) - \zeta(\gamma)\| \geq \varepsilon\} \cap M_{p,q}|}{|I^*_{p,q}(n)|} + \frac{|\{j \in \mathbb{N} : \|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|}$$

Therefore, for any $\delta > 0$ we obtain

$$\left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \|\zeta_j(\gamma) - \zeta(\gamma)\| \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} \geq \delta\right\} \subset$$

$$\left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \|\zeta_j(\gamma) - \zeta(\gamma)\| \geq \varepsilon\} \cap M_{p,q}|}{|I^*_{p,q}(n)|} \geq \delta\right\} \cup \left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \|\xi_j(\gamma) - \xi(\gamma)\| \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} \geq \delta\right\}$$

As the right hand side set belongs to I and $\gamma \in \Lambda$ was arbitrary, hence

$$\left\{n \in \mathbb{N} : \frac{|\{j \in \mathbb{N} : \|\zeta_j(\gamma) - \zeta(\gamma)\| \geq \varepsilon\} \cap I^*_{p,q}(n)|}{|I^*_{p,q}(n)|} \geq \delta\right\} \in I, \quad \forall \gamma \in \Lambda.$$

This implies that (ζ_j) is μ–deferred I–statistically convergent almost surely. □

Theorem 2.56. *Let (ξ_j) and (ζ_j) are two equivalent complex uncertain sequences with respect to (p, q) then sequence (ξ_j) is μ–deferred I–statistically convergent uniformly almost surely implies that sequence (ζ_j) is μ–deferred I–statistically convergent uniformly almost surely.*

Proof. Consider the events $\gamma \in \Gamma - (E_j)$, where (E_j) is the sequence of events such that ${}_{\mu}DS^{I}_{p,q} - \lim \mathcal{M}(E_j) = 0$ and adapt the method which is followed in the Theorem 2.55, the proof can be established easily. □

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