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Cohomology Classification of Spaces with Free $\mathbb{S}^1$ and $\mathbb{S}^3$-Actions

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Abstract. This paper gives the cohomology classification of finitistic spaces X equipped with free actions of the group $G = \mathbb{S}^3$ and the cohomology ring of the orbit space X/G is isomorphic to the integral cohomology quaternion projective space $\mathbb{HP}^n$. We have proved that the integral cohomology ring of X is isomorphic either to $\mathbb{S}^{4n+3}$ or $\mathbb{S}^3 \times \mathbb{HP}^n$. Similar results with other coefficient groups and for $G = \mathbb{S}^1$ actions are also discussed. As an application, we determine a bound of the index and co-index of cohomology sphere $\mathbb{S}^{2n+1}$ (resp. $\mathbb{S}^{4n+3}$) with respect to $\mathbb{S}^1$-actions (resp. $\mathbb{S}^3$-actions).

1. Introduction

Let G be a compact Lie group acting on a topological space X. For each $g \in G$, there exists a unique homeomorphism $\phi_g : x \mapsto g.x$. The group $\{\phi_g | g \in G\}$ of homeomorphisms is called transformation group and it is denoted by (G, X). There are interesting questions related to transformation group. One such question is whether it is possible to classify the orbit space X/G if G acts freely on X. In this generality, it is difficult to say anything. Morita et al. [9] determined the orbit space of free $G = \mathbb{Z}_2$ actions on Dold manifold $P(1, n), n$ odd. Dey et al. [2] determined the orbit spaces of free actions of $G = \mathbb{Z}_2$ or $\mathbb{S}^1$ on the real and complex Milnor manifolds. Kaur et al. [6] shown that if $G = \mathbb{S}^3$ acts freely on the mod 2 cohomology n-sphere $\mathbb{S}^n$, then $n \equiv 3 (\text{mod } 4)$ and the orbit space is the mod 2 cohomology quaternion projective space $\mathbb{HP}^n$. Some more results have been proved in the literature; for example [4, 10]. On the other hand, if the topology of the orbit space X/G is fixed, then the question becomes both tractable and interesting. In this direction, Su [12] have addressed several such problems: First, if $\mathbb{Z}_2$ acts freely on a connected space X such that the orbit space is the mod 2 cohomology $\mathbb{RP}^n$, then X is the mod 2 cohomology $\mathbb{S}^n$. Second, if $G = \mathbb{Z}_p$, p an odd prime, acts freely on a connected space X and the cohomology ring of the orbit space X/G with coefficients in $\mathbb{Z}_p$ is the Lens space L_p^{2n+1}, then X is the mod p cohomology $(2n+1)$-sphere $\mathbb{S}^{2n+1}$. He also proved that if $\mathbb{S}^1$ acts freely on a space X such that the orbit space is the integral cohomology $\mathbb{CP}^n$ and the map $\pi_2^* : H^2(X/\mathbb{S}^1) \to H^2(X)$ induced by the quotient map $\pi : X \to X/\mathbb{S}^1$ is trivial, then X is the integral cohomology $\mathbb{S}^{2n+1}$. We wish to investigate X when π_2^* is nontrivial. In this paper, it is also shown that if $G = \mathbb{S}^3$ acts freely on a finitistic space X with the orbit space X/G whose integral cohomology ring is the quaternion projective space $\mathbb{HP}^n$, then the integral cohomology ring of X is either $\mathbb{S}^{4n+3}$ or $\mathbb{S}^3 \times \mathbb{HP}^n$. A similar result with coefficients in $\mathbb{Q}$ and $\mathbb{Z}_p$, p a prime, are also discussed. We have also proved Kaur's

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result [6] with other coefficient groups. As an application of these cohomological calculations, we have determined a bound of the index and co-index [7] of cohomology spheres $\mathbb{S}^{(q+1)n+q}$ for a action of $G = \mathbb{S}^q$, where $q = 1$ or 3.

2. Preliminaries

The spaces of our concern are finitistic free G-spaces. In the year 1960, R. G. Swan introduced the idea of finitistic spaces which are more general than finite-dimensional polyhedra. Recall that a paracompact Hausdorff space X is said to be finitistic if every open cover of X has a finite dimensional open refinement. Note that all compact spaces and all finite-dimensional paracompact spaces are finitistic spaces.

Let G be a compact Lie group and $G \hookrightarrow E_G \to B_G$ be the universal principal G-bundle, where B_G is the classifying space of the group G. Suppose G acts freely on a finitistic space X. The associated bundle $X \hookrightarrow (X \times E_G)/G \to B_G$ is a fibre bundle with fibre X. Put $X_G = (X \times E_G)/G$. Then the bundle $X \hookrightarrow X_G \to B_G$ is called the Borel fibration. We consider the Leray-Serre spectral sequence for the Borel fibration. If B_G is simply connected, then the system of local coefficients on B_G is simple and the E_2-term of the Leray-Serre spectral sequence corresponding to the Borel fibration becomes

$$E_2^{k,l} = H^k(B_G; H^l(X; R)).$$

For the details about spectral sequences, we refer the reader to [8]. Let $h : X_G \to X/G$ be the map induced by the G-equivariant projection $X \times E_G \to X$. Then h is a homotopy equivalence [3].

For $G = \mathbb{S}^q, q = 1$ or 3, we assume that the associated sphere bundles $G \hookrightarrow X \to X/G$ are orientable. The following results are needed to prove our results:

Proposition 2.1. ([5]) *Let R denote a ring and $\mathbb{S}^{n-1} \to E \xrightarrow{\pi} B$ be an orientable sphere bundle. The following sequence is exact with coefficients in R*

$$\cdots \to H^i(E) \xrightarrow{\rho_i} H^{i-n+1}(B) \xrightarrow{\cup} H^{i+1}(B) \xrightarrow{\pi^*_{i+1}} H^{i+1}(E) \xrightarrow{\rho_{i+1}} H^{i-n+2}(B) \to \cdots$$

which start with

$$0 \to H^{n-1}(B) \xrightarrow{\pi^*_{n-1}} H^{n-1}(E) \xrightarrow{\rho_{n-1}} H^0(B) \xrightarrow{\cup} H^n(B) \xrightarrow{\pi^*_n} H^n(E) \to \cdots$$

where $\cup : H^i(B) \to H^{i+n}(B)$ maps $x \to x \cup u$ and $u \in H^n(B)$ denotes the Euler class of the sphere bundle. The above exact sequence is called the Gysin sequence. It is easy to observe that $\pi_i^* : H^i(B) \to H^i(E)$ is an isomorphism for all $0 \leq i < n-1$.

Proposition 2.2. ([6]) *Let A be an R-module, where R is PID, and $G = \mathbb{S}^q, q = 1$ or 3, act freely on a finitistic space X. Suppose that $H^j(X, A) = 0$ for all $j > n$, then $H^j(X/G, A) = 0$ for all $j > n$.*

Now, we recall some definitions of indices of free G-spaces:

Definition 2.3. ([7]) Let X be a finitistic free G-space, where $G = \mathbb{S}^q$, $q = 1$ or 3. The index of X is defined as

$$\text{ind}_G X = \max\{k | \text{ there exists an } G\text{-equivariant map } f : \mathbb{S}^{(q+1)k+q} \to X,\ k \geq 0\}.$$

Definition 2.4. ([7]) Let X be a finitistic free G-space, where $G = \mathbb{S}^q$, $q = 1$ or 3. The co-index of X is defined as

$$\text{co-ind}_G X = \min\{k | \text{ there exists an } G\text{-equivariant map } f : X \to \mathbb{S}^{(q+1)k+q},\ k \geq 0\}.$$

If no such k exist then $\text{co-ind}_G X = +\infty$.

Recall that [5] for any commutative ring R, we have $H^*(\mathbb{FP}^n; R) = R[a]/\langle a^{n+1} \rangle$, $H^*(\mathbb{FP}^\infty; R) = R[t]$, where $\deg a = \deg t = 2$ for $\mathbb{F} = \mathbb{C}$ and $\deg a = \deg t = 4$ for $\mathbb{F} = \mathbb{H}$. Throughout this paper, we have considered Čech cohomology with coefficients in R, where $R = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{Z}_p, p$ a prime. Note that $X \sim_R Y$ means $H^*(X; R) \cong H^*(Y; R)$.

3. Main theorems

Recall that the projective spaces $\mathbb{FP}^n$ are the orbit spaces of standard free actions of $G = \mathbb{S}^q$ on $\mathbb{S}^{(q+1)n+q}$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$ for $q = 1$ or 3, respectively. If we take a free action of $\mathbb{S}^q$ on itself and the trivial action on $\mathbb{FP}^n$, then the orbit space of this diagonal action is $\mathbb{FP}^n$. Now, the natural question: Is the converse true? If G acts freely on a finitistic space X with $X/G \sim_R \mathbb{FP}^n$, then whether $X \sim_R \mathbb{S}^{(q+1)n+q}$ or $X \sim_R \mathbb{S}^q \times \mathbb{FP}^n$. In the following theorems, we have discussed these converse statements:

Theorem 3.1. *Let $G = \mathbb{S}^3$ act freely on a finitistic connected space X with $X/G \sim_R \mathbb{HP}^n$. Then either $X \sim_R \mathbb{S}^{4n+3}$ or $X \sim_R \mathbb{S}^3 \times \mathbb{HP}^n$, where $R = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{Z}_p$, p a prime.*

Proof. Let $G \hookrightarrow X \xrightarrow{\pi} X/G$ be the principal bundle associated to the free action of G on X. By the exactness of the Gysin sequence, $H^i(X) \cong H^i(X/G)$ for $i = 0, 1, 2$; $H^{4i+1}(X) = H^{4i+2}(X) = 0$ for all $i \geq 0$ and $H^j(X) = 0$ for all $j > 4n + 3$. Let $\pi_4^* : H^4(X/G) \to H^4(X)$ be the map induced by the natural map $\pi : X \to X/G$. We consider the following cases:

If the map π_4^* is trivial, then by the exactness of the Gysin sequence ρ_{4i+3}, π^*_{4i+4} are trivial homomorphisms for all $0 \leq i < n$. This gives that $H^{4i+3}(X) = H^{4i+4}(X) = 0$ for all $0 \leq i < n$, and $H^{4n+3}(X) \cong R$. It is clear that $X \sim_R \mathbb{S}^{4n+3}$.

If the map π_4^* is an isomorphism, then ρ_{4i+3} and π^*_{4i} are isomorphisms for all $0 \leq i \leq n$. Let $a_4 \in H^4(X)$ and $b_{4i+3} \in H^{4i+3}(X)$ be such that $\pi_4^*(a) = a_4$ and $\rho_{4i+3}(b_{4i+3}) = a^i$ for all $0 \leq i \leq n$, where a denotes a generator of $H^*(X/G)$. This implies that $H^{4i+3}(X) \cong R$ with basis $\{b_{4i+3}\}$ and $H^{4i}(X) \cong R$ with basis $\{a_4^i\}$ for all $0 \leq i \leq n$. Thus, we have

$$H^i(X) = \begin{cases} R & \text{if } 0 \leq i \equiv 0 \text{ or } 3 \pmod 4 \leq 4n+3 \\ 0 & \text{otherwise.} \end{cases}$$

Now, it remains to compute the cohomology algebra of X. As B_G is simply connected and $H^*(B_G)$ is torsion free, the E_2-term of the associated Leray-Serre spectral sequence for the Borel fibration $X \hookrightarrow X_G \to B_G$ is given by $E_2^{k,l} = H^k(B_G) \otimes H^l(X)$ which converges to $H^*(X_G)$ as an algebra [8, Theorem 5.2]. Note that the only possible nontrivial differentials are $d_{4r} : E_{4r}^{*,*} \to E_{4r}^{*,*}, 1 \leq r \leq n+1$. We have $b_i b_j = 0$ for all i and j, $b_3^2 = 0$ and $a_4^{n+1} = 0$. Clearly, $d_4(1 \otimes a_4^i) = 0$ for all $i \geq 0$. Also, $d_4(1 \otimes b_3) \neq 0$, otherwise $\{t^i \otimes b_3\}$ become permanent cocycles for all $i \geq 0$, which is not possible either with coefficients in $R = \mathbb{Z}$, or with coefficients in a field $R = \mathbb{Q}$ or $\mathbb{Z}_p$, p a prime.

Now, we consider two subcases. One for coefficient groups $R = \mathbb{Q}$ or $\mathbb{Z}_p$, p a prime, and other, for $R = \mathbb{Z}$.

Let $R = \mathbb{Q}$ or $\mathbb{Z}_p$, p a prime. First, we prove that $a_4^i b_3 \neq 0$ for all $1 \leq i \leq n$. Assume otherwise. Let $a_4^k b_3 = 0$ for some $1 \leq k \leq n$. If $d_4(1 \otimes b_3) = \alpha(t \otimes 1)$ for some nonzero element $\alpha \in R$, then $\alpha t \otimes a_4^k = d_4((1 \otimes a_4^k)(1 \otimes b_3)) = 0$ which is not possible. This implies that for each $1 \leq i \leq n$, $b_{4i+3} = \alpha_i a_4^i b_3$ for some $\alpha_i \neq 0$ in R. Thus, the cohomology ring of X is $R[a_4, b_3]/\langle a_4^{n+1}, b_3^2 \rangle$, $\deg a_4 = 4$, $\deg b_3 = 3$. It is clear that $X \sim_R \mathbb{S}^3 \times \mathbb{HP}^n$. Now, let $R = \mathbb{Z}$. Here, we prove that $a_4^i b_3 = \pm b_{4i+3}$ for all $1 \leq i \leq n$. On contrary, assume that $a_4^j b_3 \neq \pm b_{4j+3}$ for some $1 \leq j \leq n$. Let $i_0 \in \mathbb{Z}$ be the largest integer such that $a_4^{i_0} b_3 \neq \pm b_{4i_0+3}$. For all $0 \leq i \leq n$, let $d_4(1 \otimes b_{4i+3}) = m_i(t \otimes a_4^i)$, where $m_i \in \mathbb{Z}$. Clearly, $m_0 \neq 0$. Then $E_\infty^{0,4} = \mathbb{Z}, E_\infty^{4,0} = \mathbb{Z}_{m_0}$ and $E_\infty^{i,4-i} = 0, 1 \leq i \leq 3$. Consider, the filtration

$$0 \subseteq F^4H^4 \subseteq F^3H^4 \subseteq F^2H^4 \subseteq F^1H^4 \subseteq F^0H^4 \subseteq H^4(X_G)$$

of $H^4(X_G)$. As $E_\infty^{p,q} \cong F^pH^{p+q}/F^{p+1}H^{p+q}$, we get $H^4(X_G) \cong \mathbb{Z} \oplus \mathbb{Z}_{m_0}$. This gives that $m_0 = \pm 1$. So, we have $E_5^{0,4j} = \mathbb{Z}, E_5^{i+1,4j} = E_5^{i,4j+3} = 0$ for all $i \geq 0$, $j = 0$ and $i_0 + 1 \leq j \leq n$. Clearly, $d_4 : E_4^{0,4j+3} \to E_4^{4,4j}$ is isomorphism for $i_0 + 1 \leq j \leq n$. If $d_4 : E_4^{0,4i_0+3} \to E_4^{4,4i_0}$ is trivial, then $\{t^i \otimes b_{4i_0+3}\}_{i \geq 0}$ are permanent cocycles, a contradiction. Now, if $d_4 : E_4^{0,4i_0+3} \to E_4^{4,4i_0}$ is nontrivial, then $d_4(1 \otimes (a_4^{i_0} b_3 \pm b_{4i_0+3})) = (m_0 \pm m_{i_0})(t \otimes a_4^{i_0})$. Consequently, $m_{i_0} \neq \pm 1$. Thus, $H^j(X_G)$ are nonzero for infinitely many values of j, again a contradiction. Therefore, $a_4^j b_3$ is b_{4j+3} or $-b_{4j+3}$ for all j. Hence, $X \sim_\mathbb{Z} \mathbb{S}^3 \times \mathbb{HP}^n$.

Finally, consider the case when π_4^* is nontrivial but not an isomorphism. This case is possible only when $R = \mathbb{Z}$ and the Euler class $u \in H^4(X/G)$ is ma, where $m \neq 0, 1, -1$. Consequently, $H^i(X) = \mathbb{Z}_m$ for $0 < i \equiv 0(\text{mod } 4) \leq 4n$, $H^i(X) = \mathbb{Z}$ for $i = 0$ or $4n+3$; and 0 otherwise. By the associated Leray-Serre spectral sequence, it is easy to see that $H^4(X_G) \cong \mathbb{Z} \oplus \mathbb{Z}_m$, a contradiction. □

By repeated application of the Gysin sequence we compute the orbit spaces of free actions of $G = \mathbb{S}^3$ on a finitistic space X with $X \sim_R \mathbb{S}^n$.

Theorem 3.2. *Let $G = \mathbb{S}^3$ act freely on a finitistic connected space X with $X \sim_R \mathbb{S}^n$. Then $n = 4k+3$, for some $k \geq 0$ and $X/G \sim_R \mathbb{HP}^k$.*

Proof. It is immediate that $H^0(X/G) \cong R$ and $H^i(X/G) = 0$, for all $1 \leq i \leq 3$ when $n > 3$. Also, we get $H^i(X/G) = 0$ for $0 < i \equiv j \ (\text{mod } 4) < n$, where $1 \leq j \leq 3$ and $H^i(X/G) \cong R$ for $0 \leq i \equiv 0(\text{mod } 4) < n$. If $n \equiv j \ (\text{mod } 4)$, then for some $0 \leq j \leq 2$, we get $H^{n-3}(X/G) = 0$ and hence $H^n(X/G) \neq 0$, which contradicts Proposition 2.2. So, $n \equiv 3 \ (\text{mod } 4)$. Let $n = 4k+3$ for some $k \geq 0$. For $n = 3$, the result is obvious. For $n > 3$ and for all $i > n$, $H^i(X/G) = 0$. So, we get $a^{k+1} = 0$, where a is a generator of $H^4(X/G)$. Consequently, $H^n(X/G) = 0$. Hence, our claim holds. □

Su [12] has proved that the orbit space of free $G = \mathbb{S}^1$-actions on the integral cohomology sphere $\mathbb{S}^{2n+1}$ is the integral cohomology complex projective space. Using Leray-Serre spectral sequence, we can easily get the similar results with coefficients in R, where $R = \mathbb{Q}$ or $\mathbb{Z}_p$, p prime.

Theorem 3.3. *Let $G = \mathbb{S}^1$ act freely on a finitistic connected space X with $X \sim_R \mathbb{S}^{2n+1}$, where $R = \mathbb{Q}$ or $\mathbb{Z}_p$, p a prime. Then $X/G \sim_R \mathbb{CP}^n$.*

Su [12] has also shown that if $G = \mathbb{S}^1$ acts freely on a space X with the orbit space $X/G \sim_{\mathbb{Z}} \mathbb{CP}^n$ and the homomorphism $\pi_2^* : H^2(X/G) \to H^2(X)$ induced by the quotient map $\pi : X \to X/G$ is trivial, then $X \sim_{\mathbb{Z}} \mathbb{S}^{2n+1}$. We are interested in discussing the case when π_2^* is nontrivial.

Theorem 3.4. *Let $G = \mathbb{S}^1$ act freely on a finitistic connected space X with $X/G \sim_{\mathbb{Z}} \mathbb{CP}^n$. If the induced map $\pi_2^* : H^2(X/G) \to H^2(X)$ is an isomorphism, then $X \sim_{\mathbb{Z}} \mathbb{S}^1 \times \mathbb{CP}^n$.*

Proof. As π_2^* is an isomorphism, we have $H^j(X) = \mathbb{Z}$ for $0 \leq j \leq 2n+1$; and 0 otherwise. Let $x \in H^1(X)$, $y \in H^2(X)$ and $b_{2i+1} \in H^{2i+1}(X)$ be such that $\rho_1(x) = 1$, $\pi_2^*(a) = y$ and $\rho_{2i+1}(b_{2i+1}) = a^i$ for all $1 \leq i \leq n$, where a is generator of $H^*(X/G)$. Now, we calculate cohomology algebra of X. Let if possible, $xy^j \neq \pm b_{2j+1}$ for some $1 \leq j \leq n$ and suppose i_0 be such an largest integer. Since $\pi_1(B_G)$ is trivial and $H^*(B_G)$ is torsion free, the E_2-term of Leray-Serre spectral sequence for the Borel fibration $X \hookrightarrow X_G \to B_G$ is $E_2^{k,l} = H^k(B_G) \otimes H^l(X)$. Note that the possible nontrivial differentials are $d_2, d_4, \cdots d_{2n+2}$. As $H^1(X_G) = 0$, we get $d_2(1 \otimes x) = m_0(t \otimes 1)$, for some $m_0 \neq 0$ in $\mathbb{Z}$. Let $d_2(1 \otimes b_{2i+1}) = m_i(t \otimes y^i)$ for all $1 \leq i \leq n$, where $m_i \in \mathbb{Z}$. Note that for $0 \leq j \leq n$, $E_3^{0,2j} = \mathbb{Z}$ and $E_3^{2i,2j} = \mathbb{Z}_{m_j}$ if $m_j \neq \pm 1$ otherwise $E_3^{2i,2j} = 0$ for $i > 0$. Also, $E_3^{2i,2j+1} = \mathbb{Z}$ if $m_j = 0$, otherwise $E_3^{2i,2j+1} = 0$ for all $i \geq 0$ and $0 \leq j \leq n$. Since $H^2(X_G) \cong \mathbb{Z}$, we have $d_2 : E_2^{0,1} \to E_2^{2,0}$ is an isomorphism and so $m_0 = 1$ or -1. Therefore, $E_3^{0,2j} = \mathbb{Z}, E_3^{i+1,2j} = E_3^{i,2j+1} = 0$ for all $i \geq 0$ and $i_0 + 1 \leq j \leq n$. If $d_2 : E_2^{0,2i_0+1} \to E_2^{2,2i_0}$ is trivial, then $\{t^i \otimes b_{2i_0+1}\}_{i \geq 0}$ are permanent cocycles, a contradiction. So, let $d_2 : E_2^{0,2i_0+1} \to E_2^{2,2i_0}$ is nontrivial. As $d_2(1 \otimes y) = 0$, we get $m_{i_0} \neq m_0$. Consequently, $E_\infty^{2i,2i_0} = \mathbb{Z}_{m_{i_0}}$ for all $i \geq 0$ which contradicts Proposition 2.2. We have $x^2 = \alpha y$ for some $\alpha \in \mathbb{Z}$. By the commutative property of cup product, α must be zero. Obviously, $y^{n+1} = 0$. Thus, we have $X \sim_{\mathbb{Z}} \mathbb{S}^1 \times \mathbb{CP}^n$. □

Note that in the above theorem, if $\pi_2^* : H^2(X/G) \to H^2(X)$ is nontrivial but not an isomorphism, then the Euler class of the bundle $G \to X \xrightarrow{\pi} X/G$ is $ma \in H^2(X/G)$, where $m \neq 0, 1, -1$. Accordingly, $H^i(X) \cong \mathbb{Z}$ for $i = 0, 2n+1$; $H^i(X) \cong \mathbb{Z}_m$ for $i = 0, 2, 4, \cdots, 2n$; and trivial otherwise. From the Leray-Serre spectral sequence $E_r^{*,*}$ for the Borel fibration $X \hookrightarrow X_G \to B_G$, we get that $H^i(X/G) \neq 0$ for some $i > 2n$, a contradiction. Therefore, in this case G cannot act freely on X.

Now, we discuss similar results with coefficients in $R = \mathbb{Z}_p, p$ a prime or $\mathbb{Q}$.

Theorem 3.5. *Let $G = \mathbb{S}^1$ act freely on a finitistic connected space X with the orbit space $X/G \sim_{\mathbb{Z}_p} \mathbb{CP}^n$, p a prime. Then $X \sim_{\mathbb{Z}_p} \mathbb{S}^{2n+1}$ or $X \sim_{\mathbb{Z}_p} \mathbb{S}^1 \times \mathbb{CP}^n$ or $X \sim_{\mathbb{Z}_p} L_p^{2n+1}$.*

Proof. As the coefficient group is $\mathbb{Z}_p$, p a prime, the map π_2^* is either trivial or an isomorphism. If π_2^* is trivial then $X \sim_{\mathbb{Z}_p} \mathbb{S}^{2n+1}$. So, let π_2^* be an isomorphism. By the exactness of the Gysin sequence for the sphere bundle $G \hookrightarrow X \to X/G$, $H^i(X) = \mathbb{Z}_p$ for all $0 \leq i \leq 2n+1$, and trivial otherwise. It is easy to see that for $1 \leq i \leq n$, basis for $H^{2i}(X)$ is $\{a_2^i\}$, where a_2 is nonzero element in $H^2(X)$. Let $\{b_{2i+1}\}$ denotes basis for $H^{2i+1}(X)$ for $0 \leq i \leq n$. In the Leray-Serre spectral sequence, we must have $d_2(1 \otimes b_1) \neq 0$ for suitable choice of generator b_1 and $d_2(1 \otimes a_2^i) = 0$ for all $0 \leq i \leq n$. This implies that $b_{2i+1} = a_2^i b_1$ for all $0 \leq i \leq n$. If $b_1^2 \neq 0$ then by commutative property of the cup product, p must be 2. In this case, $a_2 = b_1^2$ and hence $X \sim_{\mathbb{Z}_2} \mathbb{RP}^{2n+1}$. If $b_1^2 = 0$ and $\beta(b_1) = a_2$, where $\beta : H^1(X;\mathbb{Z}_p) \to H^2(X;\mathbb{Z}_p)$ is the Bockstein homomorphism associated to the coefficient sequence $0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$, then $X \sim_{\mathbb{Z}_p} L_p^{2n+1}$. Further, if $b_1^2 = 0$ and $\beta(b_1) = 0$ then $X \sim_{\mathbb{Z}_p} \mathbb{S}^1 \times \mathbb{CP}^n$. □

Remark 3.6. The above theorem also shows that the converse of [11, Theorem 1.2] is also true if the map $\pi_2^* : H^2(X/G) \to H^2(X)$ is nontrivial, the square of generator of $H^1(X)$ is zero and the associated Bockstein homomorphism is nontrivial.

Similarly, for a space with the orbit space rational cohomology the complex projective space, we get

Theorem 3.7. *Let $G = \mathbb{S}^1$ act freely on a finitistic connected space X with the orbit space $X/G \sim_{\mathbb{Q}} \mathbb{CP}^n$. Then either $X \sim_{\mathbb{Q}} \mathbb{S}^{2n+1}$ or $X \sim_{\mathbb{Q}} \mathbb{S}^1 \times \mathbb{CP}^n$.*

4. Applications

In this section, we have discussed the index and co-index of a finitistic connected space $X \sim_R \mathbb{S}^{(q+1)n+q}$ equipped with free actions of $G = \mathbb{S}^q$, where $q = 1$ or 3 and the orbit space $X/G \sim_R \mathbb{FP}^n$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$, respectively.

By Theorem 3.1, it is clear that if $X \sim_R \mathbb{S}^{(q+1)n+q}$, then the Volovikov's index $i(X)$ [13] is $(q+1)n + (q+1)$. Using [1, Theorem 1.1] and the fact that $\beta_k(B_G, R) = 1$ if $k \equiv 0(\text{mod (q+1)})$, we get there is no G-equivariant map $f : X \to S^{4j+3}$ if $0 \leq j < n$. So, we have the following result:

Theorem 4.1. *Let $G = \mathbb{S}^q, q = 1$ or 3, act freely on a finitistic path connected space X with $X \sim_R \mathbb{S}^{(q+1)n+q}$, then co-ind$_G$ $X \geq n$.*

Let $G = \mathbb{S}^q, q = 1$ or 3, act freely on a finitistic space X with $X/G \sim_R \mathbb{FP}^n$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$ respectively. Note that for the Borel fibration $X \hookrightarrow X_G \xrightarrow{\eta} B_G$, $\eta \circ h' : X/G \to B_G$ is a classifying map for the principal G-bundle $G \hookrightarrow X \to X/G$, where $h' : X/G \to X_G$ is homotopy inverse of homotopy equivalence $h : X_G \to X/G$. It is easy to see that $h'^* \circ \eta^*(t)$ is the Witney class of the principal G-bundle $G \hookrightarrow X \to X/G$. If $X \sim_R \mathbb{S}^{(q+1)n+q}$ then $h'^* \circ \eta^*(t) = a$, where a is generator of $H^*(X/G)$. Let $f : \mathbb{S}^{(q+1)k+q} \to X$ be any G-equivariant map, where $\mathbb{S}^q$ acts on $\mathbb{S}^{(q+1)k+q}$ by the standard action. Then $\eta \circ h' \circ \overline{f}$ is a classifying map for the principal bundle $G \hookrightarrow \mathbb{S}^{(q+1)k+q} \to \mathbb{FP}^k$, where $\overline{f} : \mathbb{FP}^k \to X/G$ is a continuous map induced by f. This implies that $\overline{f}^*(b) = a$, where $b \in H^{q+1}(\mathbb{FP}^k)$ denotes its generator. Therefore, $k \leq n$. So, we have the following result:

Theorem 4.2. *Let $G = \mathbb{S}^q, q = 1$ or 3, act freely on a finitistic space X with $X \sim_R \mathbb{S}^{(q+1)n+q}$ then ind$_G(X) \leq n$.*

5. Examples

We have seen that $\mathbb{FP}^n$, $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$, is the orbit space of standard free action of $G = \mathbb{S}^q, q = 1$ or 3, respectively, on $\mathbb{S}^{(q+1)n+q}$, and the diagonal action on $\mathbb{S}^q \times \mathbb{FP}^n$, where G acts freely on itself and trivially on

$\mathbb{FP}^n$. This also realizes our main theorems. It is easy to see that $\text{ind}_G(\mathbb{S}^q \times \mathbb{FP}^n) = 0$. The projection map $\mathbb{S}^q \times \mathbb{FP}^n \to \mathbb{S}^q$ is an G-equivariant map. Thus, $\text{co-ind}_G(\mathbb{S}^q \times \mathbb{FP}^n) = 0$.

Recall that the map defined by $(\lambda, (z_0, z_1, \cdots, z_n)) \to (\lambda z_0, \lambda z_1, \cdots, \lambda z_n)$, where $\lambda \in \mathbb{S}^1$ and $z_i \in \mathbb{C}, 0 \leq i \leq n$, is the standard free action of $G = \mathbb{S}^1$ on $\mathbb{S}^{2n+1}$. The orbit space X/G under this action is $\mathbb{CP}^n$. For p a prime, $H = \langle e^{2\pi i/p} \rangle$ induces a free action on $\mathbb{S}^{2n+1}$ with the orbit space $\mathbb{S}^{2n+1}/H = L_p^{2n+1}$. Consequently, $\mathbb{S}^1 = G/H$ acts freely on L_p^{2n+1} with the orbit space $\mathbb{CP}^n$. Recall that for $p = 2$, $L_p^{2n+1} = \mathbb{RP}^{2n+1}$. This realizes Theorem 3.5.

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