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Closure Operators and Connectedness in Bounded Uniform Filter Spaces

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Abstract. In this paper, we characterize both closed and strongly closed subobjects in the category of bounded uniform filter spaces and introduce two notions of closure operators which satisfy weakly hereditary, idempotent and productive properties. We further characterize each of T_j $(j = 0, 1)$ bounded uniform filter spaces using these closure operators and examine that each of them form quotient-reflective subcategories of the category of bounded uniform filter spaces. Also, we characterize connected bounded uniform filter spaces. Finally, we introduce ultraconnected objects in topological category and examine the relationship among irreducible, ultraconnected and connected bounded uniform filter spaces.

1. Introduction

There are many basic concepts of analysis that are not available in General Topology such as uniform convergence, uniform continuity, Cauchy continuity, Cartesian closedness, completeness, total boundedness, hereditary of quotients, etc. Topologists have made several approaches to deal with this inadequacy in the form of Kent convergence spaces [31]; quasiuniform spaces [22]; generalized topological spaces [20]; seminearness spaces [30] and nearness spaces [26] but failed to overcome all the above mentioned deficiencies. Then in the realm of Convenient Topology, a basic structure was introduced by Preuss in 1995, named as semi-uniform convergence space that contains almost all the above mentioned concepts [38]. In addition to it, Preuss defined pre-uniform convergence spaces by removing the symmetric condition from semi-uniform convergence spaces [39]. This idea of Preuss was further extended by Leseberg [33, 34] in 2018 and 2019 in the form of bounded uniform filter spaces. Interestingly, not only **PUConv**; the category of pre-uniform convergence spaces and uniformly continuous maps; are embedded in **b-UFIL**; the category of bounded uniform filter spaces and bounded continuous maps; but also **Born**; the category of bornological spaces and continuous maps; can easily be embedded in **b-UFIL** as its subcategories. Moreover, the category **b-UFIL** forms a strong topological universe [33].

Closure operators are one of the foremost ingredients in not only Categorical Algebra but also in Categorical Topology. In the sense of Kuratowski closure operators, coreflections have been characterized in the category **Top** are found to be much finer than the classical closure operators [25]. Epireflective subcategories

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has been defined in **Top** using closure operators by Hong [29] and Salbany [41]. Galois equivalence between conclusive factorization systems in idempotent and weakly hereditary closure operators in Top is given by Nakagawa [37]. Moreover, closure operators have played a vital role in defining diagonal theorems (referred in [13, 15, 17, 23, 24, 27, 28, 42, 44]), i.e. generalization of the renowned fact that a space S is T_2 if and only if the diagonal Δ_S is closed in $S \times S$. In a topological category, closure operators are defined by Dikranjan and Guili [16] where the epimorphisms of the full-subcategories of a topological categories are characterized by using them and suitable closure operators were formed in random topological categories (see in [5, 6, 9, 21, 40]). Both in Topology and Algebra various examples can be found where closure operators and their relations with other subcategories are studied inclusively [16, 19].

The notion of closedness and strongly closedness in arbitrary topological categories over sets were investigated by Baran [2, 3]. In addition, by the assistance of closedness, the generalization of the classical topological properties such as the notions of normal objects; Hausdorffness; compactness; perfectness; connectedness; (completely) regular and soberness are characterized in any topological categories over sets [2, 4, 6, 8, 10, 11].

The aims of this paper are stated as under:

(i) to characterize both closed and strongly closed subobjects in the category **b-UFIL** and to prove that they form favourable closure operators in the sense of [16] which satisfy the fundamental properties such as (weakly) hereditary, idempotent and productivity.

(ii) to characterize $\overline{T_0}$ and T_1 b-UFIL spaces with respect to these closure operators and examine that each of these subcategories of $\overline{T_0}$ and T_1 b-UFIL spaces are quotient-reflective and discuss the relationship among them.

(iii) to give characterization of both connected and strongly connected bounded uniform filter spaces in the sense of Baran;

(iv) to introduce ultraconnected objects in topological category, and to characterize irreducible (resp. ultraconnected) bounded uniform filter spaces and examine their relationship with connected objects.

2. Preliminaries

For arbitrary topological categories $\mathcal{G}$ and $\mathcal{H}$, the functor $\mathfrak{F} : \mathcal{G} \to \mathcal{H}$ is said to be a *topological functor* or the category $\mathcal{G}$ is said to be a topological category over $\mathcal{H}$ if (i) $\mathfrak{F}$ is concrete (amnestic and faithful), (ii) $\mathfrak{F}$ consists of small fibers, and (iii) every $\mathfrak{F}$-source has a unique initial lift, i.e., if for every source $(f_i : X \to (X_i, \zeta_i))_{i\in I}$ there exists a unique structure ζ on X such that $g : (Y, \eta) \to (X, \zeta)$ is a morphism if and only if for each $i \in I$, $f_i \circ g : (Y, \eta) \to (X_i, \zeta_i)$ is a morphism. Moreover, a topological functor is called a discrete (resp. indiscrete) if it has a left (resp. right) adjoint. In addition, a functor is called a normalized topological functor if subterminals have a unique structure ([1, 39]).

A filter ς on A is a non-empty collection such that finite intersection of elements of ς is in ς, and every superset of a set in ς is in ς. If $\emptyset \in \varsigma$ then ς is an *improper filter* otherwise it is a *proper filter*. We write $\mathcal{F}(A)$ for the set of all filters on A. Let $v \in A$, then $[v] = \dot{v} = [\{v\}] = \{W \subset A : v \in W\}$ is a filter on v. Similarly, $[U] = \{W \subset A : W \supset U\}$ is a filter on $U \subset A$.

Lemma 2.1. ([12]) *Let A be any set, η and γ be filters on $A \times A$, and $h : A \to X$ be a function. Then:*

(i) $(h \times h)(\eta \cap \gamma) = (h \times h)\eta \cap (h \times h)\gamma$ *and* $(h \times h)\eta \cup (h \times h)\gamma \subset (h \times h)(\eta \cup \gamma)$.

(ii) *If $\eta \subset \gamma$, then $(h \times h)\eta \subset (h \times h)\gamma$, and if γ is a proper filter on $X \times X$, then $\gamma \subset (hh^{-1} \times hh^{-1})\gamma$.*

Definition 2.2. ([33]) Let X be any non-empty set, $\Theta^X \subset P(X)$ be a non-empty boundedness on X with bounded subsets as elements and $\psi \subset \mathcal{F}(X \times X)$ be a non-empty set of uniform filters on cartesian product of X with itself. A pair (Θ^X, ψ) is said to be a bounded uniform filter structure (or b-UFIL structure) on X and the corresponding triplet (X, Θ^X, ψ) is known as bounded uniform filter space (or b-UFIL space) on X if the following axioms hold:

(b-UFIL 1): $E' \subset E \in \Theta^X$ implies $E' \in \Theta^X$;

(b-UFIL 2): $x \in X$ implies $\{x\} \in \Theta^X$;

(b-UFIL 3): $E \in \Theta^X \backslash \emptyset$ implies $[E] \times [E] \in \psi$;

(b-UFIL 4): $\varsigma \in \psi$ and $\varsigma \subset \varsigma' \in \mathcal{F}(X \times X)$ implies $\varsigma' \in \psi$.

A b-UFIL space (X, Θ^X, ψ) is a *symmetric b-UFIL space* provided that the following axiom holds:

(b-UFIL 5): $\varsigma \in \psi$ implies $\varsigma^{-1} \in \psi$;

A symmetric b-UFIL space (X, Θ^X, ψ) is a *symmetric bounded uniform limit space* provided that the following axiom holds:

(b-UFIL 6): $\varsigma \in \psi$ and $\varsigma' \in \psi$ implies $\varsigma \cap \varsigma' \in \psi$;

A b-UFIL space (X, Θ^X, ψ) is a *bornological b-UFIL space* provided that the following axiom holds:

(b-UFIL 7): $E, E' \in \Theta^X$ implying $E \cup E' \in \Theta^X$, and $\varsigma \in \psi$ implies $E \times E' \in \varsigma$ for some $E \in \Theta^X$;

Let (X, Θ^X, ψ_X) and (Y, Θ^Y, ψ_Y) be a pair of b-UFIL spaces and $h : X \to Y$ be a map. Then h is called *bounded uniformly continuous* (or buc) map if $E \in \Theta^X$ implies $h(E) \in \Theta^Y$; and $\varsigma \in \psi_X$ implies $(h \times h)(\varsigma) \in \psi_Y$; where $(h \times h)(\varsigma) := \{V \subset Y \times Y : \exists U \in \varsigma \mid (h \times h)[U] \subset V\}$ with $(h \times h)[U] := \{(h \times h)(x, y) : (x, y) \in U\} = \{(h(x), h(y)) : (x, y) \in U\}$.

We denote **b-UFIL** as the category of b-UFIL spaces and buc maps. Similarly, **sb-UFIL** (respectively **LIMsb-UFIL**) as the category of symmetric b-UFIL spaces (respectively the category of symmetric b-UFIL limit spaces) and buc maps. Furthermore, **BONb-UFIL** is the category of bornological b-UFIL spaces and buc maps.

Definition 2.3. (cf. [33])

(i) For given a family of b-UFIL spaces $(X_j, \Theta_{X_j}, \psi_j)_{j \in I}$ and maps $(h_j : X \to X_j)_{j \in I}$. The initial b-UFIL structure on X is represented by (Θ^X, ψ), where $\Theta^X := \{E \subset X : h_j[E] \in \Theta^{X_j},\ \forall j \in I\}$ and $\psi := \{\varsigma \in \mathcal{F}(X^2) : (h_j \times h_j)(\varsigma) \in \psi_j,\ \forall j \in I\}$ with $X^2 := X \times X$.

(ii) A b-UFIL structure on X is indiscrete if $(\Theta^X, \psi) := (P(X), \mathcal{F}(X^2))$.

(iii) For given a family of b-UFIL spaces $(X_j, \Theta^{X_j}, \psi_j)_{j \in I}$ and maps $(h_j : X_j \to X)_{j \in I}$. The final b-UFIL structure on X is represented by (Θ^X, ψ), where $\Theta^X := \{E \subset X : \exists i \in I, \exists E_j \in \Theta^{X_j} :\ E \subset h_j[E_j]\} \cup D^X := \{\emptyset\} \cup \{\{a\} : a \in X\}$ and $\psi := \{\varsigma \in \mathcal{F}(X^2) : \exists j \in I, \exists \varsigma_j \in \psi_j :\ (h_j \times h_j)(\varsigma_j) \subset \varsigma\} \cup \{\dot{x} \times \dot{x} : x \in X\} \cup \{P(X^2)\}$.

(iv) A b-UFIL structure on X is discrete if $(\Theta^X, \psi) := (D^X, \psi_{dis})$, where $\psi_{dis} := \{\dot{x} \times \dot{x} : x \in X\} \cup \{P(X^2)\}$.

Remark 2.4. 1. A bornological b-UFIL structure on X is discrete if $(\Theta^X, \psi) := (D^X_{born}, \psi_{dis})$, where $D^X_{born} := \{E \subset X : E \text{ is finite}\}$ [33].

2. The category **PUConv** is isomorphic to **DISb-UFIL** (category of discrete b-UFIL spaces and buc maps) [33].

3. The category **SUConv** is isomorphic to **DISsb-UFIL** (category of discrete symmetric b-UFIL spaces and buc maps) [33].

3. Closed and strongly closed subsets of bounded uniform filter spaces

In this section, we define notion of closedness in b-UFIL spaces by characterizing closed and strongly closed subobjects in the category **b-UFIL**.

Let X be any set and $p \in X$. We define the *wedge product of* X *at* p as the two disjoint copies of X at p and denote it as $X \bigvee_p X$. For a point $x \in X \bigvee_p X$ we write it as x_1 if x belongs to the first component of the wedge product otherwise we write x_2 that is in the second component. Moreover, X^2 is the cartesian product of X.

Definition 3.1. (cf. [2])

(i) A map $A_p : X \bigvee_p X \longrightarrow X^2$ is said to be principal p-axis map provided that

$$A_p(x_j) := \begin{cases} (x, p); & j = 1, \\ (p, x); & j = 2, \end{cases}$$

(ii) A map $\nabla_p : X \bigvee_p X \longrightarrow X$ is said to be fold map at p provided that

$$\nabla_p(x_j) := x, \qquad j = 1, 2.$$

Similarly, we define the *infinite wedge product of* X *at* p as the infinitely countable disjoint copies of X identifying at p and denote it as $\bigvee_p^\infty X$.

For a point $x \in \bigvee_p^\infty X$ we write it as x_j if it belongs to the j^{th} component of the infinite wedge product.

Definition 3.2. (cf. [2, 3])

(i) A map $A_p^\infty : \bigvee_p^\infty X \longrightarrow X^\infty$ is said to be infinite principal p axis map provided that

$$A_p^\infty(x_j) := (p, p, \cdots, p, \underbrace{x}_{j^{th}\ place}, p, \cdots), \quad \forall j \in I.$$

(ii) A map $\nabla_p^\infty : \bigvee_p^\infty X \longrightarrow X$ is said to be infinite fold map at p provided that

$$\nabla_p^\infty(x_j) := x, \quad \forall j \in I.$$

Definition 3.3. (cf. [3]) A map $Q : \mathfrak{F}X = E \to E/F$, where $F \subset E$ and $E/F = (E \backslash F) \cup \{\star\}$, is said to be the quotient map or the epi map provided that it identifies F to $\star$ and is identity at $E \backslash F$.

Definition 3.4. (cf. [2, 3]) Let $\mathfrak{F} : \mathcal{G} \longrightarrow$ **Set** be a topological functor and $X \in Obj(\mathcal{G})$ with $\mathfrak{F}(X) = E$ and $p \in E$.

(i) $\{p\}$ is closed provided that initial lift of $\mathfrak{F}$-source $\{\bigvee_p^\infty E \xrightarrow{A_p^\infty} \mathfrak{F}X^\infty = E^\infty$ *and* $\bigvee_p^\infty E \xrightarrow{\nabla_p^\infty} \mathfrak{F}DE = E\}$ is discrete.

(ii) $F \subset X$ is closed provided that $\{\star\}$ (image of F) is closed in X/F or $F = \emptyset$.

(iii) $F \subset X$ is strongly closed provided that X/F is T_1 at $\star$ or $F = \emptyset$.

(iv) If $F = E = \emptyset$ provided that F is both closed and strongly closed.

Remark 3.5. 1. In **Top**, all closed sets reduce to the classical closed sets, and set A is strongly closed provided that A is closed and for $a \notin A$ there exists an open set $\mathcal{N}_A$ containing A such that $a \notin \mathcal{N}_A$ [5].

2. In T_1**Top**, then closed sets and strongly closed sets coincide with each other [5].

3. In general, there is no relation between closed and strongly closed sets of an arbitrary topological category [3].

Theorem 3.6. (cf. [32]) *Let (X, Θ^X, ψ) be a b-UFIL space and $p \in X$. Then (X, Θ^X, ψ) is T_1 at p if and only if for all $x \in X$ with $x \neq p$, the conditions below hold.*

(i) $\{x, p\} \notin \Theta^X$,

(ii) $[x] \times [p] \notin \psi$ *and* $[p] \times [x] \notin \psi$,

(iii) $([x] \times [x]) \cap ([p] \times [p]) \notin \psi$.

Theorem 3.7. *Let (X, Θ^X, ψ) be a b-UFIL space and $p \in X$. Then $\{p\}$ is closed in X if and only if for all $x \in X$ with $x \neq p$, the conditions below hold.*

(i) $\{x, p\} \notin \Theta^X$,

(ii) $[x] \times [p] \notin \psi$ *or* $[p] \times [x] \notin \psi$,

(iii) $([x] \times [x]) \cap ([p] \times [p]) \notin \psi$.

Proof. Let $\{p\}$ be closed in X. We show that the above conditions $(i) - (iv)$ hold. Let $\{x, p\} \in \Theta^{\vee_p^\infty X}$ for $x \neq p$ and $W = \{x_1, x_2\} \in \Theta^{\vee_p^\infty X}$. Since $\nabla_p^\infty W = \{x\} \in \mathcal{D}^X$, and $\pi_1 A_p^\infty W = \pi_2 A_p^\infty W = \{x, p\} \in \Theta^X$, $\pi_k A_p^\infty W = \{p\} \in \Theta^X$ for $k \geq 3$, where $\pi_k : X^\infty \to X$ for $k \in I$ are the projection maps. By Definitions 2.2, 2.3, and 3.4(i), a contradiction. Hence, $\{x, p\} \notin \Theta^X$.

Next, suppose that $[x] \times [p] \in \psi$ for some $x \neq p$. Let $\varsigma = [x_1] \times [x_2]$. Clearly, $(\nabla_p^\infty \times \nabla_p^\infty)\varsigma = [x] \times [x] \in \psi_{dis}$, $(\pi_1 A_p^\infty \times \pi_1 A_p^\infty)\varsigma = [x] \times [p] \in \psi$, $(\pi_2 A_p^\infty \times \pi_2 A_p^\infty)\varsigma = [p] \times [x] \in \psi$, and $(\pi_k A_p^\infty \times \pi_k A_p^\infty)\varsigma = [p] \times [p] \in \psi$ for $k \geq 3$, a contradiction. It follows that either $[x] \times [p] \notin \psi$ or $[p] \times [x] \notin \psi$.

Further, suppose that $([x] \times [x]) \cap ([p] \times [p]) \in \psi$ for some $x \neq p$. Assume that $\varsigma = ([x_1] \times [x_1]) \cap ([x_2] \times [x_2])$. Since $(\nabla_p^\infty \times \nabla_p^\infty)\varsigma = [x] \times [x] \in \psi_{dis}$, $(\pi_1 A_p^\infty \times \pi_1 A_p^\infty)\varsigma = (\pi_2 A_p^\infty \times \pi_2 A_p^\infty)\varsigma = ([x] \times [x]) \cap ([p] \times [p]) \in \psi$, and $(\pi_k A_p^\infty \times \pi_k A_p^\infty)\varsigma = [p] \times [p] \in \psi$ for $k \geq 3$, a contradiction to the closedness of $\{p\}$. Thus, $([x] \times [x]) \cap ([p] \times [p]) \notin \psi$.

Conversely, let us assume that the conditions $(i) - (iv)$ hold. Let $(\Theta^{\vee_p^\infty X}, \overline{\psi})$ be the initial structure induced by $\nabla_p^\infty : \vee_p^\infty X \to (X, \mathcal{D}^X, \psi_{dis})$ and $A_p^\infty : \vee_p^\infty X \to (X^\infty, \Theta^{X^\infty}, \psi^\infty)$, where $(\mathcal{D}^X, \psi_{dis})$ and $(\Theta^{X^\infty}, \psi^\infty)$ are discrete b-UFIL structure on X and product b-UFIL structure on X^∞, respectively. We show that $(\Theta^{X \vee_p X}, \overline{\psi})$ is a discrete b-UFIL structure on $\vee_p^\infty X$. Let $W \in \Theta^{\vee_p^\infty X}$ and $\nabla_p^\infty W \in \mathcal{D}^X$.

If $\nabla_p^\infty W = \emptyset$, then $W = \emptyset$.

Suppose $\nabla_p^\infty W \neq \emptyset$, it indicates that $\nabla_p^\infty W = \{x\}$ for some $x \in X$. If $x = p$, then $W = \{p\}$. Suppose $x \neq p$. Then we show that $W = \{x_j\}$ for all $j \in I$ and the case $W \subset \{x_1, x_2, x_3, \dots\}$ can not happen. Let $W = \{x_1, x_2\}$ then, $\pi_k A_p^\infty W = \{x, p\} \notin \Theta^X$ (for $k = 1, 2$) by the assumption and by Definition 2.2(b-UFIL 1), any set containing W can not be in $\Theta^{\vee_p^\infty X}$. Hence, $W = \{x_j\}$ $(j \in I)$ and consequently, $\Theta^{\vee_p^\infty X} = \mathcal{D}^{\vee_p^\infty X}$, the discrete b-UFIL structure on $\vee_p^\infty X$.

Next, let $\varsigma \in \overline{\psi}$. By Definition 2.3(i), $(\nabla_p^\infty \times \nabla_p^\infty)\varsigma \in \mathcal{D}^X$ and $(\pi_k A_p^\infty \times \pi_k A_p^\infty)\varsigma \in \psi$ for $k \in I$. We need to show that $\varsigma = [x_j] \times [x_j]$ $(j \in I)$, $\varsigma = [p] \times [p]$ or $\varsigma = [\emptyset] = P(\vee_p^\infty X)^2$.

If $(\nabla_p^\infty \times \nabla_p^\infty)\varsigma = [\emptyset]$, then $\varsigma = [\emptyset] = P(\vee_p^\infty X)^2$.

Suppose $(\nabla_p^\infty \times \nabla_p^\infty)\varsigma = [x] \times [x]$ for some $x \in X$. If $x = p$, since $(\nabla_p^\infty)^{-1}\{p\} = \{p_j = (p, p, p, \dots)\}$, so $\varsigma = [(p, p, p, \dots)] \times [(p, p, p, \dots)]$.

If $x \neq p$, then $(\nabla_p^\infty \times \nabla_p^\infty)\varsigma = [x] \times [x]$, then either $\{x_{j_1}, x_{j_2}, \dots, x_{j_m}\} \times \{x_{j_1}, x_{j_2}, \dots, x_{j_m}\} \in \varsigma$ or $\{x_1, x_2, \dots\} \times \{x_1, x_2, \dots\} \in \varsigma$.

If $B = \{x_{j_1}, x_{j_2}, \dots, x_{j_m}\} \times \{x_{j_1}, x_{j_2}, \dots, x_{j_m}\} \in \varsigma$, there exists a finite subset N_0 of ς so that $\varsigma = [N_0]$. Clearly, $N_0 \subseteq B = \{x_{j_1}, x_{j_2}, \dots, x_{j_m}\} \times \{x_{j_1}, x_{j_2}, \dots, x_{j_m}\}$ and if $j_r \neq j_s$ $(r, s = 1, 2, \dots, m)$, then $\{\{x_{j_r}\} \times \{x_{j_s}\}\} \neq N_0$ and $\{\{x_1\} \times \{x_1\}, \{x_2\} \times \{x_2\}, \dots, \{x_{j_m}\} \times \{x_{j_m}\}\} \neq N_0$ since in particular for $k = 1$, $j_r = 1$, and $j_s = 2$, $(\pi_1 A_p^\infty \times \pi_1 A_p^\infty)([x_1] \times [x_2]) = [x] \times [p] \notin \psi$, and $(\pi_1 A_p^\infty \times \pi_1 A_p^\infty)(([x_1] \times [x_1]) \cap ([x_2] \times [x_2]) \cap \dots \cap ([x_{j_m}] \times [x_{j_m}])) = ([x] \times [x]) \cap ([p] \times [p]) \notin \psi$, using the second and the third conditions respectively.

If $B = \{x_1, x_2, \dots\} \times \{x_1, x_2, \dots\} \in \varsigma$, there exists a finite subset N_0 of ς so that $\varsigma = [N_0]$. Clearly, $N_0 \subseteq B = \{x_1, x_2, \dots\} \times \{x_1, x_2, \dots\}$. The following cases for N_0 can not happen.

(a) $\{\{x_i\}\times\{x_j\},\ i,j\in I\}\neq N_0$ since $(\pi_j A_p^\infty\times\pi_j A_p^\infty)(\varsigma)=[x]\times[p]\notin\psi$ or $[p]\times[x]\notin\psi$ (for all $j\in I$), using the condition (ii).

(b) $\{x_1,x_2,\dots\}\times\{x_1,x_2,\dots\}\neq N_0$ since $(\pi_k A_p^\infty\times\pi_k A_p^\infty)([x_1]\times[x_2])=([x]\times[x])\cap([p]\times[p])\cap([x]\times[p])\cap([p]\times[x])\notin\psi$ (for $k\in I$). By Definition 2.2(b-UFIL 3) and the condition (i) of our supposition, if $\{x,p\}\notin\Theta^X$ then $[x,p]\times[x,p]=([x]\times[x])\cap([p]\times[p])\cap([x]\times[p])\cap([p]\times[x])\notin\psi$. Otherwise, $[\{x,p\}]\times[\{x,p\}]\subset[x]\times[p]$ and by Definition 2.2(b-UFIL 4), it concludes that $[x]\times[p]\in\psi$, a contradiction to the condition (ii).

(c) For $r,s>1$ and $s\le r$, $\{x_r,x_{r+1},\dots\}\times\{x_s,x_{s+1},\dots\}\neq N_0$ as $(\pi_j A_p^\infty\times\pi_j A_p^\infty)(\varsigma)=([x]\times[x])\cap([p]\times[p])\cap([x]\times[p])\cap([p]\times[x])\notin\psi$ (for all $j\in I$), by the similar argument as in above part (b).

(d) $\{\{x_j\}\times\{x_j\},\ j\in I\}\neq N_0$ since $(\pi_j A_p^\infty\times\pi_j A_p^\infty)(\varsigma)=([x]\times[x])\cap([p]\times[p])\notin\psi$ (for all $j\in I$), using the condition (iii).

(e) For some fixed r, $\{\{x_j\}\times\{x_j\},\ j\in I\}\cup\{\{x_r\}\times\{x_{r+1}\}\}\neq N_0$ or $\{\{x_j\}\times\{x_j\},\ j\in I\}\cup\{\{x_r\}\times\{x_{r+1}\},\{x_{r+5}\}\times\{x_r\}\}\neq N_0$ or $\{\{x_j\}\times\{x_j\},\ j\in I\}\cup\{\{x_r\}\times\{x_{r+1}\},\{x_{r+1}\}\times\{x_r\},\{x_{r+5}\}\times\{x_{r+20}\}\}\neq N_0$, since $(\pi_j A_p^\infty\times\pi_j A_p^\infty)(\varsigma)=([x]\times[x])\cap([p]\times[p])\notin\psi$ (for all $j\in I$), using the condition (iii).

Therefore, we must have $\varsigma=[x_j]\times[x_j]$ $(j\in I)$ or $\varsigma=[\emptyset]$ or $\varsigma=[(p,p,p,\dots)]\times[(p,p,p,\dots)]$, and consequently, by definitions 2.2, 2.3 and 3.4(i), the singleton $\{p\}$ is closed in X. □

Theorem 3.8. *Let (X,Θ^X,ψ) be a b-UFIL space, $\emptyset\neq F\subset X$, $W\in\Theta^X$ and $\varsigma\in\psi$. For every $x,y\in X$ with $x\notin F$ and $y\in F$,*

(i) *$Q(W)\supseteq\{x,\star\}$ if and only if $W\supseteq\{x,y\}$.*

(ii) *$(Q\times Q)\varsigma\subset[x]\times[\star]$ if and only if $\varsigma\subset[x]\times[y]$ or $\varsigma\cup([x]\times[F])$ is proper.*

(iii) *$(Q\times Q)\varsigma\subset[\star]\times[x]$ if and onl if $\varsigma\subset[y]\times[x]$ or $\varsigma\cup([F]\times[x])$ is proper.*

(iv) *$(Q\times Q)\varsigma\subset([x]\times[x])\cap([\star]\times[\star])$ if and only if $\varsigma\cap([F]\times[F])\subset([x]\times[x])\cap([F]\times[F])$ and $\varsigma\cup([F]\times[F])$ is proper,*

where $Q:X\to X/F$ is a quotient map defined in Definition 3.3.

Proof. (i) Let $\{x,\star\}\subseteq Q(W)$. Then it follows that $Q^{-1}(\{x,\star\})\subseteq Q^{-1}(Q(W))\subseteq W$ and therefore, $\{x,y\}\subseteq W$, for all $x\notin F$ and $y\in F$.

Conversely, suppose that $W\supseteq\{x,y\}$, for all $x\notin F$ and $y\in F$. By Definition 3.3, it follows that $Q(W)\supseteq Q(\{x,y\})=\{x,\star\}$.

(ii) Let $(Q\times Q)\varsigma\subset[x]\times[\star]$ for $\varsigma\in\psi$ and $x\notin F$. If $\varsigma\not\subset[x]\times[y]$ and $\varsigma\cup([x]\times[F])$ is improper for some $y\in F$, then $U\cap(\{x\}\times F)=\emptyset$ for some $U\in\varsigma$. It follows that $(x,z)\notin U$ for all $z\in F$, and $(Q\times Q)(\{x\}\times\{z\})\notin(Q\times Q)(U)\in(Q\times Q)\varsigma$, which implies that $(\{x\}\times\{\star\})\notin(Q\times Q)\varsigma$. Therefore, $(Q\times Q)\varsigma\not\subset[x]\times[\star]$, a contradiction to the assumption. Thus, $\varsigma\subset[x]\times[y]$ or $\varsigma\cup([x]\times[F])$ is proper for all $x\notin F$ and $y\in F$.

Conversely, assume that $\varsigma\subset[x]\times[y]$ or $\varsigma\cup([x]\times[F])$ is proper. We claim that $(Q\times Q)\varsigma\subset[x]\times[\star]$. If $\varsigma\subset[x]\times[y]$, then we get $(Q\times Q)\varsigma\subset(Q\times Q)([x]\times[y])=(Q\times Q)([x]\times[\star])$.

If $\varsigma\cup([x]\times[F])$ is proper, then $V\cap(x\times F)\neq\emptyset$ for all $V\in\varsigma$. Let $M\in(Q\times Q)\varsigma$. Then, there exists some $U\in\varsigma$ so that $(Q\times Q)(U)\subset M$. Hence, $U\cap(x\times F)\neq\emptyset$, as $\varsigma\cup([x]\times[F])$ is proper. It follows that for some $y\in F$, $(\{x\}\times\{y\})\in U$, and $(Q\times Q)(\{x\}\times\{y\})\in(Q\times Q)(U)\subset M$, which implies that $(\{x\}\times\{\star\})\subset M$. Consequently, $M\in([x]\times[\star])$ and hence, $(Q\times Q)\varsigma\subset[x]\times[\star]$.

(iii) The proof is similar as we have done above in part (ii).

(iv) Let $(Q\times Q)\varsigma\subset([x]\times[x])\cap([\star]\times[\star])$. We first show that $\varsigma\cup([F]\times[F])$ is proper. As opposed, assume that $\varsigma\cup([F]\times[F])$ is improper, then $U\cap(F\times F)=\emptyset$ for some $U\in\varsigma$. We note that $(Q\times Q)(U)\in(Q\times Q)\varsigma\subset([x]\times[x])\cap([\star]\times[\star])\subset([\star]\times[\star])$, by the assumption. It follows that $(\{\star\}\times\{\star\})\in(Q\times Q)(U)$. Thus, for some $(\{a\}\times\{b\})\in U$, we have $(Q\times Q)(\{a\}\times\{b\})=\{\star\}\times\{\star\}$ implying that $(\{a\}\times\{b\})\in U\cap(F\times F)$, a contradiction, and it shows that $\varsigma\cup([F]\times[F])$ must be proper.

Next, we show that $\varsigma \cap ([F] \times [F]) \subset ([x] \times [x]) \cap ([F] \times [F])$. Let $U \in \varsigma \cap ([F] \times [F])$. We prove that $U' \in ([x] \times [x]) \cap ([F] \times [F])$, because $U \in \varsigma \cap ([F] \times [F])$ implies $U \in \varsigma$ and $F \times F \subset U'$. By the assumption, we get $(Q \times Q)(U') \in (Q \times Q)\varsigma \subset ([x] \times [x]) \cap ([\star] \times [\star]) = (Q \times Q)([x] \times [x]) \cap (Q \times Q)([F] \times [F]) = (Q \times Q)(([x] \times [x]) \cap ([F] \times [F]))$, hence $(Q \times Q)(U') \in (Q \times Q)(([x] \times [x]) \cap ([F] \times [F]))$. It follows that there exists some $V \in ([x] \times [x]) \cap ([F] \times [F])$ such that $(Q \times Q)(V) \subset (Q \times Q)(U')$. Further, $V \in ([x] \times [x]) \cap ([F] \times [F])$ implies that $V \in ([x] \times [x])$ and $V \in ([F] \times [F])$, i.e., $V \cap (F \times F) \neq \emptyset$, and $V \subset V \cap (F \times F)$. Also, we have $V \subset V \cap (F \times F) = (Q \times Q)^{-1}((Q \times Q)(V)) \subset (Q \times Q)^{-1}((Q \times Q)(U')) \subset U'$. Therefore, $V \subset U'$ and $U' \in ([x] \times [x]) \cap ([F] \times [F])$ and thus by the arbitrariness of U', $\varsigma \cap ([F] \times [F]) \subset ([x] \times [x]) \cap ([F] \times [F])$.

Conversely, let $\varsigma \cap ([F] \times [F]) \subset ([x] \times [x]) \cap ([F] \times [F])$ and $\varsigma \cup ([F] \times [F])$ is proper. We claim that $(Q \times Q)\varsigma \subset ([x] \times [x]) \cap ([\star] \times [\star])$. First, we show that $(Q \times Q)\varsigma \subset ([\star] \times [\star])$. As opposed assume that $(Q \times Q)\varsigma \not\subset ([\star] \times [\star])$. Then there exists some $M \subset (Q \times Q)\varsigma$ such that $(\{\star\} \times \{\star\}) \not\subset M$. Since $M \subset (Q \times Q)\varsigma$, it follows that there exists some $U \in \varsigma$ such that $(Q \times Q)(U) \subset M$. Hence, $U \cap (F \times F) \neq \emptyset$, since $\varsigma \cup ([F] \times [F])$ is proper, and we have $(Q \times Q)(U \cap (F \times F)) \subset (Q \times Q)(U) \subset M$, which implies that $(\{\star\} \times \{\star\}) \subset M$, a contradiction. Thus, we must have $(Q \times Q)\varsigma \subset ([\star] \times [\star])$. Now, $(Q \times Q)(\varsigma \cap ([F] \times [F])) = (Q \times Q)\varsigma \cap (Q \times Q)([F] \times [F]) = (Q \times Q)\varsigma \cap ([\star] \times [\star]) = (Q \times Q)\varsigma$. Also, $\varsigma \cap ([F] \times [F]) \subset ([x] \times [x]) \cap ([F] \times [F])$ by the assumption, therefore $(Q \times Q)\varsigma = (Q \times Q)(\varsigma \cap ([F] \times [F])) \subset (Q \times Q)(([x] \times [x]) \cap ([F] \times [F])) = ([x] \times [x]) \cap ([\star] \times [\star])$. □

Theorem 3.9. *Let (X, Θ^X, ψ) be a b-UFIL space, $\emptyset \neq F \subset X$ is closed if and only if for each $x, y \in X$ with $x \notin F$, $y \in F$ and $\varsigma \in \psi$, the conditions below hold:*

(i) $\{x, y\} \notin \Theta^X$,

(ii) $\varsigma \not\subseteq [x] \times [y]$ *and* $\varsigma \cup ([x] \times [F])$ *is improper (or* $\varsigma \not\subseteq [y] \times [x]$ *and* $\varsigma \cup ([F] \times [x])$ *is improper),*

(iii) $\varsigma \cap ([F] \times [F]) \not\subseteq ([x] \times [x]) \cap ([F] \times [F])$ *or* $\varsigma \cup ([F] \times [F])$ *is improper.*

Proof. Let F be non-empty closed set. Then, by Definition 3.4, $\{\star\}$ is closed in X/F since F is nonempty. By Theorem 3.7, for all $x \in X/F$ with $x \neq \star$, $\{x, \star\} \notin \Theta^{X/F}$, $[x] \times [\star] \notin \psi$ (or $[\star] \times [x] \notin \psi_{X/F}$), and $([x] \times [x]) \cap ([\star] \times [\star]) \notin \psi_{X/F}$, where $(\Theta^{X/F}, \psi_{X/F})$ is the quotient b-UFIL structure on X/F induced by $Q : X \to X/F$. By Definition 2.3(ii), for all $\varsigma \in \psi$, $x \notin F$, and $W \in \Theta^X$, we get $Q(W) \not\supseteq \{x, \star\}$, $(Q \times Q)\varsigma \not\subset [x] \times [\star]$ (or $(Q \times Q)\varsigma \not\subset [\star] \times [x]$) and $(Q \times Q)\varsigma \not\subset ([x] \times [x]) \cap ([\star] \times [\star])$ if and only if by Theorem 3.8, $\{x, y\} \notin W$, and it follows by Definition 2.2(b-UFIL 1) that $\{x, y\} \notin \Theta^X$, $\varsigma \not\subseteq [x] \times [y]$ and $\varsigma \cup ([x] \times [F])$ is improper (or $\varsigma \not\subseteq [y] \times [x]$ and $\varsigma \cup ([F] \times [x])$ is improper), and $\varsigma \cap ([F] \times [F]) \not\subseteq ([x] \times [x]) \cap ([F] \times [F])$ or $\varsigma \cup ([F] \times [F])$ is improper. □

Theorem 3.10. *Let (X, Θ^X, ψ) be a b-UFIL space, $\emptyset \neq F \subset X$ is strongly closed if and only if for each $x, y \in X$ with $x \notin F$, $y \in F$ and $\varsigma \in \psi$, the conditions below hold:*

(i) $\{x, y\} \notin \Theta^X$,

(ii) $\varsigma \not\subseteq [x] \times [y]$ *and* $\varsigma \cup ([x] \times [F])$ *is improper,*

(iii) $\varsigma \not\subseteq [y] \times [x]$ *and* $\varsigma \cup ([F] \times [x])$ *is improper,*

(iv) $\varsigma \cap ([F] \times [F]) \not\subseteq ([x] \times [x]) \cap ([F] \times [F])$ *or* $\varsigma \cup ([F] \times [F])$ *is improper.*

Proof. Let F be strongly closed. Then, by Definition 3.4, X/F is T_1 at $\star$ since F is non-empty. By Theorem 3.6, for all $x \in X/F$ with $x \neq \star$, $\{x, \star\} \notin \Theta^{X/F}$, $[x] \times [\star] \notin \psi$, $[\star] \times [x] \notin \psi_{X/F}$ and $([x] \times [x]) \cap ([\star] \times [\star]) \notin \psi_{X/F}$, where $(\Theta^{X/F}, \psi_{X/F})$ is the quotient b-UFIL structure on X/F induced by $Q : X \to X/F$. By Definition 2.3(ii), for all $\varsigma \in \psi$, $x \notin F$ and $W \in \Theta^X$, hence we get $Q(W) \not\supseteq \{x, \star\}$, $(Q \times Q)\varsigma \not\subset [x] \times [\star]$, $(Q \times Q)\varsigma \not\subset [\star] \times [x]$, and $(Q \times Q)\varsigma \not\subset ([x] \times [x]) \cap ([\star] \times [\star])$ if and only if by Theorem 3.8, $\{x, y\} \notin W$ which concludes by Definition 2.2(b-UFIL 1) that $\{x, y\} \notin \Theta^X$, $\varsigma \not\subseteq [x] \times [y]$ and $\varsigma \cup ([x] \times [F])$ is improper, $\varsigma \not\subseteq [y] \times [x]$ and $\varsigma \cup ([F] \times [x])$ is improper, and $\varsigma \cap ([F] \times [F]) \not\subseteq ([x] \times [x]) \cap ([F] \times [F])$ or $\varsigma \cup ([F] \times [F])$ is improper. □

Theorem 3.11. *1. Let $h : (X, \Theta^X, \psi_X) \to (Y, \Theta^Y, \psi_Y)$ be a buc map between two b-UFIL spaces. If $G \subset Y$ is closed, then $h^{-1}(G)$ is closed in X.*

2. *Let (X, Θ^X, ψ) be a b-UFIL space. If $F \subset X$ is closed and $E \subset F$ is closed, then $E \subset X$ is closed.*

Proof. (1) Let $G \subset Y$ be closed and for all $x, y \in X$ with $x \notin h^{-1}(G)$, $y \in h^{-1}(G)$ and $\varsigma \in \psi_X$, we show that

(i) $\{x, y\} \notin \Theta^X$.

(ii) $\varsigma \not\subseteq [x] \times [y]$ and $\varsigma \cup ([x] \times [h^{-1}(G)])$ is improper (or $\varsigma \not\subseteq [y] \times [x]$ and $\varsigma \cup ([h^{-1}(G)] \times [x])$ is improper).

(iii) $\varsigma \cap ([h^{-1}(G)] \times [h^{-1}(G)]) \not\subseteq ([x] \times [x]) \cap ([h^{-1}(G)] \times [h^{-1}(G)])$ or $\varsigma \cup ([h^{-1}(G)] \times [h^{-1}(G)])$ is improper.

Note that $h(x), h(y) \in Y$, $h(x) \notin G$, $h(y) \in G$ and $(h \times h)\varsigma \in \psi_Y$. Since G is closed, by Theorem 3.9, we have

(i) $\{h(x), h(y)\} \notin \Theta^Y$.

(ii) $(h \times h)\varsigma \not\subseteq [h(x)] \times [h(y)]$ and $(h \times h)\varsigma \cup ([h(x)] \times [G])$ is improper (or $(h \times h)\varsigma \not\subseteq [h(y)] \times [h(x)]$ and $(h \times h)\varsigma \cup ([G] \times [h(x)])$ is improper).

(iii) $(h \times h)\varsigma \cap ([G] \times [G]) \not\subseteq ([h(x)] \times [h(x)]) \cap ([G] \times [G])$ or $(h \times h)\varsigma \cup ([G] \times [G])$ is improper.

Suppose $\{h(x), h(y)\} \notin \Theta^Y$. Clearly, $\{x, y\} \notin \Theta^X$, otherwise, if $W = \{x, y\} \in \Theta^X$, then $h(W) = h(\{x, y\}) = \{h(x), h(y)\} \in \Theta^Y$, a contradiction.

Suppose $(h \times h)\varsigma \not\subseteq [h(x)] \times [h(y)]$, then by Lemma 2.1, clearly it appears that $\varsigma \not\subseteq [x] \times [y]$. Next, we conclude that $\varsigma \cup ([x] \times [h^{-1}(G)])$ is improper. On contrary, suppose that it is proper. By Lemma 2.1, $(h\times h)\varsigma\cup([h(x)]\times[G]) \subset (h\times h)\varsigma\cup([h(x)]\times[h(h^{-1}(G))]) \subset (h\times h)\varsigma\cup(h\times h)([x]\times[h^{-1}(G)])) \subset (h\times h)(\varsigma\cup([x]\times[h^{-1}(G)]))$, and consequently $(h \times h)\varsigma \cup ([h(x)] \times [G])$ is proper, a contradiction. Thus, $\varsigma \cup ([x] \times [h^{-1}(G)])$ is improper. In a similar manner, $\varsigma \not\subseteq [y] \times [x]$ and $\varsigma \cup ([h^{-1}(G)] \times [x])$ is improper.

Suppose $(h \times h)\varsigma \cap ([G] \times [G]) \not\subseteq ([h(x)] \times [h(x)]) \cap ([G] \times [G])$, then clearly $\varsigma \cap ([h^{-1}(G)] \times [h^{-1}(G)]) \not\subseteq ([x] \times [x]) \cap ([h^{-1}(G)] \times [h^{-1}(G)])$ by Lemma 2.1. Now we show that $\varsigma \cup ([h^{-1}(G)] \times [h^{-1}(G)])$ is improper. As opposed assume that it is proper. Then, $(h \times h)\varsigma \cup ([G] \times [G]) \subset (h \times h)\varsigma \cup ([h(h^{-1}(G))] \times [h(h^{-1}(G))]) \subset (h\times h)\varsigma \cup (h\times h)([h^{-1}(G)] \times [h^{-1}(G)]) \subset (h\times h)(\varsigma \cup ([h^{-1}(G)] \times [h^{-1}(G)]))$, and consequently $(h\times h)\varsigma \cup ([G] \times [G])$ is proper, a contradiction. Thus, $\varsigma \cup ([h^{-1}(G)] \times [h^{-1}(G)])$ is improper.

(2) Let $F \subset X$ and $E \subset F$ be closed, and for all $x, y \in X$ with $x \notin E$, $y \in E$, and $\varsigma \in \psi_X$, we show that

(i) $\{x, y\} \notin \Theta^X$,

(ii) $\varsigma \not\subseteq [x] \times [y]$ and $\varsigma \cup ([x] \times [E])$ is improper (or $\varsigma \not\subseteq [y] \times [x]$ and $\varsigma \cup ([E] \times [x])$ is improper).

(iii) $\varsigma \cap ([E] \times [E]) \not\subseteq ([x] \times [x]) \cap ([E] \times [E])$ or $\varsigma \cup ([E] \times [E])$ is improper.

If $x \notin F$. Since $F \subset X$ is closed, then by Theorem 3.9, we have $\{x, y\} \notin \Theta^X$, $\varsigma \not\subseteq [x] \times [y]$ and $\varsigma \cup ([x] \times [F])$ is improper (or $\varsigma \not\subseteq [y] \times [x]$ and $\varsigma \cup ([F] \times [x])$ is improper), and $\varsigma \cap ([F] \times [F]) \not\subseteq ([x] \times [x]) \cap ([F] \times [F])$ or $\varsigma\cup([F]\times[F])$ is improper. Consequently, since $E \subset F$ is closed, we get $\{x, y\} \notin \Theta^X$, $\varsigma \not\subseteq [x]\times[y]$ and $\varsigma\cup([x]\times[E])$ is improper (or $\varsigma \not\subseteq [y] \times [x]$ and $\varsigma \cup ([E] \times [x])$ is improper), and $\varsigma \cap ([E] \times [E]) \not\subseteq ([x] \times [x]) \cap ([E] \times [E])$ or $\varsigma \cup ([E] \times [E])$ is improper.

If $x \in F$. Since the inclusion map $i : (F, \Theta^F, \psi_F) \to (X, \Theta^X, \psi_X)$ is an initial lift and $\varsigma \in \psi_X$. By Definition 2.3(i), it follows that $(i \times i)^{-1}\varsigma \in \psi_F$. Note that $(i \times i)^{-1}\varsigma = \varsigma \cup ([F] \times [F])$ and $\varsigma \subset (i \times i)((i \times i)^{-1}\varsigma)$. Since $E \subset F$ is closed and $x, y \in F$ with $x \notin E$, $y \in E$, by Theorem 3.9

(i) $\{x, y\} \notin \Theta^X$,

(ii) $(i\times i)^{-1}\varsigma \not\subseteq [x]\times[y]$ and consequently $\varsigma \not\subseteq [x]\times[y]$, and $(i\times i)^{-1}\varsigma \cup ([x]\times[E]) = \varsigma \cup ([x]\times[E])$ is improper (or $\varsigma \not\subseteq [y] \times [x]$ and $\varsigma \cup ([E] \times [x])$ is improper),

(iii) $(i\times i)^{-1}\varsigma \cap ([E] \times [E]) = \varsigma \cap ([E] \times [E]) \not\subseteq ([x] \times [x]) \cap ([E] \times [E])$ or $(i\times i)^{-1}\varsigma \cup ([E] \times [E]) = \varsigma \cup ([E] \times [E])$ is improper.

Thus, $E \subset X$ is closed (since $E \subset F$). □

Theorem 3.12. *1. Let $h : (X, \Theta^X, \psi_X) \to (Y, \Theta^Y, \psi_Y)$ be a buc map between two b-UFIL spaces. If $G \subset Y$ is strongly closed, then $h^{-1}(G)$ is strongly closed in X.*

2. Let (X, Θ^X, ψ) be a b-UFIL space. If $F \subset X$ is strongly closed and $E \subset F$ is strongly closed, then $E \subset X$ is strongly closed.

Proof. The proof is analogous to the proof of Theorem 3.11 by using Theorem 3.10 instead of Theorem 3.9. □

4. Closure operators in bounded uniform filter spaces

Let $\mathcal{G}$ be a set based topological category, $X \in Obj(\mathcal{G})$ and C be the closure operator of $\mathcal{G}$ in the sense of [16, 18].

Definition 4.1. Let (X, Θ^X, ψ) be a b-UFIL space and $F \subset X$.

(i) $cl^{\textbf{b-UFIL}}(F) = \cap\{M \subset X : F \subset M \text{ and } M \text{ is closed}\}$ is known as the closure of F.

(ii) $scl^{\textbf{b-UFIL}}(F) = \cap\{M \subset X : F \subset M \text{ and } M \text{ is strongly closed}\}$ is known as the strong closure of F.

Theorem 4.2. *$scl^{\textbf{b-UFIL}}(F)$ and $cl^{\textbf{b-UFIL}}(F)$ are (weakly) hereditary, idempotent and productive closure operators of **b-UFIL**.*

Proof. The proof is straightforward by combining Theorems 3.11, 3.12, Definition 4.1, and Theorems 2.3, 2.4, Proposition 2.5 and Exercise 2.D of [18]. □

For a topological category $\mathcal{G}$ and a closure operator C of $\mathcal{G}$.

(i) $\mathcal{G}_{0C} = \{X \in \mathcal{G} : y \in C(\{x\})$ and $x \in C(\{y\})$ implies $x = y$ with $x, y \in X\}$.

(ii) $\mathcal{G}_{1C} = \{X \in \mathcal{G} : C(\{x\}) = \{x\}$ for each $x \in X\}$.

Remark 4.3. For $\mathcal{G} = \textbf{Top}$, $C = K$ (the ordinary closure operator), $\textbf{Top}_{jC}$ reduces to T_j space for $j = 1, 2$ respectively.

Theorem 4.4. *An object (X, Θ^X, ψ) is in $\textbf{b-UFIL}_{0scl}$ if and only if for all $x, y \in X$ with $x \neq y$, there exists $F_1 \subset X$ strongly closed subset of X such that $x \notin F_1$ and $y \in F_1$, or there exists $F_2 \subset X$ strongly closed subset of X such that $y \notin F_2$ and $x \in F_2$.*

Proof. Suppose that $(X, \Theta^X, \psi) \in \textbf{b-UFIL}_{0scl}$ and $x, y \in X$ with $x \neq y$. We get $y \notin scl(\{x\})$ and $x \notin scl(\{y\})$. If $x \notin scl(\{y\})$, then it follows by Definition 4.1(ii) that there exists $F_1 \subset X$ strongly closed subset of X such that $x \notin F_1$ and $y \in F_1$. Similarly, if $y \notin scl(\{x\})$, then again by Definition 4.1(ii) it follows that there exists $F_2 \subset X$ strongly closed subset of X such that $y \notin F_2$ and $x \in F_2$.

Conversely, suppose the first condition hold, i.e., for all $x, y \in X$ with $x \neq y$, there exists $F_1 \subset X$ strongly closed subset of X such that $x \notin F_1$ and $y \in F_1$. By Definition 4.1(ii), we get $x \notin scl(\{y\})$. If the later holds, i.e., for all $x, y \in X$ with $x \neq y$, there exists $F_2 \subset X$ strongly closed subset of X such that $y \notin F_2$ and $x \in F_2$. Then again by Definition 4.1(ii), it results that $y \notin scl(\{x\})$ and consequently $(X, \Theta^X, \psi) \in \textbf{b-UFIL}_{0scl}$. □

Theorem 4.5. *An object (X, Θ^X, ψ) is in $\textbf{b-UFIL}_{0cl}$ if and only if for all $x, y \in X$ with $x \neq y$, $\{x, y\} \notin \Theta^X$, there exists $F_1 \subset X$ closed subset of X such that $x \notin F_1$ and $y \in F_1$, or there exists $F_2 \subset X$ closed subset of X such that $y \notin F_2$ and $x \in F_2$.*

Proof. The proof is similar to the proof of Theorem 4.4 by using part (i) of Definition 4.1 instead of part (ii). □

Theorem 4.6. *An object* (X, Θ^X, ψ) *is in* ***b-UFIL***$_{1scl}$ *if and only if for all* $x, y \in X$ *with* $x \neq y$, $\{x, y\} \notin \Theta^X$, $[x] \times [y] \notin \psi$, $[y] \times [x] \notin \psi$ *and* $([x] \times [x]) \cap ([y] \times [y]) \notin \psi$

Proof. Suppose that $(X, \Theta^X, \psi) \in$ **b-UFIL**$_{1scl}$ and $x, y \in X$ with $x \neq y$. We get $scl(\{x\}) = \{x\}$ for all $x \in X$. It follows that $\{x\}$ is strongly closed and consequently by Theorem 3.10, for any $y \in X$ with $x \neq y$, $\{x, y\} \notin \Theta^X$, $[x] \times [y] \notin \psi$, $[y] \times [x] \notin \psi$ and $([x] \times [x]) \cap ([y] \times [y]) \notin \psi$ (for all $x \neq y$).

Conversely, suppose the condition hold, i.e., $\{x, y\} \notin \Theta^X$, $[x] \times [y] \notin \psi$, $[y] \times [x] \notin \psi$ and $([x] \times [x]) \cap ([y] \times [y]) \notin \psi$ (for all $x \neq y$). It follows that $\{x\}$ is strongly closed by Theorem 3.7. Consequently, $scl(\{x\}) = \{x\}$ for all $x \in X$ and hence $(X, \Theta^X, \psi) \in$ **b-UFIL**$_{1scl}$. □

Theorem 4.7. *An object* (X, Θ^X, ψ) *is in* ***b-UFIL***$_{1cl}$ *if and only if for all* $x, y \in X$ *with* $x \neq y$,

(i) $\{x, y\} \notin \Theta^X$,

(ii) $[x] \times [y] \notin \psi$ *(or* $[y] \times [x] \notin \psi$*)*,

(iii) $([x] \times [x]) \cap ([y] \times [y]) \notin \psi$.

Proof. The proof is analogous to the proof of Theorem 4.6 by using Theorem 3.9 instead of Theorem 3.10. □

Theorem 4.8. *Let* (X, Θ^X, ψ) *be a b-UFIL space. Then the following are equivalent:*

(i) (X, Θ^X, ψ) *is* $\overline{T_0}$.

(ii) $(X, \Theta^X, \psi) \in$ ***b-UFIL***$_{1cl}$.

(iii) *(a)* $\{x, y\} \notin \Theta^X$,
(b) $[x] \times [y] \notin \psi$ *(or* $[y] \times [x] \notin \psi$*)*,
(c) $([x] \times [x]) \cap ([y] \times [y]) \notin \psi$.

Proof. The proof can be easily deduced from Definition 3.4, Theorem 3.9 and Theorem 4.4 of [32]. □

Theorem 4.9. *Let* (X, Θ^X, ψ) *be a b-UFIL space. Then, for all* $x, y \in X$ *with* $x \neq y$.*, the following are equivalent:*

(i) (X, Θ^X, ψ) *is* T_1.

(ii) $(X, \Theta^X, \psi) \in$ ***b-UFIL***$_{1scl}$.

(iii) *(a)* $\{x, y\} \notin \Theta^X$,
(b) $[x] \times [y] \notin \psi$,
(c) $[y] \times [x] \notin \psi$,
(d) $([x] \times [x]) \cap ([y] \times [y]) \notin \psi$.

Proof. The proof can be easily deduced from Definition 3.4, Theorem 3.10 and Theorem 4.6 of [32]. □

Remark 4.10. Each of the subcategories **b-UFIL**$_{kcl}$, $k = 0, 1$ and **b-UFIL**$_{kscl}$, $k = 0, 1$ are quotient-reflective in **b-UFIL**, i.e., they are isomorphism-closed, full, closed under formation of finer structures, products and subspaces.

Corollary 4.11. *Let* (X, Θ^X, ψ) *be a bornological b-UFIL space. Then the following are equivalent:*

(i) (X, Θ^X, ψ) *is* $\overline{T_0}$.

(ii) $(X, \Theta^X, \psi) \in$ ***BONb-UFIL***$_{1cl}$.

(iii) $[x] \times [y] \notin \psi$ *(or* $[y] \times [x] \notin \psi$*) and* $([x] \times [x]) \cap ([y] \times [y]) \notin \psi$*, for all* $x, y \in X$ *with* $x \neq y$.

Proof. The proof can be easily deduced from Definition 3.4 and Corollary 4.9 of [32]. □

Corollary 4.12. *Let (X, Θ^X, ψ) be a bornological b-UFIL space. Then the following are equivalent:*

(i) (X, Θ^X, ψ) *is* T_1.

(ii) $(X, \Theta^X, \psi) \in$ ***BONb-UFIL***$_{1scl}$.

(iii) $[x] \times [y] \notin \psi$, $[y] \times [x] \notin \psi$ *and* $([x] \times [x]) \cap ([y] \times [y]) \notin \psi$, *for all* $x, y \in X$ *with* $x \neq y$.

Proof. The proof can be easily deduced from Definition 3.4 and Corollary 4.10 of [32]. □

Corollary 4.13. *Let (X, Θ^X, ψ) be a discrete symmetric b-UFIL space. Then the following are equivalent:*

(i) (X, Θ^X, ψ) *is* $\overline{T_0}$.

(ii) (X, Θ^X, ψ) *is* T_1.

(iii) $(X, \Theta^X, \psi) \in$ ***BONsb-UFIL***$_{1cl}$.

(iv) $(X, \Theta^X, \psi) \in$ ***BONsb-UFIL***$_{1scl}$.

(v) $(X, \Theta^X, \psi) \in$ ***SUConv***$_{1scl}$.

(vi) $[x] \times [y] \notin \psi$ *and* $([x] \times [x]) \cap ([y] \times [y]) \notin \psi$, *for all* $x, y \in X$ *with* $x \neq y$.

Proof. The proof can be easily deduced from Definition 3.4, Corollary 4.18 of [32] and Theorem 4.5 of [12]. □

5. Connected and strongly connected bounded uniform filter spaces

Definition 5.1. (cf. [11]) Let $\mathfrak{F} : \mathcal{G} \to$ **Set** be a topological functor, $X \in Obj(\mathcal{G})$ with $F \subset X$.

(i) F is said to be open if and only if its complement F^c is closed in X.

(ii) F is said to be strongly open if and only if its complement F^c is strongly closed in X.

Theorem 5.2. *Let (X, Θ^X, ψ) be a b-UFIL space, $\emptyset \neq F \subset X$ is open if and only if for each $x, y \in X$ with $x \in F$, $y \in F^c$, and $\varsigma \in \psi$, the conditions below hold:*

(i) $\{x, y\} \notin \Theta^X$.

(ii) $\varsigma \nsubseteq [x] \times [y]$ *and* $\varsigma \cup ([x] \times [F^c])$ *is improper (or* $\varsigma \nsubseteq [y] \times [x]$ *and* $\varsigma \cup ([F^c] \times [x])$ *is improper).*

(iii) $\varsigma \cap ([F^c] \times [F^c]) \nsubseteq ([x] \times [x]) \cap ([F^c] \times [F^c])$ *or* $\varsigma \cup ([F^c] \times [F^c])$ *is improper.*

Proof. The proof can be easily deduced from Definition 5.1 and Theorem 3.9. □

Theorem 5.3. *Let (X, Θ^X, ψ) be a b-UFIL space, $\emptyset \neq F \subset X$ is strongly open if and only if for each $x, y \in X$ with $x \in F$, $y \in F^c$, and $\varsigma \in \psi$, the conditions below hold:*

(i) $\{x, y\} \notin \Theta^X$.

(ii) $\varsigma \nsubseteq [x] \times [y]$ *and* $\varsigma \cup ([x] \times [F^c])$ *is improper.*

(iii) $\varsigma \nsubseteq [y] \times [x]$ *and* $\varsigma \cup ([F^c] \times [x])$ *is improper.*

(iv) $\varsigma \cap ([F^c] \times [F^c]) \nsubseteq ([x] \times [x]) \cap ([F^c] \times [F^c])$ *or* $\varsigma \cup ([F^c] \times [F^c])$ *is improper.*

Proof. The proof can be easily deduced from Definition 5.1 and Theorem 3.10. □

Definition 5.4. (cf. [11]) Let $\mathfrak{F} : \mathcal{G} \to \mathbf{Set}$ be a topological functor, $X \in Obj(\mathcal{G})$ with $\mathfrak{F}(X) = B$.

(i) X is said to be connected if and only if $\emptyset$ and X are the only subsets of X that are both strongly open and strongly closed.

(ii) X is said to be strongly connected if and only if $\emptyset$ and X are the only subsets of X that are both open and closed.

Remark 5.5. 1. In **Top** the notion of strongly connectedness reduce to the usual connectedness [11].
2. In T_1**Top** the notions of connectedness and strongly connectedness reduce to the usual connectedness and coincide [11].
3. In general, there is no relation between connectedness and strongly connectedness [11].

Theorem 5.6. *Let (X, Θ^X, ψ) be a b-UFIL space, $\emptyset \neq F \subset X$ be a proper subset of X. Then (X, Θ^X, ψ) is connected if and only if one of the conditions below hold:*

(i) *for some $x, y \in X$ with $x \notin F$, $y \in F$, and $\varsigma \in \psi$, either $\{x, y\} \in \Theta^X$, or $\varsigma \subseteq [x] \times [y]$ or $\varsigma \cup ([x] \times [F])$ is proper (or $\varsigma \subseteq [y] \times [x]$ or $\varsigma \cup ([F] \times [x])$ is proper), or $\varsigma \cap ([F] \times [F]) \subseteq ([x] \times [x]) \cap ([F] \times [F])$ and $\varsigma \cup ([F] \times [F])$ is proper.*

(ii) *for some $x, y \in X$ with $x \in F$, $y \in F^c$, and $\varsigma \in \psi$, either $\{x, y\} \in \Theta^X$, or $\varsigma \subseteq [x] \times [y]$ or $\varsigma \cup ([x] \times [F^c])$ is proper (or $\varsigma \subseteq [y] \times [x]$ or $\varsigma \cup ([F^c] \times [x])$ is proper), or $\varsigma \cap ([F^c] \times [F^c]) \subseteq ([x] \times [x]) \cap ([F^c] \times [F^c])$ and $\varsigma \cup ([F^c] \times [F^c])$ is proper.*

Proof. The proof can be easily deduced from Definition 5.4(ii) and Theorem 3.10. □

Theorem 5.7. *Let (X, Θ^X, ψ) be a b-UFIL space, $\emptyset \neq F \subset X$ be a proper subset of X. Then (X, Θ^X, ψ) is strongly connected if and only if one of the conditions below hold:*

(i) *for some $x, y \in X$ with $x \notin F$, $y \in F$, and $\varsigma \in \psi$, either $\{x, y\} \in \Theta^X$, or $\varsigma \subseteq [x] \times [y]$ or $\varsigma \cup ([x] \times [F])$ is proper and $\varsigma \subseteq [y] \times [x]$ or $\varsigma \cup ([F] \times [x])$ is proper, or $\varsigma \cap ([F] \times [F]) \subseteq ([x] \times [x]) \cap ([F] \times [F])$ and $\varsigma \cup ([F] \times [F])$ is proper.*

(ii) *for some $x, y \in X$ with $x \in F$, $y \in F^c$, and $\varsigma \in \psi$, either $\{x, y\} \in \Theta^X$, or $\varsigma \subseteq [x] \times [y]$ or $\varsigma \cup ([x] \times [F^c])$ is proper and $\varsigma \subseteq [y] \times [x]$ or $\varsigma \cup ([F^c] \times [x])$ is proper, or $\varsigma \cap ([F^c] \times [F^c]) \subseteq ([x] \times [x]) \cap ([F^c] \times [F^c])$ and $\varsigma \cup ([F^c] \times [F^c])$ is proper.*

Proof. The proof can be easily deduced from Definition 5.4(iii) and Theorem 3.9. □

Theorem 5.8. *Let (X, Θ^X, ψ) be a b-UFIL space. If (X, Θ^X, ψ) is strongly connected, then (X, Θ^X, ψ) is connected.*

Proof. The proof can be easily deduced from Theorems 5.6 and 5.7. □

Example 5.9. *Let $X = \{k, l, m\}$ and (Θ^X, ψ) be a b-UFIL structure on X with $\Theta^X = \{\emptyset, \{k\}, \{l\}, \{m\}\}$ and $\psi = \{[\emptyset], [k] \times [k], [l] \times [l], [m] \times [m], [k] \times [l], [k] \times [m]\}$. Then, (X, Θ^X, ψ) is connected but not strongly connected.*

6. Irreducible and ultraconnected bounded uniform filter spaces

Irreducibility or hyperconnectedness is one of the important concept of Topology and Algebraic geometry. The cofinite topology on any infinite set and the Zariski topology on a prime ideal both are irreducible spaces. However, standard topology is not irreducible.

In 2020, T.M. Baran [7] extended the classical irreducibility of topology to set based topological category.

Definition 6.1. (cf. [7]) Let $\mathfrak{F} : \mathcal{G} \to \mathbf{Set}$ be a topological functor, $X \in Obj(\mathcal{G})$.

(i) X is called irreducible if E and F are closed subobjects of X with $X = E \cup F$, then either $E = X$ or $F = X$.

(ii) X is called strongly irreducible if E and F are strongly closed subobjects of X with $X = E \cup F$, then either $E = X$ or $F = X$.

Remark 6.2. (i) In **Top** the notion of irreducibility coincides with usual irreducibility [14].

(ii) In **Top** every irreducible space is connected but converse implication is not true in general [7].

(iii) In T_1**Top** the notion of irreducibility and strongly irreducibility coincide [7].

The concept of ultraconnectedness is also one of the primary concept of Topology since it is stronger than path-connectedness, and it has been studied by several authors under the name of strongly connected [35, 36, 43].

We first introduce the notion of hyperconnectedness in a set-based topological category and examine the relationship among ultraconnectedness, strongly ultraconnectedness, connectedness and strongly connectedness in a b-UFIL space.

Definition 6.3. Let $\mathfrak{F} : \mathcal{G} \to$ **Set** be a topological functor, $X \in Obj(\mathcal{G})$.

(i) X is called ultraconnected if G and H are open subobjects of X with $X = G \cup H$, then either $G = X$ or $H = X$.

(ii) X is called strongly ultraconnected if G and H are strongly open subobjects of X with $X = G \cup H$, then either $G = X$ or $H = X$.

Remark 6.4. In **Top** the notion of ultraconnectedness coincides with classical ultraconnectedness [36].

Theorem 6.5. *Let (X, τ) be a topological space.*

(i) If (X, τ) is ultraconnected, then (X, τ) is connected but in general the converse implication is not true.

(ii) In general, there is no relationship between irreducible and ultraconnected topological spaces.

Proof. (i) It follows from Theorem 1 of [36] but the converse is not true in general. For example, $X = \{k, l, m\}$ and $\tau = \{\emptyset, \{k\}, \{k, l\}, \{k, m\}, X\}$ is connected but not ultraconnected.

(ii) Let $X = \{k, l, m\}$ and $\tau_1 = \{\emptyset, \{k\}, \{k, l\}, \{k, m\}, X\}$, and $\tau_2 = \{\emptyset, \{l\}, \{m\}, \{l, m\}, X\}$ be two topological spaces on X. Then (X, τ_1) is irreducible but not ultraconnected. Similarly, (X, τ_2) is ultraconnected but not irreducible. □

Theorem 6.6. *Let (X, Θ^X, ψ) be a b-UFIL space.*

(i) If (X, Θ^X, ψ) is irreducible (resp. ultraconnected), then (X, Θ^X, ψ) is strongly irreducible (resp. strongly ultraconnected).

(ii) If (X, Θ^X, ψ) is irreducible (resp. ultraconnected), then (X, Θ^X, ψ) is strongly connected.

(iii) If (X, Θ^X, ψ) is strongly irreducible (resp. strongly ultraconnected), then (X, Θ^X, ψ) is connected.

Proof. (i) Let (X, Θ^X, ψ) be irreducible (resp. ultraconnected). Suppose E and F are two strongly closed (resp. strongly open) subsets of X with $E \cup F = X$. By Theorems 3.9 and 3.10 (resp. Theorems 5.2 and 5.3), E and F are closed (resp. open) subsets of X. Since (X, Θ^X, ψ) is irreducible (resp. ultraconnected) and by Definition 6.1 (resp. Definition 6.3), $E = X$ or $F = X$, and consequently, (X, Θ^X, ψ) is strongly irreducible (resp. strongly ultraconnected).

(ii) Let (X, Θ^X, ψ) be irreducible (resp. ultraconnected) but not strongly connected. By the Theorem 5.7, there exists a non-empty proper subset F of X satisfying for every $x, y \in X$ with $x \notin F$, $y \in F$, and $\varsigma \in \psi$, $\{x, y\} \notin \Theta^X$, $\varsigma \nsubseteq [x] \times [y]$ and $\varsigma \cup ([x] \times [F])$ is improper, $\varsigma \nsubseteq [y] \times [x]$ and $\varsigma \cup ([F] \times [x])$ is improper, and $\varsigma \cap ([F] \times [F]) \nsubseteq ([x] \times [x]) \cap ([F] \times [F])$ or $\varsigma \cup ([F] \times [F])$ is improper, and for all $x, y \in X$ with $x \in F$, $y \in F^c$, and $\varsigma \in \psi$, $\{x, y\} \notin \Theta^X$, $\varsigma \nsubseteq [x] \times [y]$ and $\varsigma \cup ([x] \times [F^c])$ is improper, $\varsigma \nsubseteq [y] \times [x]$ and $\varsigma \cup ([F^c] \times [x])$ is improper, and $\varsigma \cap ([F^c] \times [F^c]) \nsubseteq ([x] \times [x]) \cap ([F^c] \times [F^c])$ or $\varsigma \cup ([F^c] \times [F^c])$ is improper. By Theorem 3.9 (resp. by Theorem 5.2), F and F^c are closed (resp. open) and $F \cup F^c = X$, which leads to a contradiction.

(iii) The proof is analogous to the proof of (ii). □

Example 6.7. Let $X = \{k, l, m, n\}$ and (Θ^X, ψ) be a b-UFIL structure on X with $\Theta^X = \{\emptyset, \{k\}, \{l\}, \{m\}, \{n\}\}$ and $\psi = \{[\emptyset], [k]\times[k], [l]\times[l], [m]\times[m], [n]\times[n], [k]\times[l], [k]\times[m], [k]\times[n], [l]\times[n], [k]\times[\{l,n\}], [l]\times[\{l,n\}], [\{k,l\}]\times[l], [\{k,l\}]\times[n], [\{k,l\}]\times[\{l,n\}]\}$. Then (X, Θ^X, ψ) is connected but neither strongly irreducible nor strongly ultraconnected.

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