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Coverings of Local Topological Groups

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Abstract. In this paper it is proved that local group structure of a local topological group which has a universal cover lifts to any covering space.

1. Introduction

The theory of covering spaces is concerned with differential geometry, Lie group theory [5, 6, 11] analysis and even algebra as well as topology. Covering spaces are also deeply intertwined with the study of homotopy groups and, in particular, the fundamental group. If X is a connected topological space which have a universal cover, $x_0 \in X$, and G is a subgroup of the fundamental group $\pi_1(X, x_0)$ of X at the point x_0, then we know from [12, Theorem 10.42] that there is a covering map $p\colon (\widetilde{X}_G, \widetilde{x_0}) \to (X, x_0)$ of pointed spaces, with characteristic G. In particular, If G is a singleton, then p becomes the universal covering map. Moreover, if X is a topological group, then $\widetilde{X}_G$ becomes a topological group such that p is a morphism of topological groups.

In [7], it is proved that the ring structure of a topological ring lifts to a simply connected covering space. This method is applied to topological R-modules in the case where the topological ring R is discrete and obtain a more general result than the one for the topological group case in [10]. In [9], these results are united to a large class of algebraic objects called topological groups with operations, including topological groups.

On the other hand, the result of universal covers of nonconnected topological groups was first studied in [13]. Also a similar algebraic result was given in [4] using crossed modules and group-groupoids which are internal groupoids in groups. In [1], some results on the covering morphisms of internal groupoids in groups with operations setting for an algebraic category C are given.

In [11] a *local group* is defined to be a set L with a partial composition defined on a subset $\mathcal{U}$ of $L \times L$, an identity $e \in L$ and inverse map defined on a subset V of L provided with the associativity and inverse axioms. The local group-groupoids are defined in [8] to be a local group object in the category of groupoids or equivalently internal category in local groups and the notion of local topological group-groupoid is given in Akız [2, Definition 2.6].

This study is based on the method given by Rotman in [12]. Let L and $\widetilde{L}$ be connected topological spaces and $p : \widetilde{L} \to L$ a simply connected covering. Let $p\colon (\widetilde{L}, \widetilde{e}) \to (L, e)$ be a covering map such that $\widetilde{L}$ is path connected and the characteristic group G of p is a subgroup of $\pi_1(L, e)$. Then we prove that the multiplication map $\mu\colon \mathcal{U} \to L$ and inversion map $i\colon V \to L$ lift to $\widetilde{L}$.

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2. Preliminaries

Throughout this study, all space X are assumed to be locally path-connected and semilocally 1-connected, so that each path component of X admits a simply connected cover. A covering map $p\colon \widetilde{X} \to X$ of connected spaces is called *universal* if it covers every covering of X in the sense that if $q\colon \widetilde{Y} \to X$ is another covering of X then there exists a map $r\colon \widetilde{X} \to \widetilde{Y}$ such that $p = qr$ (hence r becomes a covering). A covering map $p\colon \widetilde{X} \to X$ is called *simply connected* if $\widetilde{X}$ is simply connected. So a simply connected covering is a universal covering.

Let X be a topological space admitting a simply connected cover. A subset U of X is called liftable if it is open, path-connected and the inclusion $U \longrightarrow X$ maps each fundamental group of U trivially. If U is liftable, and $q\colon Y \longrightarrow X$ is a covering map, then for any $y \in Y$ and $x \in U$ such that $qy = x$, there is a unique map $i\colon U \longrightarrow Y$ such that $ix = y$ and qi is the inclusion $U \longrightarrow X$. A space X is called semi-locally simply connected if each point has a liftable neighborhood and locally simply connected if it has a base of simply connected sets. So a locally simply connected space is also semi-locally simply connected.

Let $p\colon (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ be a covering map of pointed topological spaces. The subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x_0}))$ of $\pi_1(X, x_0)$ is called *characteristic group* of p, where p_* is the morphism induced by p (see [3, p.379] for the characteristic group of a covering map in terms of covering morphism of groupoids). If characteristic groups of two covering maps $p\colon (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ and $q\colon (\widetilde{Y}, \widetilde{y_0}) \to (X, x_0)$ are equal, then we say p and q are equivalent, and equivalently there is a homeomorphism $f\colon (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ such that $qf = p$.

We assume that X is a topological space with base point x_0 and G a subgroup of $\pi_1(X, x_0)$. Let $P(X, x_0)$ be the set of all paths of α in X with initial point x_0. We consider an equivalence relation defined on $P(X, x_0)$ by $\alpha \simeq \beta$ if and only if $\alpha(1) = \beta(1)$ and $[\alpha \bullet \beta^{-1}] \in G$. Then the equivalence class of α is denoted by $\langle\alpha\rangle_G$ and the set of all such equivalence classes of the paths in X with initial point x_0 by $\widetilde{X}_G$. Define a function $p\colon \widetilde{X}_G \to X$ by $p(\langle\alpha\rangle_G) = \alpha(1)$. Let α_0 be the constant path at x_0 and $\widetilde{x_0} = \langle\alpha\rangle_G \in \widetilde{X}_G$. If $\alpha \in P(X, x_0)$ and U is an open neighborhood of $\alpha(1)$, then a path of the form $\alpha \bullet \lambda$, where λ is a path in U with $\lambda(0) = \alpha(1)$, is called a *continuation* of α. For an $\langle\alpha\rangle_G \in \widetilde{X}_G$ and an open neighborhood U of $\alpha(1)$, let $(\langle\alpha\rangle_G, U) = \{\langle\alpha \bullet \lambda\rangle_G : \lambda(I) \subseteq U\}$. Then the subsets $(\langle\alpha\rangle_G, U)$ form a basis for a topology on $\widetilde{X}_G$ such that the map $p\colon (\widetilde{X}_G, \widetilde{x_0}) \to (X, x_0)$ is continuous [12, p.259].

We prove the local case of the following result in Theorem 3.7.

Theorem 2.1. ([12, Theorem 10.34]) *Let (X, x_0) be a pointed topological space and G a subgroup of $\pi_1(X, x_0)$. If X is connected, locally path-connected, and semilocally simply connected, then $p\colon (\widetilde{X}_G, \widetilde{x_0}) \to (X, x_0)$ is a covering map with characteristic group G.*

Remark 2.2. Let X be a connected, locally path-connected, and semilocally simply connected topological space and $q\colon (\widetilde{X}, \widetilde{x_0}) \to (X, x_0)$ a covering map. Let G be the characteristic group of q. Then the covering map q is equivalent to the covering map $p\colon (\widetilde{X}_G, \widetilde{x_0}) \to (X, x_0)$ corresponding to G.

We obtain the following result from Theorem 2.1.

Theorem 2.3. ([12, Theorem 10.42]) *Suppose that X is a connected, locally path-connected, and semilocally simply connected topological group. Let $0 \in X$ be the identity element and $p\colon (\widetilde{X}, \widetilde{0}) \to (X, 0)$ a covering map. Then the group structure of X lifts to $\widetilde{X}$, i.e. $\widetilde{X}$ becomes a topological group such that $\widetilde{0}$ is identity and $p\colon (\widetilde{X}, \widetilde{0}) \to (X, 0)$ is a morphism of topological groups.*

3. Universal covers of local topological groups

In this section we give the definition of local topological groups. Also we give the methods of Section 2 for local topological groups and obtain some properties.

Now we emphasis the definition given in [11, Definition 2].

Definition 3.1. Let L be a set. A *local group* is a quintuple $\mathbf{L} = (L, \mu, \mathcal{U}, i, V)$, where

(1) a distinguish element $e \in L$, the identity element,
(2) a multiplication $\mu\colon \mathcal{U} \to L, (x, y) \mapsto x \circ y$ defined on a subset $\mathcal{U}$ of $L \times L$ such that $(\{e\} \times L) \cup (L \times \{e\}) \subseteq \mathcal{U}$,
(3) an inversion map $i\colon V \to L, x \mapsto \overline{x}$ defined on a subset $e \in V \subseteq L$ such that $V \times i(V) \subseteq \mathcal{U}$ and $i(V) \times V \subseteq \mathcal{U}$,
all satisfying the following properties:
  (i) Identity: $e \circ x = x = x \circ e$ for all $x \in L$
  (ii) Inverse: $i(x) \circ x = e = x \circ i(x)$, for all $x \in V$
  (iii) Associativity: If $(x, y), (y, z), (x \circ y, z)$ and $(x, y \circ z)$ all belong to $\mathcal{U}$, then

$$x \circ (y \circ z) = (x \circ y) \circ z.$$

From now on we denote such a local group by $\mathbf{L}$.

Note that if $\mathcal{U} = L \times L$ and $V = L$, then a local group becomes a group. It means that the notion of local group generalizes that of group. Now we give the following definition (see [11, Definition 5]):

Definition 3.2. Let $(L, \mu, \mathcal{U}, i, V)$ and $(\widetilde{L}, \widetilde{\mu}, \widetilde{\mathcal{U}}, \widetilde{i}, \widetilde{V})$ be local groups. A map $f\colon L \to \widetilde{L}$ is called a *local group morphism* if

(i) $(f \times f)(\mathcal{U}) \subseteq \widetilde{\mathcal{U}}$, $f(V) \subseteq \widetilde{V}$, $f(e) = \widetilde{e}$,
(ii) $f(x \circ y) = f(x) \circ f(y)$ for $(x, y) \in \mathcal{U}$,
(iii) $f(i(x)) = \widetilde{i}(f(x))$ for $x \in V$.

We study on the topological version of Definition 3.1.

Definition 3.3. ([11]) Let L be a local group, if L has a topology structure such that $\mathcal{U}$ is open in $L \times L$, V is open in L, the maps μ and i are continuous, then $(L, \mu, \mathcal{U}, i, V)$ is called a *local topological group*.

It is obvious that if $\mathcal{U} = L \times L$ and $V = L$, then a local topological group L becomes a topological group.

Example 3.4. ([11, p.26]) Let X be a topological group, L be an open neighbourhood of the identity element e. Then we obtain a local topological group taking $\mathcal{U} = (L \times L) \cap \mu^{-1}(L)$ and $V = L \cap \overline{L}$, where $\overline{L} = \{\overline{x} | x \in L\}$.

Here the group mulitlication μ and the inversion i on X are restricted to define a local group multiplication and inverse maps on L.

Further if we choose $\mathcal{U}$ and V such that

$$(\{e\} \times L) \cup (L \times \{e\}) \subseteq \mathcal{U} \subseteq (L \times L) \cap \mu^{-1}(L)$$

$$\{e\} \subseteq V \subseteq L \cap i^{-1}(L)$$

and

$$V \times i(V)) \cup (i(V) \times V) \subseteq \mathcal{U}$$

then we have a local topological group.

Definition 3.5. ([8, Definition 3.3]) Let $(L, \mu, \mathcal{U}, i, V)$ and $(\widetilde{L}, \widetilde{\mu}, \widetilde{\mathcal{U}}, \widetilde{i}, \widetilde{V})$ be local topological groups. A continuous local group morphism $f\colon L \to \widetilde{L}$ is called a *local topological group morphism*.

Theorem 3.6. *If L is a local topological group, then the fundamental group $\pi_1(L, e)$ becomes a local group.*

Proof. Let L be a local topological group with identity e. Hence we have the maps $\mu\colon \mathcal{U} \to L, \mu(x,y) = x \circ y$ and $i\colon V \to L, i(x) = \overline{x}$. Write $\widetilde{L}$ for the fundamental group $\pi_1(L,e)$. Assuming $(\alpha(t),\beta(t)) \in \mathcal{U}$, the set $\widetilde{\mathcal{U}}$ of the homotopy classes of the paths can be written by

$$\widetilde{\mathcal{U}} = \{([\alpha],[\beta]) : \alpha \circ \beta \quad is \quad defined\}$$

and considered as a subset of $\widetilde{L} \times \widetilde{L}$. Then one can define the maps

$$\widetilde{\mu}\colon \widetilde{\mathcal{U}} \to \widetilde{L}, ([\alpha],[\beta]) \mapsto [\alpha \circ \beta] \tag{1}$$

and

$$\widetilde{i}\colon \widetilde{V} \to \widetilde{L}, [\alpha] \mapsto [i(\alpha)], \tag{2}$$

where $\widetilde{V}$ is the set of homotopy classes of all paths in V.

Here since μ and i are continuous, then $\widetilde{\mu}$ and $\widetilde{i}$ are well defined. Indeed, let $\alpha \simeq \alpha'$, $\beta \simeq \beta'$ where $\alpha \circ \beta$ and $\alpha' \circ \beta'$ are defined. Since $p_1\alpha \simeq p_1\alpha'$, $p_2\alpha \simeq p_2\alpha'$ and $p_1\beta \simeq p_1\beta'$, $p_2\beta \simeq p_2\beta'$ for the projection maps p_1 and p_2, then $p_1(\alpha) \circ p_1(\beta) \simeq p_1(\alpha) \circ p_1(\beta)$ and $p_2(\alpha) \circ p_2(\beta) \simeq p_2(\alpha) \circ p_2(\beta)$. Hence we have $p_1(\alpha \circ \beta) \simeq p_1(\alpha' \circ \beta')$ and $p_2(\alpha \circ \beta) \simeq p_2(\alpha' \circ \beta')$. Then $\alpha \circ \beta \simeq \alpha' \circ \beta'$. Similarly, we assume that $\alpha \simeq \alpha'$. Since $p_1\overline{\alpha} \simeq p_1\overline{\alpha'}$ and $p_2\overline{\alpha} \simeq p_2\overline{\alpha'}$, then $\overline{\alpha'} \simeq \overline{\alpha}$.

In addition to these properties, the other details can be checked as follows:

(i) $[1_e] \circ [\alpha] = [1_e \circ \alpha][\alpha] = [\alpha \circ 1_e][\alpha] \circ [1_e]$

(ii) Inverse: $\widetilde{i}[\alpha] \circ [\alpha] = [\overline{\alpha}] \circ [\alpha] = [\overline{\alpha} \circ \alpha] = [1_0] = [\alpha \circ \overline{\alpha}] = [\alpha] \circ [\overline{\alpha}] = [\alpha] \circ \widetilde{i}[\alpha]$, for all $[\alpha] \in \widetilde{V}$, where $\alpha \circ \alpha'$ is defined,

(iii) Associativity: If $([\alpha],[\beta]), ([\beta],[\gamma]), ([\alpha \circ \beta],[\gamma])$ and $([(\alpha],[\beta \circ \gamma])$ all belong to $\widetilde{\mathcal{U}}$, then

$$[\alpha] \circ ([\beta] \circ [\gamma]) = ([\alpha] \circ [\beta]) \circ [\gamma].$$

So $\pi_1(L,0)$ becomes a local group. □

Here we give the interchange law in a local topological group L. Note that we denote the concatination of the paths by $\bullet$ and the local group multiplication by $\circ$. Also we denote the inverse path of α by α^{-1} and the local group inverse α by $\overline{\alpha}$. Assuming that $\alpha \circ \beta$, $\alpha' \circ \beta'$ and $(\alpha \bullet \beta) \circ (\alpha' \bullet \beta')$ are defined, then we have the interchange law

$$(\alpha \bullet \beta) \circ (\alpha' \bullet \beta') = (\alpha \circ \alpha') \bullet (\beta \circ \beta') \tag{3}$$

where $\bullet$ denotes the composition of the paths. Also we obtain that

$$(\alpha \circ \beta)^{-1} = \alpha^{-1} \circ \beta^{-1} \tag{4}$$

where α^{-1} is the inverse path such that $\alpha^{-1}(t) = \alpha(1-t)$ for $t \in I$. On the other hand we have that

$$(\overline{\alpha})^{-1} = \overline{\alpha^{-1}} \tag{5}$$

$$\overline{(\alpha \bullet \beta)} = \overline{\alpha} \bullet \overline{\beta} \tag{6}$$

when $\alpha(1) = \beta(0)$.

We now prove Theorem 2.3 for local topological groups.

Theorem 3.7. *Let L be a local topological group and let G be a subgroup of $\pi_1(L,e)$. Suppose that the underlying space of L is connected, locally path-connected, and semilocally simply connected. Let $p\colon (\widetilde{L}_G,\widetilde{x_0}) \to (L,x_0)$ be the covering map corresponding to G as a subgroup of the additive group of $\pi_1(L,e)$ by Theorem 2.3. Then the operations of L lift to $\widetilde{L}_G$, i.e. $\widetilde{L}_G$ is a local topological group and $p\colon \widetilde{L}_G \to L$ is a morphism of local topological groups.*

Proof. Let $P(L,e)$ be the set of all paths in L with initial point e. We know from the Section 2 that $\widetilde{L}_G$ is the set of equivalence classes via G. We have the induced multiplication

$$\widetilde{\mu}\colon \widetilde{\mathcal{U}} \to \widetilde{L}_G, (\langle\alpha\rangle_G,\langle\beta\rangle_G) \mapsto \langle\alpha\rangle_G \circ \langle\beta\rangle_G = \langle\alpha\circ\beta\rangle_G \tag{7}$$

on the subset $\widetilde{\mathcal{U}} = \widetilde{\mathcal{U}_G}$ of $\widetilde{L}_G \times \widetilde{L}_G$ such that $(\{\langle 1_e\rangle\}\times L)\cup(L\times\{\langle 1_e\rangle\}) \subseteq \widetilde{\mathcal{U}}$ and inversion map

$$\widetilde{i}\colon \widetilde{V} \to \widetilde{L}_G, \langle\alpha\rangle_G \mapsto \overline{\langle\alpha\rangle_G} = \langle\overline{\alpha}\rangle_G \tag{8}$$

such that $\langle 1_e\rangle \in \widetilde{V} = \widetilde{V}_G \subseteq \widetilde{L}_G$ such that $\widetilde{V}\times i(\widetilde{V}) \subseteq \widetilde{\mathcal{U}}$ and $i(\widetilde{V})\times\widetilde{V} \subseteq \widetilde{\mathcal{U}}$.

These maps are well defined. Indeed for $(\alpha,\alpha_1),(\beta,\beta_1) \in \mathcal{U} \subseteq L\times L$ and $(\alpha\bullet\beta,\alpha_1\bullet\beta_1)\in\mathcal{U}$ such that $\alpha(1)=\alpha_1(1)$ and $\beta(0)=\beta_1(1)$, we have that

$$\begin{aligned}
[(\alpha\circ\beta)\bullet(\alpha_1\circ\beta_1)^{-1}] &= [(\alpha\circ\beta)\bullet(\alpha_1^{-1}\circ\beta_1^{-1})] && \text{(by 4)}\\
&= [(\alpha\bullet\alpha_1^{-1})\circ(\alpha_1^{-1}\bullet\beta_1^{-1})] && \text{(by 3)}\\
&= [(\alpha\bullet\alpha_1^{-1})]\circ[(\alpha_1^{-1}\bullet\beta_1^{-1})]. && \text{(by 1)}
\end{aligned}$$

So, if $\alpha_1 \in \langle\alpha\rangle_G$ and $\beta_1\in\langle\beta\rangle_G$, then $([\alpha\bullet\alpha_1^{-1}],[\beta\bullet\beta_1^{-1}])\in\mathcal{U}_G$. Since G is a subgroup of $\pi_1(L,e)$, we have that $[\alpha\bullet\alpha_1^{-1}]\circ[\beta\bullet\beta_1^{-1}]\in G$. Hence the multiplication $\widetilde{\mu}$ is well defined. On the other hand $\alpha,\alpha_1\in V\subseteq P(L,e)$ such that $\alpha(1)=\alpha(0)$, we have that

$$\begin{aligned}
[\overline{\alpha}\bullet\overline{\alpha}_1^{-1}] &= [[\overline{\alpha}\bullet\overline{\alpha_1^{-1}}] && \text{(by 5)}\\
&= [\overline{\alpha\bullet\alpha_1^{-1}}] && \text{(by 6)}\\
&= \overline{[\alpha\bullet\alpha_1^{-1}]}. && \text{(by 2)}
\end{aligned}$$

If $[\alpha\bullet\alpha_1^{-1}]\in V_G$, then $\overline{[\alpha\bullet\alpha_1^{-1}]}\in G$. So $\widetilde{i}$ is well defined.

The other details can be checked for $\widetilde{L}_G$ and so $\widetilde{L}_G$ becomes a local group. We know from Theorem 2.1 that $p\colon(\widetilde{L}_G,\widetilde{x_0})\to(L,x_0)$ is a covering map and $\widetilde{L}_G$ is a topological group and p is a morphism of topological groups. So we need to prove that $\widetilde{L}_G$ is a local topological group and p is a local topological groups morphism. We have to show that the multiplication $\widetilde{\mu}$ and the inversion map $\widetilde{i}$ are continuous.

Let $(\langle\alpha\rangle_G,\langle\beta\rangle_G)\in\widetilde{\mathcal{U}}\subseteq\widetilde{L}_G\times\widetilde{L}_G$ and $(\widetilde{W},\langle\alpha,\beta\rangle_G)$ be a basic open neighborhood of the element $\langle\alpha,\beta\rangle_G$. Here $\widetilde{W}$ is an open neighborhood of $(\alpha\circ\beta)(1)=\alpha(1)\circ\beta(1)$. We know that the multiplication

$$\mu\colon\mathcal{U}\to L$$

is continuous. So there is an open neighborhood W of $(\alpha\circ\beta)(1)=\alpha(1)\circ\beta(1)$ such that $\mu(W)\subseteq\widetilde{W}$. Hence we have that

$$(W,(\langle\alpha\rangle_G,\langle\beta\rangle_G))\subseteq(\widetilde{W},\langle\alpha,\beta\rangle_G).$$

So the multiplication $\widetilde{\mu}$ is continuous. We now prove that the inversion map $\widetilde{i}$ is continuous. Let $(\widetilde{O},\langle\overline{\alpha}\rangle_G)$ be a base open neighborhood of $\langle\overline{\alpha}\rangle$. Then $\widetilde{O}$ is an open neighborhood of $\overline{\alpha(1)}$. Since the inversion map $i\colon V\to L$ is continuous, there is an open neighborhood O of $\alpha(1)$ such that $i(O)\subseteq\widetilde{O}$. Hence $(O,\langle\alpha\rangle_G)$ is an open neighborhood of $\langle\alpha\rangle_G$ and $i(O,\langle\alpha\rangle_G)\subseteq(\widetilde{O},\langle\overline{\alpha}\rangle_G$. Hence the inversion map i is continuous. Finally, we prove that the map $p\colon\widetilde{L}_G\to L,\langle\alpha,\beta\rangle_G\mapsto\alpha(1)$ satisfies the conditions in Definition 3.2 as follows.

(i) For the element $(\langle \alpha,\beta\rangle_G)$ of $\widetilde{\mathcal{U}}$, since $(p\times p)(\langle \alpha,\beta\rangle_G) = (p\times p)(\langle\alpha\rangle_G,\langle\beta\rangle_G) = (\alpha(1),\beta(1)) = (\alpha,\beta)(1)$, then $(p\times p)(\widetilde{\mathcal{U}}) \subseteq \mathcal{U}$. Also similarly $p(\widetilde{V}) \subseteq V$ and $p(\langle 1_e\rangle_G) = 1_e(1) = 0$.

(ii) $p(\langle \alpha\circ\beta\rangle_G) = \alpha(1)\circ\beta(1) = p(\langle\alpha\rangle_G)\circ p(\langle\beta\rangle_G)$.

(iii) $p(\widetilde{i}(\langle\alpha\rangle_G)) = p(\langle\overline{\alpha}\rangle_G = \overline{\alpha}(1) = i(\alpha(1)) = i(p\langle\alpha\rangle_G)$.

□

We now give the following result in the light of Theorem 3.7.

Theorem 3.8. *Let L be a local topological group whose underlying space is connected, locally path connected and semi locally simply connected. Let* $p\colon(\widetilde{L},\widetilde{e})\to(L,e)$ *be a covering map such that* $\widetilde{L}$ *is path connected. Then the multiplication map* $\mu\colon\mathcal{U}\to L$ *and inversion map* $i\colon V\to L$ *lift to* $\widetilde{L}$.

Proof. If we choose $\widetilde{L} = \widetilde{L}_G$ by Remark 2.2 and in the light of Theorem 3.7, then the multiplication map $\mu\colon\mathcal{U}\to L$ and inversion map $i\colon V\to L$ lift to the maps

$$\widetilde{\mu}\colon\widetilde{\mathcal{U}}\to\widetilde{L}$$

and

$$\widetilde{i}\colon\widetilde{V}\to\widetilde{L},$$

respectively, such that $(p\times p)(\widetilde{\mathcal{U}}) = \mathcal{U}$ and $p(\widetilde{V}) = V$. □

If we choose the subgroup G of $\pi_1(L,e)$ to be singleton in Theorem 3.7, then we obtain the following corollary.

Corollary 3.9. *Let L be a local topological group such that the underlying space of L is connected, locally path-connected and semi locally simply connected and* $p\colon(\widetilde{L},\widetilde{e})\to(L,e)$ *be a universal covering map. Then the multiplication map* μ *and inversion map i of L lifts to* $\widetilde{L}$.

Before giving Theorem 3.12, we prove the following proposition.

Proposition 3.10. *Let L be a local topological group and B is a liftable neighborhood of e in L. Then there is a liftable neighborhood A of e in L such that* $A\times A\subseteq\mathcal{U}$ *and* $\mu(A,A)\subseteq B$*, where* $\mu\colon\mathcal{U}\to L$.

Proof. If L is a local topological group and hence the multiplication map

$$\mu\colon\mathcal{U}\to L$$

is continuous, then there is an open neighborhood B of e in L such that $A\times A\subseteq\mathcal{U}$ and $\mu(A,A)\subseteq B$. Moreover, if B is liftable, then A can be chosen as liftable. If B is liftable, then for each $x\in A$, the fundamental group $\pi_1(A,x)$ is mapped to the singleton by the morphism induced by the inclusion map $\iota\colon A\to L$. Consider that A is not necessarily path-connected and hence not necessarily liftable. But here, since the path component $C_e(A)$ of e in A is liftable and satisfies these conditions, A can be replace by the path component $C_e(A)$ of e in A and A is assumed to be liftable. □

Definition 3.11. Let $(L,\mu,\mathcal{U},i,V)$ and $(L',\mu',\mathcal{U}',i',V')$ be local topological groups and A is an open neighborhood of e such that $A\times A\subseteq\mathcal{U}$. A continuous map $\varphi\colon A\to B$ is called a local morphism of local topological groups, if $\varphi(a\circ b) = \varphi(a)\circ\phi(b)$ when $a,b\in A$ such that $a\circ b\in A$.

Theorem 3.12. *Let* $(L,\mu,\mathcal{U},i,V)$ *and* $(L',\mu',\mathcal{U}',i',V')$ *be local topological groups and* $q\colon\widetilde{L}\to L$ *a local topological group morphism, which is a covering map. Let A be an open, path-connected neighbourhood where* $A\times A\subseteq\mathcal{U}$ *such that* $\mu(A,A)\subseteq U$ *is contained in a liftable neighborhood B of e in L. Then the inclusion map* $\iota\colon A\to L$ *lifts to a local morphism* $\widetilde{\iota}\colon A\to\widetilde{L}$ *in local topological groups.*

Proof. Assuming that B is liftable, A lifts to $\widetilde{L}$ by $\widehat{\iota}\colon A \to \widetilde{L}$. We need to prove that $\widehat{\iota}$ is a local morphism of local topological groups. By the lifting lemma we know that $\widehat{\iota}$ is continuous. We choose $a, b \in A$ such that $a \circ b \in A$. Considering that A is path connected, we also choose the paths α and β in A from e to a and b, respectively. If we assume that $\rho = \alpha \circ \beta$, then ρ is a path from e to $a \circ b$. Here since $\mu(A, A) \subseteq B$, $\rho = \alpha \circ \beta$ is a path in B. Hence the paths α, β and ρ lift to $\widetilde{\alpha}$, $\widetilde{\beta}$ and $\widetilde{\rho}$, respectively. Since q is a local grup morphism, we have that

$$q(\widetilde{\rho}) = \rho = \alpha \circ \beta = q(\widetilde{\alpha}) \circ q(\widetilde{\beta}) = q(\widetilde{\alpha} \circ \widetilde{\beta}).$$

By the unique path lifting,

$$\widetilde{\rho} = \widetilde{\alpha} \circ \widetilde{\beta},$$

since $\widetilde{\alpha} \circ \widetilde{\beta}$ and $\widetilde{\rho}$ have the initial point $\widetilde{0}$ in $\widetilde{L}$. If we evaluate that these paths at $1 \in I$, then we have

$$\widehat{\iota}(a \circ b) = \widehat{\iota}(a) \circ \widehat{\iota}(b),$$

thus $\widehat{\iota}$ is a local morphism in local topological groups. □

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