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S-Zariski Topology on S-Spectrum of Modules

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Abstract. Let R be a commutative ring with nonzero identity and M be an R-module. In this paper, first we give some relations between S-prime and S-maximal submodules that are generalizations of prime and maximal submodules, respectively. Then we construct a topology on the set of all S-prime submodules of M , which is generalization of prime spectrum of M. We investigate when $Spec_S(M)$ is T_0 and T_1-space. We also study on some continuous maps and irreducibility on $Spec_S(M)$. Moreover, we introduce the notion of S-radical of a submodule N of M and use it to show the irreducibility of S-variety $V_S(N)$.

1. Introduction

Throughout the paper, R denotes a commutative ring with identity, M denotes an R-module. $Spec(R)$, $Spec(M)$ and $Max(R)$ denote the set of all prime ideals of R, prime submodules of M and maximal ideals of R, respectively. For ideals I, J of R the residual of I by J denoted by $(I :_R J)$ is the set of elements a of R such that $aJ \subseteq I$. For a submodule of N of M the residual of N by M denoted by $(N :_R M)$ is the set of elements a of R such that $aM \subseteq N$. If no confusion arises, we can omit R and write $(I : J)$ instead of $(I :_R J)$.

In [9], the author constructed a topology on $Spec(M)$ which is the set of all prime submodules of M. He proved some results that are known for $Spec(R)$. Also he defined absolutely flat R-module. In 1995, Chin-Pi Lu investigated some properties of $Spec(M)$. She gave a relation between $Spec(M)$ and $Spec(S^{-1}M)$. She showed that the statement "$Spec(M) \neq \emptyset$ if and only if $M \neq 0$" is not necessarily true for all modules by giving an example of a nonzero module M with $Spec(M) = \emptyset$. She also showed $Spec(M) \neq \emptyset$ for some special modules such as multiplication modules. Moreover, she proved the existence of a surjective map between $Spec(M)$ and $Spec(R/Ann(M))$ where M is a finitely generated R module. This map is bijective if and only if M is multiplication [13]. In [16], the authors investigated when $Spec(M)$ has a Zariski topology. Let M be a finitely generated R-module. They proved that $Spec(M)$ has a Zariski topology if and only if M is a multiplication module. After that, in [14], Chin-Pi Lu continued to investigate topological properties of $Spec(M)$. She obtained conditions when $Spec(M)$ is a spectral space. Furthermore, she showed that the map $\phi : Spec(M) \to Spec(R/Ann(M))$ plays a significant role for $Spec(M)$ being spectral space. Currently, Sevim et al. introduced the notion of S-prime submodules which is a generalization of prime submodules [19]. Let P be a submodule of an R-module M such that $(P : M) \cap S = \emptyset$. Then P is said to be S-prime submodule if there exists $s \in S$ such that $am \in P$ implies $sa \in (P : M)$ or $sm \in P$. They gave many features of S-prime

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submodules and characterized some prime submodules by using S-prime submodules. More recently, Yildiz et al. constructed a topology on the set of all S-prime ideals denoted by $Spec_S(R)$ and this topology is a generalization of classical Zariski topology [23]. They investigated some topological properties of $Spec_S(R)$ such as connectedness, compactness and separation axioms.

In this paper, firstly we define S-maximal submodules and give some relations between S-maximal and S-prime submodules (See, Lemma 2.5, Proposition 2.6, Proposition 2.7). Then we introduce a topology on the set of all S-prime submodules of R-module M. We define the set $V_S(N) = \{P \in Spec_S(N) : s(N : M) \subseteq (P : M)$for some s $\in S\}$. The collection of $V_S(N)$ for every submodule N of M satisfies the axioms of closed sets in a topological space (See, Theorem 3.6). Then $Spec_S(M)$ with these closed sets induces a topology and we call it as S-Zariski topology. Further, we illustrate that S-Zariski topology and classical Zariski topology are two different concepts with examples (See, Example 3.1, Example 3.2). Starting from this, we give a basis for S-Zariski topology (See, Theorem 3.7) and investigate some properties of this topological space such as T_0, T_1 axioms and continuity of some maps on the space (See, Theorem 3.13, Proposition 3.15, Theorem 3.16, Theorem 3.17). Also we define the closure of a subset of $Spec_S(M)$ (See, Theorem 3.10).The last section is dedicated to the irreducibility of the topology. We define the notion of S-radical that is a generalization of the radical of a submodule and use it to investigate irreducibility.

2. S-maximal and S-prime submodules

Definition 2.1. Let $\emptyset \neq S \subseteq R$ such that $0 \notin S$. Then S is called a *multiplicatively closed set* if $1 \in S$ and for all $s, s' \in S,\ ss' \in S$.

Let P be an ideal of R such that $P \cap S = \emptyset$. Then P is called an S-prime ideal if there exists an $s \in S$ and $ab \in P$ for some $a, b \in R$, implies either $sa \in P$ or $sb \in P$ [10].

Let N be a submodule M such that $(N : M) \cap S = \emptyset$. Then N is called an S-prime submodule if there exists an $s \in S$ such that $am \in N$ for some $a \in R, m \in M$ implies that $sa \in (N : M)$ or $sm \in N$ [19].

Definition 2.2. Let S be a multiplicatively closed subset of R and P be an ideal of R that is disjoint from S. Then P is said to be an S-maximal ideal if there exists a fixed $s \in S$ such that $P \subseteq Q$ for some ideal Q of R implies either $sQ \subseteq P$ or $Q \cap S \neq \emptyset$ [23].

A submodule N of M with $(N : M) \cap S = \emptyset$ is said to be an S-maximal submodule if there exists a fixed $s \in S$ and $N \subseteq K$ for some submodule K of M, implies either $sK \subseteq N$ or $(K : M) \cap S \neq \emptyset$.

Proposition 2.3. ([23, Proposition 10]) *Every S-maximal ideal is an S-prime ideal.*

The converse of Proposition 2.3 is not true in general. See the following example.

Example 2.4. Let $R = \mathbb{Z}[X]$ and $S = \{(X+2)^n : n \in \mathbb{N}\} \cup \{1\}$. Take the ideal $P = (X^2+2X)$. Here $P \cap S = \emptyset$. Now choose $f(X)g(X) \in P \subseteq (X)$. Since X is a prime ideal, $f(X) \in (X)$ or $g(X) \in (X)$. This gives that $sf(X) \in P$ or $sg(X) \in P$ where $s = X + 2$. Hence P is an S-prime ideal of R. If we choose $K = (X, 3)$. Then $P \subseteq K$ and $K \cap S = \emptyset$. Also for any $s' = (X + 2)^n \in S$, $s'K \nsubseteq P$ since $3(X + 1)^n \notin P$. Therefore, we conclude that P is not an S-maximal ideal of R.

Lemma 2.5. *Let R be a ring, M be a finitely generated R-module, S be a multiplicatively closed subset of R and K, N be finitely generated submodules of M. Then $S^{-1}K = S^{-1}N$ if and only if $sK \subseteq N$ and $s'N \subseteq K$ for some $s, s' \in S$.*

Proof. Assume that $S^{-1}K = S^{-1}N$. Since K is finitely generated, we can write $K = \sum_{i=1}^{n} = Rm_i$ for some $m_1, m_2, \ldots, m_n \in K$. This gives that $\frac{m_i}{1} \in S^{-1}K = S^{-1}N$. Then there exists $s_i \in S$ such that $s_i m_i \in N$. Put $s = s_1 s_2 \ldots s_n \in S$. Thus we have $sK \subseteq N$. Similarly $s'N \subseteq K$ for some $s' \in S$. For the reverse direction, suppose that $sK \subseteq N$ and $s'N \subseteq K$ for some $s, s' \in S$. Let $\frac{a}{u} \in S^{-1}K$. Then there exists u' such that $u'a \in K$. Since $sK \subseteq N$ for some $s \in S$, we have $su'a \in sK \subseteq N$. Then $\frac{a}{u} = \frac{su'a}{su'u} \in S^{-1}N$ which implies that $S^{-1}K \subseteq S^{-1}N$. Similarly, one can show that $S^{-1}N \subseteq S^{-1}K$, as required. □

Recall from [3] that a module M is called S-Noetherian if for each submodule N of M, $sN \subseteq K \subseteq N$ for some $s \in S$ and some finitely generated submodule K.

Proposition 2.6. *Let M be a finitely generated R-module, S be a multiplicatively closed subset of R. If a submodule P such that $(P : M) \cap S = \emptyset$ is S-maximal submodule of M, then $S^{-1}P$ is a maximal submodule of M. The converse is also true when M is an S-Noetherian module and $P \in Spec_S(M)$.*

Proof. Assume that P is S-maximal submodule. Choose a maximal submodule $S^{-1}Q$ such that $S^{-1}P \subseteq S^{-1}Q$ where Q is prime submodule and $(Q : M) \cap S = \emptyset$. Then $P \subseteq Q$. Since P is S-maximal, $sQ \subseteq P$ for some $s \in S$. So $S^{-1}(sQ) = S^{-1}Q \subseteq S^{-1}P$ which completes the proof.

Now suppose $S^{-1}P$ is a maximal submodule of $S^{-1}M$. Let $P \subseteq Q$. Then $S^{-1}P \subseteq S^{-1}Q \subseteq S^{-1}M$. As $S^{-1}P$ is maximal, $S^{-1}P = S^{-1}Q$ or $S^{-1}Q = S^{-1}M$.

Case 1: Assume that $S^{-1}P = S^{-1}Q$. Since Q is S-finite, there exists $m_1, m_2, \ldots m_n \in Q$ such that $sQ \subseteq \sum_{i=1}^{n} Rm_i$. As $\frac{m_i}{1} \in S^{-1}Q = S^{-1}P$, there exists $s_i \in S$ such that $s_i m_i \in P$. Now put $s' = ss_1s_2 \ldots s_n \in S$. Then we have $s'Q \subseteq P$. Since P is S-prime, there exists a fixed $t \in S$ such that $tQ \subseteq P$.

Case 2: Assume that $S^{-1}Q = S^{-1}M$. Since M is S-finite, by a similar argument in Case 1, we get $tM \subseteq Q$ for some $t \in S$. Thus $t \in (Q : M) \cap S$; that is, $(Q : M) \cap S \neq \emptyset$, as required.

Consequently, P is an S-maximal submodule of M. □

Recall that an R-module M is called multiplication if $(N : M)M = N$ for every submodule N of M [7]. An R-module M is said to be a cancellation module if $IM = JM$ implies $I = J$ for all ideals I, J of R [4]. One can easily see that, in a cancellation module M, $(IM : M) = I$ for any ideal I of R. We call here a multiplication module that is cancellation module as a cancellation multiplication module.

Proposition 2.7. *Let M be a cancellation multiplication R-module. Then P is an S-maximal submodule of M if and only if $(P : M)$ is an S-maximal ideal of R.*

Proof. Assume that P is S-maximal submodule and let $(P : M) \subseteq I$. Then $(P : M)M \subseteq IM$ implying $P \subseteq IM$. Since P is S-maximal, either $sIM \subseteq P$ for some $s \in S$ or $(IM : M) \cap S \neq \emptyset$. This implies that $sI \subseteq (P : M)$ or $I \cap S \neq \emptyset$, as needed.

Now suppose $(P : M)$ is an S-maximal ideal of R. Let $P \subseteq Q \subseteq M$. Then $(P : M) \subseteq (Q : M)$. As $(P : M)$ is S-maximal, there exists $s \in S$ such that $s(Q : M) \subseteq (P : M)$ or $(Q : M) \cap S \neq \emptyset$. If the former case holds, then $s(Q : M)M \subseteq (P : M)M$ showing that $sQ \subseteq P$. If the latter case holds, then we are done. □

3. Topologies on $Spec_S(M)$

Let R be a ring, S be a multiplicatively closed subset of R and I be an ideal of R. Define the set $V_S(I) = \{P \in Spec_S(R) : sI \subseteq P \text{ for some } s \in S\}$ which is called S-variety of I. Then the collection of $V_S(I)$ for any ideal I of R satisfies the axioms for closed sets in a topological space and so induces a topology. This topology is known as the S-Zariski topology on $Spec_S(R)$ [23].

The set of all S-prime submodules of M is denoted by $Spec_S(M)$. For any submodule N of M, we have two different types of S-varieties denoted by $V^*_S(N)$ and $V_S(N)$.

Define $V^*_S(N) = \{P \in Spec_S(M) : sN \subseteq P \text{ for some } s \in S\}$. Then:

(i) $V^*_S(M) = \emptyset$ and $V^*_S((0)) = Spec_S(M)$.

(ii) $\bigcap_{i \in I} V^*_S(N_i) = V^*_S(\sum_{i \in I} N_i)$ where $N_i \leq M$ and I is an index set.

(iii) $V^*_S(K) \cup V^*_S(N) \subseteq V^*_S(K \cap N)$ for any submodules K, N of M.

Next define $V_S(N) = \{P \in Spec_S(M) : s(N :_R M) \subseteq (P :_R M) \text{ for some } s \in S\}$.

(i) $V_S(M) = \emptyset$ and $V_S((0)) = Spec_S(M)$.

(ii) $\bigcap_{i \in I} V_S(N_i) = V_S(\sum_{i \in I}(N_i :_R M)M)$ for any submodule N_i of M.

(iii) $V_S(K) \cup V_S(N) = V_S(K \cap N)$ for any submodules K, N of M.

In order to construct a topology on $Spec_S(M)$, we address the above sets $V_S^*(N)$ and $V_S(N)$. The collection of $V_S^*(N)$ where $N \leq M$ induces a topology, say τ_S^*, if and only if finite union of $V_S^*(N)$ where $N \leq M$ is closed. In this case, the induced topology is called S-quasi Zariski topology on $Spec_S(M)$. A module is said to be S-top module if $\tau_S^*(M)$ is a topology. A module is not necessarily to be an S-top module. On the other hand, the collection of $V_S(N)$ and $V_S^*(IM)$ always induces a topology, say τ_S, on $Spec_S(M)$. This topology is called S-Zariski topology.

Note that if $P \in Spec(M)$ with $(P : M) \cap S = \emptyset$ then $P \in Spec_S(M)$. But the following example shows that the converse is not true in general.

Example 3.1. Let $M = \mathbb{Z}_3 \times \mathbb{Z}$, $R = \mathbb{Z}$. Consider the submodule $P = \overline{0} \times 0$ of M. Here $(\overline{0} \times 0 : \mathbb{Z}_3 \times \mathbb{Z}) = 0$. Though $3(\overline{1}, 0) = (\overline{0}, 0) \in P$, neither $3 \in (\overline{0} \times 0 : \mathbb{Z}_3 \times \mathbb{Z})$ nor $(\overline{1}, 0) \in \overline{0} \times 0$. Thus P is not a prime submodule of M. On the other hand, take $S = \mathbb{Z} - \{0\}$. Then $(P : M) \cap S = \emptyset$. Choose $r(\overline{a}, b) = (r\overline{a}, rb) \in P$. This gives $r\overline{a} = \overline{0}$ and $rb = 0$. If $r = 0$, we are done. So assume that $r \neq 0$. Then $b = 0$ and this implies $3(\overline{a}, b) = (\overline{0}, 0) \in P$ where $s = 3$. Hence P is an S-prime submodule of M. Since $P \in Spec_S(M)$ but $P \notin Spec(M)$, we conclude that $Spec_S(M)$ is strictly bigger than $Spec(M)$.

Example 3.2. Let $M = \mathbb{Z} \times \mathbb{Z}$ and $R = \mathbb{Z}$. Take $N = 6\mathbb{Z} \times 5\mathbb{Z}$. Here $(N : M) = 30\mathbb{Z}$. Then $V(N) = \{2\mathbb{Z} \times \mathbb{Z}, 3\mathbb{Z} \times \mathbb{Z}, \mathbb{Z} \times 5\mathbb{Z}\}$. On the other side, $V_S(N) = \emptyset$ where $S = \mathbb{Z} - \{0\}$. Now suppose $N = 0 \times 5\mathbb{Z}, 6\mathbb{Z} \times 0$ or 0×0. Then $V_S(N) = Spec_S(M)$. Therefore, though $Spec(M)$ has many varieties, $Spec_S(M)$ has only $\emptyset$ and $Spec_S(M)$ itself.

Theorem 3.3. *Let R be a ring, $S \subseteq R$ be a multiplicatively closed subset and M be an R-module. Then the following statements hold:*

(*i*) $V_S(A) = V_S((A))$ *for any subset A of M where (A) denotes the submodule generated by the subset $A \subseteq M$.*

(*ii*) $V_S(M) = \emptyset$ *and* $V_S((0)) = Spec_S(M)$.

(*iii*) $\bigcap_{i \in I} V_S(N_i) = V_S(\sum_{i \in I}(N_i :_R M)M)$ *for any submodule N_i of M.*

(*iv*) $V_S(K) \cup V_S(N) = V_S(K \cap N)$ *for any submodules K, N of M.*

Proof. (*i*) It is clear.

(*ii*) Let $P \in V_S(M)$. Then there exists $s \in S$ such that $s(M :_R M) \subseteq (P :_R M)$. This gives $s \in (P :_R M) \cap S$, a contradiction. So $V_S(M) = \emptyset$. Now choose $P \in V_S(0)$. Then we have $s(0 :_R M) \subseteq (P :_R M)$ for some $s \in S$. This is true for all $P \in Spec_S(M)$. Thus $V_S((0)) = Spec_S(M)$.

(*iii*) Take $P \in \bigcap_{i \in I} V_S(N_i)$. Then $P \in V_S(N_i)$ for all i. Since P is S-prime submodule, there exists a fixed $s \in S$ such that $s(N_i : M) \subseteq (P : M)$ for each N_i.

$$\begin{aligned}
& s(N_i : M) \subseteq (P : M) \Rightarrow s(N_i : M)M \subseteq (P : M)M \\
& \Leftrightarrow (N_i : M)M \subseteq ((P : M)M : s) \\
& \Leftrightarrow ((N_i : M)M : M) \subseteq (((P : M)M : s) : M) = (((P : M)M : M) : s) \\
& \Leftrightarrow s((N_i : M)M : M) \subseteq ((P : M)M : M) = (P : M) \\
& \Leftrightarrow s(\sum_{i \in I}(N_i : M)M : M) \subseteq (P : M) \\
& \Leftrightarrow P \in V_S(\sum_{i \in I}(N_i : M)M).
\end{aligned}$$

(*iv*) Take $P \in V_S(N) \cup V_S(L)$. Then $P \in V_S(N)$ or $V_S(L)$. This means that $s(N :_R M) \subseteq (P :_R M)$ or $s(L :_R M) \subseteq (P :_R M)$ for some $s \in S$. Thus $s(N \cap L :_R M) \subseteq (P :_R M)$ giving that $P \in V_S(N \cap L)$.

Let $P \in V_S(N \cap L)$. Then there exists $s \in S$ such that $s((N :_R M) \cap (L :_R M)) = s(N \cap L :_R M) \subseteq (P :_R M)$. So, $s(N :_R M) \subseteq (P :_R M)$ or $s(L :_R M) \subseteq (P :_R M)$. This gives either $P \in V_S(N)$ or $P \in V_S(L)$. Hence $P \in V_S(N) \cup V_S(L)$, as needed. □

From the previous theorem, there exists a topology on $Spec_S(M)$ having the collection of $V_S(N)$ for $N \leq M$ as the family of all closed sets. This topology is called S-Zariski topology on $Spec_S(M)$. It can be seen that any open set on S-Zariski topology has the form $Spec_S(M) - V_S(N)$ for $N \leq M$.

Proposition 3.4. *Let M be an R-module and N be a submodule of M. Then $V_S(N) = V_S((N : M)M) = V_S^*((N : M)M)$.*

Proof. Let $P \in V_S(N)$. Then $s(N : M) \subseteq (P : M)$ for some $s \in S$. This implies that $s(N : M)M \subseteq (P : M)M$. So we have $s((N : M)M : M) \subseteq (s(N : M)M : M) \subseteq ((P : M)M : M) = (P : M)$. Thus we conclude that $P \in V_S((N : M)M)$. For the other inclusion, take $P \in V_S((N : M)M)$. Then $s((N : M)M : M) \subseteq (P : M)$ for some $s \in S$. Since $((N : M)M : M) = (N : M)$, we get $s(N : M) \subseteq (P : M)$ showing that $P \in V_S(N)$.

Let $P \in V_S^*((N : M)M)$. Then $s(N : M)M \subseteq P$ for some $s \in S$. This gives $s(N : M) = s((N : M)M : M) \subseteq (s(N : M)M : M) \subseteq (P : M)$. Hence $P \in V_S(N)$. Choose $P \in V_S(N) = V_S((N : M)M)$. Then $s((N : M)M : M) \subseteq (P : M)$ for some $s \in S$ implying $s(N : M)M \subseteq P$ and this gives $P \in V_S^*((N : M)M)$, as desired. □

Define the set $Spec_S^p(M) = \{P \in Spec_S(M) : S^{-1}(P : M) = S^{-1}p, p \in Spec_S(R)\}$.

Proposition 3.5. *Let M be an R-module and N be a submodule of M. Then,*

$$V_S(N) = \bigcup_{p \in V_S((N:M))} Spec_S^p(M).$$

Proof. Choose $P \in V_S(N)$. Then $s(N : M) \subseteq (P : M)$ for some $s \in S$ implying $S^{-1}(N : M) \subseteq S^{-1}(P : M) = S^{-1}p$. Here, $s'(N : M) \subseteq p$ for some $s' \in S$. Thus $p \in V_S((N : M))$. This means that $P \in \bigcup_{p \in V_S((N:M))} Spec_S^p(M)$.

On the other hand, let $Q \in \bigcup_{p \in V_S((N:M))} Spec_S^p(M)$. Then $Q \in Spec_S^p(M)$ for some $p \in V_S((N : M))$. So $S^{-1}(Q : M) = S^{-1}p$ where $s(N : M) \subseteq p$ for some $s \in S$. This implies that $S^{-1}(N : M) \subseteq S^{-1}p = S^{-1}(Q : M)$. Hence we have $s'(N : M) \subseteq (Q : M)$ showing $Q \in V_S(N)$. □

Lemma 3.6. *Let R be a ring, M be an R-module, S be a multiplicatively closed subset of R and K, N be submodules of M. If $S^{-1}(K : M) = S^{-1}(N : M)$, then $V_S(K) = V_S(N)$. The converse is also true when K and N are S-prime.*

Proof. Assume that $S^{-1}(K : M) = S^{-1}(N : M)$. Take $P \in V_S(K)$. Then there exists $s \in S$ such that $s(K : M) \subseteq (P : M)$. Choose $r \in (N : M)$ implying $\frac{r}{s} \in S^{-1}(N : M) = S^{-1}(K : M)$. So $s'r \in (K : M)$ for some $s' \in S$. Then we get $ss'r \in s(K : M) \subseteq (P : M)$. Since $(P : M)$ is S-prime ideal, there exists $t \in S$ such that $tr \in (P : M)$ and so $t(N : M) \subseteq (P : M)$, that is, $P \in V_S(N)$. Similar argument shows that $V_S(N) \subseteq V_S(K)$, as desired.

On the other hand, suppose that $V_S(K) = V_S(N)$. Choose $\frac{a}{s} \in S^{-1}(K : M)$. Then there exists $u \in S$ such that $ua \in (K : M)$. Since $s'(K : M) \subseteq (N : M)$ for some $s' \in S$, we get $s'ua \in s'(K : M) \subseteq (N : M)$. Then $\frac{a}{s} = \frac{s'ua}{s'us} \in S^{-1}(N : M)$. This shows that $S^{-1}(K : M) \subseteq S^{-1}(N : M)$. For the converse, take $\frac{b}{s} \in S^{-1}(N : M)$. Then there exists $u \in S$ such that $ub \in (N : M)$. Since $s'(N : M) \subseteq (K : M)$ for some $s' \in S$, we get $s'ub \in s'(N : M) \subseteq (K : M)$. Then $\frac{b}{s} = \frac{s'ub}{s'us} \in S^{-1}(K : M)$. This shows that $S^{-1}(N : M) \subseteq S^{-1}(K : M)$ which proves the equality. □

Theorem 3.7. *The collection of $D_a^S = \{P \in Spec_S(M) : s(aM : M) \nsubseteq (P : M)$ for all $s \in S\}$ where $a \in R$ is a basis for S-Zariski topology.*

Proof. First, we will show that D_a^S is open for any $a \in R$. Let $P \in Spec_S(M) - D_a^S$. Then $P \notin D_a^S$ implying that $s(aM :_R M) \subseteq (P :_R M)$. Hence $P \in V_S(aM)$ which gives $Spec_S(M) - D_a^S \subseteq V_S(aM)$. For the reverse inclusion, take $P \in V_S(aM)$. Then $s(aM :_R M) \subseteq (P :_R M)$. This means that $P \notin D_a^S$ and so $P \in Spec_S(M) - D_a^S$. Since $Spec_S(M) - D_a^S = V_S(aM)$ and $V_S(aM)$ is closed in $Spec_S(M)$, we conclude that D_a^S is open.

Now we will show that any open set $Spec_S(M) - V_S(N)$ can be written as a union of D_a^S, that is, $Spec_S(M) - V_S(N) = \bigcup D_a^S$. If $P \in Spec_S(M) - V_S(N)$. Then $P \notin V_S(N)$ which means that $s(N :_R M) \nsubseteq (P :_R M)$

for all $s \in S$. So $(N :_R M) \nsubseteq ((P :_R M) :_R s)$. Then there exists $s' \in S$ such that $((P :_R M) :_R s) \subseteq ((P :_R M) :_R s')$ for all $s \in S$ by [19, Lemma 2.16]. Let $N = \sum_{\{i \in \Delta\}} a_i M$ and $\Delta' = \{i \in \Delta : (a_i M :_R M) \nsubseteq ((P :_R M) :_R s')\}$. Then for each $s \in S$, $i \in \Delta'$, we have $s(a_i M :_R M) \nsubseteq (P :_R M)$. This gives $P \in D^S_{a_i}$ implying that $P \in \bigcup_{i \in \Delta'} D^S_{a_i}$. Conversely, choose $P \in \bigcup_{i \in \Delta'} D^S_{a_i}$. Then $P \in D^S_{a_i}$ for some $i \in \Delta'$, $a_i \in R$. So $s(a_i M :_R M) \nsubseteq (P :_R M)$ for all $s \in S$. Then we get $a(N :_R M) \nsubseteq (P :_R M)$ for all $s \in S$. Thus $P \notin V_S(N)$ giving $P \in Spec_S(M) - V_S(N)$, as desired. □

Lemma 3.8. ([17]) *Let $\mathcal{B}$ and $\mathcal{B}'$ be basis for topologies τ and τ', respectively, on X. Then τ' is finer that τ if and only if for each $x \in X$ and each basis element $x \in B \in \mathcal{B}$, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.*

Recall from [2] that a module M is called S-multiplication if for each submodule N of M there exists an $s \in S$ such that $sN \subseteq (N : M)M \subseteq N$.

Note that the collection of $D^{*S}_a = \{P \in Spec_S(M) : saM \nsubseteq P \text{ for all } s \in S\}$ where $a \in R$ is a basis for quasi S-Zariski topology in an S-top module. In this case, quasi S-Zariski topology is finer than S-Zariski topology. In particular, in an S-multiplication module, they are coincide.

Proposition 3.9. *Let M be a S-multiplication R-module. Then $\tau^*_S = \tau_S$.*

Proof. We already know that $\tau_S \subseteq \tau^*_S$ by Proposition 3.4. For the other inclusion, since M is an S-multiplication module, we have $sN \subseteq (N : M)M \subseteq N$ for some $s \in S$ and it implies that $V^*_S(N) \subseteq V^*_S((N : M)M) \subseteq V^*_S(N)$. Thus we obtain $V^*_S(N) = V^*_S((N : M)M) = V_S(N)$ by Proposition 3.4. Thus $\tau^*_S = \tau_S$, as desired. □

By [19, Lemma 2.16], there exists an $s \in S$ such that $(P :_M s') \subseteq (P :_M s)$ for each $s' \in S$ and $(P :_M s)$ is a prime submodule. From this time forth, we denote this $s \in S$ for $P \in Spec_S(M)$ by s_P. The following theorem illustrates a relationship between the closure of any subset of $Spec_S(R)$ and closed sets.

Theorem 3.10. *Let M be a finitely generated R-module and $Y \subseteq Spec_S(M)$. Then,*

$$\overline{Y} = V_S(\bigcap_{P \in Y}(P :_M s_P)).$$

Proof. Let $Q \in Y$. Then $\bigcap_{P \in Y}(P :_M s_P) \subseteq (Q :_M S_Q)$ and this implies that $s_Q \bigcap_{P \in Y}(P :_M s_P) \subseteq Q$.

$$(s_Q \bigcap_{P \in Y}(P :_M s_P) :_R M) \subseteq (Q :_R M) \Rightarrow s_Q(\bigcap_{P \in Y}(P :_M s_P) :_R M) \subseteq (Q :_R M) \Rightarrow Q \in V_S(\bigcap(P :_M s_P))$$

$$\Rightarrow Y \subseteq V_S(\bigcap_{P \in Y}(P :_M s_P))$$
$$\Rightarrow \overline{Y} \subseteq V_S(\bigcap_{P \in Y}(P :_M s_P)).$$

Conversely, suppose $Y \subseteq V_S(N)$. If $P \in Y$ then $P \in V_S(N)$. This gives that $s(N :_R M) \subseteq (P :_R M)$ for some $s \in S$.

$$(N :_R M) \subseteq ((P :_R M) :_R s) \subseteq ((P :_R M) :_R s_P) = ((P :_M s_P) :_R M)$$

$$\Rightarrow (N :_R M) \subseteq \bigcap_{P \in Y}((P :_M s_P) :_R M) = (\bigcap_{P \in Y}(P :_M s_P) :_R M).$$

Let $Q \in V_S(\bigcap_{P \in Y}(P :_M s_P))$. Then we get $s(N : M) \subseteq s(\bigcap_{P \in Y}(P :_M s_P) : M) \subseteq (Q : M)$, that is, $Q \in V_S(N)$, as desired. □

Let R be a ring, M be an R-module and S be a multiplicatively closed subset of R. Define the set $\theta = \{S^{-1}(P : M) : P \in Spec_S(M)\}$. $S^{-1}(P : M)$ is a maximal element of θ if $S^{-1}(P : M) \subseteq S^{-1}(Q : M)$ implies that $S^{-1}(P : M) = S^{-1}(Q : M)$ where $Q \in Spec_S(M)$.

Theorem 3.11. *Let M be an R-module and $P \in Spec_S(M)$. Then we have the following:*

(i) $\overline{\{P\}} = V_S(P) = V_S((P :_M s_P))$.

(ii) For any $Q \in Spec_S(M)$, $Q \in \overline{\{P\}}$ iff $s(P : M) \subseteq (Q : M)$ for some $s \in S$ iff $V_S(Q) \subseteq V_S(P)$.

(iii) $\{P\}$ is closed in $Spec_S(M)$ if and only if $S^{-1}(P : M)$ is a maximal element of θ and $Spec^p_S(M) = \{P\}$ where $S^{-1}(P : M) = S^{-1}p$, that is, $|Spec^p_S(M)| = 1$.

Proof. (i) $\overline{\{P\}} = V_S(\bigcap_{P\in\{P\}} (P :_M s_P)) = V_S((P :_M s_P))$. Since $P \subseteq (P :_M s_P)$, it is clear that $V_S((P :_M s_P)) \subseteq V_S(P)$. Now choose $Q \in V_S(P)$. Then $s(P : M) \subseteq (Q : M)$ for some $s \in S$. This implies that $ss_P((P :_M s_P) : M) = ss_P((P : M) : s_P) \subseteq s(P : M) \subseteq (Q : M)$ and so $Q \in V_S((P :_M s_P))$ which completes the proof.

(ii) Take $Q \in \overline{\{P\}} = V_S(P)$. Then $s(P : M) \subseteq (Q : M)$ for some $s \in S$. Let $N \in V_S(Q)$. Then there exists $s' \in S$ such that $s'(Q : M) \subseteq (N : M)$. This gives $s's(P : M) \subseteq s'(Q : M) \subseteq (N : M)$. Thus $N \in V_S(P)$ implying $V_S(Q) \subseteq V_S(P)$.

For the converse, let $Q \in V_S(Q) \subseteq V_S(P)$. Then $s(P : M) \subseteq (Q : M)$ for some $s \in S$. Hence $Q \in V_S(P) = \overline{\{P\}}$ which completes the proof.

(iii) Suppose $\{P\}$ is closed. Then $\{P\} = \overline{\{P\}} = V_S(P)$. Since $S^{-1}(P : M) \subseteq S^{-1}(Q : M)$ where $Q \in Spec_S(M)$ implies $s(P : M) \subseteq (Q : M)$ for some $s \in S$, we have $Q \in V_S(P) = \{P\}$. Thus $Q = P$ and this means that $S^{-1}(P : M)$ is a maximal element of θ. Also, we have $Spec^p_S(M) \subseteq V_S(P) = \{P\}$.

On the other hand, choose $Q \in \overline{\{P\}}$. Then there exists $s \in S$ such that $s(P : M) \subseteq (Q : M)$. It means that $S^{-1}(P : M) \subseteq S^{-1}(Q : M)$. Since $S^{-1}(P : M)$ is a maximal element of θ, we have $S^{-1}(P : M) = S^{-1}(Q : M) = S^{-1}p$. So $Q \in Spec^p_S(M)$. As $|Spec^p_S(M)| = 1$, $P = Q$. Then we conclude that $\overline{\{P\}} = \{P\}$ and so $\{P\}$ is closed. □

Theorem 3.12. *Let M be an R-module and $P, Q \in Spec_S(M)$. Then the following statements are equivalent:*

(i) The natural map $\phi : Spec_S(M) \to Spec(S^{-1}R/Ann(S^{-1}M))$ is injective.

(ii) If $V_S(P) = V_S(Q)$, then $P = Q$.

(iii) $|Spec^p_S(M)| \leq 1$.

Proof. (i) ⇒ (ii) Assume that $V_S(P) = V_S(Q)$. Then we have $S^{-1}(P : M) = S^{-1}(Q : M)$ by Lemma 3.6. This gives that $\overline{S^{-1}(P : M)} = \overline{S^{-1}(Q : M)}$ implying $\phi(P) = \phi(Q)$. Since ϕ is injective, we get $P = Q$.

(ii) ⇒ (iii) Let $P, Q \in Spec^p_S(M)$. Then $S^{-1}(P : M) = S^{-1}p = S^{-1}(Q : M)$. This implies that $V_S(P) = V_S(Q)$ and so $P = Q$, as desired.

(iii) ⇒ (i) Let $\phi(P) = \phi(Q)$. Then $\overline{S^{-1}(P : M)} = \overline{S^{-1}(Q : M)} = \overline{S^{-1}p}$. Thus we have $P, Q \in Spec^p_S(M)$. As $|Spec^p_S(M)| \leq 1$, $P = Q$ which shows ϕ is injective. □

Theorem 3.13. *Let M be an R-module. Then the following are equivalent:*

(i) $Spec_S(M)$ is T_0-space.

(ii) If $V_S(P) = V_S(Q)$, then $P = Q$ for any $P, Q \in Spec_S(M)$.

Proof. (i) ⇒ (ii) Assume that $V_S(P) = V_S(Q)$. Then we have $\overline{\{P\}} = \overline{\{Q\}}$. Since $Spec_S(M)$ is T_0, we conclude that $P = Q$.

(ii) ⇒ (i) Assume that $V_S(P) = V_S(Q)$ implies $P = Q$. Since $\overline{\{P\}} = V_S(P)$, $\overline{\{P\}} = \overline{\{Q\}}$ means that $V_S(P) = V_S(Q)$ and so $P = Q$ by the assumption. Thus $Spec_S(M)$ is T_0-space. □

Corollary 3.14. *If M is a multiplication module, then $Spec_S(M)$ is a T_0-space for both S-Zariski topology τ_S and quasi S-Zariski topology τ^*_S.*

Proof. Suppose $V_S(P) = V_S(Q)$ for $P, Q \in Spec_S(M)$. Then $S^{-1}(P : M) = S^{-1}(Q : M)$ by Lemma 3.5. This implies that $(P : M) = (Q : M)$. As M is a multiplication module, we have $P = (P : M)M = (Q : M)M = Q$. Thus $Spec_S(M)$ is T_0 by Theorem 3.13. The rest follows from the fact that $\tau_S \leq \tau^*_S$. □

Proposition 3.15. *Let M be an R-module whose $Spec_S(M)$ may be empty. $Spec_S(M)$ is a T_1-space if and only if $S^{-1}(P : M)$ is a maximal element of θ and $|Spec_S^p(M)| \leq 1$ for every $p \in Spec_S(R)$.*

Proof. If $Spec_S(M) = \emptyset$, it is clear that the statement is true. Now suppose $Spec_S(M) \neq \emptyset$. If $Spec_S(M)$ is a T_1-space, then $S^{-1}(P : M)$ is a maximal element of θ and $Spec_S^p(M) = \{P\}$ where $S^{-1}(P : M) = S^{-1}p$, that is, $|Spec_S^p(M)| = 1$ by Theorem 3.11 (iii).

On the other hand, $|Spec_S^p(M)| = 1$ for every $S^{-1}p \in \theta$. Then $\{P\}$ is closed for every $P \in Spec_S(M)$ by Theorem 3.11. Therefore, $Spec_S(M)$ is T_1. □

Theorem 3.16. *Let M be a S-multiplication R-module. Then the map $\phi : Spec_S(M) \to Spec_S(R)$ defined by $\phi(N) = (N : M)$ is continuous.*

Proof. Let F be any closed set in $Spec_S(R)$. We will show that $\phi^{-1}(F)$ is closed in $Spec_S(M)$. Since F is closed in $Spec_S(R)$, we have $F = V_S(I)$ where $I \trianglelefteq R$. For any $N \in Spec_S(M)$, $N \in \phi^{-1}(F)$ and so $\phi(N) \in V_S(I)$. Since $(N : M) \in V_S(I)$, there exists $s \in S$ such that $sI \subseteq (N : M)$. Then $sIM \subseteq (N : M)M$.

$$(sIM : M) \subseteq ((N : M)M : M) \Rightarrow s(IM : M) \subseteq (N : M).$$

This gives $N \in V_S(IM)$.

Conversely, take $N \in V_S(IM)$. Then $s(IM : M) \subseteq (N : M)$ for some $s \in S$.

Since M is S-multiplication, there exists $s' \in S$ such that $s'IM \subseteq (IM : M)M$. This implies that $ss'IM \subseteq s(IM : M)M \subseteq N$, that is, $ss'I \subseteq (N : M)$. Then $(N : M) \in V_S(I)$, that is, $N \in \phi^{-1}(F)$. Then we have $V_S(IM) \subseteq \phi^{-1}(F)$. Therefore, we conclude that $\phi^{-1}(F) = V_S(IM)$ proving that $\phi^{-1}(F)$ is closed in $Spec_S(M)$. □

Theorem 3.17. *Let M, M' be R-modules, $X = Spec_S(M)$ and $X' = Spec_S(M')$. If $f : M \to M'$ is epimorphism, then $\phi : X' \to X$ defined by $P' \mapsto f^{-1}(P')$ is continuous.*

Proof. For any $P' \in X'$ and any closed set $V_S(N)$ where $N \leq M$. Choose $P' \in \phi^{-1}(V_S(N)) = \phi^{-1}(V_S^*((N : M)M))$. Then $\phi(P') = f^{-1}(P) \in V_S^*((N : M)M)$. This implies that $s(N : M)M \subseteq \phi(P') = f^{-1}(P')$. Hence we obtain $f(s(N : M)M) \subseteq f(\phi(P')) = P'$. Then

$$s(N : M)M' \subseteq P' \Rightarrow P' \in V_S^*((N : M)M') = V_S((N : M)M').$$

Conversely, take $P' \in V_S^*((N : M)M') = V_S((N : M)M')$. Then

$$s(N : M)M' \subseteq P' \Rightarrow sf((N : M)M) \subseteq P' \Rightarrow s(N : M)M \subseteq f^{-1}(P') = \phi(P').$$

$$\phi(P') \in V_S^*((N : M)M) \Rightarrow P' \in \phi^{-1}(V_S^*(N : M)M) = \phi^{-1}(V_S(N)).$$

□

4. Irreducibility in $Spec_S(M)$

Proposition 4.1. *Let P be an S-prime submodule of an R-module M. Then $V_S(P)$ is an irreducible closed subset of $Spec_S(M)$.*

Proof. Assume that $V_S(P) = V_S(K) \cup V_S(L)$ for some submodules N, L of M. It is clear that $V_S(K) \subseteq V_S(P)$. Since $P \in V_S(P)$, $P \in V_S(K)$ or $P \in V_S(L)$. Without loss of generality, suppose $P \in V_S(K)$. Then there exists $s \in S$ such that $s(K : M) \subseteq (P : M)$. Choose $Q \in V_S(P)$. Then $s'(P : M) \subseteq (Q : M)$. This implies that $s's(K : M) \subseteq s'(P : M) \subseteq (Q : M)$. This gives $Q \in V_S(K)$ implying $V_S(P) \subseteq V_S(K)$. Thus we get $V_S(P) = V_S(K)$ which completes the proof. □

Proposition 4.2. *Let M be an R-module and Y be a subset of $Spec_S(M)$. Assume that $S^{-1}(\bigcap_{P \in Y} (P :_M s_P) : M) = S^{-1}p$ is a prime ideal of R. If $Spec_S^p(M) \neq \emptyset$, then Y is irreducible.*

Proof. Let $Q \in Spec_S^p(M)$. Then $S^{-1}(Q:M) = S^{-1}p = S^{-1}(\bigcap_{P\in Y}(P:_M s_P):M)$. Hence $V_S(Q) = V_S(\bigcap_{P\in Y}(P:_M s_P) = \overline{Y}$. Since $V_S(Q)$ is irreducible for S-prime submodule Q of M, $\overline{Y}$ is irreducible. So Y is also irreducible. □

Corollary 4.3. *Let M be an R-module and Y be a subset of $Spec_S(M)$. If $\bigcap_{P\in Y} P$ is an S-prime submodule of M, then Y is irreducible.*

Proof. If $\bigcap_{P\in Y} P$ is an S-prime submodule of M, $V_S(\bigcap_{P\in Y} P) = \overline{Y}$ is irreducible. So Y is irreducible. □

Corollary 4.4. *Let $Spec_S^p(M) \neq \emptyset$ for some $p \in Spec_S(R)$. If p is S-maximal ideal of R, then $Spec_S^p(M)$ is irreducible closed subset of $Spec_S(M)$.*

Proof. One can easily see that $p \subseteq (pM:M)$. Since p is S-maximal, we have either $s(pM:M) \subseteq p$ or $(pM:M)\cap S \neq \emptyset$. If we assume $(pM:M)\cap S \neq \emptyset$, then there exists $s \in S$ such that $s \in (pM:M)$. Let $P \in Spec_S^p(M)$. This gives $S^{-1}(P:M) = S^{-1}p$ by the definition. So $s'p \subseteq (P:M)$ for some $s' \in S$. Then,

$$s'pM \subseteq (P:M)M \Rightarrow s'(pM:M) \subseteq (s'pM:M) \subseteq ((P:M)M:M) = (P:M).$$

So, $ss' \in (P:M)$, a contradiction. If the former case holds, then $S^{-1}(pM:M) \subseteq S^{-1}p$. So we have $S^{-1}(pM:M) = S^{-1}p$. Now we claim that $Spec_S^p(M) = V_S(pM)$ where p is S-maximal ideal of R. Let $P \in Spec_S^p(M)$. Then $S^{-1}(P:M) = S^{-1}p = S^{-1}(pM:M)$. This gives $V_S(P) = V_S(pM)$. So $P \in V_S(pM)$. Now take $Q \in V_S(pM)$. Then $s(pM:M) \subseteq (Q:M)$ for some $s \in S$. This gives $S^{-1}p = S^{-1}(pM:M) \subseteq S^{-1}(Q:M)$. Since p is S-maximal, $S^{-1}p$ is maximal. So we have $S^{-1}p = S^{-1}(Q:M)$ showing $Q \in Spec_S^p(M)$. □

Definition 4.5. Let M be an R-module and N be a submodule of M. Then, S-radical of N is defined as

$$\sqrt[S]{N} = \{r \in R : sr^nM \subseteq N, \exists s \in S, \exists n \in \mathbb{Z}^+\}.$$

Proposition 4.6. *Let M be a finitely generated multiplication module and N be a submodule of M. Then,*

$$\sqrt[S]{N} = \bigcap_{P\in V_S(N)} ((P:M):s_P).$$

Proof. Let $a \in \sqrt[S]{N}$. Then $sa^nM \subseteq N$ for some $s \in S$ and $n \in \mathbb{Z}^+$ implying $sa^n \in (N:M)$. Take $P \in V_S(N)$. Then $s'(N:M) \subseteq (P:M)$. So we have $s'sa^n \in s'(N:M) \subseteq (P:M)$. This gives $a^n \in ((P:M):s's) \subseteq ((P:M):s_P)$. Since $((P:M):s_P)$ is a prime ideal, $a \in ((P:M):s_P)$ for all $P \in V_S(N)$.

Conversely, choose $b \in \bigcap_{P\in V_S(N)} ((P:M):s_P)$. Then $b \in ((P:M):s_P)$ for all $P \in V_S(N)$. Suppose $b \notin \sqrt[S]{N}$. So $sb^nM \nsubseteq N$ for all $s \in S$ and $n \in \mathbb{Z}^+$. Then $\frac{b^n}{1} = (\frac{b}{1})^n S^{-1}M \nsubseteq S^{-1}N$. This means that $\frac{b}{1} \notin \sqrt{(S^{-1}N : S^{-1}M)}$. There exists a prime submodule P^* of $S^{-1}M$ with $S^{-1}N \subseteq P^*$ such that $\frac{b}{1}S^{-1}M \nsubseteq P^* = S^{-1}P'$ for some prime submodule P' of M. As $S^{-1}N \subseteq S^{-1}P'$, $sN \subseteq P'$ implying $s(N:M) \subseteq (sN:M) \subseteq (P':M)$ and so $P' \in V_S(N)$. Since $b \in \cap_{P\in V_S(N)}((P:M):s_P)$, $b \in ((P':M):s'_P) = (P':M)$. Thus $\frac{b}{1} \in S^{-1}(P':M)$ and this implies $\frac{b}{1}S^{-1}M \subseteq S^{-1}P' = P^*$, a contradiction. □

Proposition 4.7. *Let M be a finitely generated multiplication module and N be a submodule of M. Then,*

$$V_S(N) = V_S(\sqrt[S]{N}M).$$

Proof. Since $N = (N:M)M \subseteq \sqrt[S]{N}M$, we have $V_S(\sqrt[S]{N}) \subseteq V_S(N)$. For the converse, suppose $Q \in V_S(N)$. As $\sqrt[S]{N} = \bigcap_{P\in V_S(N)} ((P:M):s_P) \subseteq ((Q:M):s_Q)$, we obtain $s_Q\sqrt[S]{N} \subseteq (Q:M)$ implying $s_Q\sqrt[S]{N}M \subseteq Q$. Then $s_Q(\sqrt[S]{N}M:M) \subseteq (Q:M)$ and so $Q \in V_S(\sqrt[S]{N}M)$, as desired. □

Proposition 4.8. *Let M be a finitely generated multiplication module and N be a submodule of M. If $V_S(N)$ is irreducible, then $\sqrt[S]{N}$ is a prime ideal.*

Proof. Take $ab \in \sqrt[S]{N}$ but $a \notin \sqrt[S]{N}$ and $b \notin \sqrt[S]{N}$. Then there exist $P, Q \in V_S(N)$ such that $a \notin ((P : M) : s_P)$ and $b \notin ((Q : M) : s_Q)$. This implies that $sa \notin (P : M)$ and $sb \notin (Q : M)$ for all $s \in S$. So $s(aM : M) \nsubseteq (P : M)$ and $s(bM : M) \nsubseteq (Q : M)$. So we conclude that $P \in D_a^S$ and $Q \in D_b^S$ which imply $P \in D_a^S \cap V_S(N)$ and $Q \in D_b^S \cap V_S(N)$. Thus $D_a^S \cap V_S(N)$ and $D_b^S \cap V_S(N)$ are nonempty open sets in subspace topology. Since $V_S(N)$ is irreducible, $(D_a^S \cap V_S(N)) \cap (D_b^S \cap V_S(N)) \neq \emptyset$. Suppose $U \in (D_a^S \cap V_S(N)) \cap (D_b^S \cap V_S(N))$. As $U \in V_S(N) = V_S(\sqrt[S]{N}M)$ by Proposition 4.7, we get $s(\sqrt[S]{N}M : M) \subseteq (U : M)$. Also, $U \in D_a^S \cap D_b^S = D_{ab}^S$ implies $s'(abM : M) \nsubseteq (U : M)$ for all $s' \in S$. But we have $sab \in (U : M)$ that gives $sabM \subseteq (U : M)M$. Then we have $s(abM : M) \subseteq ((U : M)M : M) = (U : M)$, a contradiction. Thus $\sqrt[S]{N}$ is a prime ideal. □

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