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Fixed Points of Mappings on Extended Cone b-Metric Space Over Real Banach Algebra

Wahid Ullah[a], Maria Samreen[a], Tayyab Kamran[a]

[a]*Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan*

Abstract. In this article, we introduce a new geometrical structure which is the hybrid of cone metric space over Banach algebra and extended b-metric space. We prove analogues of Banach, Kannan and Reich type fixed point theorems in our introduced space. We also furnish with various concrete examples to establish the validity of our results. The obtained results generalize many well-known results in literature, especially the main results due to Vujakovic *et al*., Hussain *et al*., Huang, Radenovic, Xu become special cases of our results.

1. Introduction

In 2007, Huang and Zhang [2] initiated the concept of cone metric space over a Banach space as the generalization of metric spaces. They used ordered Banach space E instead of $\mathbb{R}$ as the range set of metric ρ, *i.e.* they used $\rho : \mathfrak{X} \times \mathfrak{X} \to E$. They also discussed Banach type contraction and proved some fixed point results. After that, many researcher concentrated to investigate such spaces and proved a number of fixed point theorems. According to imperfect statistics, by using cone metric spaces, more than six hundred articles have been published [3]. But recently some scholars obtained the equivalent results of usual metric space $(\mathfrak{X}, d^*)$ and that of cone metric space $(\mathfrak{X}, \rho)$. They defined the real valued metric function d^* as the non-linear scalarization function ξ [4, 5]. However, this circumstances changed when Liu and Xu [6] in 2013 introduced cone metric space by using a real Banach algebra instead of Banach space and defined generalized Lipschitz mapping. They presented an example which furnish that results of fixed point in metric spaces are not equivalent to that of results in cone metric spaces over Banach algebras. Later on in 2016, Huang and Radenovic [7] extended the idea of cone metric space over Banach algebras to cone b-metric spaces over Banach algebras. They proved Banach and Kannan type theorems for such spaces.
In 1993, Czerwik [8] introduced the notion of a b-metric space by replacing the triangular property of a metric space with $\rho(p,t) \leq b[\rho(p,q) + \rho(q,t)]$, where $b \geq 1$. Later on, in 2017 Kamran *et al*. [9] further extended the concept of b-metric space by introducing extended b-metric spaces. They introduce a function $\theta : \mathfrak{X} \times \mathfrak{X} \to [1, \infty)$ instead of b in triangular inequality condition. They established a Banach like contraction and proved some fixed point results in such spaces. This shows that the class of such type of spaces is much more larger than the class of b-metric spaces and the class of metric spaces. In this context by using the

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Email addresses: wahidmathqau@gmail.com (Wahid Ullah), maria.samreen@hotmail.com, msamreen@qau.edu.pk (Maria Samreen), tayyabkamran@gmail.com (Tayyab Kamran)

generalized triangle inequality, several authors have published a number of papers in different direction (see e.g [10–14]). In 1971, Reich [15] introduced a new type of contraction which we call Reich contraction. It generalizes the two eminent contractions (i.e. Banach contraction and Kannan contraction). On the other hand, Samet *et al*. in 2012 initiated the idea of α-admissible mapping in metric spaces. Recently in 2015 and 2017, Malhotra *et al*. [16, 17] used the idea of α-admissibility in cone metric spaces by using Banach algebras and proved Banach and Kannan type theorems. Later on in 2017, Hussain *et al*. [18] used the concept of α-admissible mapping in cone b-metric spaces over Banach algebras and proved the Banach type results for such spaces.

In this article, we presented the definition of an extended cone b-metric space over Banach algebra and then proved some fixed point results in such spaces. We also furnish with an example to show the validity of our obtained results. The last section of this paper consists of some important consequences of our results and application in the existence of solution of integral equations. Throughout the paper, we used only real Banach algebras with identity e.

2. Preliminaries

Let $\mathcal{A}$ be a real Banach algebra with zero element ϑ. A cone $\mathcal{K}$ in $\mathcal{A}$ is a nonempty closed subset of $\mathcal{A}$ such that $\mathcal{K}\mathcal{A}p(-\mathcal{K}) = \vartheta$, $\mathcal{K} + \mathcal{K} \subseteq \mathcal{K}$, $\mathcal{K} \cdot \mathcal{K} \subseteq \mathcal{K}$ and $\mu\mathcal{K} \subseteq \mathcal{K}$ for all $\mu \geq 0$. If the interior of $\mathcal{K}$ denoted by int$\mathcal{K}$ is nonempty, then the cone $\mathcal{K}$ is called a solid cone . If we define a relation $\preceq$ on $\mathcal{A}$ by $\varsigma \preceq \varpi$ iff $\varpi - \varsigma \in \mathcal{K}$, then $\preceq$ is a partial order on $\mathcal{A}$. We write $\varsigma \preceq c\varpi$ iff $\varpi - \varsigma \in \mathcal{K}$ and $\varsigma \neq \varpi$. Define another partial order $\ll$ on $\mathcal{A}$ by $\varsigma \ll \varpi$ iff $\varpi - \varsigma \in \text{int}\mathcal{K}$. A cone $\mathcal{K}$ in $\mathcal{A}$ is said to be a normal cone if for all $\varsigma, \varpi \in \mathcal{A}$ with $\vartheta \preceq \varsigma \preceq \varpi$, there exists a real number $M > 0$ such that $\|\varsigma\| \leq M\|\varpi\|$. The normal constant of $\mathcal{K}$ is the least positive constant M for which the above inequality holds.

Consider a unital Banach algebra $\mathcal{A}$ with identity element e. An element ς in $\mathcal{A}$ is said to be invertible if there exists ϖ in $\mathcal{A}$ such that $\varsigma\varpi = \varpi\varsigma = e$. A complex number $\mu \in \mathbb{C}$ is said to be spectral value of $\varpi \in \mathcal{A}$ if $\varpi - \mu e$ is non-invertible in $\mathcal{A}$. The set of all spectral values of $\varpi \in \mathcal{A}$ denoted by $\sigma(\varpi)$ is called the spectrum of ϖ. The number $r_\sigma(\varpi)$ (or $r(\varpi)$) defined by $r_\sigma(\varpi) = \sup\{|\mu| \ : \ \mu \in \sigma(\varpi)\}$ is called the spectral radius of $\varpi \in \mathcal{A}$.

Lemma 2.1. *([1]) Let $\mathcal{A}$ be a Banach algebra with identity e. Then the spectral radius $r(\varpi)$ of $\varpi \in \mathcal{A}$ satisfies:*

$$r(\varpi) = \lim_{n\to\infty} \|\varpi^n\|^{1/n}. \tag{1}$$

Furthermore, if $r(\varpi) < |\mu|$ for some $\varpi \in \mathcal{A}$, then $(\mu e - \varpi)$ is invertible,

$$(\mu e - \varpi)^{-1} = \sum_{i=0}^{\infty} \frac{\varpi^i}{\mu^{i+1}} \quad \text{and} \quad r[(\mu e - \varpi)^{-1}] \leq \frac{1}{|\mu| - r(\varpi)}.$$

Lemma 2.2. *[1] Let $\mathcal{A}$ be a Banach algebra and $\varpi_1, \varpi_2 \in \mathcal{A}$. If ϖ_1 and ϖ_2 commute, then*

$$r(\varpi_1 + \varpi_2) \leq r(\varpi_1) + r(\varpi_2) \qquad r(\varpi_1\varpi_2) \leq r(\varpi_1)r(\varpi_2).$$

Definition 2.3. *([19]) Let $\mathcal{A}$ be a Banach algebra with solid cone $\mathcal{K}$. A c-sequence is a sequence $\{\varpi_i\}$ in $\mathcal{K}$ such that for every $c \in \mathcal{A}$ with $c \gg \vartheta$, there exists $K \in \mathbb{N}$ such that $\varpi_i \ll c$ for all $i \geq K$.*

Lemma 2.4. *([7]) Let α, $\beta \in \mathcal{K}$ be any two arbitrary vectors and $\{u_n\}$, $\{q_n\}$ be two c-sequences in a solid cone $\mathcal{K}$ of a Banach algebra $\mathcal{A}$. Then $\{\alpha u_n + \beta q_n\}$ is a c-sequence.*

Lemma 2.5. *([20]) Let $\mathcal{K}$ be a cone in a Banach algebra $\mathcal{A}$ (not necessary a normal cone). Then the following assertions hold:*

(u_1) *If for each c with $c \gg \vartheta$ and $\vartheta \preceq \varpi \ll c$, implies that $\varpi = \vartheta$.*

(u_2) *If $\varpi \in \mathcal{K}$ is such that $r(\varpi) < 1$, then $\|\varpi^j\| \to 0$ as $j \to \infty$.*

(u_3) *Let $c \in int\mathcal{K}$ and $\varpi_j \to \vartheta$ in $\mathcal{A}$ as $j \to \infty$. Then there exists $M \in \mathbb{N}$ such that for all $j \geq M$, $\varpi_j \ll c$.*

(u_4) *If $\varpi \preceq \varpi k$, where $\varpi, k \in \mathcal{K}$ and $r(k) < 1$, then $\varpi = \vartheta$.*

Definition 2.6. *[7] For a nonempty set $\mathfrak{X}$ and a constant $b \geq 1$. A mapping $d_b : \mathfrak{X} \times \mathfrak{X} \to \mathcal{A}$ is called a cone b-metric over a Banach algebra $\mathcal{A}$ if the following axioms hold:*

B_1 : $\forall\ \eta, \xi \in \mathfrak{X}$, $d_b(\eta, \xi) \succeq \vartheta$ and $d_b(\eta, \xi) = \vartheta$ iff $\eta = \xi$;

B_2 : $\forall\ \eta, \xi \in \mathfrak{X}$, $d_b(\eta, \xi) = d_b(\xi, \eta)$;

B_3 : $\forall\ \eta, \xi, \zeta \in \mathfrak{X}$, $d_b(\eta, \zeta) \preceq b[d_b(\eta, \xi) + d_b(\xi, \zeta)]$.

The pair $(\mathfrak{X}, d_b)$ is called a cone b-metric space over a Banach algebra $\mathcal{A}$ (in short CbMS over $\mathcal{A}$).

Remark 2.7. *If $b = 1$, then we say that d_1 is a cone metric over a Banach algebra $\mathcal{A}$. So we can say that cone b-metric is the generalization of a cone metric.*

Example 2.8. *Consider the Banach algebra $\mathcal{A} = C([0,1])$ with unit element $e(t) = 1$ and supremum norm where multiplication is defined point wise. Let $\mathfrak{X} = \mathbb{R}$ and $\mathcal{K} = \{f \in \mathcal{A} : f(h) \geq 0 ; \forall h \in [0,1]\}$. Define $d_b : \mathfrak{X} \times \mathfrak{X} \to \mathcal{A}$ by*

$$d_b(\eta, \xi)(\varpi) = |\eta - \xi|^a e^{\varpi} \quad \forall \eta, \xi \in \mathfrak{X} \ \&\ a > 1.$$

Then d_b is a CbMS over $\mathcal{A}$ with $b = 2^{a-1}$ but it is not a cone metric on $\mathfrak{X}$.

Definition 2.9. *([7]) Let $\{\varpi_k\}$ be a sequence in $\mathfrak{X}$ where $(\mathfrak{X}, d_b)$ is a CbMS over $\mathcal{A}$. Then $\{\varpi_k\}$ is said to be:*

(i) *a convergent sequence which converges to $\varpi \in \mathfrak{X}$ if for every $c \in int\mathcal{K}$ (i.e.$\vartheta \ll c$), there exists a natural number N such that $d_b(\varpi_k, \varpi) \ll c$ for all $k \geq N$;*

(ii) *a Cauchy sequence if for every $c \in int\mathcal{K}$ (i.e.$\vartheta \ll c$), there exists a natural number N such that $d_b(\varpi_k, \varpi_i) \ll c$ for all k, $i \geq N$.*

If every Cauchy sequence in $\mathfrak{X}$ is convergent in $\mathfrak{X}$, then the space $(\mathfrak{X}, d_b)$ is called a complete *CbMS* over $\mathcal{A}$.

Remark 2.10. *[8, 21] 1. If $\{\varpi_n\}$ converges to ϖ in $\mathfrak{X}$, then $\{d_b(\varpi_k, \varpi)\}$ and $\{d_b(\varpi_k, \varpi_{k+i})\}$ are c-sequences for any $i \in \mathbb{N}$.*
2. If $\|\varpi_k\| \to 0$ as $k \to \infty$, then for any $c \gg \vartheta$, there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $\varpi_k \ll c$.

Definition 2.11. *[9] Let $\mathfrak{X}$ be a non empty set and $s : \mathfrak{X} \times \mathfrak{X} \to [1, \infty)$. A function $d_s : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ is called an extended b-metric if for all $\eta, \zeta, \xi \in \mathfrak{X}$ it satisfies:*

(i) $d_s(\eta, \zeta) = 0$ *iff* $\eta = \zeta$;

(ii) $d_s(\eta, \zeta) = d_s(\zeta, \eta)$;

(iii) $d_s(\eta, \xi) \leq s(\eta, \xi)[d_s(\eta, \zeta) + d_s(\zeta, \xi)]$.

An extended b-metric space is the pair $(\mathfrak{X}, d_s)$ with d_s an extended b-metric on $\mathfrak{X}$.

Remark 2.12. *If $\forall \eta, \xi \in \mathfrak{X}, s(\eta, \xi) = b$ for some $b \geq 1$, then the Definition 2.11 coincides with the definition of a b-metric space.*

Theorem 2.13. *[9] Let $(\mathfrak{X}, d_s)$ be a complete extended b-metric space with d_s continuous. Let F be a self-map on $\mathfrak{X}$ which satisfy*

$$d_s(F\eta, F\xi) \leq \kappa d_s(\eta, \xi) \quad \text{for all } \eta, \xi \in \mathfrak{X}, \tag{2}$$

where $\kappa \in [0,1)$ be such that for each $t_0 \in \mathfrak{X}$, $\lim_{j,i\to\infty} s(t_{j+1}, t_i) < \frac{1}{\kappa}$, here $t_j = F^j t_0$, $j = 1, 2, \cdots$. Then F has precisely one fixed point ϱ. Moreover for each $y \in \mathfrak{X}$, the iterative sequence $F^j y$ converges to ϱ.

Now we want to recall the definition of generalized α-admissible, α-regular and generalized Reich type mapping in the setting of cone b-metric spaces over Banach algebras.

Definition 2.14. *[21] Let $(\mathfrak{X}, d_b)$ be a CbMS over $\mathcal{A}$ with $\mathcal{K}$ an underlying solid cone. Let $\alpha : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ and F be a self-map on $\mathfrak{X}$. Then:*

(i) *F is said to be a generalized α-admissible mapping if for $\eta, \xi \in \mathfrak{X}$, $\alpha(\eta, \xi) \geq b$ implies that $\alpha(F\eta, F\xi) \geq b$;*

(ii) *$(\mathfrak{X}, d_b)$ is said to be α-regular if any sequence $\{\varpi_k\} \in \mathfrak{X}$ with $\alpha(\varpi_k, \varpi_{k+1}) \geq b$ for all $k \in \mathbb{N}$ and $\varpi_k \to \varpi$ implies that $\alpha(\varpi_k, \varpi) \geq b$.*

Definition 2.15. *[21] Let $\mathcal{A}$ be a Banach algebra with underlying solid cone $\mathcal{K}$, $(\mathfrak{X}, d_b)$ a CbMS over $\mathcal{A}$ with coefficient b, and $\alpha : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ be a mapping. Then the map F on $\mathfrak{X}$ is called a generalized Reich type contraction if there exists $\varpi_1, \varpi_2, \varpi_3 \in \mathcal{K}$ such that $\forall\ \eta, \xi \in \mathfrak{X}$ with $\alpha(\eta, \xi) \geq b$:*

(i) $2br(\varpi_1) + (b + 1)r(\varpi_2 + \varpi_3) < 2$;

(ii) $d(F\eta, F\xi) \preceq \varpi_1 d(\eta, \xi) + \varpi_2 d(\eta, F\eta) + \varpi_3 d(\xi, F\xi)$.

Remark 2.16. *We noted in [21] that Vujakovic et al. used the idea of α-admissible mapping $\alpha : \mathfrak{X} \times \mathfrak{X} \to \mathcal{A}$ by $\alpha(p, q) \geq b \implies \alpha(Fp, Fq) \geq b$ for $b \geq 1$. But this is possible only when we take the Banach algebra with identity element $1 \in \mathbb{R}$. Otherwise $\alpha(x, y) \geq b$ does not make sense, because $\alpha(x, y) \in \mathcal{A}$ and $b \in \mathbb{R}$.*

3. Main results

In the following, we introduce a new type of metric space over a real Banach algebra which we call an extended cone b-metric space over a Banach algebra. By using such spaces we prove some fixed point theorems for generalized Reich type contraction and generalized Lipschitz mapping.

Definition 3.1. *Let $\mathcal{A}$ be a real Banach algebra with cone $\mathcal{K}$, $\mathfrak{X}$ be a non empty set and $s : \mathfrak{X} \times \mathfrak{X} \to [1, \infty)$ be a mapping. An extended cone b-metric on $\mathfrak{X}$ over $\mathcal{A}$ is a function $d_s : \mathfrak{X} \times \mathfrak{X} \to \mathcal{A}$ such that:*

(E_1) *$d_s(\eta, \xi) \succeq \vartheta$ and $d_s(\eta, \xi) = \vartheta$ iff $\eta = \xi$ for all $\eta, \xi \in \mathfrak{X}$;*

(E_2) *$d_s(\eta, \xi) = d_s(\xi, \eta)$ for all $\eta, \xi \in \mathfrak{X}$;*

(E_3) *$d_s(\eta, \zeta) \preceq s(\eta, \zeta)[d_s(\eta, \xi) + d_s(\xi, \zeta)]$ for all $\eta, \xi, \zeta \in \mathfrak{X}$.*

The pair$(\mathfrak{X}, d_s)$ is then called an extended CbMS over $\mathcal{A}$.

Remark 3.2. *It is clear that the class of extended cone b-metric space over a Banach algebra $\mathcal{A}$ is larger than the classes of b-metric spaces and metric spaces over Banach algebras.*
The definition of Cauchy sequences, convergent sequences and completeness for extended cone b-metric space over a Banach algebra are similar to that of cone b-metric spaces over Banach algebras defined in the Definition 2.9.
In general d_s is not necessarily a continuous function but in this paper, d_s will always mean a continuous function $d_s : \mathfrak{X} \times \mathfrak{X} \to \mathcal{A}$.

Example 3.3. *Let $s : \mathfrak{X} \times \mathfrak{X} \to [1, \infty)$ be defined as $s(p, q) = 1 + p + q$ for $\mathfrak{X} = \{1, 2, 3\}$. Consider the real Banach algebra $\mathcal{A} = \mathbb{R}^2$ with solid cone $\mathcal{K} = \{(a, b) \in \mathbb{R}^2\ :\ a,\ b \geq 0\}$. If we define $d_s : \mathfrak{X} \times \mathfrak{X} \to \mathcal{A}$ by:*

$$d_s(1, 2) = d_s(2, 1) = (80, 80);$$
$$d_s(1, 3) = d_s(3, 1) = (1000, 1000);$$
$$d_s(3, 2) = d_s(2, 3) = (600, 600);$$
$$d_s(1, 1) = d_s(2, 2) = d_s(3, 3) = (0, 0) = \vartheta.$$

Clearly the first and second conditions of an extended CbMS over $\mathcal{A}$ are satisfied. For the third condition we have:

$$s(1,2)[d_s(1,3)+d_s(3,2)]-d_s(1,2)=4[(1000,1000)+(600,600)]-(80,80)=(6320,6320)\in\mathcal{K};$$

$$s(1,3)[d_s(1,2)+d_s2,3)]-d_s(1,3)=5[(80,80)+(600,600)]-(1000,1000)=(2400,2400)\in\mathcal{K};$$

$$s(2,3)[d_s(2,1)+d_s1,3)]-d_s(2,3)=6[(80,80)+(1000,1000)]-(600,600)=(5880,5880)\in\mathcal{K}.$$

Hence for all $\eta,\xi,\zeta\in\mathfrak{X}$,

$$d_s(\eta,\xi)\preceq s(\eta,\xi)[d_s(\eta,\zeta)+d_s(\zeta,\xi)].$$

Thus $(\mathfrak{X},d_s)$ is an extended CbMS over $\mathcal{A}=\mathbb{R}^2$.

Remark 3.4. *Let $(\mathfrak{X},d_s)$ be an extended CbMS over $\mathcal{A}$ with $s:\mathfrak{X}\times\mathfrak{X}\to[1,\infty)$. If $\mathcal{A}=\mathbb{R}$ and $\mathcal{K}=[0,\infty)$, then $(\mathfrak{X},d_s)$ is an extended b-metric space.*

We now define generalized α-admissible mapping and α-regular space in term of extended cone b-metric spaces over Banach algebras.

Definition 3.5. *Consider $(\mathfrak{X},d_s)$ an extended CbMS over $\mathcal{A}$ with $\mathcal{K}$ an underlying solid cone in $\mathcal{A}$. Let F be a self-map on $\mathfrak{X}$ and $\alpha:\mathfrak{X}\times\mathfrak{X}\to[0,\infty)$. Then:*

(*i*) *F is said to be a generalized α-admissible mapping if for $\eta,\xi\in\mathfrak{X},\alpha(\eta,\xi)\geq s(\eta,\xi)$ implies that $\alpha(F\eta,F\xi)\geq s(F\eta,F\xi)$;*

(*ii*) *$(\mathfrak{X},d_s)$ is said to be α-regular if any sequence $\{\varpi_k\}\in\mathfrak{X}$ with $\alpha(\varpi_k,\varpi_{k+1})\geq s(\varpi_k,\varpi_{k+1})$ for all $k\in\mathbb{N}$ and $\varpi_k\to\varpi$ implies that $\alpha(\varpi_k,\varpi)\geq s(\varpi_k,\varpi)$.*

We are now in a position to define a generalized Reich type contraction by using the extended cone b-metric spaces over Banach algebras.

Definition 3.6. *Let $(\mathfrak{X},d_s)$ be an extended CbMS over $\mathcal{A}$ with $\mathcal{K}$ an underlying solid cone and $\alpha:\mathfrak{X}\times\mathfrak{X}\to[0,\infty)$ be a mapping. Then a self-map F on $\mathfrak{X}$ is called a generalized Reich type contraction if there exists three vectors $\varpi_1,\varpi_2,\varpi_3$ in $\mathcal{K}$ such that for all $\eta,\xi\in\mathfrak{X}$ with $\alpha(\eta,\xi)\geq s(\eta,\xi)$:*

(*i*) *$2s(\eta,\xi)r(\varpi_1)+(s(\eta,\xi)+1)r(\varpi_2+\varpi_3)<2$ and for each $u_0\in\mathfrak{X}$ with $u_j=F^ju_0$,*

$$\lim_{k,i\to\infty}s(u_{j+1},u_i)<\frac{1}{\|\kappa\|}\ \textit{where}\ \kappa=(2e-\varpi)^{-1}(2\varpi_1+\varpi)\ \textit{for}\ \varpi=\varpi_2+\varpi_3;$$

(*ii*) *$d_s(F\eta,F\xi)\preceq\varpi_1d_s(\eta,\xi)+\varpi_2d_s(\eta,F\eta)+\varpi_3d_s(\xi,F\xi)$.*

The main result of our paper is given as follows:

Theorem 3.7. *Let $(\mathfrak{X},d_s)$ be a complete extended CbMS over $\mathcal{A}$ with $\alpha:\mathfrak{X}\times\mathfrak{X}\to[0,\infty)$ be a mapping and $\mathcal{K}$ an underlying solid cone. Suppose that the self-map F on $\mathfrak{X}$ is a generalized Reich type contraction with vectors $v_1,v_2,v_3\in\mathcal{K}$ such that:*

1. *F is a generalized α-admissible;*
2. *there exists an element $u_0\in\mathfrak{X}$ such that $\alpha(u_0,Fu_0)\geq s(u_0,Fu_0)$;*
3. *$(\mathfrak{X},d_s)$ is regular or F is continuous.*

Then there exists a point ϱ in $\mathfrak{X}$ which is fixed under the map F.

Proof. Let u_0 be a point in $\mathfrak{X}$ such that $\alpha(u_0, Fu_0) \geq s(u_0, Fu_0)$. For $u_0 \in \mathfrak{X}$, if we define $u_1 = Fu_0,\ u_2 = Fu_1 = F(Fu_0) = T^2u_0, \cdots,\ u_{n+1} = Fu_n = F^{n+1}u_0$, then

$$\alpha(u_0, u_1) \geq s(u_0, u_1).$$

But F is generalized α-admissible, so

$$\alpha(Fu_0, Fu_1) = \alpha(u_1, u_2) \geq s(u_1, u_2),$$

and so by induction we get

$$\alpha(u_n, u_{n+1}) \geq s(u_n, u_{n+1}).$$

By using Definition 3.6, we have

$$\begin{aligned} d_s(u_n, u_{n+1}) &= d_s(Fu_{n-1}, Fu_n) \\ &\preceq v_1 d_s(u_{n-1}, u_n) + v_2 d_s(u_{n-1}, Fu_{n-1}) + v_3 d_s(u_n, Fu_n), \text{ i.e.} \end{aligned}$$

$$(e - v_3) d_s(u_n, u_{n+1}) \preceq (v_1 + v_2) d_s(u_{n-1}, u_n). \tag{3}$$

Similarly

$$\begin{aligned} d_s(u_{n+1}, u_n) &= d_s(Fu_n, Fu_{n-1}) \\ &\preceq v_1 d_s(u_n, u_{n-1}) + v_2 d_s(u_n, Fu_n) + v_3 d_s(u_{n-1}, Fu_{n-1}), \text{ i.e.} \end{aligned}$$

$$(e - v_2) d_s(u_{n+1}, u_n) \preceq (v_1 + v_3) d_s(u_{n-1}, u_n). \tag{4}$$

Adding (3) and (4), we obtain

$$(2e - v_2 - v_3) d_s(u_n, u_{n+1}) \preceq (2v_1 + v_2 + v_3) d_s(u_{n-1}, u_n).$$

If we take $v = v_2 + v_3$, then we obtain

$$(2e - v) d_s(u_{n+1}, u_n) \preceq (2v_1 + v) d_s(u_{n-1}, u_n). \tag{5}$$

Note that

$$2r(v) \leq (s(u_n, u_{n+1}) + 1) r(v) \leq 2r(v_1) + (s(u_n, u_{n+1}) + 1) r(v) < 2.$$

Hence $r(v) < 1 < 2 \implies r(v) < 2$. Thus by using Lemma 2.1, we presume that $2e - v$ is invertible and $(2e - v)^{-1} = \sum_{n=0}^{\infty} \frac{v^n}{2^{n+1}}$, $r((2e - v)^{-1}) < \frac{1}{2 - r(v)}$.
Hence (5) becomes

$$d_s(u_n, u_{n+1}) \preceq \kappa d_s(u_{n-1}, u_n), \tag{6}$$

where $\kappa = (2e - v)^{-1}(2v_1 + v)$. The inequality (6) then implies that for all $n \in \mathbb{N}$

$$\begin{aligned} d_s(u_n, u_{n+1}) &\preceq \kappa d_s(u_{n-1}, u_n) \\ &\preceq \kappa^2 d_s(u_{n-1}, u_n) \\ &\ \vdots \\ &\preceq \kappa^n d_s(u_0, u_1). \end{aligned} \tag{7}$$

Now if we take $m > n$, then by using (7) and Definition 3.1, (*iii*) we have

$$\begin{aligned} d_s(u_n, u_m) &\preceq s(u_n, u_{n+1})d_s(u_n, u_{n+1}) + s(u_n, u_{n+1})s(u_{n+1}, u_{n+2})d_s(u_{n+1}, u_{n+2}) + \cdots + \\ &s(u_n, u_{n+1})s(u_{n+1}, u_{n+2})\ldots s(u_{m-1}, u_m)(d_s(u_{m-1}, u_m)) \\ &\preceq s(u_n, u_m)\kappa^n d_s(u_0, u_1) + s(u_n, u_m)s(u_{n+1}, u_m)\kappa^{n+1}d_s(u_0, u_1) + \cdots + \\ &s(u_n, u_m)s(u_{n+1}, u_m)s(u_{n+2}, u_m)...s(u_{m-2}, u_m)s(u_{m-1}, u_m)\kappa^{m-1}d_s(u_0, u_1) \\ &\preceq d_s(u_0, u_1)\Big[s(u_1, u_m)s(u_2, u_m)\ldots s(u_{n-1}, u_m)s(u_n, u_m)\kappa^n + \\ &s(u_1, u_m)s(u_2, u_m)\ldots s(u_n, u_m)s(u_{n+1}, u_m)\kappa^{n+1} + \cdots + \\ &\{s(u_1, u_m)s(u_2, u_m)\ldots s(u_n, u_m)s(u_{n+1}, u_m)\ldots s(u_{m-2}, u_m)s(u_{m-1}, u_m)\}\kappa^{m-1}\Big] \\ &= d_s(u_0, u_1)\Big[\kappa^n \prod_{j=1}^{n} s(u_j, u_m) + \kappa^{n+1}\prod_{j=1}^{n+1} s(u_j, u_m) + \cdots + \kappa^{m-1}\prod_{j=1}^{m-1} s(u_j, u_m)\Big]. \end{aligned}$$

Let $a_n = \kappa^n \prod_{j=1}^{n} s(u_j, u_m)$ and $S = \sum_{n=1}^{\infty} a_n$.
Since by Definition 3.6, $\|\kappa\| \lim_{n,m\to\infty} s(u_{n+1}, u_m) < 1$, so the series S converges absolutely. Because by using ratio test we have

$$\lim_{n\to\infty} \frac{\|a_{n+1}\|}{\|a_n\|} \leq \lim_{n\to\infty} \frac{\|\kappa\|\|\kappa^n\|s(u_{n+1}, u_m)}{\|\kappa^n\|} = \|\kappa\| \lim_{n,m\to\infty} s(u_{n+1}, u_m) < 1.$$

But $\mathcal{A}$ is a Banach algebra and the series S is absolutely convergent, so it converges in $\mathcal{A}$. Thus $S_{m-1} - S_n = \Big[\kappa^n \prod_{j=1}^{n} s(u_j, u_m) + \cdots + \kappa^{m-1}\prod_{j=1}^{m-1} s(u_j, u_m)\Big] \to \vartheta$ as $n, m \to \infty$ and so is $d_s(u_0, u_1)(S_{m-1} - S_n)$. By Lemma 2.5, for every $c \gg \vartheta$, there exists a natural number n_0 such that for all $n \geq n_0$, $d_s(u_n, u_m) \ll c$. Thus by Definition 2.9 $\{u_n\}$ is a Cauchy sequence in $\mathfrak{X}$. But $\mathfrak{X}$ is complete so there exists $\varrho \in \mathfrak{X}$ such that $u_n \to \varrho$ as $n \to \infty$. We show that ϱ is fixed under the map F.
Suppose that F is continuous. It follows that $u_{n+1} = Fu_n \to F\varrho$ as $n \to \infty$. But limit of a sequence is unique, so we must have $F\varrho = \varrho$. Hence ϱ is fixed under the map F in this case.
However, if $(\mathfrak{X}, d_s)$ is α-regular, then by Definition 3.5 we have

$$\alpha(u_n, \varrho) \geq s(u_n, \varrho), \qquad \text{for all } n \in \mathbb{N}.$$

$$\begin{aligned} d_s(\varrho, F\varrho) &\preceq s(\varrho, F\varrho)\Big[d_s(\varrho, Fu_n) + d_s(Fu_n, F\varrho)\Big] \\ &\preceq s(\varrho, F\varrho)d_s(\varrho, Fu_n) + s(\varrho, F\varrho)\Big[v_1 d_s(u_n, \varrho) + v_2 d_s(u_n, Fu_n) + v_3 d_s(\varrho, F\varrho)\Big] \\ &\preceq s(\varrho, F\varrho)d_s(\varrho, Fu_n) + s(\varrho, F\varrho)v_1 d_s(u_n, \varrho) + s(\varrho, F\varrho)v_3 d_s(\varrho, F\varrho) \\ &\quad + s(\varrho, F\varrho)s(u_n, u_{n+1})v_2\Big[d_s(u_n, \varrho) + d_s(\varrho, u_{n+1})\Big] \\ &= s(\varrho, F\varrho)(e + s(u_n, u_{n+1})v_2)d_s(\varrho, u_{n+1}) + s(\varrho, F\varrho)v_3 d_s(\varrho, F\varrho) \\ &\quad + s(\varrho, F\varrho)(v_1 + s(u_n, u_{n+1})v_2)d_s(u_n, \varrho), \end{aligned}$$

which further implies that

$$(e - s(\varrho, F\varrho)v_3)d_s(\varrho, F\varrho) \preceq s(\varrho, F\varrho)(e + s(u_n, u_{n+1})v_2)d_s(u_{n+1}, \varrho) + s(\varrho, F\varrho)(v_1 + s(u_n, u_{n+1})v_2)d_s(u_n, \varrho) \tag{8}$$

Similarly,

$$\begin{aligned}
d_s(\varrho, F\varrho) &\preceq s(\varrho, F\varrho)\Big[d_s(\varrho, Fu_n) + d_s(Fu_n, F\varrho)\Big]\\
&= s(\varrho, F\varrho)d_s(\varrho, Fu_n) + s(\varrho, F\varrho)d_s(F\varrho, Fu_n)\\
&\preceq s(\varrho, F\varrho)d_s(\varrho, Fu_n) + s(\varrho, F\varrho)\Big[v_1 d_s(\varrho, u_n) + v_2 d_s(\varrho, F\varrho) + v_3 d_s(u_n, Fu_n)\Big]\\
&\preceq s(\varrho, F\varrho)d_s(\varrho, Fu_n) + s(\varrho, F\varrho)v_1 d_s(\varrho, u_n) + s(\varrho, F\varrho)v_2 d_s(\varrho, F\varrho)\\
&\quad + s(\varrho, F\varrho)s(u_n, u_{n+1})v_3\Big[d_s(u_n, \varrho) + d_s(\varrho, u_{n+1})\Big]\\
&= s(\varrho, F\varrho)(e + s(u_n, u_{n+1})v_3)d_s(\varrho, u_{n+1}) + s(\varrho, F\varrho)v_2 d_s(\varrho, F\varrho)\\
&\quad + s(\varrho, F\varrho)(v_1 + s(u_n, u_{n+1})v_3)d_s(u_n, \varrho),
\end{aligned}$$

which further implies that

$$(e - s(\varrho, F\varrho)v_2)d_s(\varrho, F\varrho) \preceq s(\varrho, F\varrho)(e + s(u_n, u_{n+1})v_3)d_s(u_{n+1}, \varrho) + s(\varrho, F\varrho)(v_1 + s(u_n, u_{n+1})v_3)d_s(u_n, \varrho) \tag{9}$$

Therefore, by combining (8) and (9), we get

$$\begin{aligned}
(2e - s(\varrho, F\varrho)v_2 - s(\varrho, F\varrho)v_3)d_s(\varrho, F\varrho) &\preceq s(\varrho, F\varrho)(2e + s(\varrho, F\varrho)v_2 + s(\varrho, F\varrho)v_3)d_s(u_{n+1}, \varrho)\\
&\quad + s(\varrho, F\varrho)(2v_1 + s(\varrho, F\varrho)v_2 + s(\varrho, F\varrho)v_3)d_s(u_n, \varrho), \text{ i.e.}
\end{aligned}$$

$$\begin{aligned}
(2e - s(\varrho, F\varrho)v)d_s(\varrho, F\varrho) &\preceq s(\varrho, F\varrho)(2e + s(\varrho, F\varrho)v)d_s(u_{n+1}, \varrho)\\
&\quad + s(\varrho, F\varrho)(2v_1 + s(\varrho, F\varrho)v)d_s(u_n, \varrho)
\end{aligned} \tag{10}$$

We also note that

$$r(s(\varrho, F\varrho)v) = s(\varrho, F\varrho)r(v) \leq 2s(\varrho, F\varrho)r(v_1) + (s(\varrho, F\varrho) + 1)r(v) < 2$$

Thus by Lemma 2.1, $2e - s(\varrho, F\varrho)v$ is invertible and so (10) implies that

$$\begin{aligned}
d_s(\varrho, F\varrho) &\preceq (2e - s(\varrho, F\varrho)v)^{-1}\Big[s(\varrho, F\varrho)(2e + s(\varrho, F\varrho)v)d_s(u_{n+1}, \varrho)\\
&\quad + s(\varrho, F\varrho)(2v_1 + s(\varrho, F\varrho)v)d_s(u_n, \varrho)\Big]
\end{aligned} \tag{11}$$

By using Remark 2.10 the sequences $\{d_s(u_{n+1}, \varrho)\}$ and $\{d_s(u_n, \varrho)\}$ are c-sequences. Hence by Lemma 2.4, the sequence $\{\tau_1 d_s(u_{n+1}, \varrho) + \tau_2 d_s(u_n, \varrho)\}$ is a c-sequence (where $\tau_1 = (2e - s(\varrho, F\varrho)v)^{-1}s(\varrho, F\varrho)(2e + s(\varrho, F\varrho)v)$ and $\tau_2 = (2e - s(\varrho, F\varrho)v)^{-1}s(\varrho, F\varrho)(2v_1 + s(\varrho, F\varrho)v)$). Therefore, for any $c \in int(\mathcal{K})$, there exists $n_0 \in \mathbb{N}$ such that

$$d_s(\varrho, F\varrho) \preceq \tau_1 d_s(u_{n+1}, \varrho) + \tau_2 d_s(u_n, \varrho) \ll c.$$

Which further implies by using Lemma 2.5 that $d_s(\varrho, F\varrho) = \vartheta$. Therefore, $F\varrho = \varrho$ and this complete the proof. □

Example 3.8. *Let $\mathcal{A} = C^1_{\mathbb{R}}[0,1]$ and $\|f\| = \|f\|_\infty + \|f'\|_\infty$. If we define point wise multiplication of functions on $\mathcal{A}$, then $\mathcal{A}$ becomes a real Banach algebra with identity $e(t) = 1$. If we take $\mathcal{K} = \{f \in \mathcal{A} \ : \ f(t) \geq 0,\ t \in [0,1]\}$, then $\mathcal{K}$ is a non-normal cone (see [3]). Let $\mathfrak{X} = [0,\infty)$ and $s : \mathfrak{X} \times \mathfrak{X} \to [1,\infty)$ be defined as*

$$s(x, y) = \begin{cases} p + q + 2 & \text{if } p, q \in [0,1];\\ 2 & \text{elsewhere.} \end{cases}$$

Define $d_s : \mathfrak{X} \times \mathfrak{X} \to \mathcal{A}$ by

$$d_s(p, q)(t) = (p - q)^2 e^t.$$

Then d_s is an extended CbMS over $\mathcal{A}$. Also note that $\mathfrak{X}$ is complete with respect to d_s. Define two maps $\alpha : \mathfrak{X} \times \mathfrak{X} \to [0,\infty)$ and $F : \mathfrak{X} \to \mathfrak{X}$ by:

$$\alpha(x,y) = \begin{cases} s(p,q) & \textit{if } p,q \in [0,1]; \\ 0 & \textit{elsewhere.} \end{cases}$$

$$F(p) = \begin{cases} \frac{\sqrt{5}}{3}p & \textit{if } p \in [0,1]; \\ p-1 & \textit{if } p > 1 \end{cases}$$

Note that for every $p \in [0,1]$, $Fp \in [0,1]$. By choosing $v_1(t) = \frac{1}{36} + \frac{1}{36}t$, $v_2(t) = \frac{1}{18} + \frac{1}{18}t$ and $v_3(t) = \frac{1}{24} + \frac{1}{24}t$ we obtain that $r(v_1) = \frac{2}{9}$, $r(v) = r(v_2 + v_3) = \frac{7}{36}$. Simple calculations show that $2(2+2)r(v_1) + (2+2+1)r(v) = \frac{51}{36} < 2$ and so F is a generalized Reich type contraction as;

$$2s(x,y)r(v_1) + (s(x,y)+1)r(v) \leq 2(2+2)r(v_1) + (2+2+1)r(v) = \frac{51}{36} < 2.$$

Also for each $u_0 \in \mathfrak{X}$, the limit $\lim_{n,m\to\infty} s(u_{n+1}, u_m) = 2$ and $\|\kappa\| = \|(2e - v)^{-1}(2v_1 + v)\| \leq \left(\frac{72}{130}\right)\left(\frac{46}{72}\right) = \frac{23}{65} < \frac{1}{2} = \frac{1}{\lim_{n,m\to\infty} s(u_{n+1},u_m)}$. Similarly by easily calculation one can show that

$$d_s(Fp,Fq) \leq v_1 d_s(p,q) + v_2 d_s(p,Fp) + v_3 d_s(q,Fq).$$

Next we show that there is a point u_0 in $\mathfrak{X}$ such that $\alpha(u_0,Fu_0) \geq s(u_0,Fu_0)$. Indeed, for $u_0 = 1$, we have

$$\alpha(1,F1) = \alpha(1, \frac{\sqrt{5}}{3}) \geq s(1, \frac{\sqrt{5}}{3}) = s(1,F1).$$

Next we show that F is a generalized α-admissible mapping. In fact, if $p,q \in \mathfrak{X}$ are such that $\alpha(p,q) \geq s(p,q)$, then by definition of α, the points p,q is in $[0,1]$. Therefore, $Fp,Fq \in [0,1]$ and so

$$\alpha(Fp,Fq) \geq s(Fp,Fq).$$

Finally we show that $(\mathfrak{X}, d_s)$ is α-regular. If we assume a sequence $\{p_n\}$ in $\mathfrak{X}$ such that $\alpha(p_n, p_{n+1}) \geq s(p_n, p_{n+1})$ for all $n \in \mathbb{N}$ and $p_n \to q \in \mathfrak{X}$ (as $n \to \infty$), then $\{p_n\} \subseteq [0,1]$. But $[0,1]$ is closed, so $q \in [0,1]$. This implies that $\alpha(p_n, q) \geq s(p_n, q)$ for all $n \in \mathbb{N}$. Hence all the conditions of Theorem 3.7 are satisfied, so there is a point $\varrho = 0$ (say) which is fixed under the map F.

Theorem 3.9. *Let $\mathcal{K}$ be a solid cone in a Banach algebra $\mathcal{A}$. Let $(\mathfrak{X}, d_s)$ be a complete extended CbMS over $\mathcal{A}$ with $\alpha : \mathfrak{X} \times \mathfrak{X} \to [0,\infty)$ a mapping. Suppose that the self-map F on $\mathfrak{X}$ is a generalized Reich type contraction with vectors v_1, v_2, v_3 in $\mathcal{K}$ such that v_1 commutes with $v_2 + v_3$ and:*

1. *F is a generalized α-admissible;*
2. *there exists $u_0 \in \mathfrak{X}$ such that $\alpha(u_0,Fu_0) \geq s(u_0,Fu_0)$;*
3. *F is continuous or $(\mathfrak{X}, d_s)$ is regular;*
4. *for any two fixed points ϖ, ζ of F, there exists z in $\mathfrak{X}$ such that $\alpha(\varpi, z) \geq s(\varpi, z)$ and $\alpha(\zeta, z) \geq s(\zeta, z)$.*

Then there exists a unique point ϱ in $\mathfrak{X}$ which is fixed under the map F.

Proof. Using Theorem 3.7 and the first three given condition we can say that there exists a point $\varrho \in \mathfrak{X}$ which is fixed under the map F. We show that this point is unique and for this let $\zeta \in Fix(F)$ such that $\varrho \neq \zeta$. Then by using condition 4, there exists $z \in \mathfrak{X}$ with

$$\alpha(\varrho, z) \geq s(\varrho, z) \quad \text{and} \quad \alpha(\zeta, z) \geq s(\zeta, z) \tag{12}$$

Since F is a generalized α-admissible mapping and $\varrho, \zeta \in Fix(F)$ so by (12) we get

$$\alpha(\varrho, F^i z) \geq s(\varrho, F^i z) \quad \text{and} \quad \alpha(\zeta, F^i z) \geq s(\zeta, F^i z)\ , \quad \text{for all } i \in \mathbb{N}. \tag{13}$$

By using Definition 3.6 and (13) we obtain

$$\begin{aligned} d_s(\varrho, F^i z) &= d_s(F\varrho, F(F^{i-1}z)) \\ &\leq v_1 d_s(\varrho, F^{i-1}z) + v_2 d_s(\varrho, F\varrho) + v_3 d_s(F^{i-1}z, F^i z) \\ &\leq v_1 d_s(\varrho, F^{i-1}z) + v_3 s(F^{i-1}z, F^i z)[d_s(F^{i-1}z, \varrho) + d_s(\varrho, F^i z)] \end{aligned}$$

which further implies that

$$(e - s(F^{i-1}z, F^i z)v_3) d_s(\varrho, F^i z) \leq (v_1 + s(F^{i-1}z, F^i z))v_3) d_s(\varrho, F^{i-1}z). \tag{14}$$

Similarly,

$$\begin{aligned} d_s(F^i z, \varrho) &= d_s(F(F^{i-1}z), F\varrho) \\ &\leq v_1 d_s(F^{i-1}z, \varrho) + v_2 d_s(F^{i-1}z), F^i z)) + v_3 d_s(\varrho, F\varrho) \\ &\leq v_1 d_s(F^{i-1}z, \varrho) + v_2 s(F^{i-1}z, F^i z)[d_s(F^{i-1}z, \varrho) + d_s(\varrho, F^i z)] \end{aligned}$$

which further implies that

$$(e - s(F^{i-1}z, F^i z)v_2) d_s(F^i z, \varrho) \leq (v_1 + s(F^{i-1}z, F^i z))v_2) d_s(F^{i-1}z, \varrho). \tag{15}$$

Adding (14) and (15) we have

$$(2e - s(F^{i-1}z, F^i z)v_2 - s(F^{i-1}z, F^i z)v_3) d_s(\varrho, F^i z) \leq (2v_1 + s(F^{i-1}z, F^i z)v_2 + s(F^{i-1}z, F^i z)v_3) d_s(\varrho, F^{i-1}z)$$

$$(2e - s(F^{i-1}z, F^i z)v) d_s(\varrho, F^i z) \leq (2v_1 + s(F^{i-1}z, F^i z)v) d_s(\varrho, F^{i-1}z).$$

Note that $2r(s(F^{i-1}z, F^i z)v) \leq (s(u_n, u_{n+1}) + 1) r(s(F^{i-1}z, F^i z)v) \leq 2r(v_1) + (s(u_n, u_{n+1}) + 1) r(s(F^{i-1}z, F^i z)v) < 2$. Which implies that $r(s(F^{i-1}z, F^i z)v) < 1 < 2$. Thus by Lemma 2.1, we can say that $2e - s(F^{i-1}z, F^i z)v$ is invertible and $(2e - s(F^{i-1}z, F^i z)v)^{-1} = \sum_{n=0}^{\infty} \frac{(s(F^{i-1}z, F^i z)v)^n}{2^{n+1}}$,
$r((2e - s(F^{i-1}z, F^i z)v)^{-1}) < \frac{1}{2 - r(s(F^{i-1}z, F^i z)v)}$. Thus we have

$$d_s(\varrho, F^i z) \leq (2e - s(F^{i-1}z, F^i z)v)^{-1} (2v_1 + s(F^{i-1}z, F^i z)v) d_s(\varrho, F^{i-1}z)\ , \ \textit{i.e.}$$

$$d_s(\varrho, F^i z) \leq \tau d_s(\varrho, F^{i-1}z) \tag{16}$$

where $\tau = (2e - s(F^{i-1}z, F^i z)v)^{-1}(2v_1 + s(F^{i-1}z, F^i z)v)$. Therefore, we have

$$\begin{aligned} d_s(\varrho, F^i z) &\leq \tau d_s(\varrho, F^{i-1}z) \\ &\leq \tau^2 d_s(\varrho, F^{i-2}z) \\ &\ \vdots \\ &\leq \tau^i d_s(\varrho, z) \ \text{ for all } i \in \mathbb{N}. \end{aligned}$$

Since v_1 commutes with $v_2 + v_3 = v$, so

$$(2e - s(F^{i-1}z, F^i z)v)^{-1}(2v_1 + s(F^{i-1}z, F^i z)v) = \Big(\sum_{n=0}^{\infty} \frac{(s(F^{i-1}z, F^i z)v)^n}{2^{n+1}}\Big)(2v_1 + s(F^{i-1}z, F^i z)v)$$
$$= 2v_1\Big(\sum_{n=0}^{\infty} \frac{(s(F^{i-1}z, F^i z)v)^n}{2^{n+1}}\Big) + s(F^{i-1}z, F^i z)v\Big(\sum_{n=0}^{\infty} \frac{(s(F^{i-1}z, F^i z)v)^n}{2^{n+1}}\Big)$$
$$= (2v_1 + s(F^{i-1}z, F^i z)v)(2e - s(F^{i-1}z, F^i z)v)^{-1}.$$

Which shows that $(2e - s(F^{i-1}z, F^i z)v)^{-1}$ commutes with $(2v_1 + s(F^{i-1}z, F^i z)v)$. Hence by applying Lemma 2.1 and Lemma 2.2 we obtain that;

$$\begin{aligned} r(\tau) &= r((2e - s(F^{i-1}z, F^i z)v)^{-1}(2v_1 + s(F^{i-1}z, F^i z)v)) \\ &\leq r((2e - s(F^{i-1}z, F^i z)v)^{-1}) \cdot r((2v_1 + s(F^{i-1}z, F^i z)v)) \\ &\leq \frac{1}{2 - r(s(F^{i-1}z, F^i z)v)}(2r(v_1) + r(s(F^{i-1}z, F^i z)v)) \\ &< \frac{1}{s(u_n, u_{n+1})} < 1 \end{aligned}$$

By Lemma 2.5 it follows that $\|\tau^i\| \to 0$ as $i \to \infty$ and so

$$\|\tau^i d_s(\varrho, z)\| \leq \|\tau^i\|\|d_s(\varrho, z)\| \to 0 \quad (i \to \infty).$$

By Remark 2.10 we conclude that for any $c \in \mathcal{A}$ with $c \gg \vartheta$, there exists a natural number M such that

$$d_s(\varrho, F^i z) \preceq \tau^i d_s(\varrho, z) \preceq c \quad \forall\, i \geq M.$$

Thus by Lemma 2.5 $F^i z \to \varrho$ as $i \to \infty$. Similarly we obtain that $F^i z \to \zeta$ as $i \to \infty$. Now by uniqueness of limit, we conclude that $\varrho = \zeta$.

□

Theorem 3.10. *Let $(\mathfrak{X}, d_s)$ be a complete extended CbMS over $\mathcal{A}$ with $\mathcal{K}$ an associated cone in $\mathcal{A}$. Let F be a self-map on $\mathfrak{X}$ such that for all $p, q \in \mathfrak{X}$;*

$$d_s(Fp, Fq) \preceq \kappa d_s(p, q) \tag{17}$$

where $\kappa \in \mathcal{K}$ be such that $r(\kappa) < 1$ and for each $u_0 \in \mathfrak{X}$, $\lim_{n,m\to\infty} s(u_{n+1}, u_m) < \frac{1}{\|\kappa\|}$. Then there exists a unique point $\varrho \in \mathfrak{X}$ which is fixed under the map F. Furthermore for each $u_0 \in \mathfrak{X}$, the iterative sequence $u_n = F(u_{n-1}) = F^n u_0$ converges to ϱ.

Proof. If we take $v_1 = \kappa$, $v_2 = v_3 = \vartheta$ and $\alpha(p, q) = s(p, q)$, then all the conditions of Theorem 3.7 are satisfied, i.e. F satisfies the condition of Definition 3.6, F is generalized α-admissible, $(\mathfrak{X}, d_s)$ is regular and for every $u_0 \in \mathfrak{X}$ $\alpha(u_0, Fu_0) \geq s(u_0, Fu_0)$. Hence there exists ϱ in $\mathfrak{X}$ which is fixed under the map F. Now it remains only to show that this fixed point is unique. For this, let there is ζ in $\mathfrak{X}$ such that $F\zeta = \zeta$. Then we have;

$$d_s(\varrho, \zeta) = d_s(F\varrho, F\zeta) \preceq \kappa d_s(\varrho, \zeta)$$

But $r(\kappa) < 1$, so by Lemma 2.1, $e - \kappa$ is invertible. Thus by Lemma 2.5 $d_s(\varrho, \zeta) = \vartheta$.

□

Theorem 3.11. *Let $(\mathfrak{X}, d_s)$ be a complete extended CbMS over $\mathcal{A}$ and $\mathcal{K}$ be the associated cone in $\mathcal{A}$. Let F be a self-map on $\mathfrak{X}$ satisfies the generalized Lipschitz condition, i.e. for all $p, q \in \mathfrak{X}$;*

$$d_s(Fp, Fq) \preceq \kappa[d_s(Fp, p) + d_s(Fq, q)] \tag{18}$$

where $\kappa \in \mathcal{K}$ be such that $r(\kappa) < \frac{1}{s(p, q) + 1}$ and for each $u_0 \in \mathfrak{X}$, $\lim_{n,m\to\infty} s(u_{n+1}, u_m) < \frac{1}{\|\tau\|}$ with $\tau = (e - \kappa)^{-1}\kappa$. Then there exists a unique point $\varrho \in \mathfrak{X}$ which is fixed under the map F.

Proof. If we take $v_1 = \vartheta$, $v_2 = v_3 = \kappa$ and $\alpha(p,q) = s(p,q)$, then all the condition of Theorem 3.7 are satisfied. Hence there exists ϱ in $\mathfrak{X}$ which is fixed under the map F. Finally we show that ϱ is a unique point which is fixed under the map F. For this if ζ is another fixed point of F, then

$$d_s(\varrho,\zeta) = d_s(F\varrho,F\zeta) \preceq \kappa[d_s(\varrho,F\varrho) + d_s(\zeta,F\zeta) = \vartheta.$$

Therefore, $\varrho = \zeta$. □

Following is the result of generalized Lipschitz mappings on *CbMS* over Banach algebras [7] which can be directly proved by using our results, Theorem 3.10 and Theorem 3.11 when we define $s(\eta,\xi) = b$ for some $b \geq 1$.

Theorem 3.12. *[7] Let $(\mathfrak{X},d)$ be a complete CbMS over $\mathcal{A}$ with coefficient $b \geq 1$ and $\mathcal{K}$ be the associated solid cone (not necessary normal) in $\mathcal{A}$. Suppose that F is a self-map on $\mathfrak{X}$ such that for all p, $q \in \mathfrak{X}$ one of the following conditions hold:*

(i) $d(Fp,Fq) \preceq \kappa d(p,q)$ where $\kappa \in \mathcal{K}$ be such that $r(\kappa) < \dfrac{1}{b}$.

(ii) $d(Fp,Fq) \preceq \kappa(d(Fp,p) + d(Fq,q))$ where $\kappa \in \mathcal{K}$ be such that $r(\kappa) < \dfrac{1}{1+b}$.

Then there exists a unique point $\varrho \in \mathfrak{X}$ which is fixed under the mapF.

Corollary 3.13. *Let $\mathcal{K}$ be the associated cone in a Banach algebra $\mathcal{A}$ and $(\mathfrak{X},d_s)$ be a complete cone metric space over $\mathcal{A}$. Let F be a self-map on $\mathfrak{X}$ such that for all $p,q \in \mathfrak{X}$;*

$$d_s(Fp,Fq) \preceq \kappa d_s(p,q) \tag{19}$$

where $\kappa \in \mathcal{K}$ be such that $r(\kappa) < 1$. Then for every $u_0 \in \mathfrak{X}$, the iterative sequence $u_n = F(u_{n-1}) = F^n u_0$ converges to a unique fixed point of F.

Proof. Take $b = 1$ in Theorem 3.12, we get the required result. □

Remark 3.14. *1. If we take $s(x,y) = b$ for some $b \geq 1$ in Theorem 3.10 and in Theorem 3.11, we get the main results of [7] for cone b-metric spaces over Banach algebras.*
2. By using Remark 3.4, we obtain Theorem 2.13 as a corollary of our Theorem 3.10.
3. If we take $s(x,y) = b$ for some $b \geq 1$ in Theorem 3.7 and in Theorem 3.9, we get the main results of [21] for cone b-metric spaces over Banach algebra.

4. Consequences

In this section, we have listed some important consequences of our results which generalizes the results of Hussain *et* al. [18], Xu and Radenovic [20], Malhotra *et* al. [16], Malhotra *et* al. [17] and the results of Liu and Xu [6].

Definition 4.1. *Let $\mathfrak{X}$ be a non-empty set and $\alpha : \mathfrak{X} \times \mathfrak{X} \to [0,\infty)$ be a function. A mapping $F : \mathfrak{X} \to \mathfrak{X}$ is said to be an α-admissible mapping if $\alpha(\eta,\xi) \geq 1 \implies \alpha(F\eta,F\xi) \geq 1$.*

Definition 4.2. *Let $(\mathfrak{X},d_s)$ be a complete extended CbMS over $\mathcal{A}$ and $\mathcal{K}$ be the underlying solid cone in $\mathcal{A}$. A self-map F on $\mathfrak{X}$ is said to be generalized α-Lipschitz contraction if for all $\eta,\xi \in \mathfrak{X}$ with $\alpha(\eta,\xi) \geq 1$ satisfies the following:*

$$d_s(F\eta,F\xi) \preceq \kappa d_s(\eta,\xi),$$

where $\kappa \in \mathcal{K}$ is such that $r(\kappa) < \frac{1}{s(\eta,\xi)}$ and for each $\varpi_0 \in \mathfrak{X}$, $\lim_{n,m\to\infty} s(\varpi_{n+1},\varpi_m) < \dfrac{1}{\|\kappa\|}$.

The following theorem becomes special case of Theorem 3.7 if we define $\alpha : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ by $\alpha(\eta, \xi) = s(\eta, \xi) \geq 1$ for all $\eta, \xi \in \mathfrak{X}$ and take $\kappa = \varpi_1, \varpi_2 = \varpi_3 = \vartheta$.

Theorem 4.3. *Let $(\mathfrak{X}, d_s)$ be a complete extended CbMS over $\mathcal{A}$ and $\mathcal{K}$ be the associated solid cone. Let $F : \mathfrak{X} \to \mathfrak{X}$ satisfies the generalized α-Lipschitz contraction with Lipschitz constant κ such that:*

1. *F is α-admissible;*
2. *there exists $\varpi_0 \in \mathfrak{X}$ such that $\alpha(\varpi_0, F\varpi_0) \geq 1$;*
3. *F is continuous or if a sequence $\{\varpi_n\} \in \mathfrak{X}$ with $\alpha(\varpi_n, \varpi_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\varpi_n \to \varpi$ implies that for every $n \in \mathbb{N}$, $\alpha(\varpi_n, \varpi) \geq 1$.*

Then there is a point ϱ in $\mathfrak{X}$ which is fixed under the map F.

For uniqueness of this point, we use the following extra condition:

$$\forall \varrho, \zeta \in Fix(F), \text{ there exists } \eta \in \mathfrak{X} \text{ such that } \alpha(\varrho, \eta) \geq 1 \text{ and } \alpha(\zeta, \eta) \geq 1. \tag{20}$$

Theorem 4.4. *If we add the condition (20) in the assumption of Theorem 4.3, then the fixed point is unique.*

Proof. The assertion follows simply by using Theorem 4.3 and Theorem 3.9. □

Remark 4.5. *1. If we take $s(\eta, \xi) = b$ for some $b \geq 1$, then we obtain the main results due to Hussain et al. [18, Theorems 3.1 and 3.2].*
2. Results due to in Malhotra et al. [16, Theorems 3.1, 3.2 and 3.5] become special cases of Theorems 4.3 and 4.4 for $s(\eta, \xi) = 1$, $\varpi_1 = 1$ and $\varpi_2 = \varpi_3 = \vartheta$.
3. Results due to Malhotra et al. [17, Theorems 3.1, 3.2 and 3.3] become special cases of Theorems 4.3 and 4.4 for $s(\eta, \xi) = 1$, $\varpi_1 = \vartheta$ and $\varpi_2 = \varpi_3$.

If the given extended *CbMS* over $\mathcal{A}$ is a partially ordered, then we can use the following theorem.

Theorem 4.6. *Let $s : \mathfrak{X} \times \mathfrak{X} \to [1, \infty)$ be a map for a partially ordered set $(\mathfrak{X}, \trianglerighteq)$. Let $(\mathfrak{X}, d_s)$ be a complete extended CbMS over $\mathcal{A}$ with underlying solid cone $\mathcal{K}$. Assume a self-map F on $\mathfrak{X}$ which is non-decreasing with respect to $\trianglerighteq$ and satisfies the following conditions:*

(1) *there exists vectors $\varpi_1, \varpi_2, \varpi_3 \in \mathcal{K}$ such that $2s(\eta, \xi)r(\varpi_1) + (s(\eta, \xi) + 1)r(\varpi_2 + \varpi_3) < 2$, $d_s(F\eta, F\xi) \preceq \varpi_1 d_s(\eta, \xi) + \varpi_2 d_s(\eta, F\eta) + \varpi_3 d_s(\xi, F\xi)$ for all $\eta, \xi \in \mathfrak{X}$ with $\eta \trianglerighteq \xi$ and for each $u_0 \in \mathfrak{X}$ with $u_n = F^n u_0$,*

$$\lim_{n,m\to\infty} s(u_{n+1}, x_m) < \frac{1}{\|\kappa\|} \text{ where } \kappa = (2e - \varpi)^{-1}(2\varpi_1 + \varpi) \text{ for } \varpi = \varpi_2 + \varpi_3;$$

(2) *$\exists\, \varpi_0 \in \mathfrak{X}$ such that $\varpi_0 \trianglerighteq F\varpi_0$;*

(3) *F is continuous or if $\{\varpi_n\}$ is a non-decreasing sequence in $\mathfrak{X}$ with respect to $\trianglerighteq$ such that $\varpi_n \to \varpi \in \mathfrak{X}$ as $(n \to \infty)$, then $\varpi_n \trianglerighteq \varpi$ for all $n \in \mathbb{N}$.*

Then there exists a point ϱ in $\mathfrak{X}$ which is fixed under the map F.

Proof. Define a function $\alpha : \mathfrak{X} \times \mathfrak{X} \to [0, \infty)$ by

$$\alpha(\eta, \xi) = \begin{cases} s(\eta, \xi) & \text{if } \eta \trianglerighteq \xi; \\ 0 & \text{elsewhere.} \end{cases}$$

By condition (1), The mapping F is a generalized Reich type contraction. Since F is non-decreasing mapping, so F is a generalized α-admissible mapping. Definition of α and condition (2) implies that there exists $\varpi_0 \in \mathfrak{X}$ such that $\alpha(\varpi_0, F\varpi_0) = s(\varpi_0, F\varpi_0)$. By condition (3) we can see that either F is continuous or $(\mathfrak{X}, d_s)$ is regular. It follows that all the necessary conditions of Theorem 3.7 are satisfied, so we conclude that there exists a point in $\mathfrak{X}$ which is fixed under the map F. □

Corollary 4.7. *Let $(\mathfrak{X}, \trianglerighteq)$ be a partially ordered set and $s : \mathfrak{X} \times \mathfrak{X} \to [1, \infty)$. Let $(\mathfrak{X}, d_s)$ be a complete extended CbMS over $\mathcal{A}$ with underlying solid cone $\mathcal{K}$. Let F be a self-map on $\mathfrak{X}$ which is non-decreasing with respect to $\trianglerighteq$ and the following assumptions hold:*

(1) *there exists vectors $\kappa \in \mathcal{K}$ such that $r(\kappa) < \frac{1}{s(\eta,\xi)}$, $d_s(F\eta, F\xi) \preceq \kappa d_s(\eta, \xi)$ for all $\eta, \xi \in \mathfrak{X}$ with $\eta \trianglerighteq \xi$ and for each $u_0 \in \mathfrak{X}$ with $u_n = F^n u_0$,*

$$\lim_{n,m\to\infty} s(u_{n+1}, x_m) < \frac{1}{\|\kappa\|};$$

(2) there exists $\varpi_0 \in \mathfrak{X}$ such that $\varpi_0 \trianglerighteq F\varpi_0$;

(3) F is continuous or if $\{\varpi_n\}$ is a non-decreasing sequence in $\mathfrak{X}$ with respect to $\trianglerighteq$ such that $\varpi_n \to \varpi \in \mathfrak{X}$ as $(n \to \infty)$, then $\varpi_n \trianglerighteq \varpi$ for all $n \in \mathbb{N}$.

Then there exists a unique point ϱ in $\mathfrak{X}$ which is fixed under the map F.

Proof. The assertion follows directly if we take $\varpi_1 = \kappa$ and $\varpi_2 = \varpi_3 = \vartheta$ in Theorem 4.6.
□

Remark 4.8. *1. Theorem 4.6 reduces the main result due to Vujakovic [21, Theorem 3.6] for $s(p, q) = b$ and $b \geq 1$.*
2. Corollary 4.7 reduces to the main results due to Hussain et al. [18, Theorems 4.2 and 4.3] for $s(p, q) = b$ and $b \geq 1$.
3. Corollary 4.7 reduces to the results due to Nieto and Rodreguez-Lopez [22, Theorems 2.1 and 2.2] for $s(p, q) = 1$ and $\mathcal{A} = \mathbb{R}$.

5. Applications

Following is given a lemma which is proved for cone b-metric spaces in [23] and the proof for extended cone b-metric spaces over Banach algebras are same.

Lemma 5.1. *Let Ψ be a Lebesgue measurable function defined on $[0, 1]$ with $k \geq 1$. Then we have*

$$\left| \int_0^1 \Psi(s)ds \right|^k \leq \int_0^1 |\Psi(s)|^k ds.$$

Example 5.2. *Let $\mathcal{A} = \mathfrak{X} = C^1_{\mathbb{R}}[0, 1]$ be the space of all real valued differentiable functions with continuous derivative defined on $[0, 1]$. If we take $\mathcal{K} = \{h \in \mathcal{A} : h(a) \geq 0 : \forall a \in [0, 1]\}$, then $\mathcal{K}$ is a cone in $\mathcal{A}$. Define a map $d_s : \mathfrak{X} \times \mathfrak{X} \to \mathcal{A}$ by:*

$$d_s(\eta, \xi)(t) = \|\eta - \xi\|^p_\infty e^t.$$

Then d_s is an extended cone b-metric over $\mathcal{A}$ with $s : \mathfrak{X}\times\mathfrak{X} \to [1, \infty)$ defined as $s(\eta, \xi)(t) = \max |\eta(t)| + \max |\xi(t)| + 2^p$.

Consider the following nonlinear integral equation

$$f(t) = \int_0^1 F(t, f(\eta))ds, \tag{21}$$

where F satisfies the following:

(a) $F : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous;

(b) there exists a constant $M \in [0, \frac{1}{2})$ such that for each $f_0 \in \mathfrak{X}$ we have that: $M^p < \frac{1}{\lim_{n,m\to\infty} s(f_{n+1}, f_m)}$ and for all $t \in [0, 1]$ and $\eta, \xi \in \mathbb{R}$, $|F(t, \eta) - F(t, \xi)| \leq M|\eta - \xi|$.

Theorem 5.3. *The equation (21) has a unique solution in $\mathfrak{X} = C^1_{\mathbb{R}}$.*

Proof. To show that (21) has a unique solution, define $F : \mathfrak{X} \to \mathfrak{X}$ by

$$F(f)(t) = \int_0^1 F(t, f(s))ds.$$

By using Lemma 5.1 we have

$$\begin{aligned}
d_s(F(f), F(g))(t) &= e^t \|F(f) - F(g)\|_\infty^p \\
&= e^t \max_{0\le x\le 1} |F(f)(x) - F(g)(x)|^p \\
&= e^t \max_{0\le x\le 1} \left| \int_0^1 F(x, f(s))ds - \int_0^1 F(x, g(s))ds \right|^p \\
&= e^t \max_{0\le x\le 1} \left| \int_0^1 \left(F(x, f(s)) - F(x, g(s))\right) ds \right|^p \\
&\le e^t \max_{0\le x\le 1} \int_0^1 \left| F(x, f(s)) - F(x, g(s)) \right|^p ds \\
&\le e^t \int_0^1 (M \left| f(s) - g(s) \right|)^p ds \\
&= e^t M^p \int_0^1 \left| f(s) - g(s) \right|^p ds \\
&\le e^t M^p \max_{0\le s\le 1} \left| f(s) - g(s) \right|^p ds \\
&= M^p d_s(f, g).
\end{aligned}$$

If we take $\kappa = M^p e$, then $r(\kappa) \le \|M^p e\| = M^p < \frac{1}{\lim_{n,m\to\infty} s(f_{n+1}, f_m)}$. So all the conditions of Theorem 3.10 and thus there is a unique point in $\mathfrak{X}$ which is fixed under the map F. Equivalently, 21 has a unique solution in $\mathfrak{X} = C^1_{\mathbb{R}}$. □

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