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Conditions Under Which Convergence of a Sequence or its Certain Subsequences Follows From Deferred Cesàro Summability

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Abstract. Let $(u_n : n = 1, 2, ...)$ be a sequence of real or complex numbers. We aim in this paper to determine necessary and/or sufficient conditions under which convergence of a sequence (u_n) or its certain subsequences follows from summability by deferred Cesàro means. We also investigate the limiting behavior of deferred moving averages of (u_n). The conditions in our theorems are one-sided if (u_n) is a sequence of real numbers, and two-sided if (u_n) is a sequence of complex numbers. The theory developed in this paper should be useful for developing more interesting and useful results in connection with other sophisticated summability means as well as to extend to other spaces like ordered linear spaces.

1. Introduction

Let (u_n) be a sequence of real or complex numbers. The deferred Cesàro mean $D_n^{p,q}(u)$ of (u_n) is defined by

$$D_n^{p,q}(u) = \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} u_k, \quad n = 1, 2, ..., \tag{1}$$

where (p_n) and (q_n) are sequences of non-negative integers such that

$$p_n < q_n, \quad n = 1, 2, ..., \tag{2}$$

and

$$\lim_{n\to\infty} q_n = \infty. \tag{3}$$

Note that, if $p_n = n-1$ and $q_n = n$, we have the identity transformation and in the case $p_n = 0$ and $q_n = n$, the corresponding deferred Cesàro mean is the well known $(C, 1)$ mean.

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We say that (u_n) is deferred Cesàro summable to ℓ if

$$\lim_{n\to\infty} D_n^{p,q}(u) = \ell. \tag{4}$$

The concept of the deferred Cesàro mean is first defined by Agnew [1]. This concept has drawn more attention of mathematicians in recent years due to its applications in summability theory and approximation theory. In the year 2016, the deferred statistical convergence of sequences was introduced in [18]. Later, this concept is extended to statistical deferred Cesàro summability and statistical deferred weighted summability and extensively studied from different aspects in the papers [13, 14, 26–28]. Recently, Et et al. [7] generalized the concept of deferred Cesàro summability to Lebesgue measurable real-valued functions, and investigated the relations between the set of strong deferred Cesàro summable and μ-deferred statistical convergent functions.

The deferred Cesàro summability method is regular under the conditions (2) and (3) (see Agnew [1]). Namely, the set of conditions (2) and (3) is sufficient that every convergent sequence to be deferred Cesàro summable to the same limit. However, it is shown by the following example that convergence does not follow from deferred Cesàro summability in general.

Choose $p_n = 2n-1$ and $q_n = 4n-1$ and define the sequence (see Dutta et al. [6]) by

$$u_n = \begin{cases} 0, & n \text{ is even} \\ 1, & n \text{ is odd}. \end{cases}$$

The sequence (u_n) is deferred Cesàro summable to $1/2$, but it is not convergent in the usual sense.

The main aim of this paper is to derive converse conclusions, i.e., Tauberian results. But, this can only be true under additional assumptions, the so-called Tauberian conditions. Here, we are interested in finding Tauberian conditions on (u_n) under which the convergence of $(D_n^{p,q}(u))$ implies that of (u_n) or its certain subsequences. Also, we investigate the limiting behavior of deferred moving averages of (u_n). We present one-sided or two-sided conditions in the case of real or complex sequences, respectively. The theory in this paper is suitable to obtain analogous results for sequences in ordered linear spaces. For the special case of the Cesàro method $(C,1)$, our results contain Tauberian theorems by Móricz [21].

"Tauberian Theory" began in 1897 with Alfred Tauber's [32] two theorems for the conditional converse of Abel's theorem and has been developed over the years. Tauberian theorems for various summability methods have an extensive literature; for example, we refer to the papers [21, 29, 31] which we inspired throughout the study and the classical books [11, 16].

2. Preliminaries

Throughout this paper, we assume that (q_n) is a strictly increasing sequence of positive integers. Define $\gamma_n := [\gamma n]$ for $\gamma > 0$, where $[\gamma n]$ denotes the integral part of the product γn. We need following lemmas and definitions which are essential in the proofs of our main results.

Lemma 2.1. (i) *For $\gamma > 1$ and sufficiently large n,*

$$u_{q_n} - D_n^{p,q}(u) = \frac{q_{\gamma_n} - p_n}{q_{\gamma_n} - q_n}\left(D_n^{p,q_\gamma}(u) - D_n^{p,q}(u)\right) - \frac{1}{q_{\gamma_n} - q_n}\sum_{k=q_n+1}^{q_{\gamma_n}}(u_k - u_{q_n}). \tag{5}$$

(ii) *For $0 < \gamma < 1$ and sufficiently large n,*

$$u_{q_n} - D_n^{p,q}(u) = \frac{q_{\gamma_n} - p_n}{q_n - q_{\gamma_n}}\left(D_n^{p,q}(u) - D_n^{p,q_\gamma}(u)\right) + \frac{1}{q_n - q_{\gamma_n}}\sum_{k=q_{\gamma_n}+1}^{q_n}(u_{q_n} - u_k). \tag{6}$$

Proof. (i) By definition,

$$\begin{aligned} D_n^{p,q_\gamma}(u) &= \frac{1}{q_{\gamma_n} - p_n} \sum_{k=p_n+1}^{q_{\gamma_n}} u_k \\ &= \frac{1}{q_{\gamma_n} - p_n} \sum_{k=p_n+1}^{q_n} u_k + \frac{1}{q_{\gamma_n} - p_n} \sum_{k=q_n+1}^{q_{\gamma_n}} u_k \\ &= \frac{q_n - p_n}{q_{\gamma_n} - p_n} D_n^{p,q}(u) + \frac{1}{q_{\gamma_n} - p_n} \sum_{k=q_n+1}^{q_{\gamma_n}} u_k. \end{aligned}$$

Therefore,

$$\frac{q_{\gamma_n} - p_n}{q_{\gamma_n} - q_n} \left(D_n^{p,q_\gamma}(u) - D_n^{p,q}(u) \right) - \frac{1}{q_{\gamma_n} - q_n} \sum_{k=q_n+1}^{q_{\gamma_n}} u_k = -D_n^{p,q}(u),$$

that is equivalent to (5).

(ii) The proof of (6) is similar. □

Definition 2.2. We say that a sequence (q_n) of positive numbers is regularly varying of index $\rho \in \mathbb{R}$, in the sense of Karamata [15] if

$$\lim_{n\to\infty} \frac{q_{\gamma_n}}{q_n} = \gamma^\rho \quad \text{for every } \gamma > 0.$$

Since then, much literature have been devoted to Karamata's theory of regular variation (see [3], [23] and references therein).

One of the possible extensions of regular variation, due to Avakumović [2], is O-regular variation.

Definition 2.3. We say that a sequence (q_n) of positive numbers is O-regularly varying if

$$\limsup_{n\to\infty} \frac{q_{\gamma_n}}{q_n} < \infty \quad \text{for every } \gamma > 0.$$

The above two classes of sequences have an important role in the theory of Tauberian theorems [5, 12, 17] and in qualitative analysis of divergent sequential processes [9, 30].

Lemma 2.4. ([4]) *Let (q_n) be a non-decreasing sequence of positive numbers, then the following assertions are equivalent:*

$$\liminf_{n\to\infty} \frac{q_{\gamma_n}}{q_n} > 1 \quad (\gamma > 1), \tag{7}$$

$$\limsup_{n\to\infty} \frac{q_{\gamma_n}}{q_n} < 1 \quad (0 < \gamma < 1), \tag{8}$$

$$\liminf_{n\to\infty} \frac{q_n}{q_{\gamma_n}} > 1 \quad (0 < \gamma < 1), \tag{9}$$

$$\limsup_{n\to\infty} \frac{q_n}{q_{\gamma_n}} < 1 \quad (\gamma > 1). \tag{10}$$

The conditions (7)–(10) are satisfied by those regularly varying sequences of index $\rho > 0$ (see [4]). These conditions have been widely used in the formulation of Tauberian theorems (see e.g. [5, 8, 22] and our previous papers [24, 25]).

3. Main results

In the case of real sequences, we prove the following one-sided theorems.

First, we give necessary and sufficient Tauberian conditions under which convergence of a certain subsequence of a sequence of real numbers follows from its deferred Cesàro summability.

Theorem 3.1. *Let condition (7) be satisfied. If a sequence (u_n) of real numbers is deferred Cesàro summable to a finite limit ℓ, then*

$$\lim_{n\to\infty} u_{q_n} = \ell \tag{11}$$

if and only if

$$\limsup_{\gamma\downarrow 1}\liminf_{n\to\infty}\frac{1}{q_{\gamma_n}-q_n}\sum_{k=q_n+1}^{q_{\gamma_n}}(u_k-u_{q_n})\geq 0 \tag{12}$$

and

$$\limsup_{\gamma\uparrow 1}\liminf_{n\to\infty}\frac{1}{q_n-q_{\gamma_n}}\sum_{k=q_{\gamma_n}+1}^{q_n}(u_{q_n}-u_k)\geq 0, \tag{13}$$

in which case we necessarily have

$$\lim_{n\to\infty}\frac{1}{q_{\gamma_n}-q_n}\sum_{k=q_n+1}^{q_{\gamma_n}}(u_k-u_{q_n})=0 \tag{14}$$

for all $\gamma>1$, and

$$\lim_{n\to\infty}\frac{1}{q_n-q_{\gamma_n}}\sum_{k=q_{\gamma_n}+1}^{q_n}(u_{q_n}-u_k)=0 \tag{15}$$

for all $0<\gamma<1$.

We can reformulate conditions (12) and (13) as follows: To every $\epsilon>0$ and $\gamma_0>1$, there exist $n_0(\epsilon)>0$ and $\gamma=\gamma(\epsilon)$ with $1<\gamma<\gamma_0$ such that for every $n\geq n_0$ we have

$$\frac{1}{q_{\gamma_n}-q_n}\sum_{k=q_n+1}^{q_{\gamma_n}}(u_k-u_{q_n})\geq -\epsilon$$

and for another $1<\gamma<\gamma_0$ we have

$$\frac{1}{q_n-q_{\gamma_n^{-1}}}\sum_{k=q_{\gamma_n^{-1}}+1}^{q_n}(u_{q_n}-u_k)\geq -\epsilon,$$

where $\gamma_n^{-1}:=[\gamma^{-1}n]$.

The symmetric counterparts of conditions (12) and (13) are the following:

$$\liminf_{\gamma\downarrow 1}\limsup_{n\to\infty}\frac{1}{q_{\gamma_n}-q_n}\sum_{k=q_n+1}^{q_{\gamma_n}}(u_k-u_{q_n})\leq 0 \tag{16}$$

and

$$\liminf_{\gamma\uparrow 1}\limsup_{n\to\infty}\frac{1}{q_n-q_{\gamma_n}}\sum_{k=q_{\gamma_n}+1}^{q_n}(u_{q_n}-u_k)\leq 0, \tag{17}$$

respectively.

Theorem 3.1 remains valid if conditions (12) and (13) are replaced by (16) and (17), respectively.

Motivated by the definition of a slowly decreasing sequence (see, e.g., [21]), we say that a sequence (u_n) of real numbers is deferred slowly decreasing if

$$\lim_{\gamma\downarrow 1}\liminf_{n\to\infty}\min_{q_n<k\leq q_{\gamma_n}}(u_k-u_{q_n})\geq 0. \tag{18}$$

Using ϵ's and γ's this is: For given $\epsilon>0$, there exist $n_0=n_0(\epsilon)>0$ and $\gamma=\gamma(\epsilon)>1$ such that $u_k-u_{q_n}\geq -\epsilon$ whenever $n_0\leq n$ and $q_n<k\leq q_{\gamma_n}$.

An equivalent reformulation of (18) is the following:

$$\lim_{\gamma\uparrow 1}\liminf_{n\to\infty}\min_{q_{\gamma_n}<k\leq q_n}(u_{q_n}-u_k)\geq 0. \tag{19}$$

Conditions (12) and (13) are trivially satisfied if (u_n) is deferred slowly decreasing.

Next, we show that condition of being deferred slowly decreasing is sufficient for a deferred Cesàro summable sequence to be convergent.

Theorem 3.2. *Let condition (7) be satisfied. If a sequence (u_n) of real numbers is deferred Cesàro summable to a finite limit ℓ and deferred slowly decreasing, then (u_n) converges to ℓ.*

It is easy to verify that if the classical one-sided Tauberian condition

$$n(u_n-u_{n-1})\geq -H$$

of Landau [19] is satisfied for some $H>0$ and all $n=1,2,...$, then (u_n) is deferred slowly decreasing, provided that (q_n) is regularly varying of positive index ρ.

Indeed, in this case we have

$$u_k-u_{q_n}=\sum_{j=q_n+1}^{k}(u_j-u_{j-1})\geq -H\sum_{j=q_n+1}^{k}\frac{1}{j}\geq -H\left(\frac{k-q_n}{q_n}\right). \tag{20}$$

It follows from (20) that

$$\min_{q_n<k\leq q_{\gamma_n}}(u_k-u_{q_n})\geq -H\left(\frac{q_{\gamma_n}}{q_n}-1\right)$$

and then

$$\liminf_{n\to\infty}\min_{q_n<k\leq q_{\gamma_n}}(u_k-u_{q_n})\geq -H\left(\gamma^{\rho}-1\right).$$

Since γ can be chosen as close to 1 as we want, (18) easily follows.

In the case of complex sequences, we prove the following two-sided theorems.

First, we give necessary and sufficient Tauberian conditions under which convergence of a certain subsequence of a sequence of complex numbers follows from its deferred Cesàro summability.

Theorem 3.3. *Let condition (7) be satisfied. If a sequence (u_n) of complex numbers is deferred Cesàro summable to a finite limit ℓ, then (u_{q_n}) converges to ℓ if and only if at least one of the following two conditions is satisfied:*

$$\liminf_{\gamma\downarrow 1}\limsup_{n\to\infty}\left|\frac{1}{q_{\gamma_n}-q_n}\sum_{k=q_n+1}^{q_{\gamma_n}}(u_k-u_{q_n})\right|=0 \tag{21}$$

and

$$\liminf_{\gamma\uparrow 1}\limsup_{n\to\infty}\left|\frac{1}{q_n-q_{\gamma_n}}\sum_{k=q_{\gamma_n}+1}^{q_n}(u_{q_n}-u_k)\right|=0, \tag{22}$$

in which case we necessarily have (14) *for all* $\gamma>1$, *and* (15) *for all* $0<\gamma<1$.

We can reformulate conditions (21) and (22) as follows: To every $\epsilon>0$ and $\gamma_0>1$, there exist $n_0(\epsilon)>0$ and $\gamma=\gamma(\epsilon)$ with $1<\gamma<\gamma_0$ such that for every $n\geq n_0$ we have

$$\left|\frac{1}{q_{\gamma_n}-q_n}\sum_{k=q_n+1}^{q_{\gamma_n}}(u_k-u_{q_n})\right|\leq\epsilon$$

and for another $1<\gamma<\gamma_0$ we have

$$\left|\frac{1}{q_n-q_{\gamma_n^{-1}}}\sum_{k=q_{\gamma_n^{-1}}+1}^{q_n}(u_{q_n}-u_k)\right|\leq\epsilon,$$

where $\gamma_n^{-1}:=[\gamma^{-1}n]$.

The symmetric counterparts of conditions (21) and (22) are the following:

$$\limsup_{\gamma\downarrow 1}\liminf_{n\to\infty}\left|\frac{1}{q_{\gamma_n}-q_n}\sum_{k=q_n+1}^{q_{\gamma_n}}(u_k-u_{q_n})\right|=0 \tag{23}$$

and

$$\limsup_{\gamma\uparrow 1}\liminf_{n\to\infty}\left|\frac{1}{q_n-q_{\gamma_n}}\sum_{k=q_{\gamma_n}+1}^{q_n}(u_{q_n}-u_k)\right|=0, \tag{24}$$

respectively.

Theorem 3.3 remains valid if conditions (21) and (22) are replaced by (23) and (24), respectively.

Motivated by the definition of a slowly oscillating sequence (see, e.g., [21]), we say that a sequence (u_n) of complex numbers is deferred slowly oscillating if

$$\lim_{\gamma\downarrow 1}\limsup_{n\to\infty}\max_{q_n<k\leq q_{\gamma_n}}|u_k-u_{q_n}|=0. \tag{25}$$

Using ϵ's and γ's this is: For given $\epsilon>0$, there exist $n_0=n_0(\epsilon)>0$ and $\gamma=\gamma(\epsilon)>1$ such that $|u_k-u_{q_n}|\leq\epsilon$ whenever $n_0\leq n$ and $q_n<k\leq q_{\gamma_n}$.

An equivalent reformulation of (25) is the following:

$$\lim_{\gamma\uparrow 1}\limsup_{n\to\infty}\min_{q_{\gamma_n}<k\leq q_n}|u_{q_n}-u_k|=0. \tag{26}$$

Conditions (21) and (22) are trivially satisfied if (u_n) is deferred slowly oscillating.

Next, we show that condition of being deferred slowly oscillating is sufficient for a deferred Cesàro summable sequence to be convergent.

Theorem 3.4. *Let condition (7) be satisfied. If a sequence (u_n) of complex numbers is deferred Cesàro summable to a finite limit ℓ and deferred slowly oscillating, then (u_n) converges to ℓ.*

If the classical two-sided Tauberian condition

$$n|u_n - u_{n-1}| \leq H$$

of Hardy [10] is satisfied for some $H > 0$ and all $n = 1, 2, ...$, then (u_n) is deferred slowly oscillating, provided that (q_n) is regularly varying of positive index ρ.

Indeed, in this case we have

$$|u_k - u_{q_n}| \leq \sum_{j=q_n+1}^{k} |u_j - u_{j-1}| \leq H \sum_{j=q_n+1}^{k} \frac{1}{j} \leq H\left(\frac{k - q_n}{q_n}\right). \tag{27}$$

It follows from (27) that

$$\max_{q_n<k\leq q_{\gamma_n}} |u_k - u_{q_n}| \leq H\left(\frac{q_{\gamma_n}}{q_n} - 1\right)$$

and then

$$\limsup_{n\to\infty} \max_{q_n<k\leq q_{\gamma_n}} |u_k - u_{q_n}| \leq H\left(\gamma^\rho - 1\right).$$

Since γ can be chosen as close to 1 as we want, (25) easily follows.

In the special case of $p_n = 0$ and $q_n = n$, we have $(C, 1)$ summability of (u_n). In this case, Theorems 3.1 and 3.3 were proved in [21].

Furthermore, we determine the limiting behavior of deferred moving averages of complex sequences.

Theorem 3.5. *Let condition (7) be satisfied. If a sequence (u_n) of complex numbers is deferred Cesàro summable to a finite limit ℓ, then for each $\gamma > 1$,*

$$\lim_{n\to\infty} \frac{1}{q_{\gamma_n} - q_n} \sum_{k=q_n+1}^{q_{\gamma_n}} u_k = \ell, \tag{28}$$

and for each $0 < \gamma < 1$,

$$\lim_{n\to\infty} \frac{1}{q_n - q_{\gamma_n}} \sum_{k=q_{\gamma_n}+1}^{q_n} u_k = \ell. \tag{29}$$

4. Proofs of Theorems

Proof of Theorem 3.1. Necessity. It follows from (4) and (11) that

$$\lim_{n\to\infty} \left(u_{q_n} - D_n^{p,q}(u)\right) = 0. \tag{30}$$

Assuming (7) we have

$$\begin{aligned}
\limsup_{n\to\infty} \frac{q_{\gamma_n} - p_n}{q_{\gamma_n} - q_n} &\leq \limsup_{n\to\infty} \frac{q_{\gamma_n}}{q_{\gamma_n} - q_n} \\
&= \left\{\liminf_{n\to\infty}\left(1 - \frac{q_n}{q_{\gamma_n}}\right)\right\}^{-1} \\
&= \left\{1 - \left(\liminf_{n\to\infty} \frac{q_{\gamma_n}}{q_n}\right)^{-1}\right\}^{-1} < \infty.
\end{aligned} \tag{31}$$

By (4) and (31), for each $\gamma > 1$,

$$\lim_{n\to\infty} \frac{q_{\gamma_n} - p_n}{q_{\gamma_n} - q_n} \left(D_n^{p,q_\gamma}(u) - D_n^{p,q}(u)\right) = 0. \tag{32}$$

The same holds for each $0 < \gamma < 1$. Then, (14) (respectively (15)) is obtained from (5) (respectively (6)), (30) and (32).

Sufficiency. Assume that (4), (12) and (13) are satisfied. It follows from (12) that there exists a sequence $\gamma_j \downarrow 1$ satisfying

$$\lim_{j\to\infty} \liminf_{n\to\infty} \frac{1}{q_{\gamma_{jn}} - q_n} \sum_{k=q_n+1}^{q_{\gamma_{jn}}} (u_k - u_{q_n}) \geq 0, \tag{33}$$

where $\gamma_{jn} := [\gamma_j n]$.

By (5), we have

$$\limsup_{n\to\infty} \left(u_{q_n} - D_n^{p,q}(u)\right) \leq \lim_{j\to\infty} \limsup_{n\to\infty} \frac{q_{\gamma_{jn}} - p_n}{q_{\gamma_{jn}} - q_n} \left(D_n^{p,q_{\gamma_j}}(u) - D_n^{p,q}(u)\right) + \lim_{j\to\infty} \limsup_{n\to\infty} \left(-\frac{1}{q_{\gamma_{jn}} - q_n} \sum_{k=q_n+1}^{q_{\gamma_{jn}}} (u_k - u_{q_n})\right).$$

Considering (4), (32) and (33), we obtain

$$\limsup_{n\to\infty} \left(u_{q_n} - D_n^{p,q}(u)\right) = -\lim_{j\to\infty} \liminf_{n\to\infty} \left(\frac{1}{q_{\gamma_{jn}} - q_n} \sum_{k=q_n+1}^{q_{\gamma_{jn}}} (u_k - u_{q_n})\right) \leq 0. \tag{34}$$

It follows from (13) that for some sequence $\gamma_j \uparrow 1$, we have

$$\lim_{j\to\infty} \liminf_{n\to\infty} \frac{1}{q_n - q_{\gamma_{jn}}} \sum_{k=q_{\gamma_{jn}}+1}^{q_n} (u_{q_n} - u_k) \geq 0.$$

In a similar way, we find that

$$\liminf_{n\to\infty} \left(u_{q_n} - D_n^{p,q}(u)\right) \geq \lim_{j\to\infty} \liminf_{n\to\infty} \frac{q_{\gamma_{jn}} - p_n}{q_n - q_{\gamma_n}} \left(D_n^{p,q}(u) - D_n^{p,q_\gamma}(u)\right) + \lim_{j\to\infty} \liminf_{n\to\infty} \frac{1}{q_n - q_{\gamma_{jn}}} \sum_{k=q_{\gamma_{jn}}+1}^{q_n} (u_{q_n} - u_k) \geq 0. \tag{35}$$

Combining (34) and (35) yields (30), which implies (11) since (4). □

Proof of Theorem 3.2. Assume (18) is satisfied, then so is (19). It is clear that conditions (18) and (19) imply (12) and (13), respectively. Then, from Theorem 3.1, we have convergence of (u_{q_n}) to ℓ: For given $\epsilon > 0$, there exists $N = N(\epsilon) > 0$ such that

$$-\frac{\epsilon}{2} \leq u_{q_n} - \ell \leq \frac{\epsilon}{2} \tag{36}$$

whenever $n \geq N$.

It follows from the equivalent form of (18) that for given $\epsilon > 0$, there exist $n_0 = n_0(\epsilon) > 0$ and $\gamma = \gamma(\epsilon) > 1$ such that

$$u_k - u_{q_n} \geq -\frac{\epsilon}{2} \tag{37}$$

whenever $n \geq n_0$ and $q_n < k \leq q_{\gamma_n}$.

It follows from the equivalent form of (19) that for given $\epsilon > 0$, there exist $n_1 = n_1(\epsilon) > 0$ and $0 < \gamma = \gamma(\epsilon) < 1$ such that

$$u_{q_n} - u_k \geq -\frac{\epsilon}{2} \tag{38}$$

whenever $n \geq n_1$ and $q_{\gamma_n} < k \leq q_n$.

Taking (36) and (37) into account, we have

$$u_k - \ell = u_k - u_{q_n} + u_{q_n} - \ell \geq -\frac{\epsilon}{2} - \frac{\epsilon}{2} = -\epsilon \tag{39}$$

whenever $k \geq n \geq N_1 = \max\{n_0, N\}$.

Also, taking (36) and (38) into account, we have

$$u_k - \ell = u_k - u_{q_n} + u_{q_n} - \ell \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{40}$$

whenever $k \geq n \geq N_2 = \max\{n_1, N\}$.

By (39) and (40), we have for given $\epsilon > 0$ there exists $N_3 = \max\{N_1, N_2\}$ such that

$$-\epsilon \leq u_n - \ell \leq \epsilon$$

whenever $n \geq N_3$. This completes the proof. □

Proof of Theorem 3.3. Necessity. The proof runs along similar lines to the proof of the necessity part in Theorem 3.1.

Sufficiency. Assume that (4) and (21) are satisfied. It follows from (21) that there exists a sequence $\gamma_j \downarrow 1$ satisfying

$$\lim_{j\to\infty} \limsup_{n\to\infty} \left| \frac{1}{q_{\gamma_{jn}} - q_n} \sum_{k=q_n+1}^{q_{\gamma_{jn}}} (u_k - u_{q_n}) \right| = 0. \tag{41}$$

By (5), we have

$$\limsup_{n\to\infty} \left| u_{q_n} - D_n^{p,q}(u) \right| \leq \lim_{j\to\infty} \limsup_{n\to\infty} \frac{q_{\gamma_{jn}} - p_n}{q_{\gamma_{jn}} - q_n} \left| D_n^{p,q_{\gamma_j}}(u) - D_n^{p,q}(u) \right| + \lim_{j\to\infty} \limsup_{n\to\infty} \left| \frac{1}{q_{\gamma_{jn}} - q_n} \sum_{k=q_n+1}^{q_{\gamma_{jn}}} (u_k - u_{q_n}) \right|.$$

Taking (4), (32) and (41) into account, we obtain

$$\limsup_{n\to\infty} \left| u_{q_n} - D_n^{p,q}(u) \right| \leq \lim_{j\to\infty} \limsup_{n\to\infty} \left| \frac{1}{q_{\gamma_{jn}} - q_n} \sum_{k=q_n+1}^{q_{\gamma_{jn}}} (u_k - u_{q_n}) \right| = 0,$$

which concludes the proof of convergence of (u_{q_n}) to ℓ.

A similar proof can be given if (22) is satisfied. □

Proof of Theorem 3.4. Assume that (u_n) is deferred Cesàro summable to ℓ and condition (25) is satisfied. By Theorem 3.2, we have convergence of (u_{q_n}) to ℓ : For given $\epsilon > 0$ there exists $N = N(\epsilon) > 0$ such that

$$|u_{q_n} - \ell| \leq \frac{\epsilon}{2} \tag{42}$$

whenever $n \geq N$.

It follows from the equivalent form of (25) that for given $\epsilon > 0$, there exist $n_0 = n_0(\epsilon) > 0$ and $\gamma = \gamma(\epsilon) > 1$ such that

$$|u_k - u_{q_n}| \leq \frac{\epsilon}{2} \tag{43}$$

whenever $n \geq n_0$ and $q_n < k \leq q_{\gamma_n}$.

Taking (42) and (43) into account, we have

$$|u_k - \ell| \leq |u_k - u_{q_n}| + |u_{q_n} - \ell| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

whenever $k \geq n \geq N_1 = \max\{n_0, N\}$.

A similar proof can be given if (26) is satisfied. □

Proof of Theorem 3.5. If $\gamma > 1$ and n is large enough such that $q_{\gamma_n} > q_n$, then from Lemma 2.1

$$\frac{1}{q_{\gamma_n} - q_n} \sum_{k=q_n+1}^{q_{\gamma_n}} u_k = D_n^{p,q}(u) + \frac{q_{\gamma_n} - p_n}{q_{\gamma_n} - q_n} \left(D_n^{p,q_\gamma}(u) - D_n^{p,q}(u) \right).$$

Thus (28) is obtained from (31) and the assumed convergence of $(D_n^{p,q}(u))$.

If $0 < \gamma < 1$ and n is large enough such that $q_n > q_{\gamma_n}$, then from Lemma 2.1

$$\frac{1}{q_n - q_{\gamma_n}} \sum_{k=q_{\gamma_n}+1}^{q_n} u_k = D_n^{p,q}(u) + \frac{q_{\gamma_n} - p_n}{q_n - q_{\gamma_n}} \left(D_n^{p,q}(u) - D_n^{p,q_\gamma}(u) \right).$$

By (8) we have

$$\begin{aligned}
\limsup_{n\to\infty} \frac{q_{\gamma_n} - p_n}{q_n - q_{\gamma_n}} &\leq \limsup_{n\to\infty} \frac{q_{\gamma_n}}{q_n - q_{\gamma_n}} \\
&= \left\{ \liminf_{n\to\infty} \left(\frac{q_n}{q_{\gamma_n}} - 1 \right) \right\}^{-1} \\
&= \left\{ \left(\limsup_{n\to\infty} \frac{q_{\gamma_n}}{q_n} \right)^{-1} - 1 \right\}^{-1} < \infty. \qquad (44)
\end{aligned}$$

Thus (29) is obtained from (44) and the assumed convergence of $(D_n^{p,q}(u))$. □

Following Maddox [20], the theory developed in Section 3 can be extended to sequences in ordered linear spaces. We leave this discussion to the readers.

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